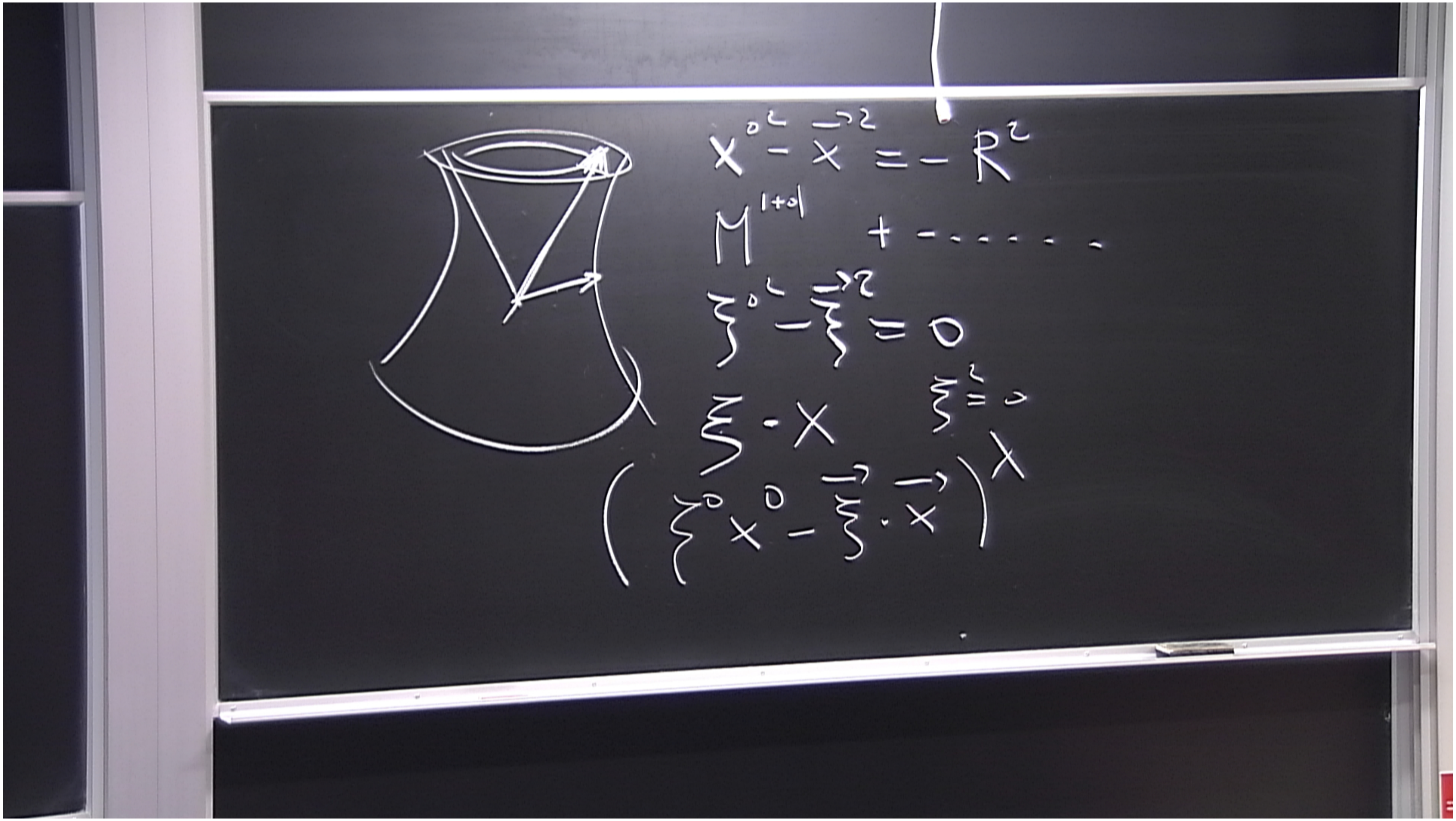
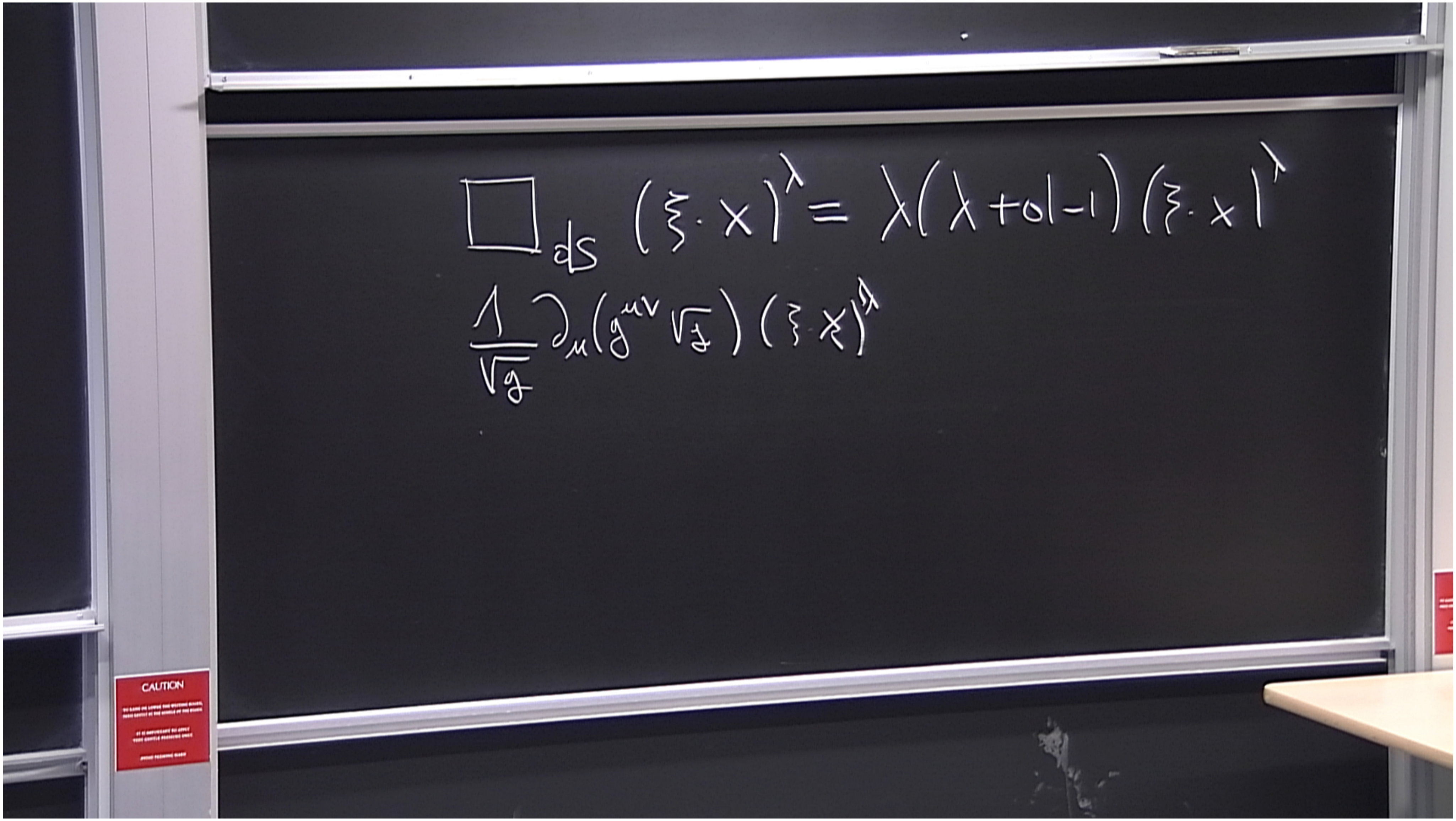
Title: Topics in QFT on Flat and Curved Spacetimes - Lecture 10

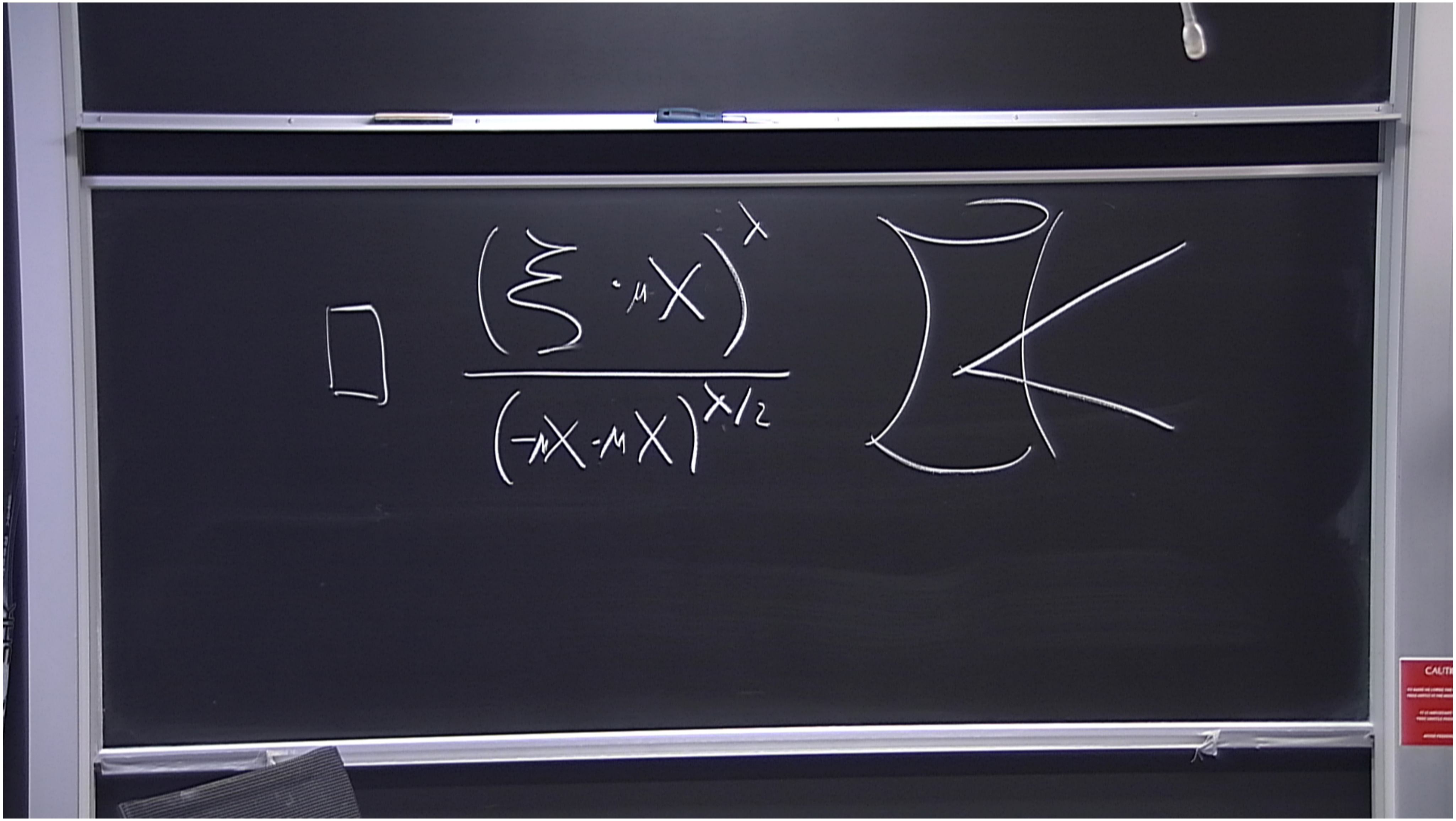
Date: Oct 23, 2013 10:00 AM

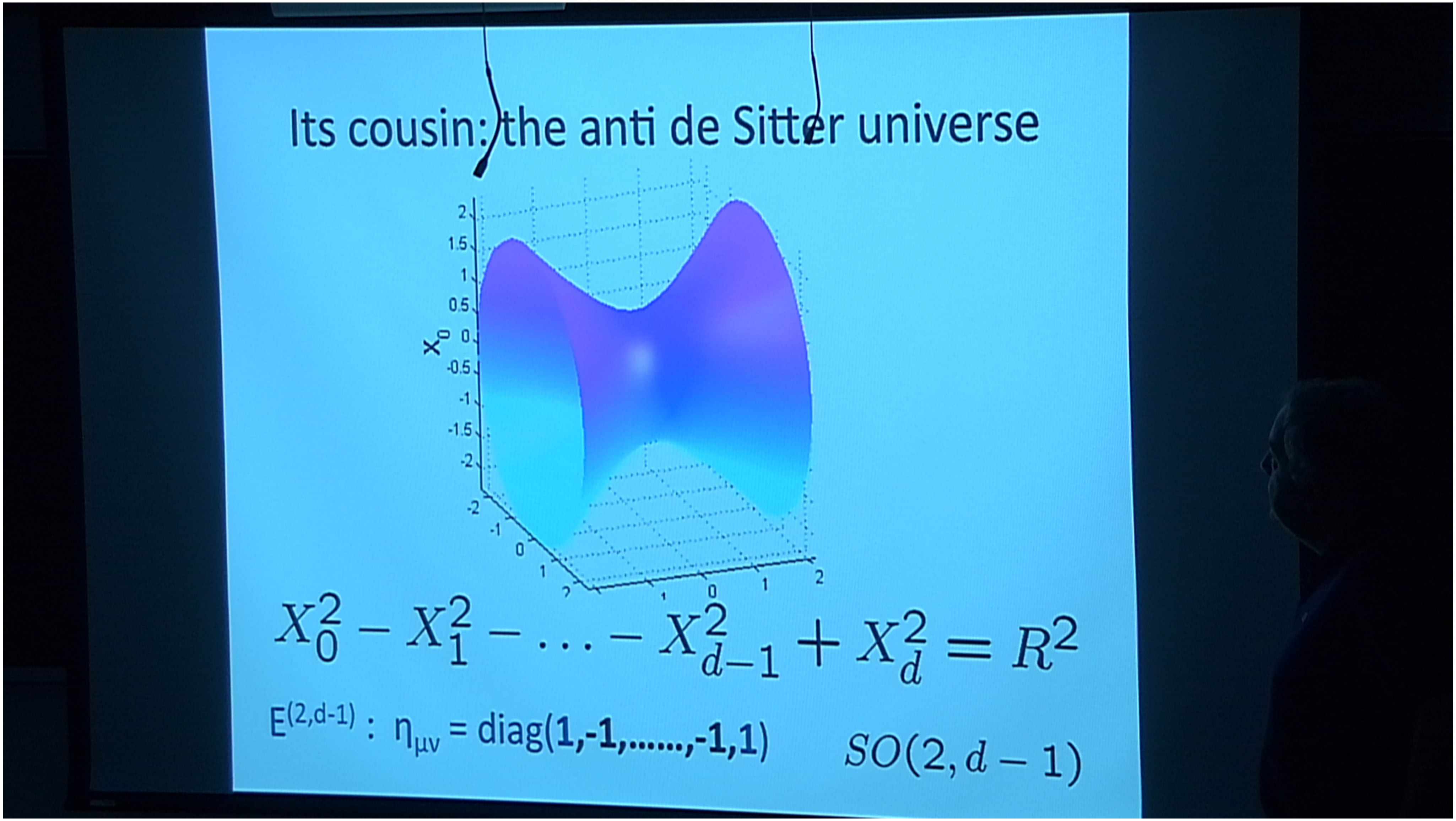
URL: http://pirsa.org/13100018

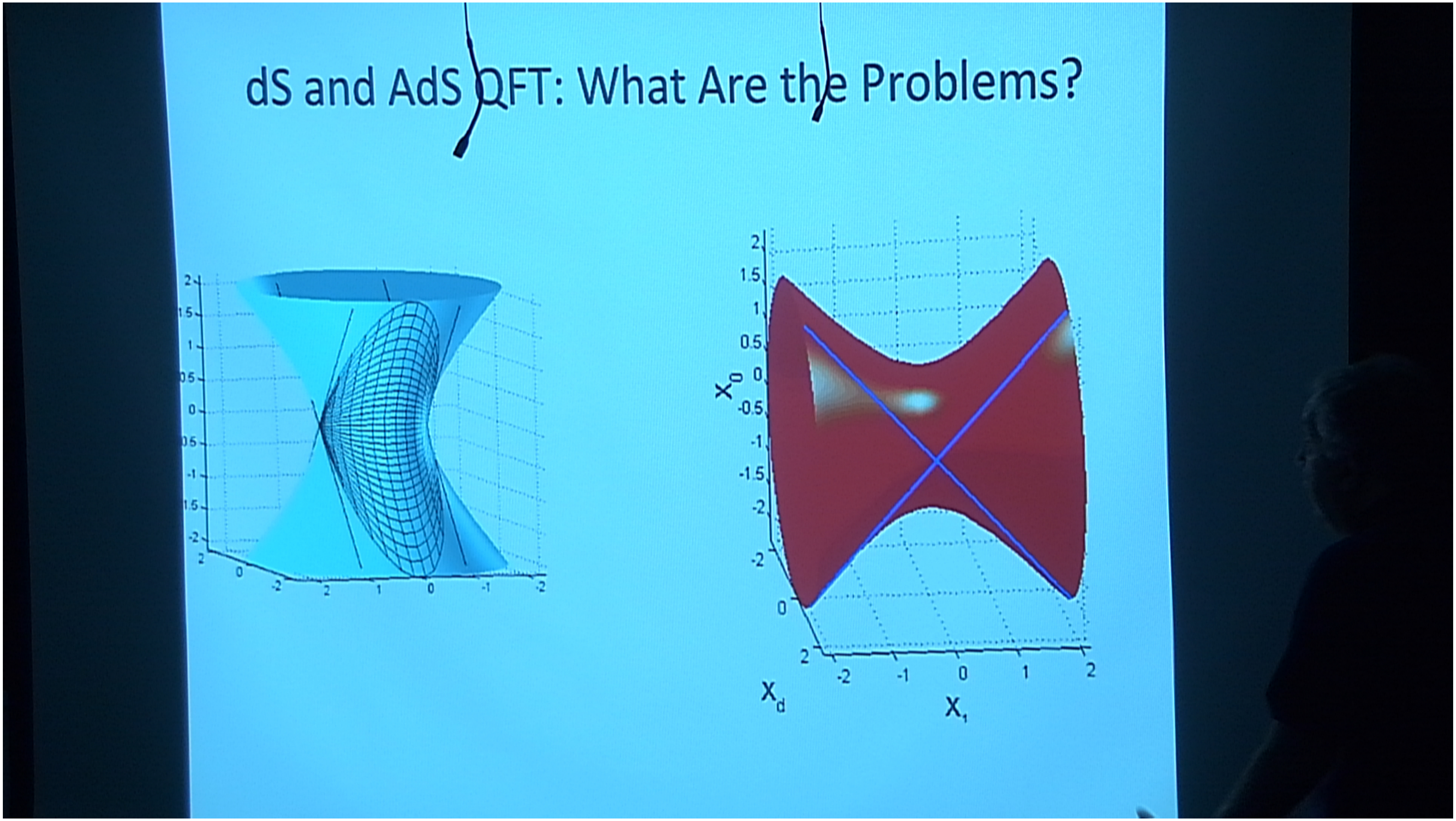
Abstract:


X^0² − X⃗² = −R²
M^{1+d} + ...
ξ^0² − ξ⃗² = 0
ξ · X
ξ² = 0
(ξ^0 X^0 − ξ⃗ · X⃗)^λ

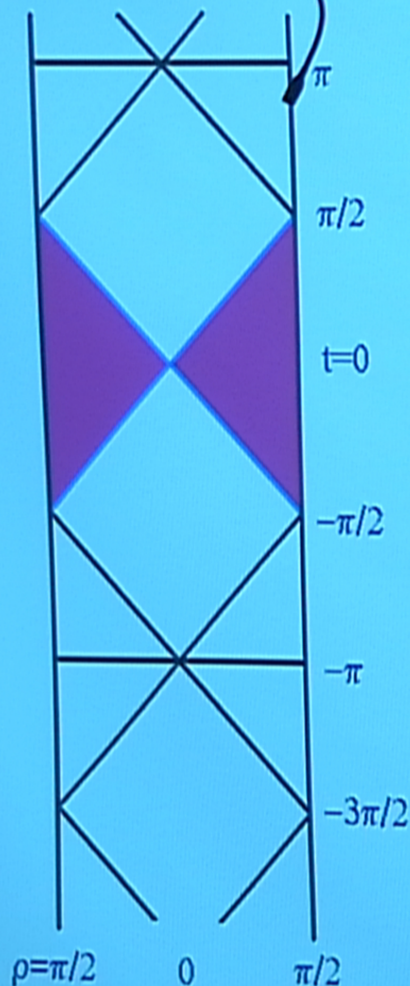

$\Box_{dS} (\xi \cdot x)^\lambda = \lambda(\lambda + d - 1)(\xi \cdot x)^\lambda$
$\frac{1}{\sqrt{g}} \partial_\mu (g^{\mu\nu} \sqrt{g}) (\xi \cdot x)^\lambda$
CAUTION

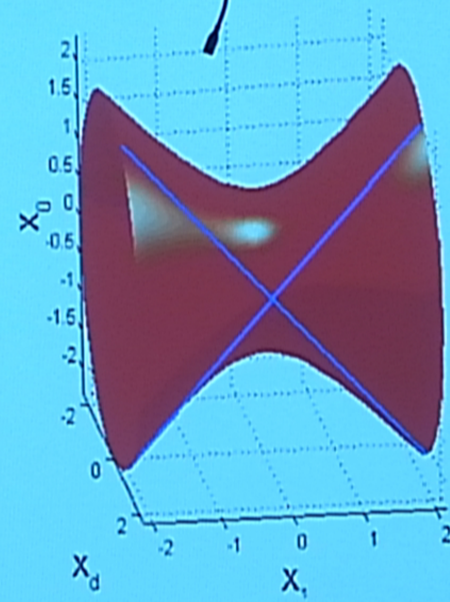

$\Box \; \frac{(\xi \cdot_{\mu} X)^{\Delta}}{(-_{\mu}X \cdot_{\mu} X)^{\Delta/2}}$


Its cousin: the anti de Sitter universe
X_0
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
0
1
2
$X_0^2 - X_1^2 - \ldots - X_{d-1}^2 + X_d^2 = R^2$
$E^{(2,d-1)}$: $\eta_{\mu\nu}$ = diag(**1,-1,……,-1,1**)
$SO(2, d-1)$


dS and AdS QFT: What Are the Problems?
X0
Xd
X1

Penrose Diagrams



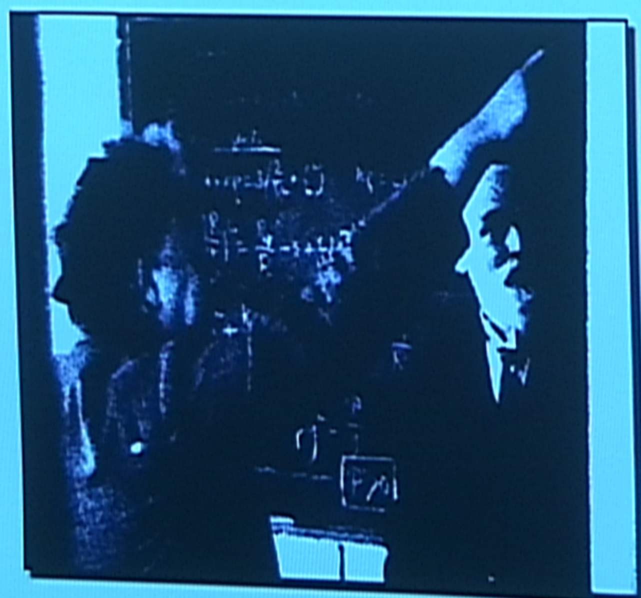


$t \in \mathbb{R}$

$$\begin{cases} X_0 = \frac{\sin t}{\cos \rho} & 0 \le \rho < \frac{\pi}{2} \\ X_i = \tan \rho\, \omega_i & |\vec{\omega}|^2 = 1 \\ X_d = \frac{\cos t}{\cos \rho} & \end{cases}$$

The truly revolutionary year 1917: The birth of cosmology as a science

1916-1918: the famous debate between Einstein, De Sitter, Weyl and Klein over the relativity of inertia.



1917: The birth of cosmology as a science

The first relativistic cosmological models were proposed in this debate :

- The "Einstein cylinder world" filled with a uniform static mass distribution

  A. Einstein, Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie. K. Preuss. Akad. Wiss. Sitz. 142–152 (1917)

- The empty "de Sitter hyperboloid world" (name given by Weyl in his 1923 book).

  W. de Sitter "On the Relativity of Inertia. Remarks Concerning Einstein's Latest Hypothesis." Koninklijke Akademie van Wetenschappen te Amsterdam. Section of Sciences. Proceedings 19 (1916–17): 1217–1225.

"The rough and winding road"

Einstein's requirements:

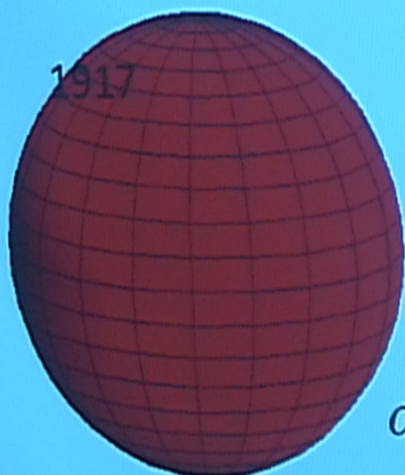
1) The universe is static
2) The metric structure is fully determined by the matter content (Einstein's Mach principle): *"In a consistent theory of relativity there can be no inertia relatively to 'space,' but only an inertia of masses relatively to one another"* A. Einstein. Ibidem (1917)

- Boundary conditions are a remnant of Newton's absolute space and violate Mach's principle. How to get rid of them?

A (spatially) spherical universe!

Einstein's crazy idea: *"I have again perpetrated something relating to the theory of gravitation that might endanger me of being committed to a madhouse".*

Letter to P. Ehrenfest, 4 February 1917



$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2$$

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2\Big|_{\mathbf{S}^3}$$

$$= \sum_{i,j=1}^{3} \left(\delta_{ij} + \frac{x_i x_j}{a^2 - x_1^2 - x_2^2 - x_3^2} \right) dx^i dx^j$$

The Cosmological Constant

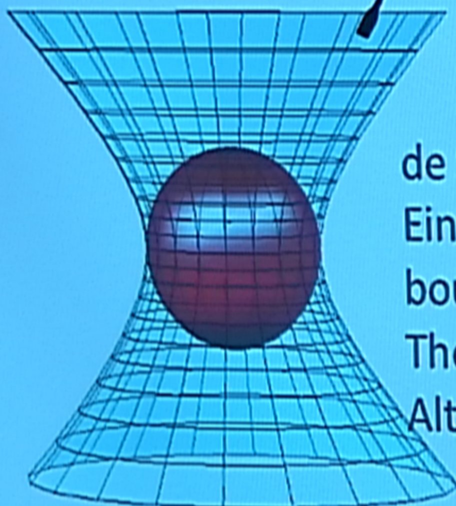
"The history of science provides many instances of discoveries which have been made for reasons which are no longer considered satisfactory. It may be that the discovery of the cosmological constant is such a case."

George E. Lemaître, article in the book "Albert Einstein: Philosopher–Scientist", 1949

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad (1915 - 1916)$$

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad (1917)$$

de Sitter's analogy



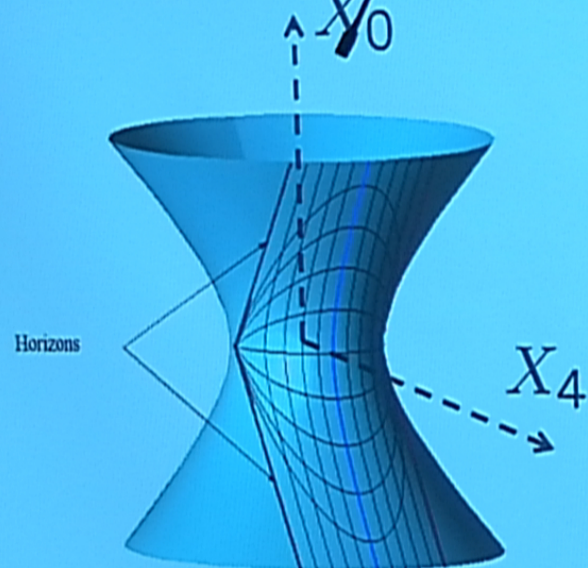
de Sitter says:
Einstein's static model does not solve the boundary condition problem in an invariant way. There is a sort of 'absolute space' remaining in it.
Alternative proposal:

$$(iX_0)^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = R^2$$

$$ds^2 = dX_0^2 - dX_1^2 - dX_2^2 - dX_3^2 - dX_4^2\Big|_{dS_4} =$$

$$= \sum_{\mu,\nu=0}^{3} \left(\eta_{\mu\nu} - \frac{x_\mu x_\nu}{R^2 - X_1^2 - X_2^2 - X_3^2 + X_0^2} \right) dX^\mu dX^\nu$$

Static BH coordinates (de Sitter 1917)



$$
\begin{cases}
X_0 = \sqrt{R^2 - r^2}\,\sinh(t/R) \\
X_1 = r \sin\theta \sin\phi \\
X_2 = r \sin\theta \cos\phi \\
X_3 = r \cos\theta \\
X_4 = \sqrt{R^2 - r^2}\,\cosh(t/R)
\end{cases}
$$

$$
ds^2 = dX_0^2 - dX_1^2 - dX_2^2 - dX_3^2 - dX_4^2\Big|_{dS_4} =
$$

$$
= (1 - \frac{r^2}{R^2})dt^2 - \frac{1}{1-\frac{r^2}{R^2}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)
$$

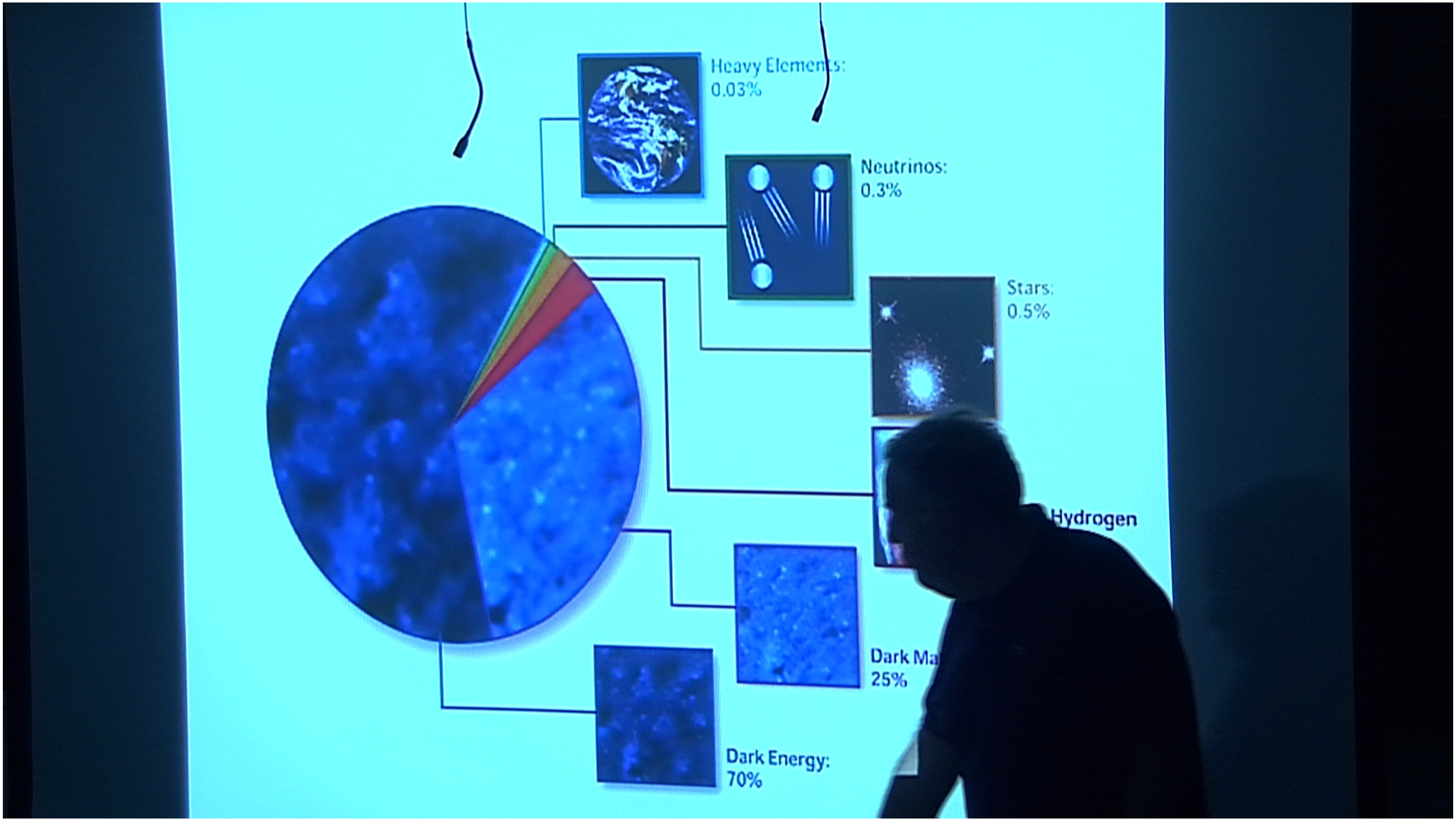
Einstein was not happy!

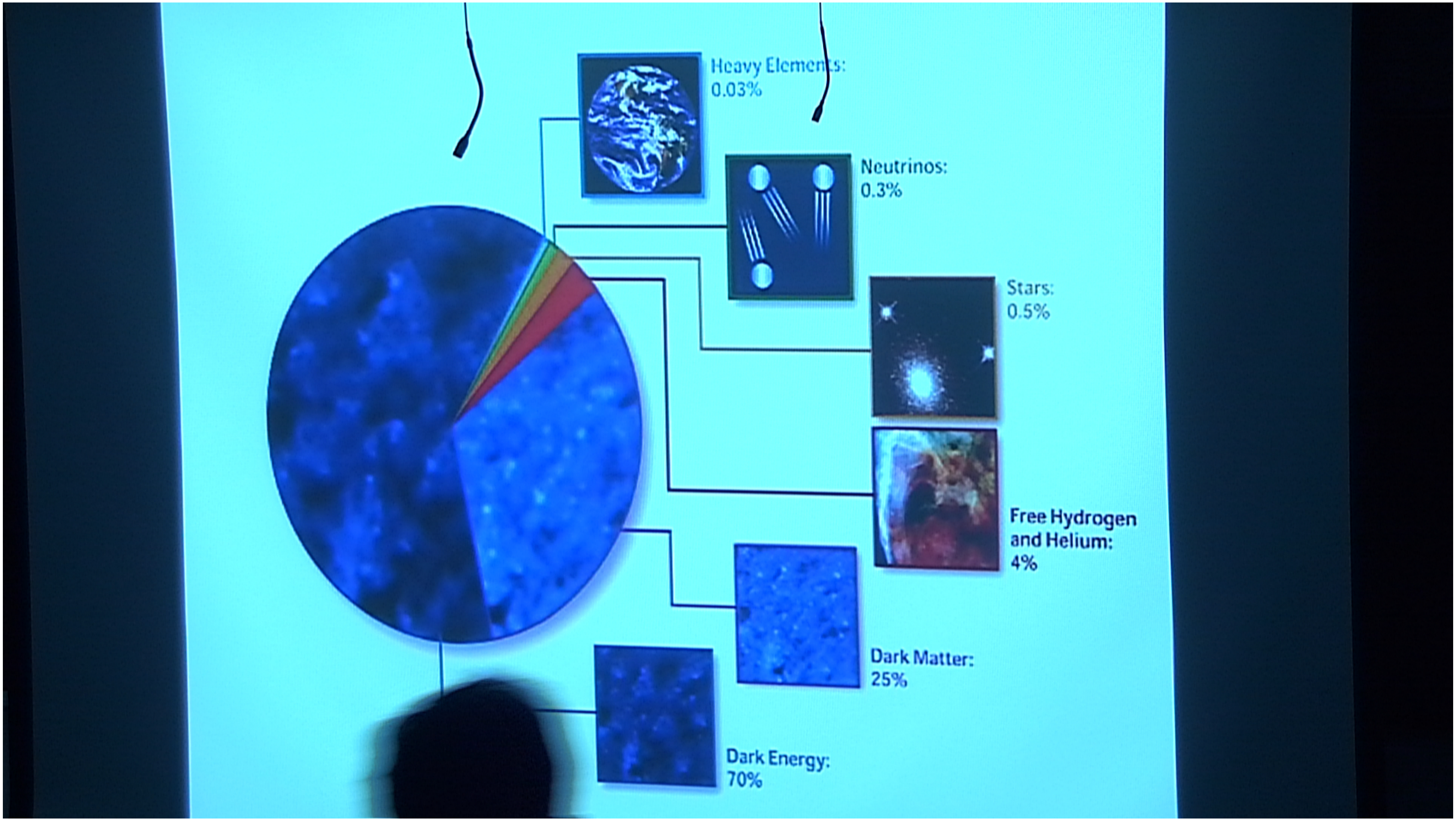
- The de Sitter metric **is a vacuum solution of the new equations**; it is a counterexample to Einstein's requirements
- It is anti-Machian (but more 'relativistic' than Einstein's own solution)
- Einstein's strategy: try to kill it (*it is wrong, it is singular, it is not empty*)

$$ds^2 = (1-\frac{r^2}{R^2})dt^2 - \frac{1}{1-\frac{r^2}{R^2}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

End of the *querelle*

- In the course of the debate, Einstein, de Sitter and even Weyl make several (interesting) mistakes, also due to the many possible coordinate system.
- Klein finally solves the problem: the de Sitter solution is a fully regular solution of the new Einstein's cosmological equations *in vacuo*.
- Einstein criticisms were all wrong but in the end he dismisses the de Sitter metric as unphysical because non globally static.


Heavy Elements:
0.03%
Neutrinos:
0.3%
Stars:
0.5%
Hydrogen
Dark Ma
25%
Dark Energy:
70%

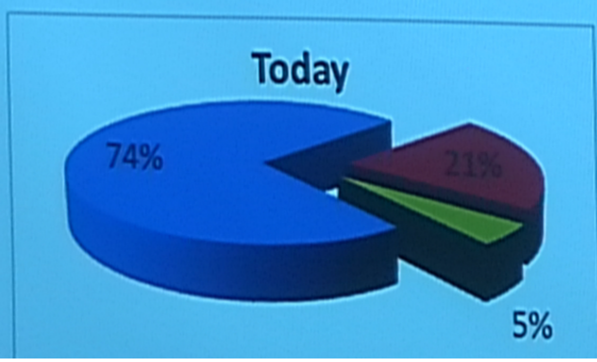

Heavy Elements:
0.03%
Neutrinos:
0.3%
Stars:
0.5%
Free Hydrogen
and Helium:
4%
Dark Matter:
25%
Dark Energy:
70%

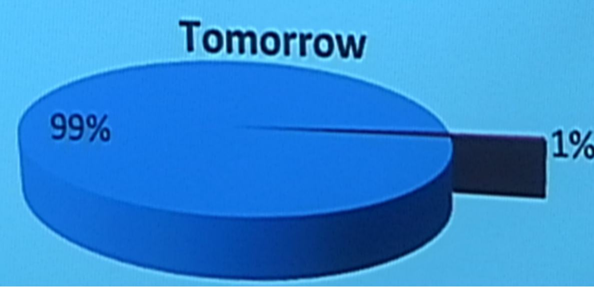
The cosmological constant now and forever

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G(\rho_M + \rho_R + \rho_\Lambda) - \frac{K}{a^2}$$

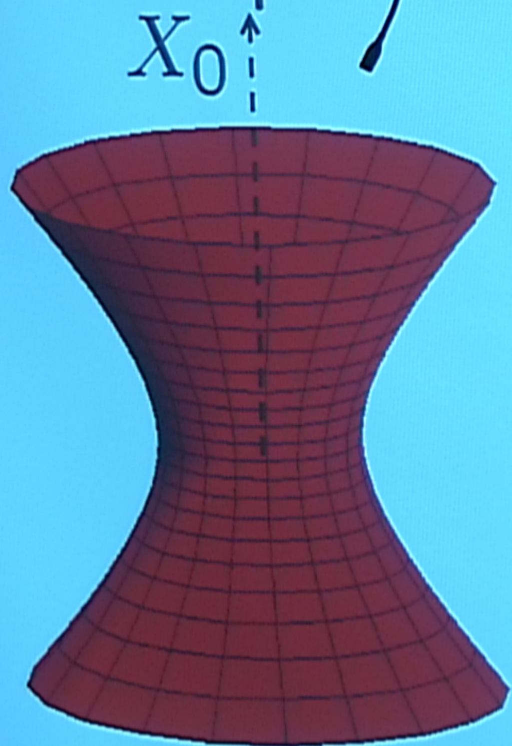
$$p_M = 0, \quad p_R = \frac{1}{3}\rho_R, \quad p_\Lambda = -\rho_\Lambda,$$

$$H^2 = \frac{\Omega_M}{a(t)^3} + \frac{\Omega_R}{a(t)^4} + \Omega_\Lambda + \frac{\Omega_K}{a(t)^2}$$





Spherical de Sitter model



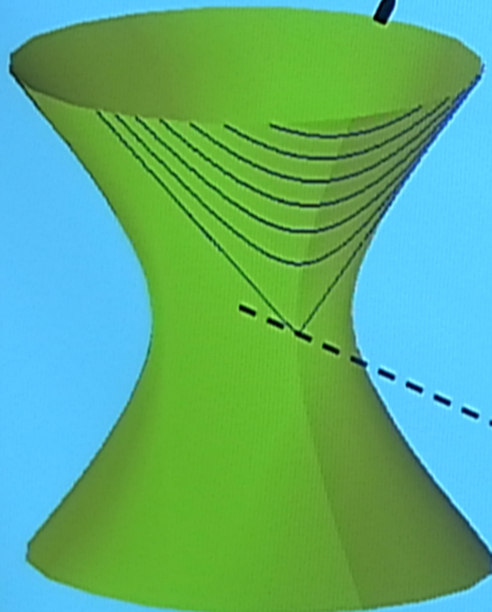
$$\begin{cases} X_0 &= R\sinh(t/R) \\ X_1 &= R\cosh(t/R)\sin\theta\sin\chi\sin\phi \\ X_2 &= R\cosh(t/R)\sin\theta\sin\chi\cos\phi \\ X_3 &= R\cosh(t/R)\sin\theta\cos\chi \\ X_4 &= R\cosh(t/R)\cos\theta \end{cases}$$

$$R = \sqrt{\frac{3}{\Lambda}}$$

$$ds^2 = dX_0^2 - dX_1^2 - \ldots dX_4^2\big|_{dS} =$$

$$= dt^2 - R^2\cosh^2\frac{t}{R}\left(d\theta^2 + \sin^2\theta(d\chi^2 + \sin^2\chi d\phi^2)\right)$$

Open de Sitter model (de Sitter 1917)



$$\begin{cases} X_0 = R\sinh\frac{t}{R}\cosh\chi \\ X_1 = R\sinh\frac{t}{R}\sinh\chi\sin\theta\sin\phi \\ X_2 = R\sinh\frac{t}{R}\sinh\chi\sin\theta\cos\phi \\ X_3 = R\sinh\frac{t}{R}\sinh\chi\cos\theta \\ X_4 = R\cosh\frac{t}{R} \end{cases}$$

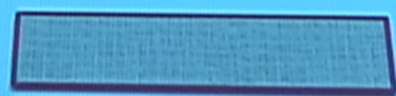
$$ds^2 = dX_0^2 - dX_1^2 - \dots dX_4^2\Big|_{dS} =$$

$$= dt^2 - R^2\sinh^2\frac{t}{R}\left(d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right)$$

Only Λ=0

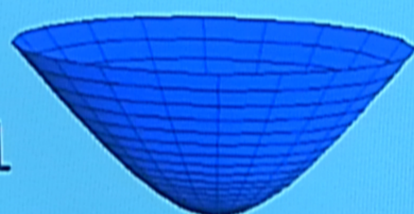
$$\ddot{a} = 0 \qquad \dot{a}^2 = -K$$

$K = 0$



$a(t) = 1$

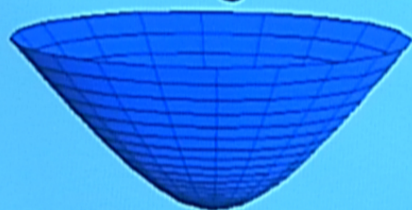
$K = -1$



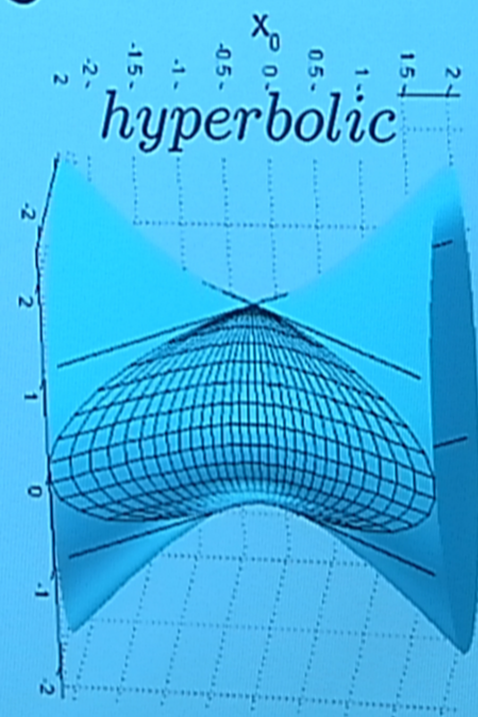
$a(t) = t$

Only Λ<0

$$\ddot{a} = -\frac{1}{3}|\Lambda|\,a \qquad \dot{a}^2 = -\frac{1}{3}|\Lambda|\,a^2 - K$$



$$K = -1$$



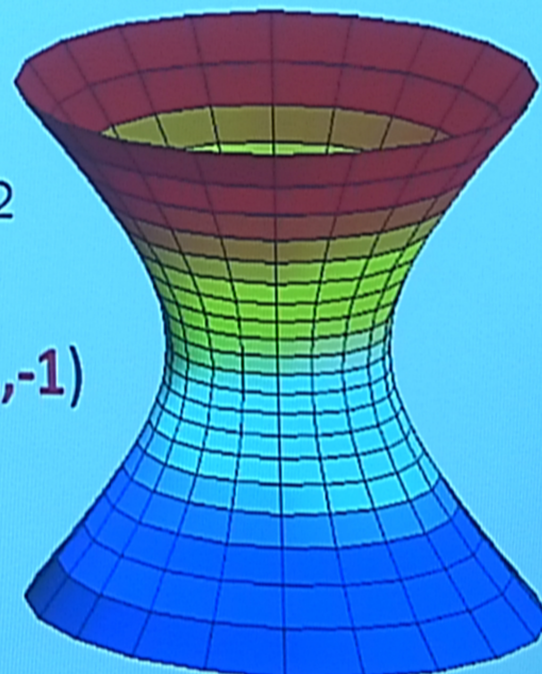
$$a(t) = \sqrt{\frac{3}{|\Lambda|}}\sin\sqrt{\frac{|\Lambda|}{3}}\,t$$

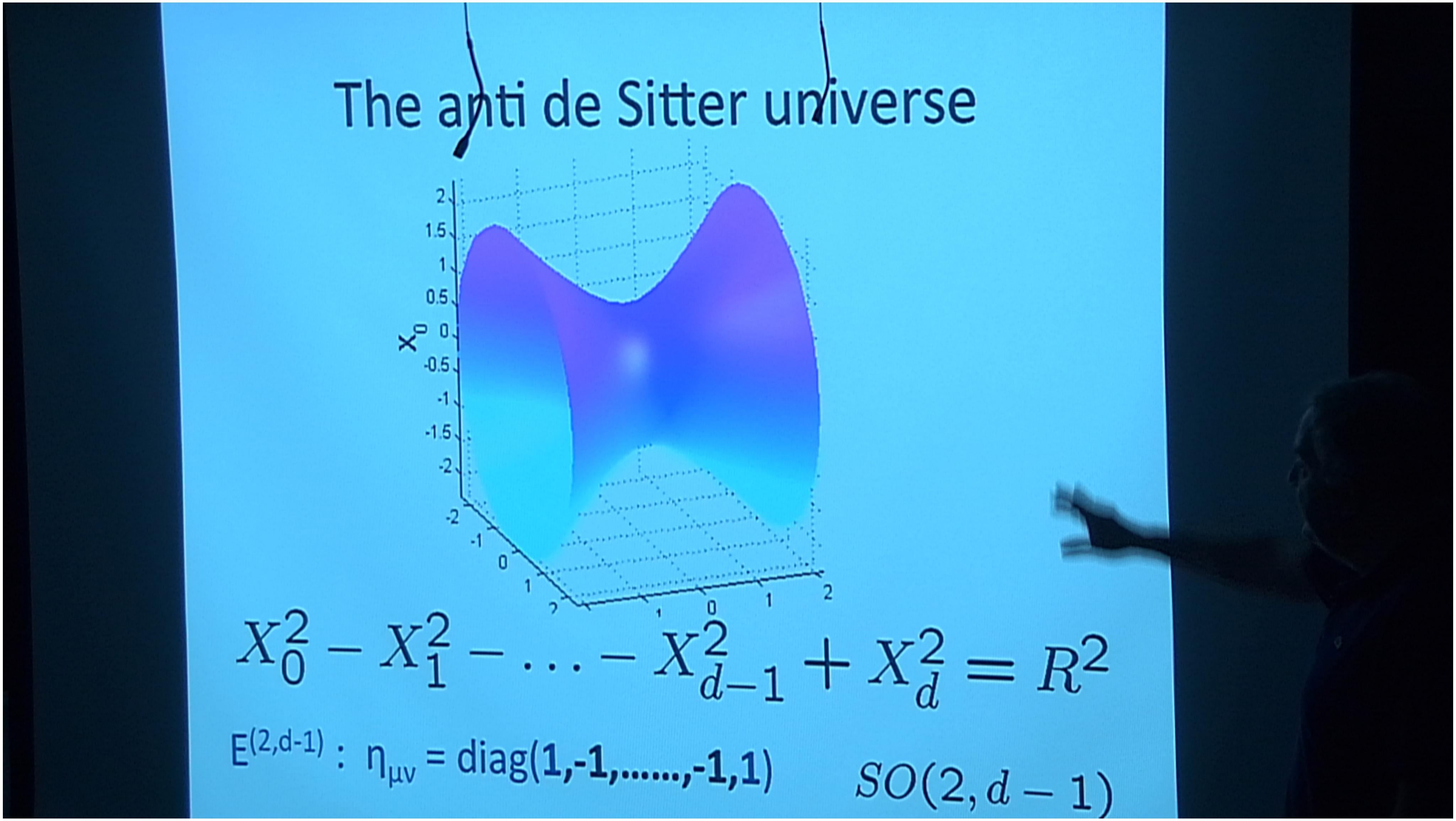
Once more the de Sitter universe

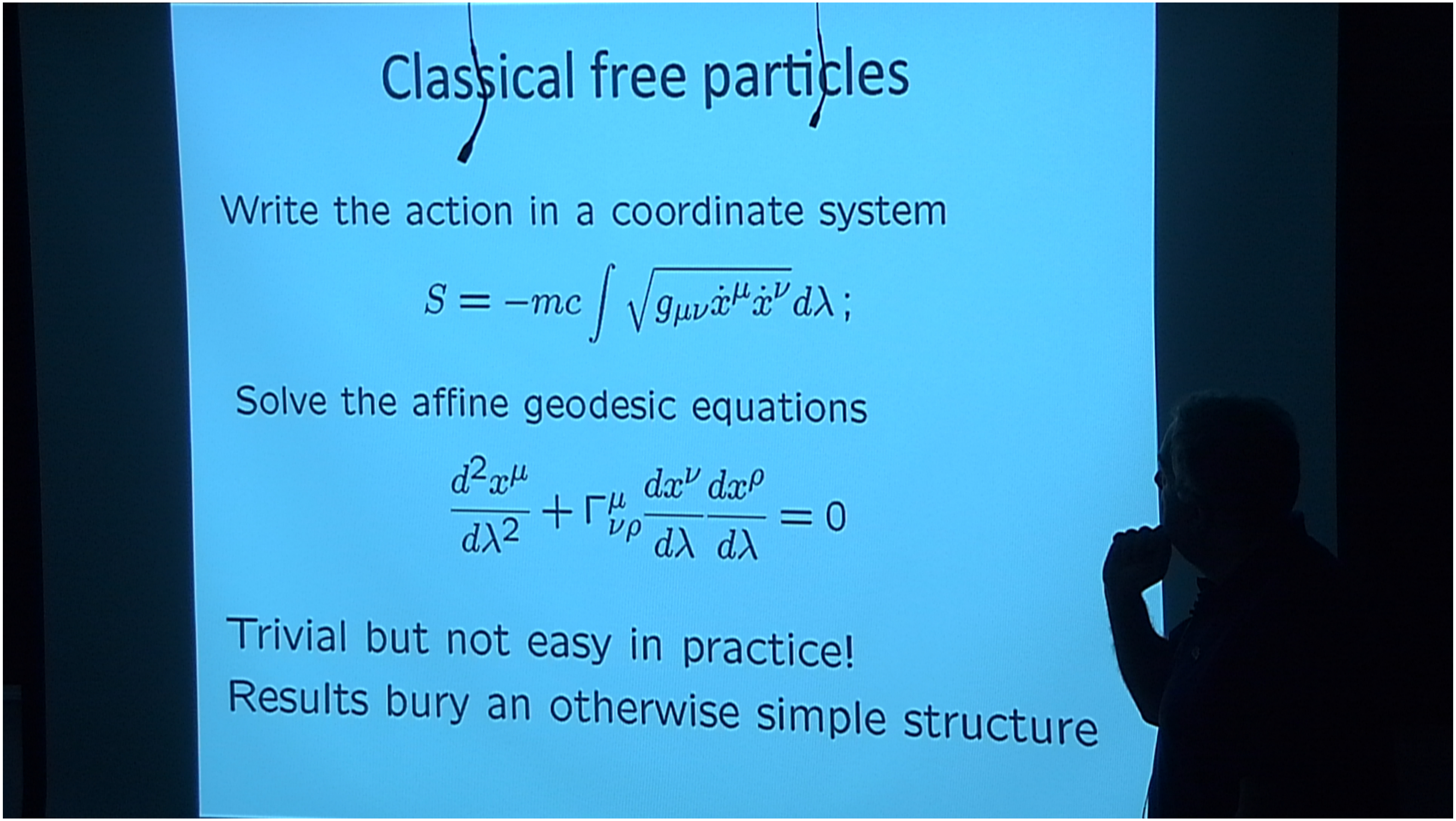
$$X_0^2 - X_1^2 - \dots X_d^2 = -R^2$$

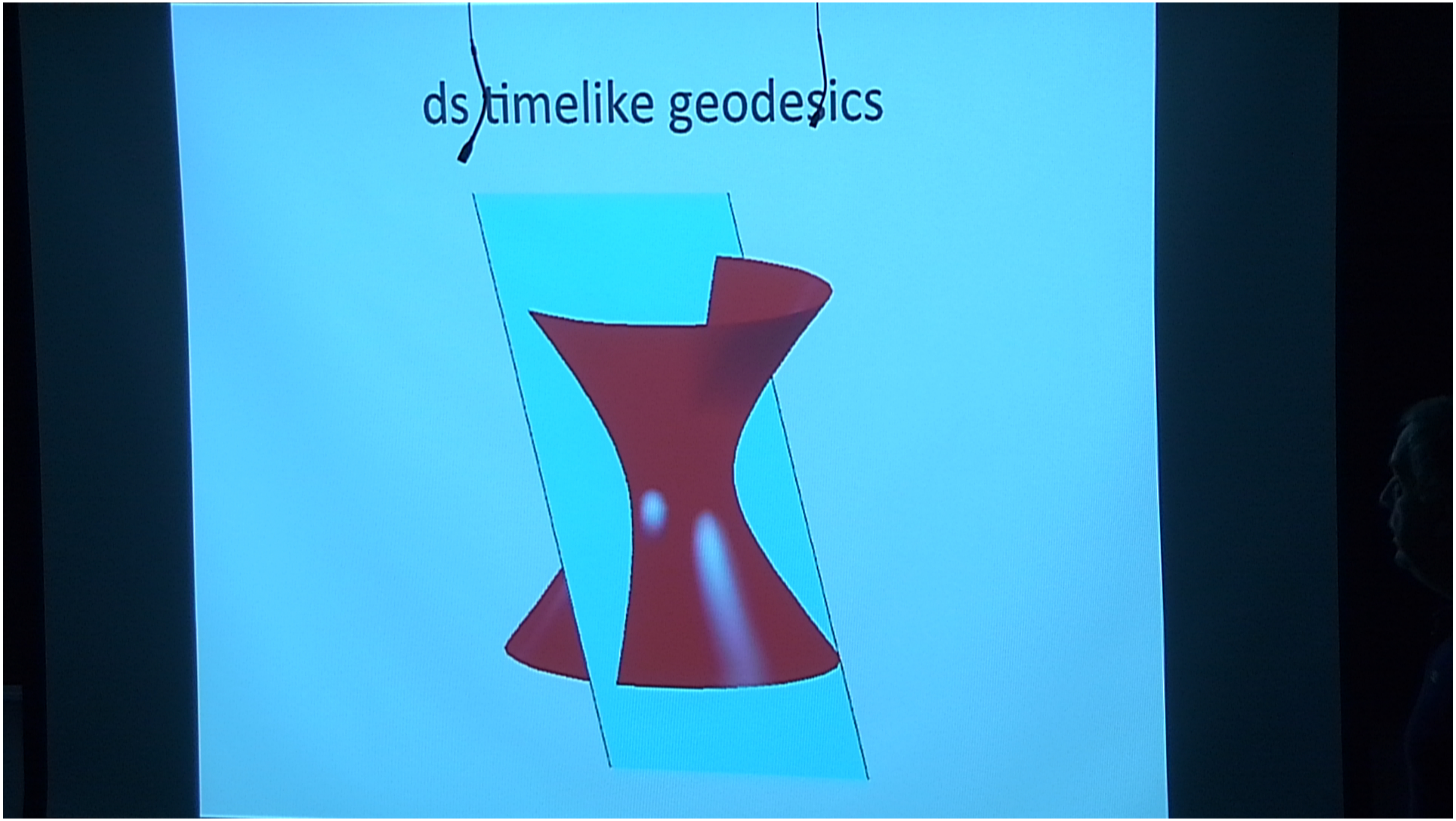
$M^{(d+1)}$: $\eta_{\mu\nu} = \mathrm{diag}(\mathbf{1},\mathbf{-1},\mathbf{\ldots\ldots},\mathbf{-1})$

$$G = SO(1,d)$$

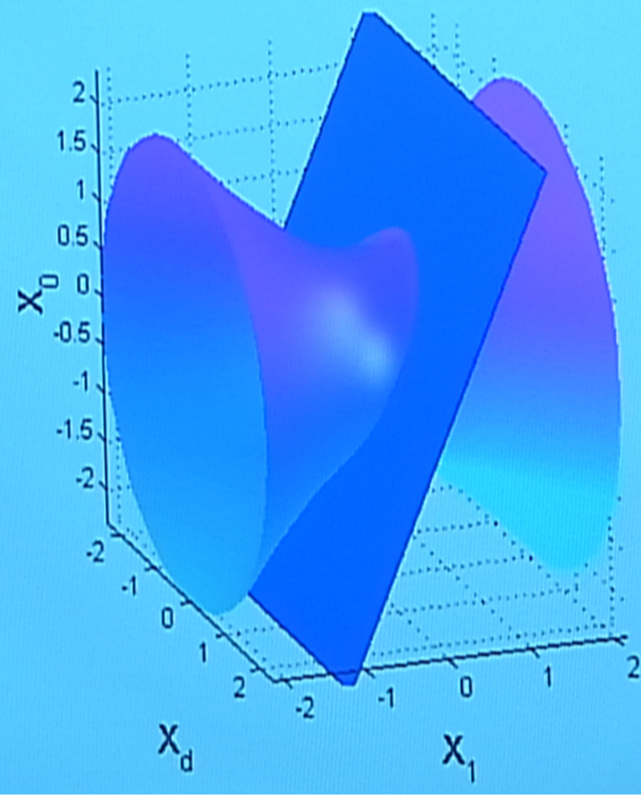



The anti de Sitter universe
X_0
$X_0^2 - X_1^2 - \ldots - X_{d-1}^2 + X_d^2 = R^2$
$E^{(2,d-1)}$: $\eta_{\mu\nu}$ = diag(**1**,**-1**,……,**-1**,**1**)
$SO(2, d-1)$

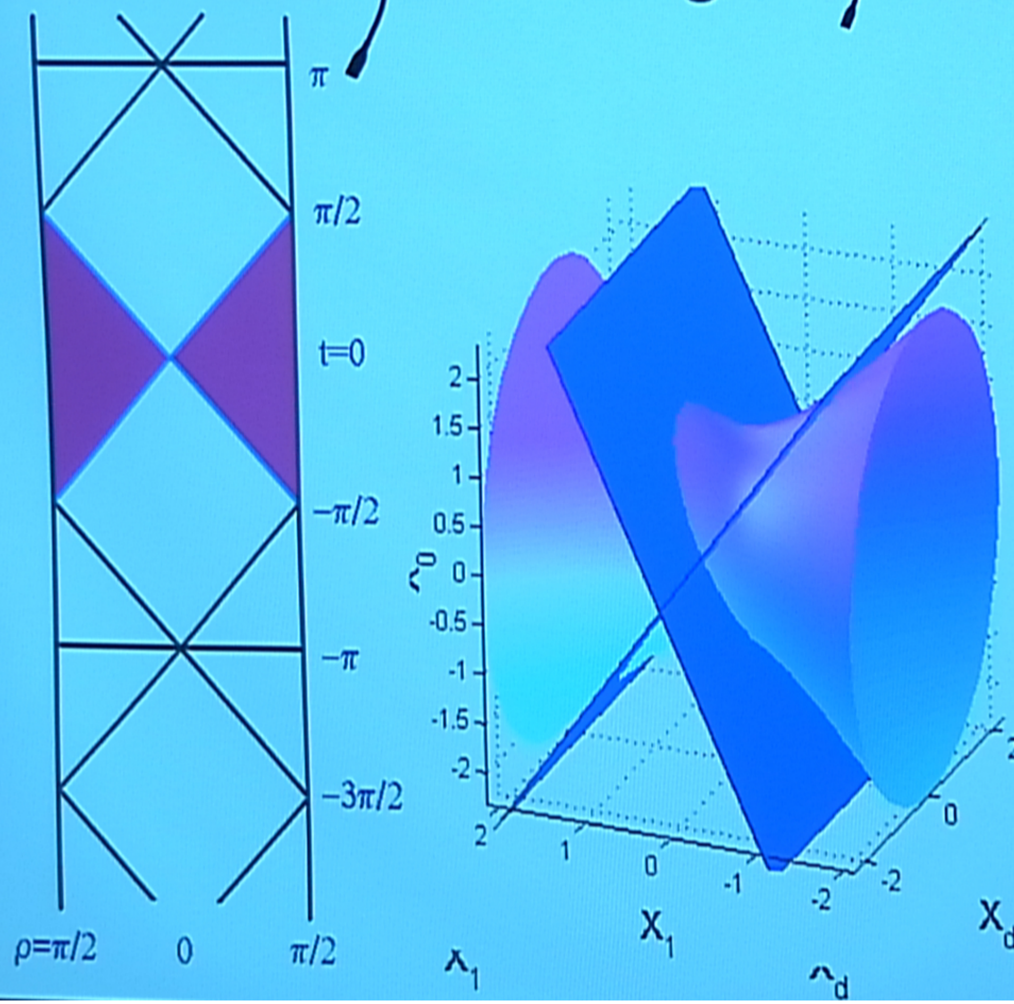

Classical free particles
Write the action in a coordinate system
$S = -mc \int \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\, d\lambda\,;$
Solve the affine geodesic equations
$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0$
Trivial but not easy in practice!
Results bury an otherwise simple structure


ds timelike geodesics

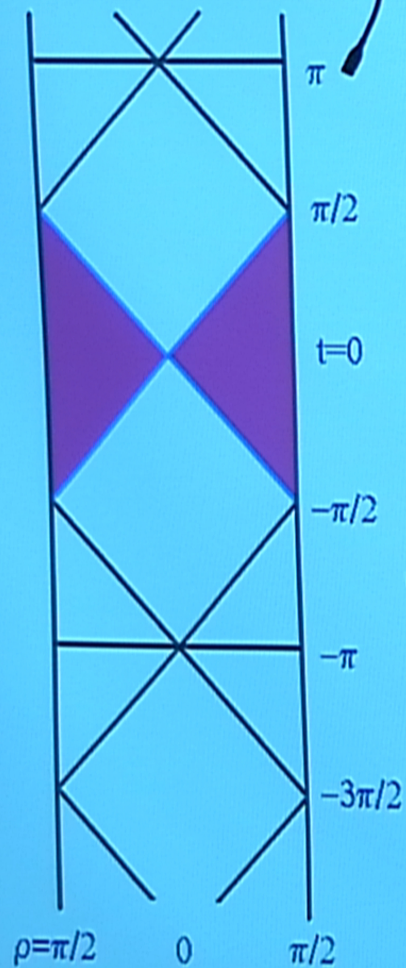
AdS: timelike geodesics

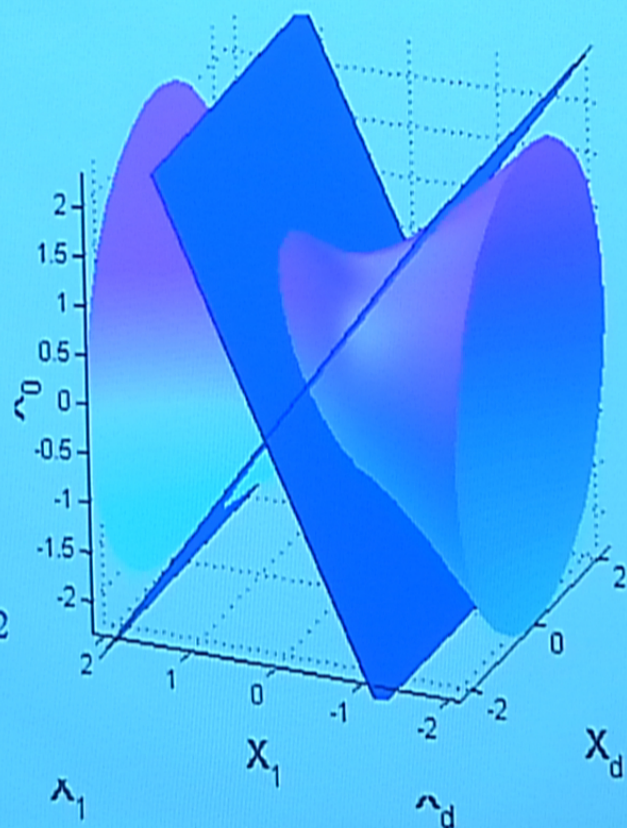


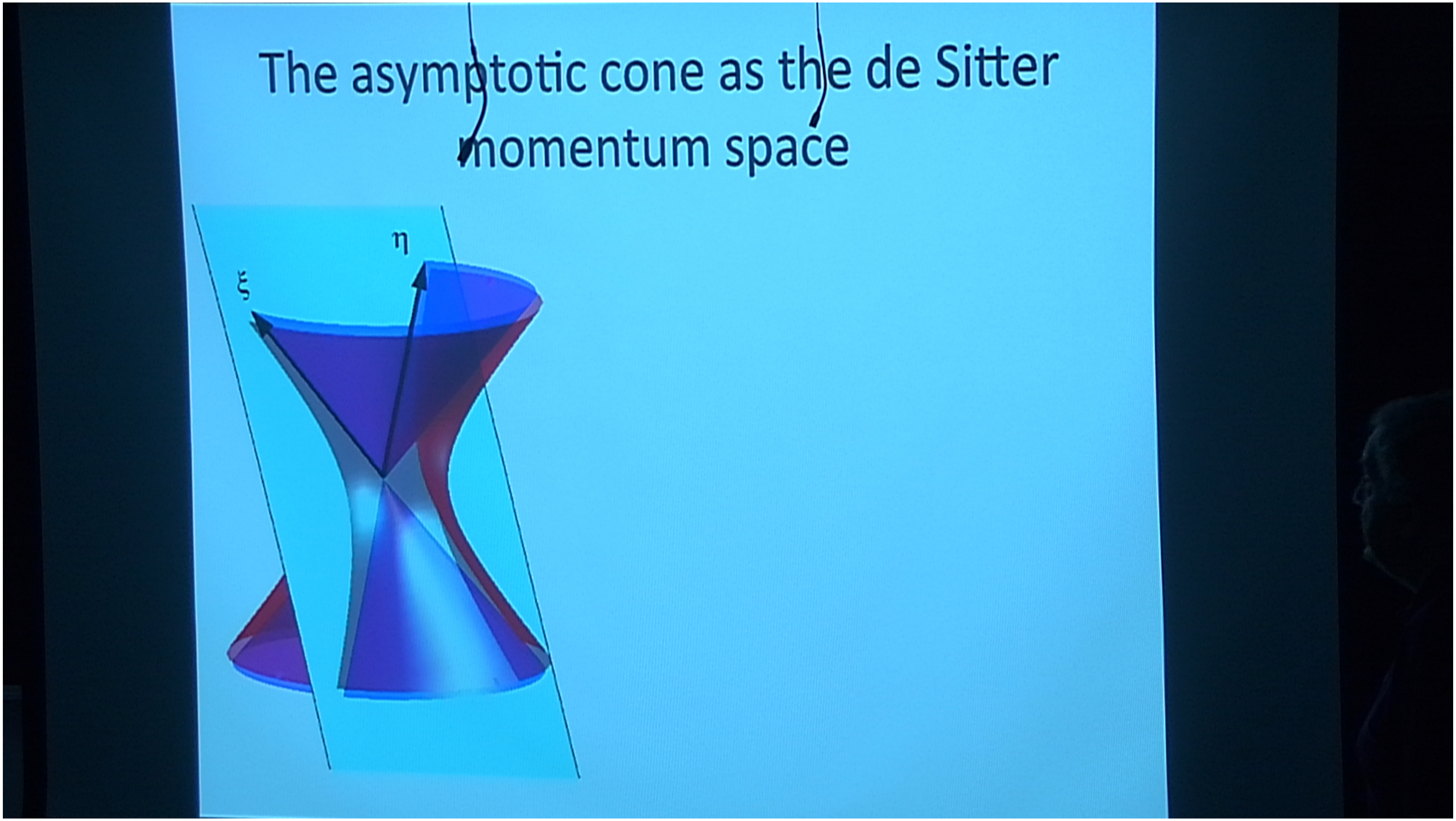
AdS: timelike geodesics



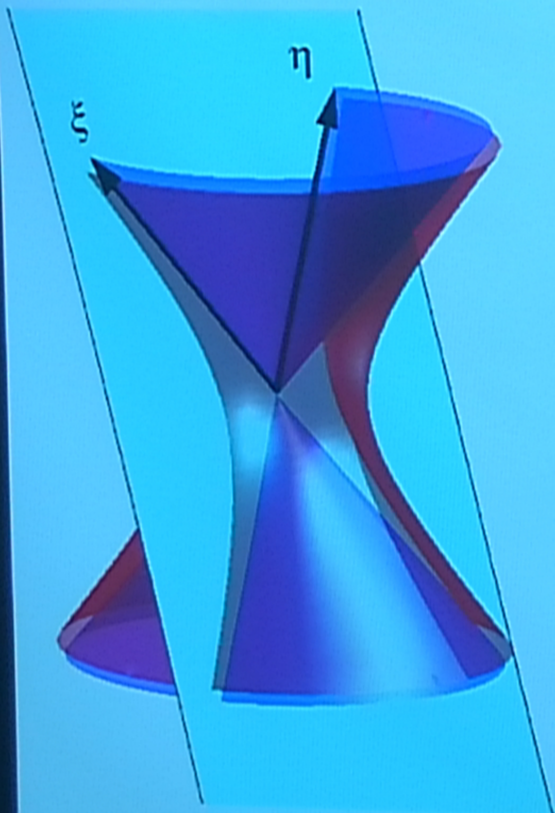
AdS: timelike geodesics






The asymptotic cone as the de Sitter momentum space
η
ξ

The asymptotic cone as the de Sitter momentum space



Geodesics: de Sitter

$$X_\mu(\tau) = \frac{R}{\sqrt{2\xi\cdot\eta}}\left(\xi_\mu e^{\frac{c\tau}{R}} - \eta_\mu e^{-\frac{c\tau}{R}}\right)$$

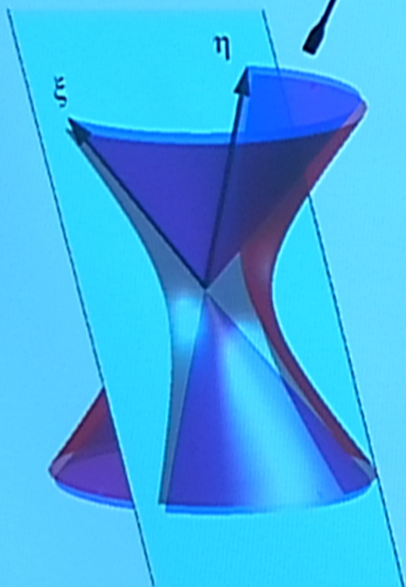
Minkowski

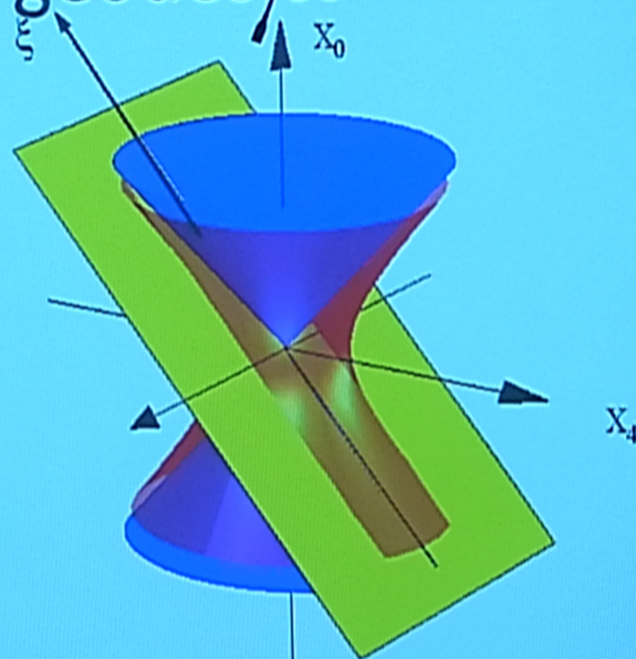
$$x_\mu(\tau) = x_\mu(0) + \frac{p_\mu \tau}{mc}$$

$$X_\mu(0) = \frac{R}{\sqrt{2\xi\cdot\eta}}\left(\xi_\mu - \eta_\mu\right)$$

$$X(\tau) = X(0)e^{-\frac{c\tau}{R}} + \frac{kR\xi}{m}\sinh\frac{c\tau}{R}.$$

Lightlike geodesics





$$X_\mu(\lambda) = x_0 + \xi_\mu \lambda, \quad \text{with } \xi \cdot x_0 = 0$$

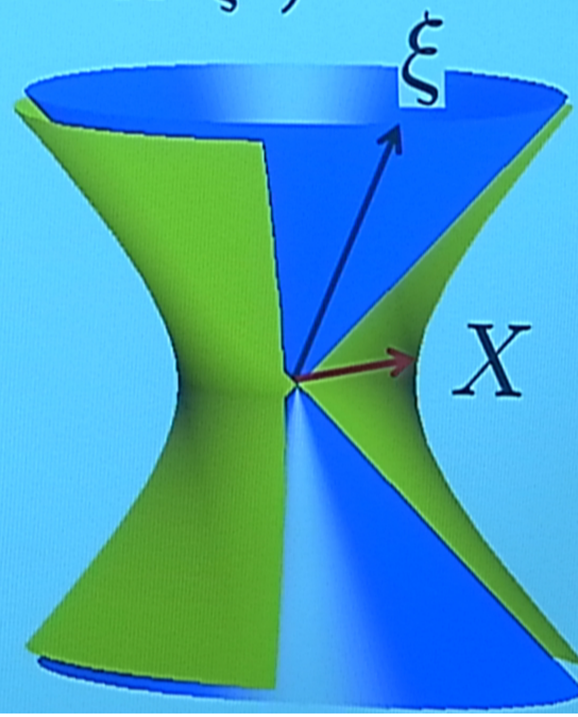
to be compared with

de Sitter plane waves

$$\psi_\lambda(X,\xi) = (X \cdot \xi)^\lambda =$$

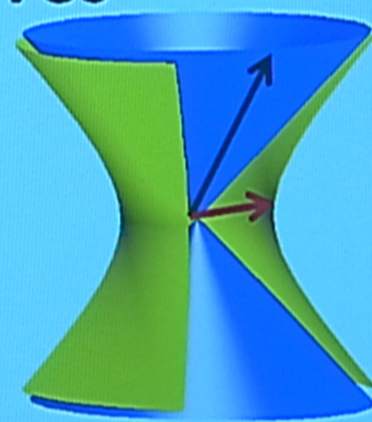
$$= (X^0\xi^0 - X^1\xi^1 - \ldots - X^d\xi^d)^\lambda$$

$$\lambda \in \mathbf{C}, \quad \xi^2 = 0$$



de Sitter plane waves

$$\Box (X \cdot \xi)^\lambda = \lambda(\lambda + d - 1)(X \cdot \xi)^\lambda$$

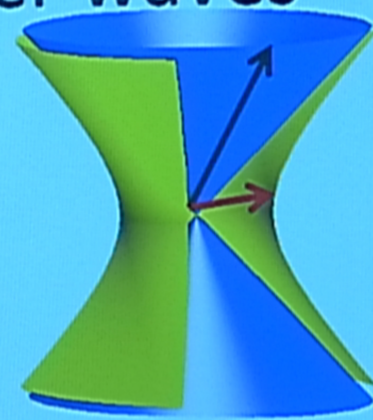


Involution:

$$\lambda \longrightarrow \bar{\lambda} = -\lambda - (d-1)$$

$$\lambda + \bar{\lambda} = -(d-1)$$

Complementary de Sitter waves



$$\lambda = -\frac{d-1}{2} + \nu\,, \quad \nu \in \mathbf{R}$$

$$\psi_\lambda(X,\xi) = (X\cdot\xi)^{-\frac{d-1}{2}+\nu}$$

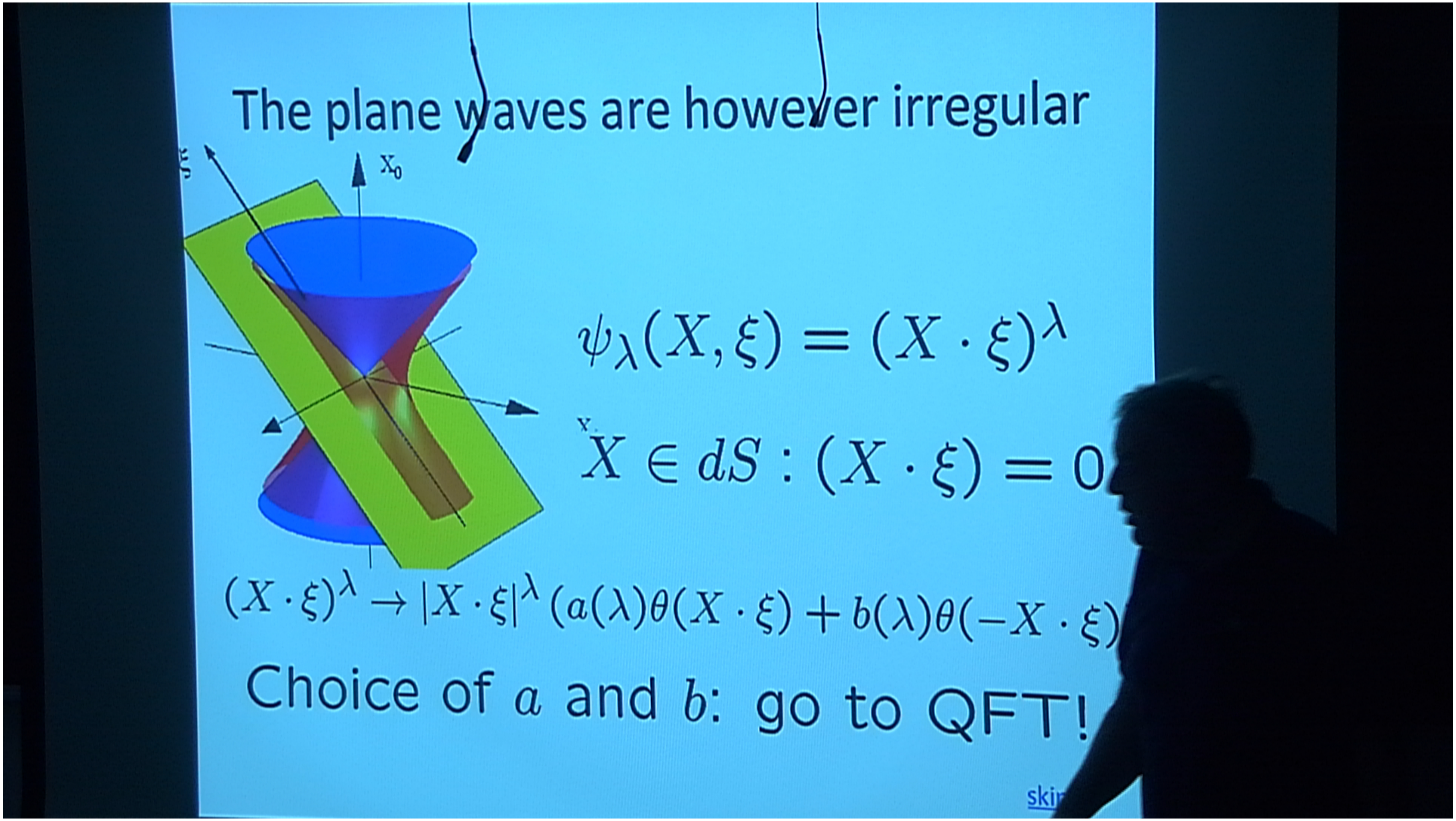
These waves do not oscillate!

$$\bar{\lambda} = -\lambda - (d-1) = -\frac{d-1}{2} - \nu$$

$$\psi_{\bar{\lambda}}(X,\xi) = (X\cdot\xi)^{-\frac{d-1}{2}-\nu} \neq \overline{\psi_\lambda(X,\xi)} = \psi_\lambda(X,\xi)$$

$$m^2 = \lambda\bar{\lambda} = \left(\frac{d-1}{2}\right)^2 - \nu^2$$

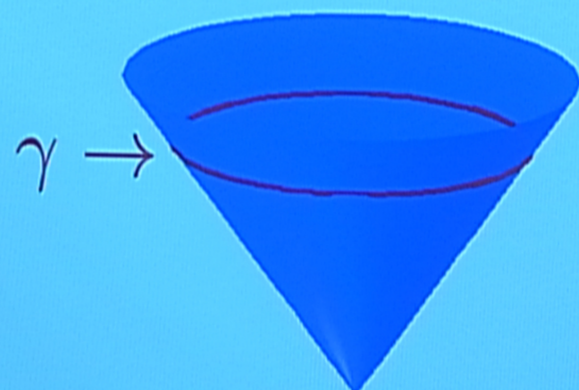
$$-\left(\frac{d-1}{2}\right) < \nu < \left(\frac{d-1}{2}\right)$$

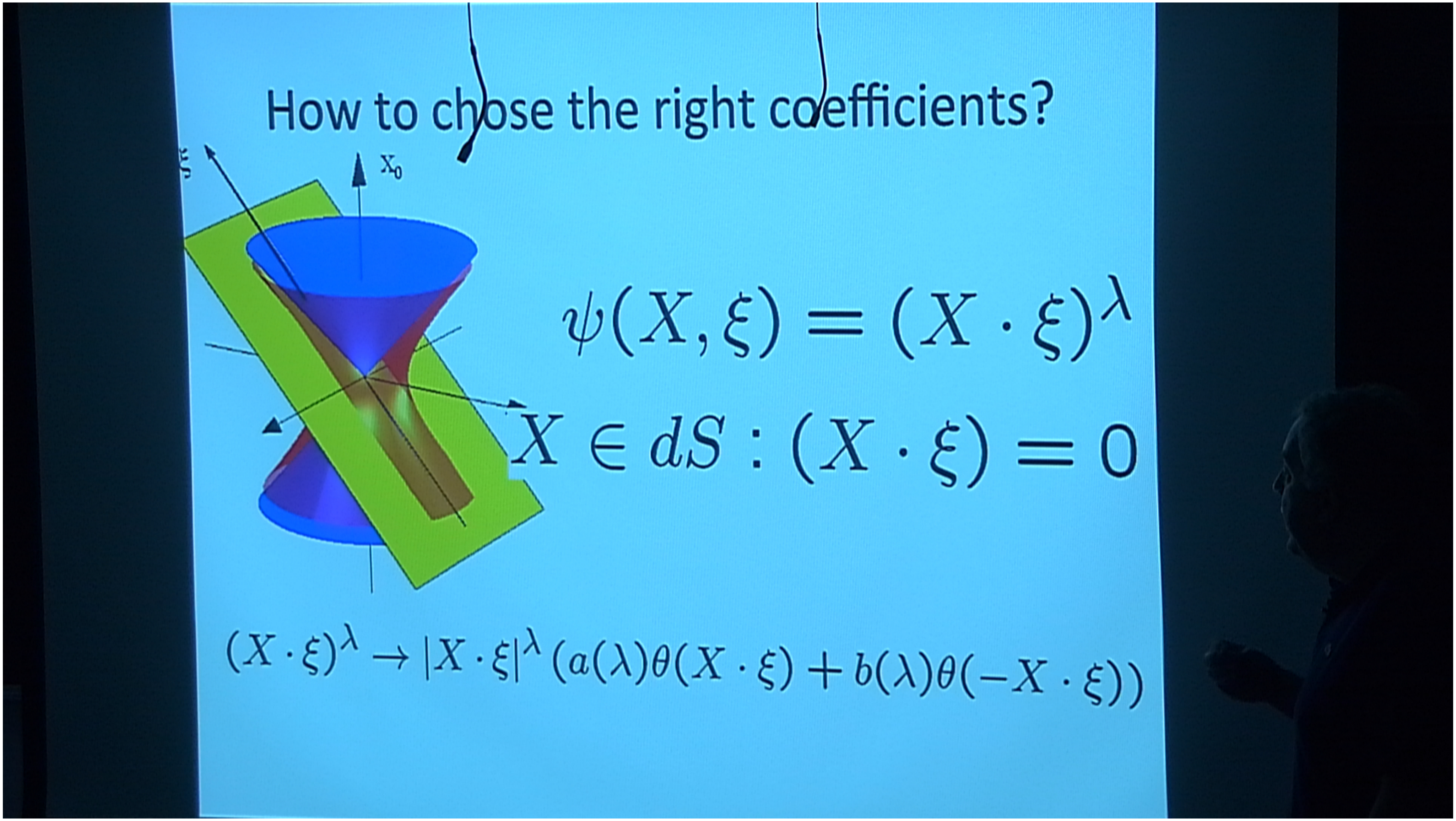

The plane waves are however irregular
ξ
X_0
$\psi_\lambda(X,\xi) = (X\cdot\xi)^\lambda$
$X \in dS : (X\cdot\xi) = 0$
$(X\cdot\xi)^\lambda \to |X\cdot\xi|^\lambda\,(a(\lambda)\theta(X\cdot\xi) + b(\lambda)\theta(-X\cdot\xi)$
Choice of a and b: go to QFT!

dS: construction of two-point functions

$$W(x_1, x_2) = \int e^{-ip\cdot x_1} e^{ip\cdot x_2} \theta(p^0)\delta(p^2 - m^2)dp$$

$$F_{\lambda,\gamma}(X_1, X_2) = \int (X_1 \cdot \xi)^{\lambda}(\xi \cdot X_2)^{-\lambda-d+1} d\mu_{\gamma}(\xi)$$




How to chose the right coefficients?
ξ
X_0
$\psi(X,\xi) = (X \cdot \xi)^\lambda$
$X \in dS : (X \cdot \xi) = 0$
$(X \cdot \xi)^\lambda \to |X \cdot \xi|^\lambda \left(a(\lambda)\theta(X \cdot \xi) + b(\lambda)\theta(-X \cdot \xi)\right)$

Recall: spectral property of QFT

There exists a complete set of nonegative energy states
(The energy-momentum spectrum is in the closed future cone)

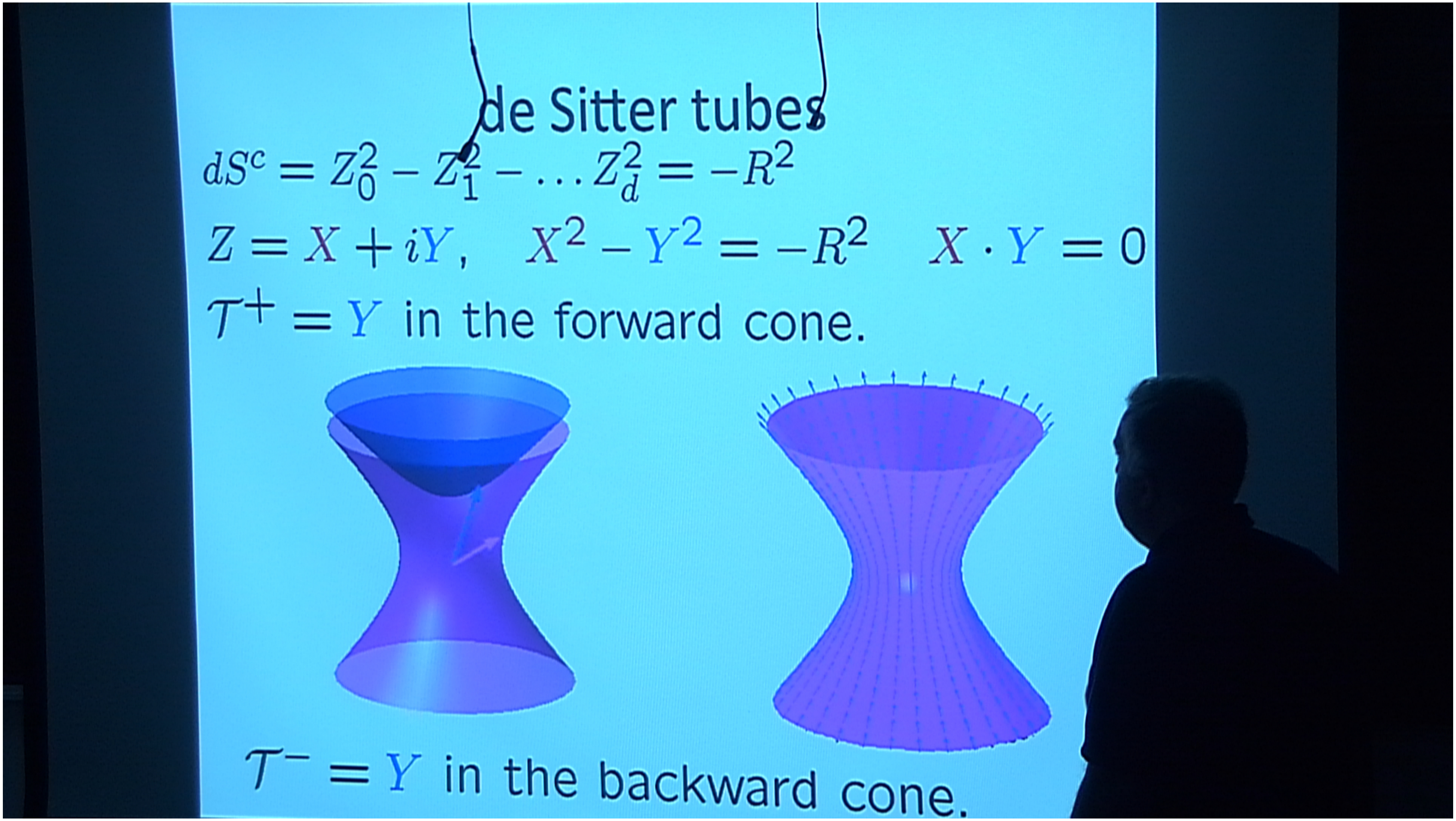
equivalent to

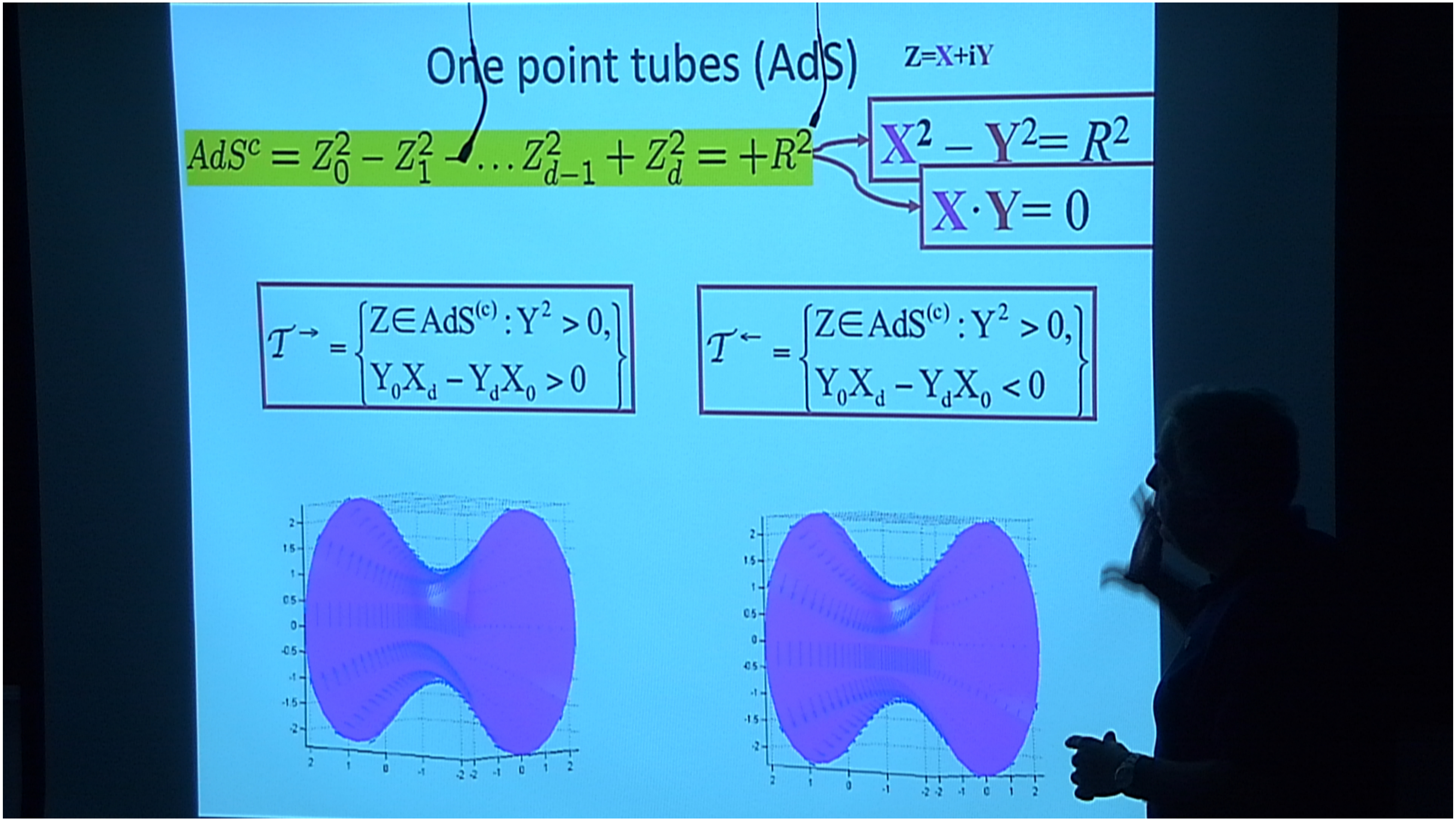
The n point-function $W(x_1, \ldots, x_n)$ is the boundary value of a function $W(z_1, \ldots, z_n)$ holomorphic in a "tube" of the complex Minkowski spacetime

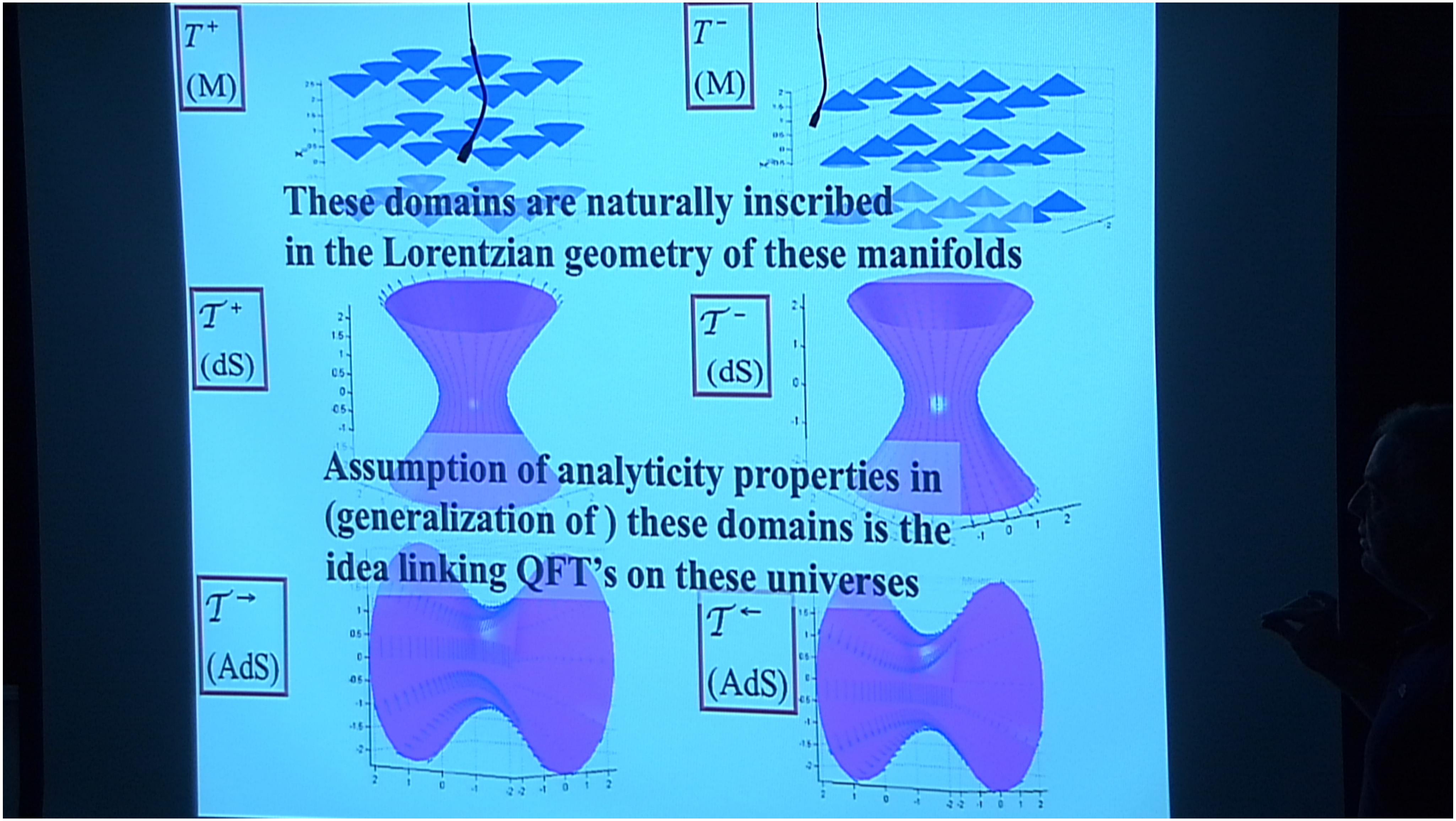
(tube = {Im $(z_{k+1} - z_k)$ contained in the closed future cone})

Consequences

Unique determination of the vacuum. All the common wisdom of perturbative renormalizable local and covariant QFT follows.


de Sitter tubes
$dS^c = Z_0^2 - Z_1^2 - \ldots Z_d^2 = -R^2$
$Z = X + iY, \quad X^2 - Y^2 = -R^2 \quad X \cdot Y = 0$
$\mathcal{T}^+ = Y$ in the forward cone.
$\mathcal{T}^- = Y$ in the backward cone.


One point tubes (AdS)
Z=X+iY
AdS^c = Z_0^2 − Z_1^2 − … Z_{d−1}^2 + Z_d^2 = +R^2
X^2 − Y^2 = R^2
X·Y = 0
T^→ = {Z∈AdS^(c) : Y^2 > 0, Y_0X_d − Y_dX_0 > 0}
T^← = {Z∈AdS^(c) : Y^2 > 0, Y_0X_d − Y_dX_0 < 0}

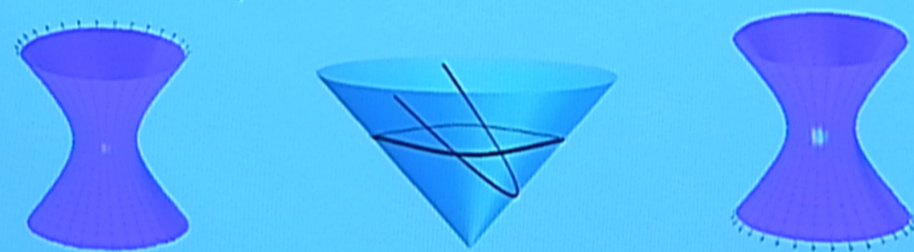


T^+ (M)
T^- (M)
These domains are naturally inscribed
in the Lorentzian geometry of these manifolds
$\mathcal{T}^+$ (dS)
$\mathcal{T}^-$ (dS)
Assumption of analyticity properties in
(generalization of) these domains is the
idea linking QFT's on these universes
$\mathcal{T}^\rightarrow$ (AdS)
$\mathcal{T}^\leftarrow$ (AdS)

Fourier representation for Bunch-Davies aka Euclidean aka ......two-point functions

For $Z_1 \in \mathcal{T}^-$ e $Z_2 \in \mathcal{T}^+$

$$W_\lambda(Z_1, Z_2) = \int_\gamma (Z_1 \cdot \xi)^\lambda \; (\xi \cdot Z_2)^{-\lambda-(d-1)} d\mu(\xi)$$



To be compared with the standard flat case:

$$W(z_1 - z_2) = \int e^{-ip\cdot z_1} e^{ip\cdot z_2} \theta(p^0) \delta(p^2 - m^2) d^4 p$$

$z_1 \in T^- \quad z_2 \in T^+$

$$W_\lambda(Z_1,\ Z_2) \ = \ \int_\gamma (Z_1\cdot\xi)^\lambda\ (\xi\cdot Z_2)^{-\lambda-(d-1)} d\mu(\xi)$$

$$= \frac{\Gamma(-\lambda)\Gamma(\lambda+d-1)}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\ {}_2F_1\left(-\lambda,\ \lambda+d-1;\ \frac{d}{2};\ \frac{1-\zeta}{2}\right)\ .$$

$$= \frac{\Gamma(-\lambda)\,\Gamma(\lambda+d-1)}{2(2\pi)^{\frac{d}{2}}}\ (\zeta^2-1)^{-\frac{d-2}{4}}\ P^{-\frac{d-2}{2}}_{-\lambda-\frac{d}{2}}(\zeta)$$

a) $W(Z_2, Z_2)$ is invariant under the complex de Sitter group

$W(Z_1, Z_2)$ is maximally analytic.

The cut reflects causality

$$\zeta = Z_1\cdot Z_2$$

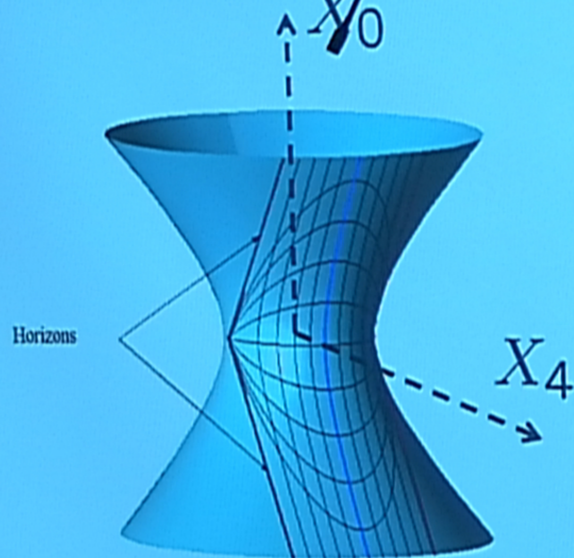
ζ = -1

b) $W(X_1, X_2)$ is b.v.of $W(Z_1, Z_2)$ from $\mathcal{T}^-\times\mathcal{T}^+$

The permuted function $W(X_2, X_1)$

is b.v.of the same $W(Z_1, Z_2)$ from $\mathcal{T}^+\times\mathcal{T}^-$

Static BH coordinates (de Sitter 1917)



$$
\begin{cases}
X_0 = \sqrt{R^2 - r^2}\,\sinh(t/R) \\
X_1 = r\sin\theta\sin\phi \\
X_2 = r\sin\theta\cos\phi \\
X_3 = r\cos\theta \\
X_4 = \sqrt{R^2 - r^2}\,\cosh(t/R)
\end{cases}
$$

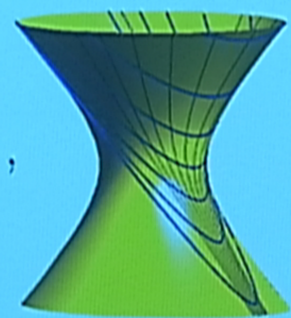
$$
ds^2 = dX_0^2 - dX_1^2 - dX_2^2 - dX_3^2 - dX_4^2\Big|_{dS_4} =
$$

$$
= (1 - \frac{r^2}{R^2})dt^2 - \frac{1}{1-\frac{r^2}{R^2}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)
$$

$$\psi^{\pm}_{i\nu}(z,\xi) = (x \pm iy\cdot\xi)^{-\frac{d-1}{2}+i\nu},$$

are globally well-defined in the tubes.

$$\begin{cases} x^0 = R\sinh\frac{t}{R} + \frac{1}{2R}\mathrm{x}^2\exp\frac{t}{R} \\ x^i = \mathrm{x}^\mathrm{i}\ \exp\frac{t}{R}, \\ x^d = R\cosh\frac{t}{R} - \frac{1}{2R}\mathrm{x}^2\ \exp\frac{t}{R}. \end{cases},$$

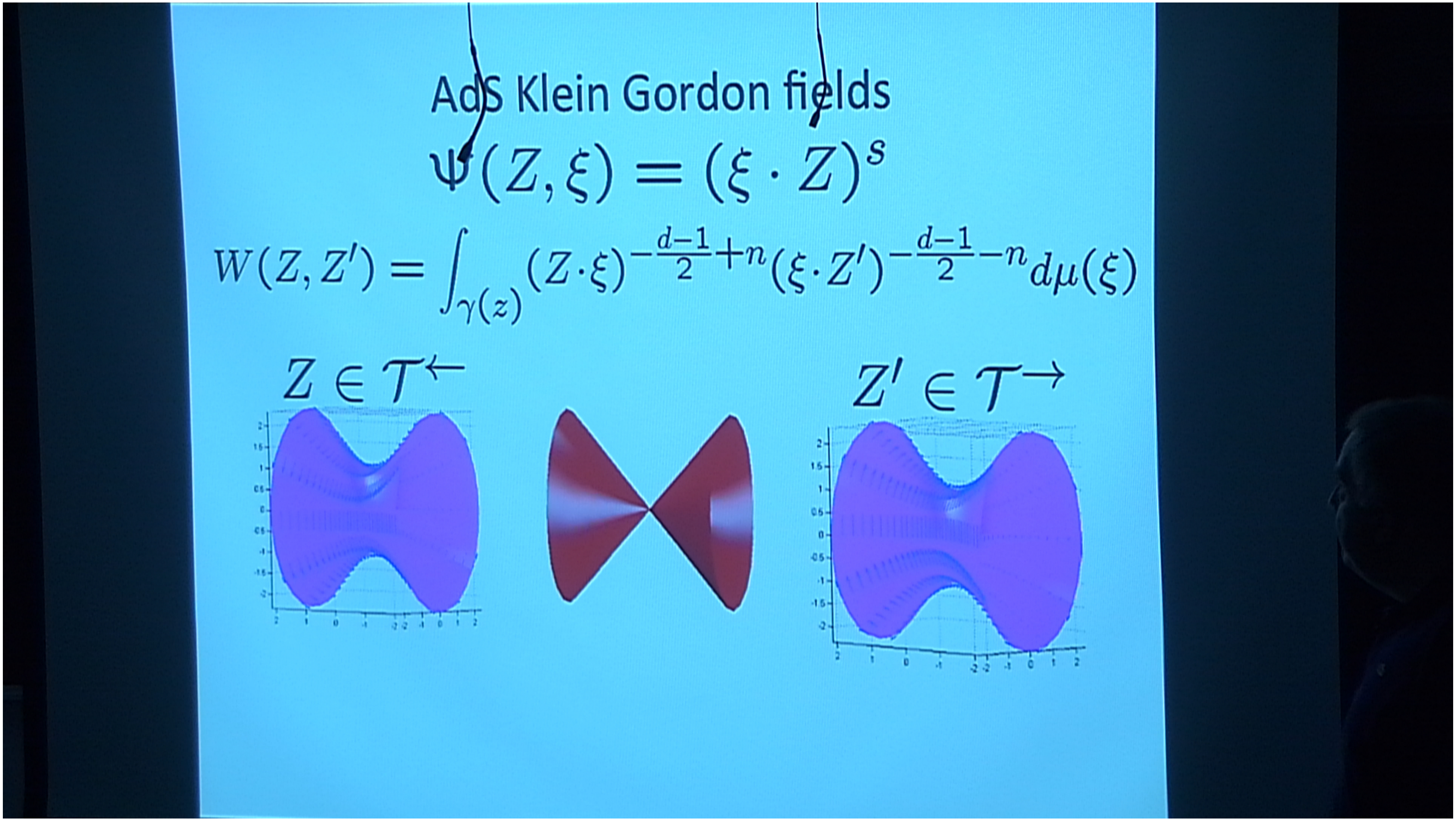


$$\begin{cases} \xi^0 = \frac{1}{2}(1+\eta^2) \\ \xi^i = \eta \\ \xi^d = \frac{1}{2}(1-\eta^2) \end{cases},$$

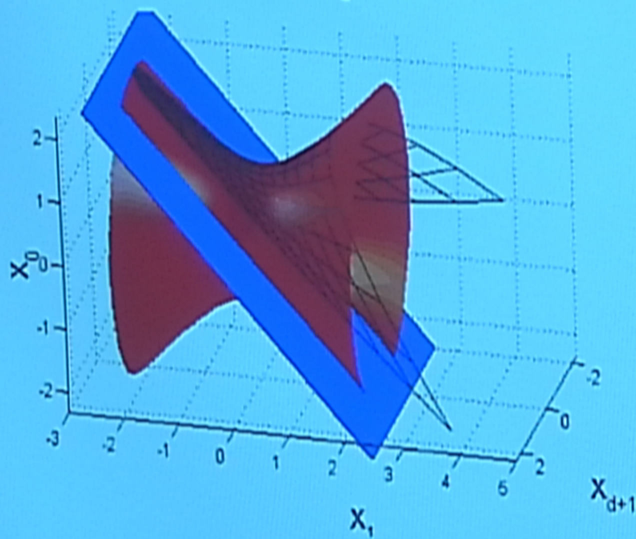
$$\widetilde{\psi^{\pm}_{i\nu}}(t,\mathbf{k}) = \int [\mathbf{x}(\mathbf{t}\pm\mathbf{i}\epsilon,\mathbf{x})\cdot\xi(\eta)]^{-\frac{3}{2}+\mathbf{i}\nu}\mathbf{e}^{\mathbf{ikx}}\mathbf{dx}$$

$$\widetilde{\psi^{+}_{i\nu}}(t,\mathbf{k}) = \frac{i\pi}{\Gamma\left(\frac{3}{2}-\nu\right)}(2\pi\mathrm{e}^{-\mathrm{t}})^{\frac{3}{2}}\exp(\mathbf{ik}\eta)\,\mathbf{k}^{-\nu}\,\mathbf{H}^{(2)}_{\mathbf{i}\nu}(\mathbf{ke}^{-\mathbf{t}})$$

$$\widetilde{\psi^{-}_{i\nu}}(t,\mathbf{k}) = \frac{i\pi}{\Gamma\left(\frac{3}{2}-\nu\right)}(2\pi\mathrm{e}^{-\mathrm{t}})^{\frac{3}{2}}\exp(\mathbf{ik}\eta)\,\mathbf{k}^{-\nu}\,\mathbf{H}^{(1)}_{\mathbf{i}\nu}(\mathbf{ke}^{-\mathbf{t}})$$


AdS Klein Gordon fields
$\Psi(Z,\xi) = (\xi \cdot Z)^s$
$W(Z,Z') = \int_{\gamma(z)} (Z\cdot\xi)^{-\frac{d-1}{2}+n} (\xi\cdot Z')^{-\frac{d-1}{2}-n} d\mu(\xi)$
$Z \in \mathcal{T}^{\leftarrow}$
$Z' \in \mathcal{T}^{\rightarrow}$

AdS: Poincaré coordinates



$$X_d + X_{d+1} = \exp v$$

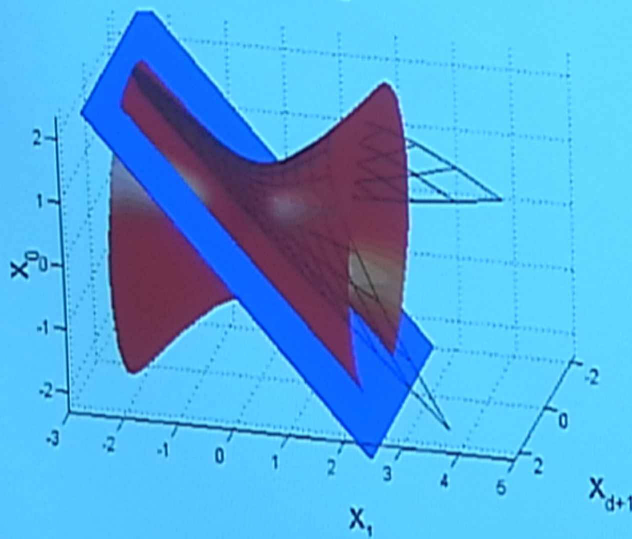
$$\begin{cases} X_\mu &= e^v x_\mu \\ X_d &= \sinh v + \frac{1}{2} e^v x^2 \\ X_{d+1} &= \cosh v - \frac{1}{2} e^v x^2 \end{cases}$$

$$x^2 = x_0^2 - x_1^2 - \ldots - x_{d-1}^2$$

$$ds^2 = e^{2v} dx^\mu dx_\mu - dv^2$$

$$X_0^2 - X_1^2 - \ldots - X_d^2 + X_{d+1}^2 = 1$$

AdS: Poincaré coordinates



$$X_d + X_{d+1} = \exp v$$

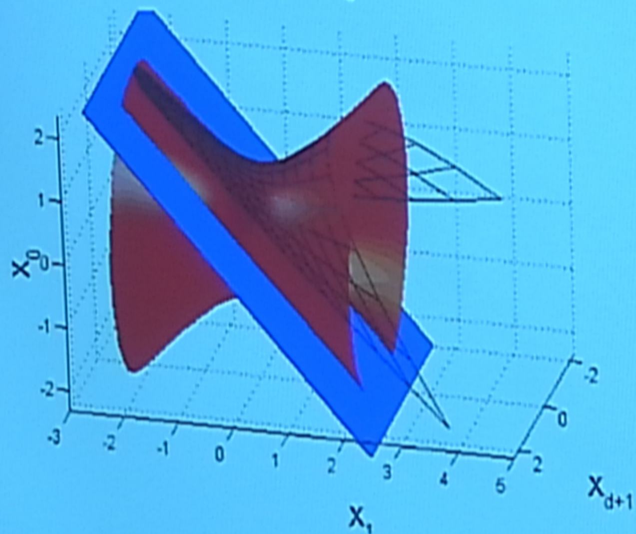
$$\begin{cases} X_\mu &= e^v x_\mu \\ X_d &= \sinh v + \frac{1}{2} e^v x^2 \\ X_{d+1} &= \cosh v - \frac{1}{2} e^v x^2 \end{cases}$$

$$x^2 = x_0^2 - x_1^2 - \ldots - x_{d-1}^2$$

$$ds^2 = e^{2v} dx^\mu dx_\mu - dv^2$$

$$X_0^2 - X_1^2 - \ldots - X_d^2 + X_{d+1}^2 = 1$$

AdS: Poincaré coordinates



$$X_d + X_{d+1} = \exp v$$

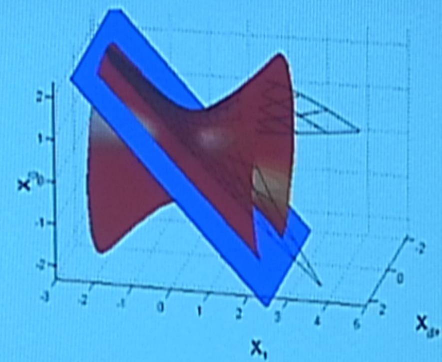
$$\begin{cases} X_\mu &= e^v x_\mu \\ X_d &= \sinh v + \frac{1}{2} e^v x^2 \\ X_{d+1} &= \cosh v - \frac{1}{2} e^v x^2 \end{cases}$$

$$x^2 = x_0^2 - x_1^2 - \ldots - x_{d-1}^2$$

$$ds^2 = e^{2v} dx^\mu dx_\mu - dv^2$$

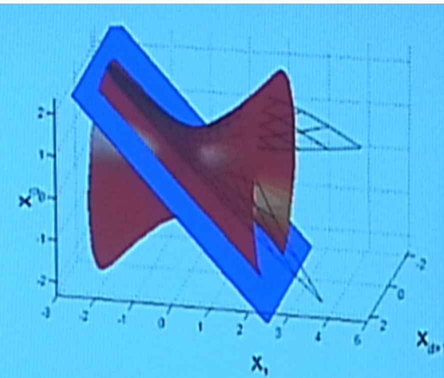
$$X_0^2 - X_1^2 - \ldots - X_d^2 + X_{d+1}^2 = 1$$

a) On the fiber v=const (brane)
AdS causality ⇔ Minkowski causality



$$ds^2 = e^{2v} dx^\mu dx_\mu - dv^2$$

a) On the fiber v=const (brane)
AdS causality ⇔ Minkowski causality



b) If $z \in T_M^+ \Rightarrow Z(v,z) \in \mathcal{T}^{\rightarrow}_{AdS}$

If $z \in T_M^- \Rightarrow Z(v,z) \in \mathcal{T}^{\leftarrow}_{AdS}$

$$ds^2 = e^{2v} dx^\mu dx_\mu - dv^2$$

Fibre Complexification

$$\begin{cases} Z_\mu & = & e^v z_\mu \\ Z_d & = & \sinh v + \frac{1}{2} e^v z^2 \\ Z_{d+1} & = & \cosh v - \frac{1}{2} e^v z^2 \end{cases}$$

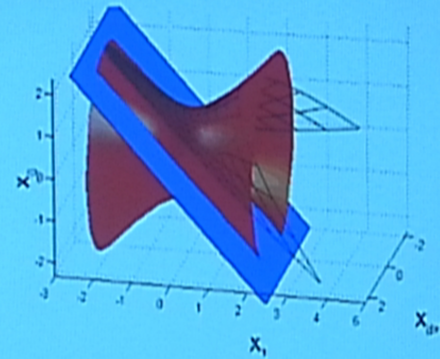
$W_v(z,z') \equiv W(Z(v,z), Z'(v,z'))$

is local, Poincare' invariant, and satisfies the spectral condition. It defines an acceptable Minkowski QFT

a) On the fiber v=const (brane)
AdS causality ⇔ Minkowski causality

b) If $z \in T_M^+ \Rightarrow Z(v,z) \in \mathcal{T}_{AdS}^{\rightarrow}$

If $z \in T_M^- \Rightarrow Z(v,z) \in \mathcal{T}_{AdS}^{\leftarrow}$



$$ds^2 = e^{2v} dx^\mu dx_\mu - dv^2$$

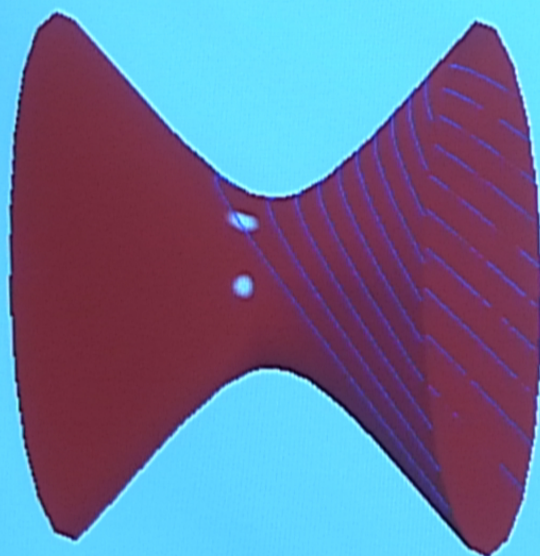
Fibre Complexification

$$\begin{cases} Z_\mu &= e^v z_\mu \\ Z_d &= \sinh v + \frac{1}{2} e^v z^2 \\ Z_{d+1} &= \cosh v - \frac{1}{2} e^v z^2 \end{cases}$$

$W_v(z,z') \equiv W(Z(v,z), Z'(v,z'))$

is local, Poincare' invariant, and satisfies the spectral condition.
It defines an acceptable Minkowski QFT

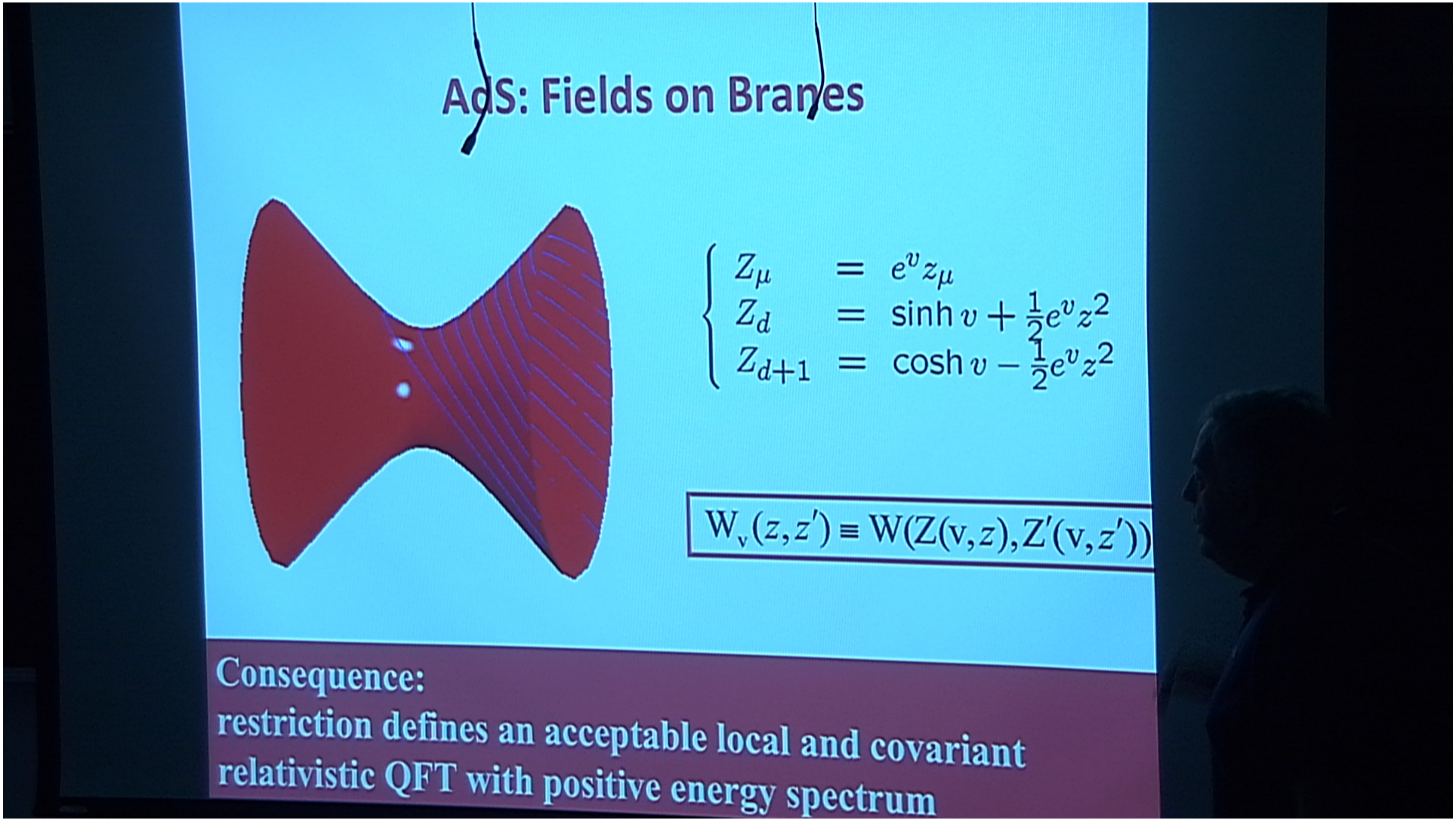
AdS: Fields on Branes

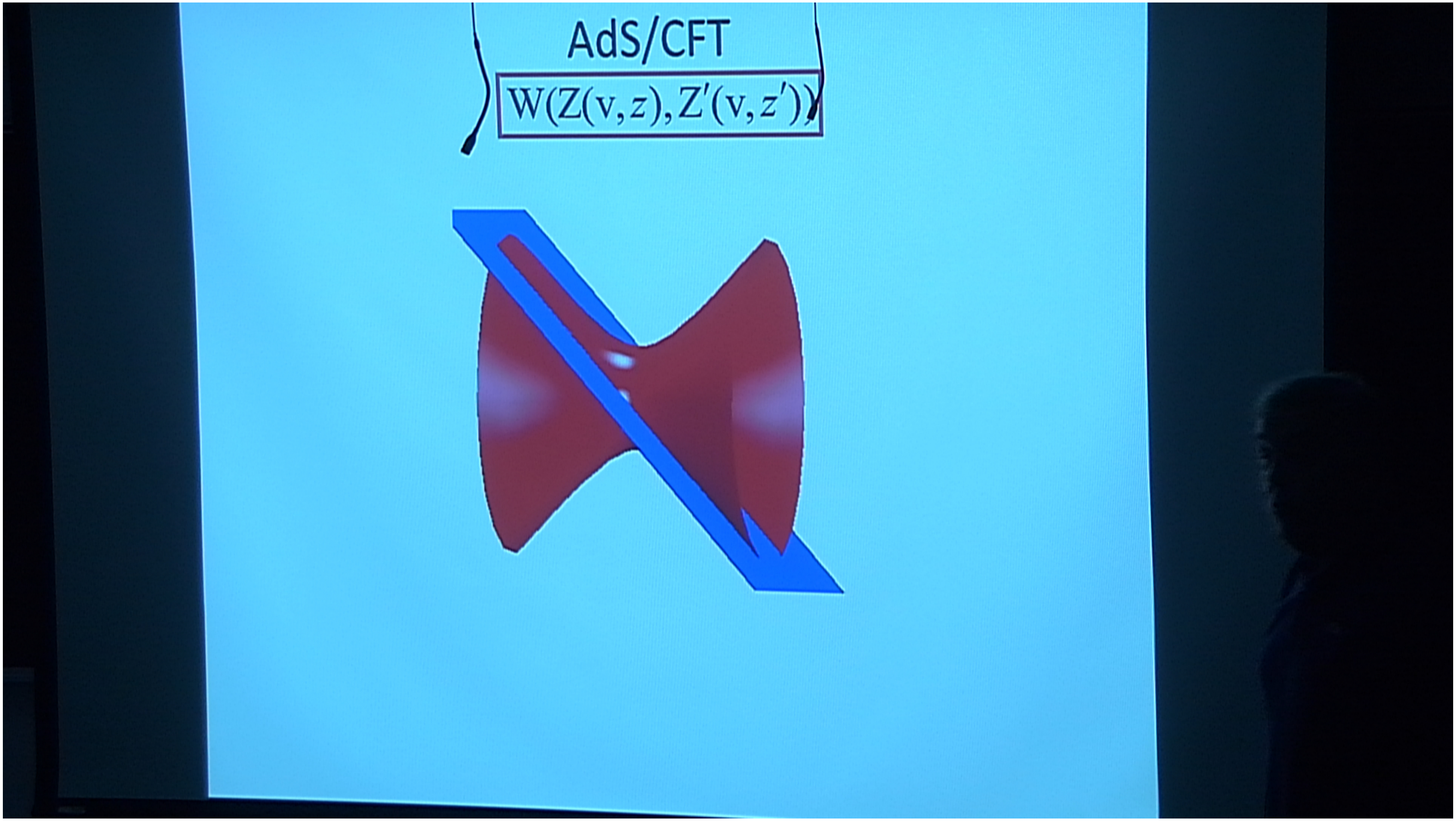


$$\begin{cases} Z_\mu & = & e^v z_\mu \\ Z_d & = & \sinh v + \frac{1}{2} e^v z^2 \\ Z_{d+1} & = & \cosh v - \frac{1}{2} e^v z^2 \end{cases}$$

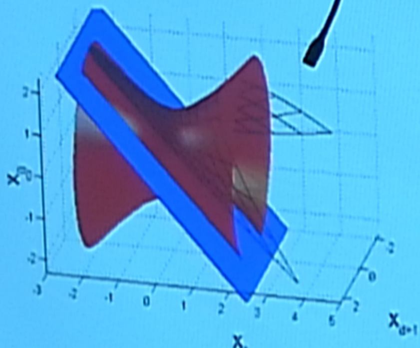
$$\boxed{W_v(z, z') \equiv W(Z(v, z), Z'(v, z'))}$$

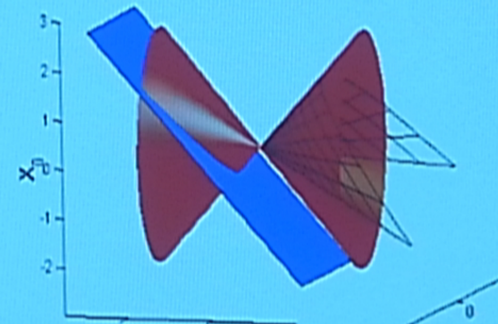
Consequence:
restriction defines an acceptable local and covariant relativistic QFT with positive energy spectrum


AdS: Fields on Branes
$Z_\mu = e^v z_\mu$
$Z_d = \sinh v + \frac{1}{2} e^v z^2$
$Z_{d+1} = \cosh v - \frac{1}{2} e^v z^2$
$W_v(z, z') \equiv W(Z(v,z), Z'(v,z'))$
Consequence:
restriction defines an acceptable local and covariant relativistic QFT with positive energy spectrum


AdS/CFT
W(Z(v,z),Z′(v,z′))

AdS/CFT: The Klein-Weyl-Dirac construction





$$\begin{cases} X_\mu &= e^v x_\mu \\ X_d &= \sinh v + \frac{1}{2} e^v x^2 \\ X_{d+1} &= \cosh v - \frac{1}{2} e^v x^2 \end{cases}$$

$$\begin{cases} \eta_\mu &= e^v x_\mu \\ \eta_d &= \frac{1}{2} e^v (1 + x^2) \\ \eta_{d+1} &= \frac{1}{2} e^v (1 - x^2) \end{cases}$$

$$x^2 = x_0^2 - x_1^2 - \ldots - x_{d-1}^2$$

$$\begin{cases} \eta_d + \eta_{d+1} &= e^v \\ \eta_d - \eta_{d+1} &= e^v x^2 \end{cases}$$

$$x_\mu = \frac{\eta_\mu}{\eta_d + \eta_{d+1}}$$

Projector identity

- Proiector identity: non trivial holds only for the pricipal series

$$\int w_\nu(z,\, x)\, w_{\nu'}(x,\, y)\, dx = 2\pi \coth \pi\nu\delta(\nu^2 - \nu'^2) w_\nu(z,\, y)$$

Computing the KL weight

$$\rho(s;\ m_1, m_2) = (2\pi)^{1-d} \int \delta(P - p_1 - p_2)\, \delta(p_1^2 - m_1^2)\theta(p_1^0)\, \delta(p_2^2 - m_2^2)\theta(p_2^0)\, d^d p_1\, d^d p_2\,,$$

where $P^0 = \sqrt{s},\ \vec{P} = 0,\ s \geq 0.$

$$\begin{aligned}
\rho(s;\ m_1, m_2) &= (2\pi)^{1-d} \int \delta(p_1^2 - m_1^2)\theta(p_1^0)\, \delta\left(s - 2\sqrt{s}p_1^0 + m_1^2 - m_2^2\right) d^d p_1 \\
&= \frac{(2\pi)^{1-d}\Omega_{d-1}}{2\sqrt{s}} \int_0^\infty \delta\left(r^2 + m_1^2 - \frac{(s + m_1^2 - m_2^2)^2}{4s}\right) r^{d-2}\, dr \\
&= \frac{(2\pi)^{1-d}\Omega_{d-1}}{4\sqrt{s}} \left(\frac{(s - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2}{4s}\right)^{\frac{d-3}{2}} \\
&= \frac{(2\pi)^{1-d}\Omega_{d-1}}{4\sqrt{s}} \left(\frac{(s - (m_1 + m_2)^2)(s - (m_1 - m_2)^2)}{4s}\right)^{\frac{d-3}{2}}
\end{aligned}$$

$$\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad \forall\ n \geq 1$$

KL weight

Evaluate the Mehler-Fock transform

$$h_d(\kappa,\nu,\lambda) = \int_1^\infty P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\kappa}(u)\, P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u) P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u)(u^2-1)^{-\frac{d-2}{4}}\, du$$

which provides the Kallen-Lehmann weight

$$\rho(\kappa^2,\nu,\lambda) \;=\; \frac{\Gamma\left(\frac{d-1}{2}+i\nu\right)\Gamma\left(\frac{d-1}{2}-i\nu\right)\Gamma\left(\frac{d-1}{2}+i\lambda\right)\Gamma\left(\frac{d-1}{2}-i\lambda\right)}{2(2\pi)^{1+\frac{d}{2}}}\sinh(\pi\kappa)\;h_d(\kappa,\nu,\lambda),$$

Another Integral representation

Let us evaluate the 2 pt function at purely imaginary events of the past e future tubes

$$\mathcal{W}_\nu(-iy, iy') = \frac{\Gamma(\frac{d-1}{2} + i\nu)\Gamma(\frac{d-1}{2} - i\nu)}{2^{d+1}\pi^d} \int_\gamma (y\cdot\xi)^{-\frac{d-1}{2}+i\nu} \left(\xi\cdot y'\right)^{-\frac{d-1}{2}-i\nu} d\mu_\gamma(\xi) =$$

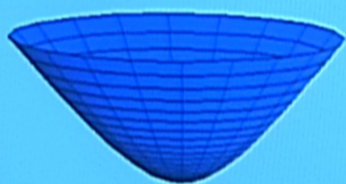
$$= \frac{\Gamma\left(\frac{d-1}{2} + i\nu\right)\Gamma\left(\frac{d-1}{2} - i\nu\right)}{2(2\pi)^{\frac{d}{2}}} \left(\left(y\cdot y'\right)^2 - 1\right)^{-\frac{d-2}{4}} P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}\left(y\cdot y'\right)$$

Another Integral representation

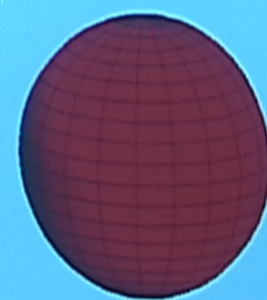
Let us evaluate the 2 pt function at purely imaginary events of the past e future tubes

$$\mathcal{W}_\nu(-iy, iy') = \frac{\Gamma(\frac{d-1}{2}+i\nu)\Gamma(\frac{d-1}{2}-i\nu)}{2^{d+1}\pi^d}\int_\gamma (y\cdot\xi)^{-\frac{d-1}{2}+i\nu}\left(\xi\cdot y'\right)^{-\frac{d-1}{2}-i\nu} d\mu_\gamma(\xi) =$$

$$= \frac{\Gamma\left(\frac{d-1}{2}+i\nu\right)\Gamma\left(\frac{d-1}{2}-i\nu\right)}{2(2\pi)^{\frac{d}{2}}}\left(\left(y\cdot y'\right)^2-1\right)^{-\frac{d-2}{4}} P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}\left(y\cdot y'\right)$$



$$y = (u, \sqrt{u^2-1}, \vec{\omega})$$
$$\xi = (1, \vec{\Omega})$$
$$y' = (1, 0, \ldots, 0)$$
$$y\cdot y' = y^0 = u \geq 1$$
$$y'\cdot\xi = 1$$

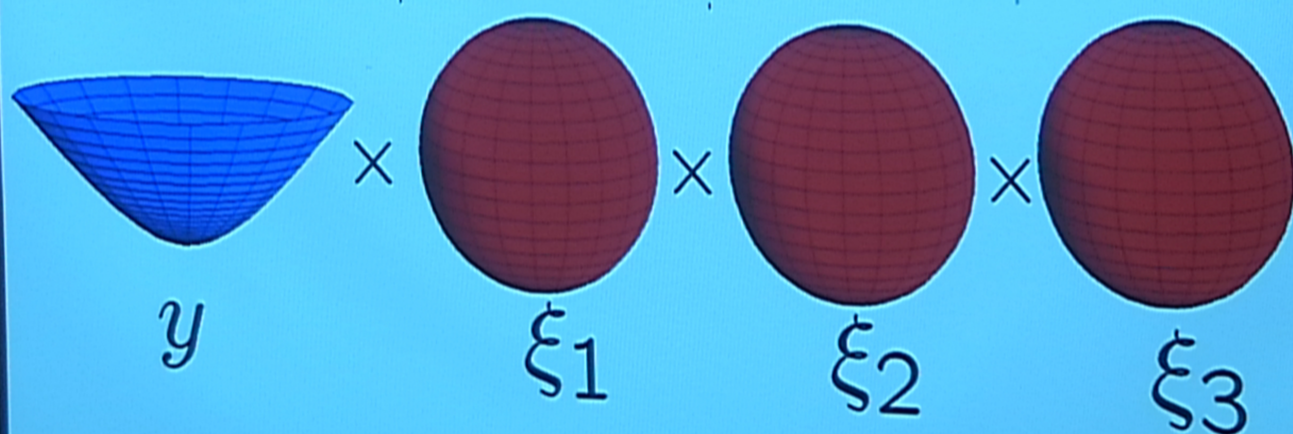


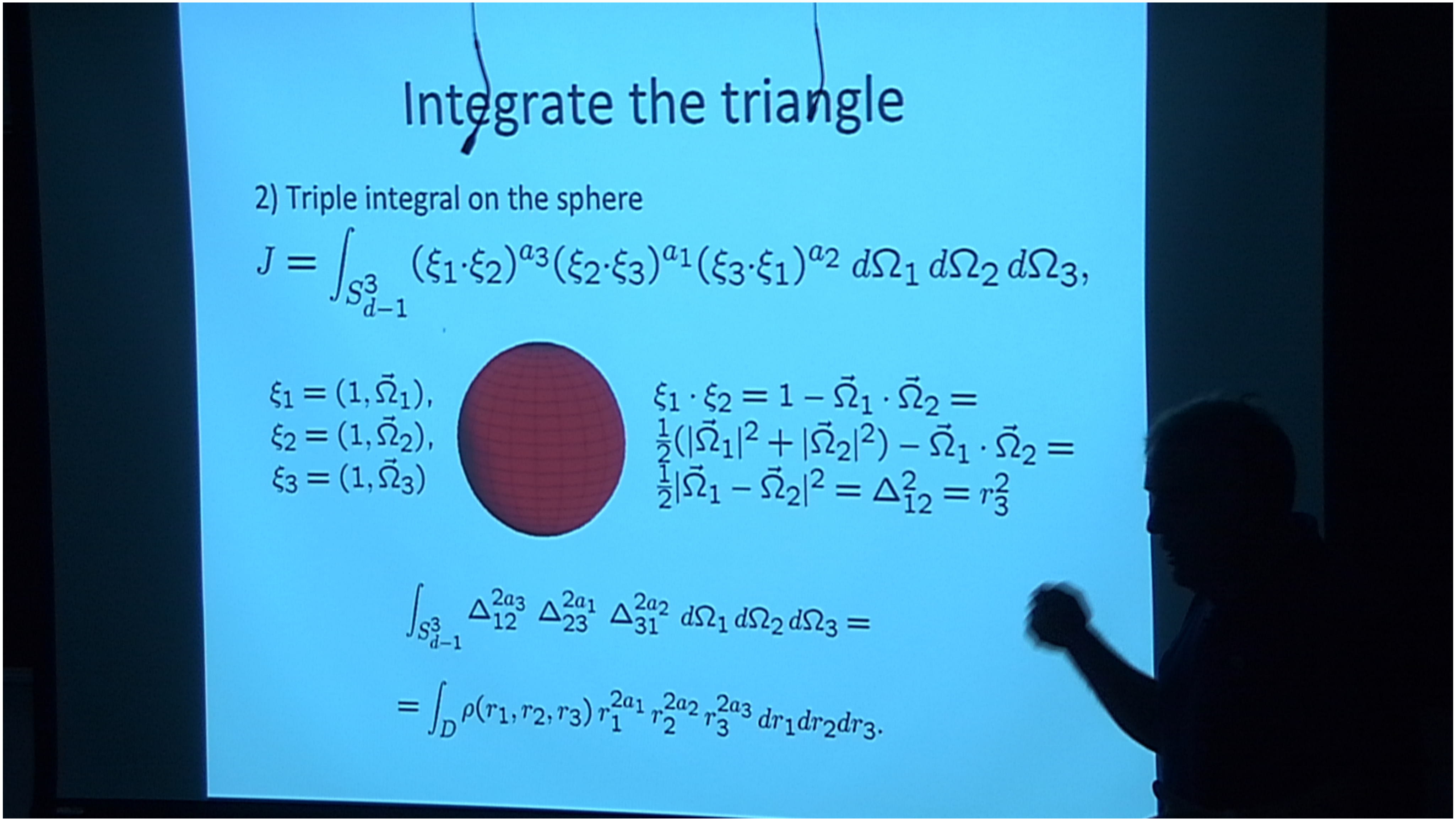
$$\left(u^2-1\right)^{-\frac{d-2}{4}} P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\gamma_0} (y\cdot\xi)^{-\frac{d-1}{2}-i\nu} d\mu_\gamma(\xi)$$

Fourier-like representation

$$h_d(\kappa,\nu,\lambda) = \int_1^\infty P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\kappa}(u)\, P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u) P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u)(u^2-1)^{-\frac{d-2}{4}}\, du =$$

$$\int_{H_d}\int_{S^3_{d-1}} (y\cdot\xi_1)^{-\frac{d-1}{2}-i\kappa}\,(y\cdot\xi_2)^{-\frac{d-1}{2}-i\nu}\,(y\cdot\xi_3)^{-\frac{d-1}{2}-i\lambda}\,dy\,d\Omega_1\,d\Omega_2\,d\Omega_3$$

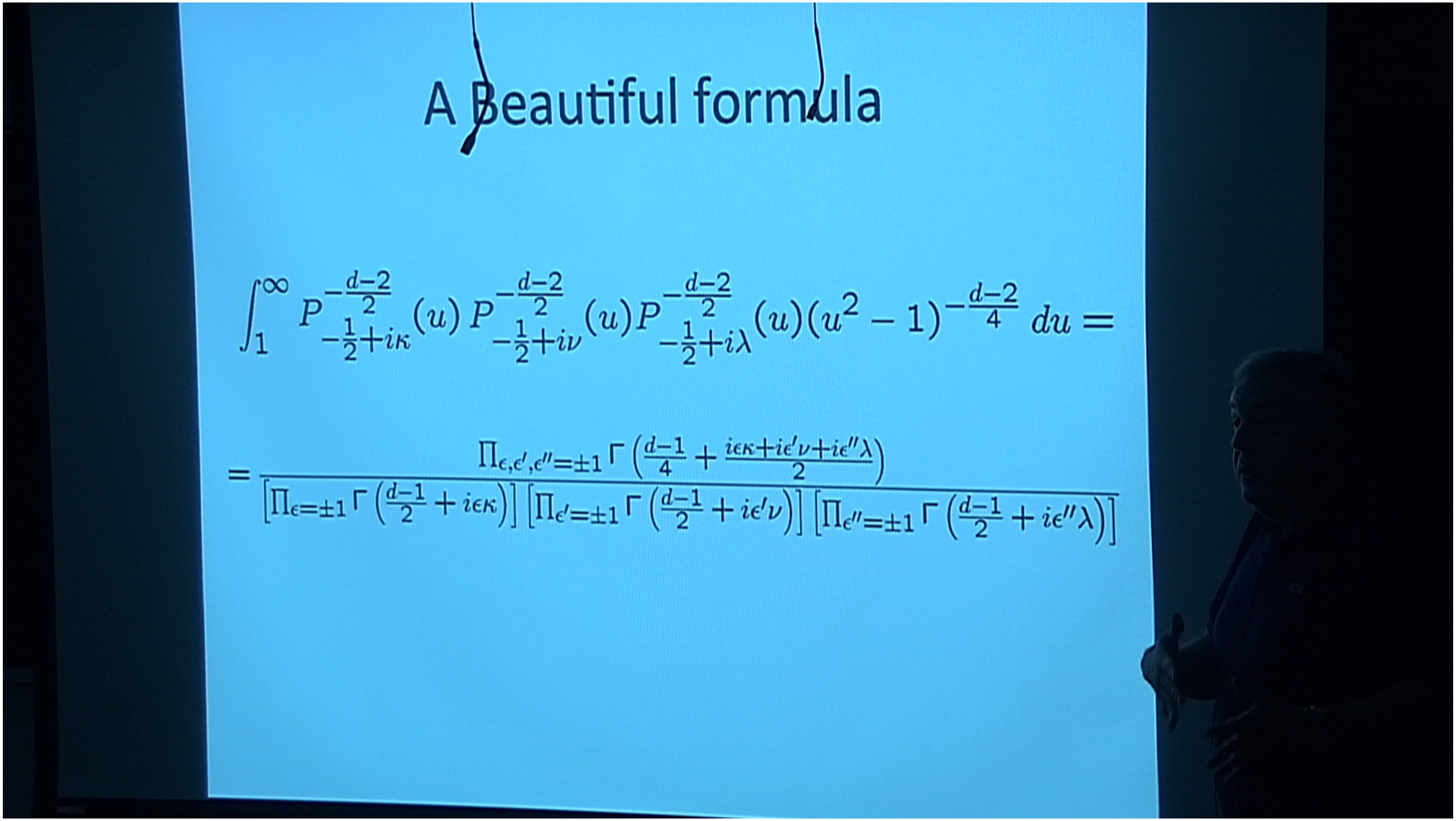



Integrate the triangle
2) Triple integral on the sphere
$J = \int_{S^3_{d-1}} (\xi_1 \cdot \xi_2)^{a_3} (\xi_2 \cdot \xi_3)^{a_1} (\xi_3 \cdot \xi_1)^{a_2} \, d\Omega_1 \, d\Omega_2 \, d\Omega_3,$
$\xi_1 = (1, \vec{\Omega}_1),$
$\xi_2 = (1, \vec{\Omega}_2),$
$\xi_3 = (1, \vec{\Omega}_3)$
$\xi_1 \cdot \xi_2 = 1 - \vec{\Omega}_1 \cdot \vec{\Omega}_2 =$
$\frac{1}{2}(|\vec{\Omega}_1|^2 + |\vec{\Omega}_2|^2) - \vec{\Omega}_1 \cdot \vec{\Omega}_2 =$
$\frac{1}{2}|\vec{\Omega}_1 - \vec{\Omega}_2|^2 = \Delta_{12}^2 = r_3^2$
$\int_{S^3_{d-1}} \Delta_{12}^{2a_3} \, \Delta_{23}^{2a_1} \, \Delta_{31}^{2a_2} \, d\Omega_1 \, d\Omega_2 \, d\Omega_3 =$
$= \int_D \rho(r_1, r_2, r_3) \, r_1^{2a_1} \, r_2^{2a_2} \, r_3^{2a_3} \, dr_1 dr_2 dr_3.$

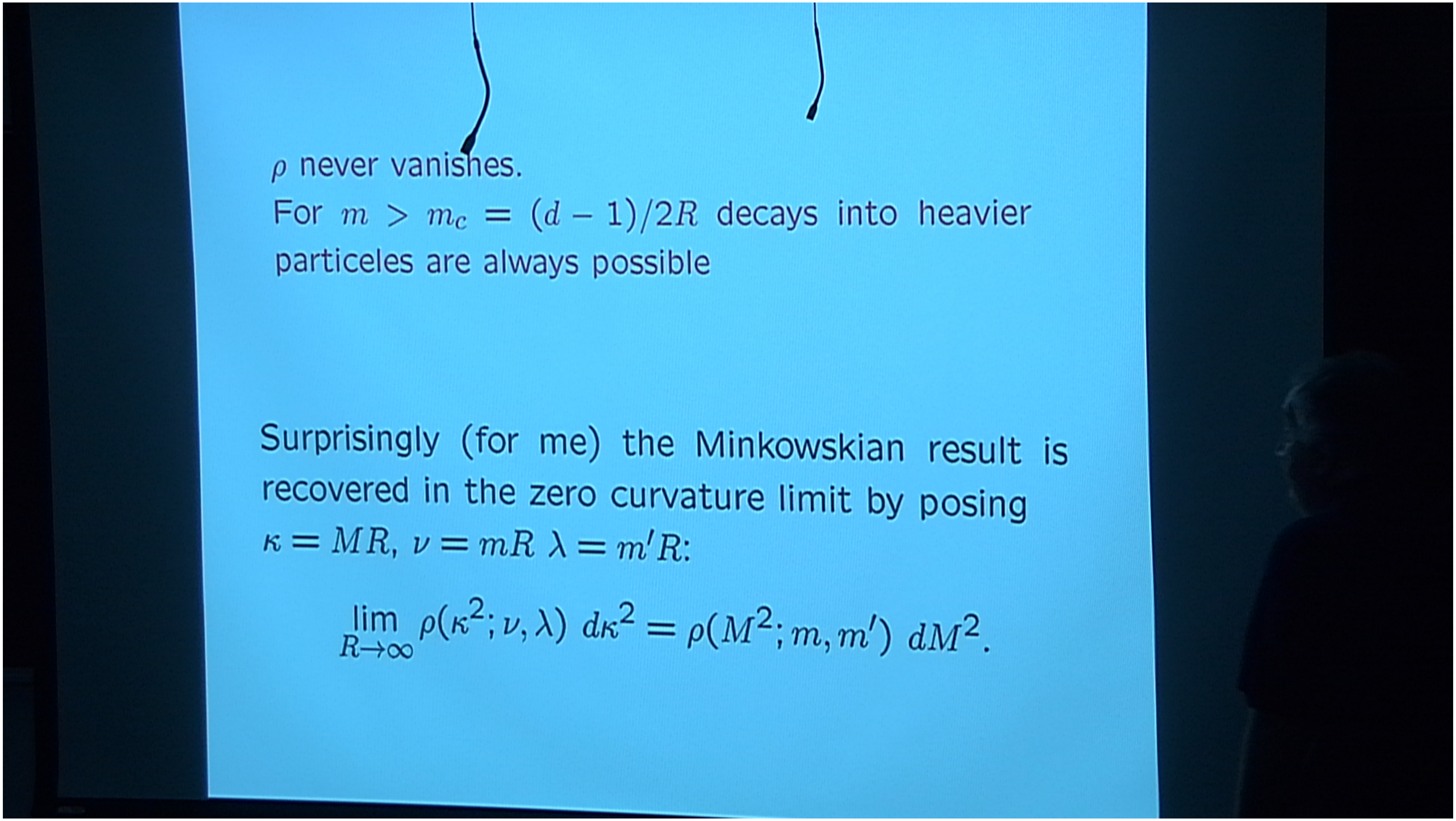
A Beautiful formula

$$\int_1^\infty P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\kappa}(u)\, P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u) P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u)(u^2-1)^{-\frac{d-2}{4}}\, du =$$

$$= \frac{\prod_{\epsilon,\epsilon',\epsilon''=\pm 1} \Gamma\left(\frac{d-1}{4} + \frac{i\epsilon\kappa + i\epsilon'\nu + i\epsilon''\lambda}{2}\right)}{\left[\prod_{\epsilon=\pm 1}\Gamma\left(\frac{d-1}{2} + i\epsilon\kappa\right)\right]\left[\prod_{\epsilon'=\pm 1}\Gamma\left(\frac{d-1}{2} + i\epsilon'\nu\right)\right]\left[\prod_{\epsilon''=\pm 1}\Gamma\left(\frac{d-1}{2} + i\epsilon''\lambda\right)\right]}$$



A Beautiful formula
$$\int_1^\infty P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\kappa}(u)\, P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u) P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u)(u^2-1)^{-\frac{d-2}{4}}\, du =$$
$$= \frac{\prod_{\epsilon,\epsilon',\epsilon''=\pm1} \Gamma\left(\frac{d-1}{4} + \frac{i\epsilon\kappa + i\epsilon'\nu + i\epsilon''\lambda}{2}\right)}{\left[\prod_{\epsilon=\pm1}\Gamma\left(\frac{d-1}{2} + i\epsilon\kappa\right)\right]\left[\prod_{\epsilon'=\pm1}\Gamma\left(\frac{d-1}{2} + i\epsilon'\nu\right)\right]\left[\prod_{\epsilon''=\pm1}\Gamma\left(\frac{d-1}{2} + i\epsilon''\lambda\right)\right]}$$


ρ never vanishes.
For $m > m_c = (d-1)/2R$ decays into heavier particeles are always possible
Surprisingly (for me) the Minkowskian result is recovered in the zero curvature limit by posing $\kappa = MR$, $\nu = mR$ $\lambda = m'R$:
$$\lim_{R\to\infty} \rho(\kappa^2; \nu, \lambda)\ d\kappa^2 = \rho(M^2; m, m')\ dM^2.$$