

Title: General Relativity for Cosmology - Lecture 14

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Abstract:



GR for Cosmology, Achim Kempf, Fall 2013, Lecture 14

Note Title

On $T^{\mu\nu}$, continued:

Recall: \square We defined $T^{\mu\nu}$ as that tensor which obeys for all $\delta g_{\mu\nu}(\lambda, x)$:

$$\frac{dS'}{d\lambda}\Big|_{\lambda=0} = \frac{1}{2} \int_{\mathcal{B}} T^{\mu\nu} \delta g_{\mu\nu} \sqrt{g} d^4x$$

$\leftarrow = \frac{dg_{\mu\nu}(\lambda, x)}{d\lambda} \Big|_{\lambda=0}$

\leftarrow (we choose $T^{\mu\nu}$ symmetric because $g_{\mu\nu}$ is symmetric)

\square The above is meant when writing:

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

$$\frac{dS'}{d\lambda}\Big|_{\lambda=0} = \frac{1}{2} \int_{\mathcal{B}} T^{\mu\nu} \delta g_{\mu\nu} \sqrt{g} d^4x$$

(we choose $T^{\mu\nu}$ symmetric because $g_{\mu\nu}$ is symmetric)

□ The above is meant when writing:

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

□ We found that $T^{\mu\nu}{}_{;\nu} = 0$ always holds.

(Since is consequence of diffeomorphism invariance)

However: □ $T^{\mu\nu}{}_{;\nu} = 0$ is not conservation law!

However: $\Delta T^{\mu\nu}_{;\nu} = 0$ is not conservation law!

Why? $T^{\mu\nu}_{;\nu} \nabla_{\mu}$ is not a divergence, unlike $K^{\mu}_{;\nu} \nabla_{\mu} = \text{div}_{\Omega} K$

Except: if space-time possesses isometries, i.e., covariant so-called Killing fields, ξ , i.e., fields obeying:

$$L_{\xi} g = 0, \text{ i.e., } \xi_{\mu;\nu} = -\xi_{\nu;\mu}$$

Because then: $P^{\mu} := T^{\mu\nu} \xi_{\nu}$ obeys $P^{\mu}_{;\mu} = 0$

Thus: $\int_B P^{\mu}_{;\mu} \nabla_{\mu} d^4x \stackrel{\text{Stokes (brau\ss)}}{=} \int_{\partial B} i_P \Omega = 0$
 $\text{div}_{\Omega} P = d i_P \Omega$ a conservation law

Proposition: maximal number of indep. Killing vector fields on spacetime: **10**

Actual spacetime has no Killing vector fields, but realistic simplified models of parts or all of spacetime often do:

Definition: A space-time (M, g) is called "stationary" if it possesses energy conservation, i.e., a time-like Killing vector field ξ , which obeys:

$$L_{\xi}g = 0 ; \quad g(\xi, \xi) < 0$$

(= 0 would be called "null" or light-like
> 0 would be called space-like)

i.e., if:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0, \quad \xi^{\mu}\xi_{\mu} < 0$$

→ In this case, observers can travel along the integral curves of ξ and they will see no change in the prevailing-

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→ In this case, observers can travel along the integral curves of ξ and they will see no change in the prevailing local curvature, i.e., following the observer, one can find a (so-called comoving) coordinate system, in which:

$$\frac{\partial}{\partial x^{\alpha}} g_{\mu\nu}(x^0, x^1, x^2, x^3) = 0$$

□ But: This does not imply that there is a cds in which

$$g = \begin{pmatrix} g_{00}, 0, 0, 0 \\ 0, & \times \\ 0, & \times \\ 0, & \times \end{pmatrix} \quad (\times)$$

Example: The g of a stationary black hole ("Kerr metric")

Definition: A space-time is called "static", if the time-like Killing field ξ , viewed as a 1-form,

$$\xi = \xi_\mu dx^\mu$$

obeys the "Frobenius condition":

$$\xi \wedge d\xi = 0 \quad (\mathcal{F})$$

↑ Exercise: write it out in coordinates

Significance? (\mathcal{F}) holds $\Leftrightarrow g = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & \text{diag}(\ast) \end{pmatrix}$ in suitable c.d.s.

- o If, in a suitable coordinate system, g is of the form (\ast) and the time-like ξ is $\xi = g_{00} dt$, then $\xi \wedge d\xi = 0$ trivially.

$$\xi \wedge d\xi = 0 \quad (\mathcal{F})$$

↑ Exercise: write it out in coordinates

Significance?

(\mathcal{F}) holds $\Leftrightarrow g = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{pmatrix}$ in suitable cds.

- o If, in a suitable coordinate system, g is of the form (\mathcal{X}) and the time-like ξ is $\xi = g_{00} dt$, then $\xi \wedge d\xi = 0$ trivially.
- o One can also show that, conversely, (\mathcal{F}) implies existence of cds in which (\mathcal{X}) holds.

Generic properties of $T^{r\nu}$:

$T^{r\nu}$

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Generic properties of $T^{\mu\nu}$:

- $T^{\mu\nu}$ has contributions from known and also from as yet unknown matter fields (e.g., from dark matter).
- Thus, in order to draw generic conclusions about, e.g.,
 - a.) the occurrence of singularities, or (Note: black hole formation stops energy dropping)
 - b.) the overall positivity of the energy (despite universal attraction!)
 one needs plausible conjectures about the full $T_{\mu\nu}$:

The "weak energy condition":

$$T_{\mu\nu} v^\mu v^\nu \geq 0 \quad \text{for all timelike } v: g(v,v) < 0$$

(by continuity it then also holds for lightlike v) 5/23

The "weak energy condition":

$$T_{\mu\nu} v^\mu v^\nu \geq 0 \quad \text{for all timelike } v: g(v,v) < 0$$

Why assume it?

(by continuity it then also holds for lightlike v)

(Note: Negative energy would be anti-gravitating i.e. repulsive.)



All observers travel with a time-like tangent v .

They then see a positive local energy density: $T_{\mu\nu} v^\mu v^\nu \geq 0$

The "dominant energy condition":

$$T_{\mu\nu} v^\mu v^\nu \geq 0 \quad \text{for all timelike } v \quad (\text{i.e., weak energy condition})$$

and

$$K_\mu := T_{\mu\nu} v^\nu \quad \text{obeys} \quad K_\mu K^\mu \leq 0 \quad (\text{i.e., } T_{\mu\nu} v^\nu \text{ is non-space-like})$$

The "dominant energy condition":

$T_{\mu\nu} v^\mu v^\nu \geq 0$ for all timelike v (i.e., weak energy condition)

and

$K_\mu := T_{\mu\nu} v^\nu$ obeys $K_\mu K^\mu \leq 0$ (i.e., $T_{\mu\nu} v^\nu$ is non-space-like)

Why assume it?

□ The local energy-momentum flow vector, K , may not be conserved but should be non-space-like:
"All flow should be into the future."

□ In an orthonormal basis, the dominant energy condition takes the form:

(Note: This is all intuition)

Why assume it?

□ The local energy-momentum flow vector, K , may not be conserved but should be non-space-like: "All flow should be into the future."

□ In an orthonormal basis, the dominant energy condition takes the form:

$$T^{00} \geq |T^{ab}|$$

i.e. "energy dominates over momentum."

(Note: This is all intuition from fluid mechanics analogy. Quantum fields may or may not behave this way.)

The dynamics of space-time!

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□ Consider the full matter action:

$$S[g, \psi] = \int_M L(g, \psi) \sqrt{|g|} d^4x$$

all matter fields: e^- , photons, quarks, gluons etc.

□ The equations of motion of matter fields are

$$\frac{\delta S}{\delta \psi^{a \dots b}_{(i) \dots d}} = 0$$

$$\Delta [g, \Psi] = \int_M L(g, \Psi) \sqrt{g} d^4x$$

all matter fields: e^- , photons, quarks, gluons etc.

The equations of motion of matter fields are

$$\frac{\delta S}{\delta \Psi_{(i)}^{a\dots b}{}_{c\dots d}} = 0$$

i.e.:

$$\frac{\partial L}{\partial \Psi_{(i)}^{a\dots b}{}_{c\dots d}} = \left(\frac{\partial L}{\partial \Psi_{(i)}^{a\dots b}{}_{c\dots d;e}} \right)_{;e}$$

Do we obtain suitable equations of motion for e by setting

□ Do we obtain suitable equations of motion for $g_{\mu\nu}$ by setting

$$\frac{\delta S'}{\delta g_{\mu\nu}} = 0 \quad ?$$

Apparently not, because it would mean:

$$\frac{\delta S'}{\delta g_{\mu\nu}} = \frac{1}{2} T^{\mu\nu} \sqrt{g} = 0 !$$

Thus, the universe would have to be empty of matter (assuming all matter has positive energy).

□ Andrei Sakharov (1968):

The quantum effects of matter induce suitable

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□ Andrei Sakharov (1968):

The quantum effects of matter induce suitable extra terms in the action!

Sakharov's reasoning: (modernized version)

□ Classical deterministic evolution of matter obeys:

$$\frac{\delta S}{\delta \psi_{(i)}} = 0$$

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- Classical deterministic evolution of matter obeys:

$$\frac{\delta S}{\delta \Psi_{(i)}} = 0$$

- But quantum theory allows every evolution $\Psi_{(i)}(x,t)$ to happen "virtually", with probability amplitudes:

$$\text{prob. ampl. } [\Psi_{(i)}] = N e^{\frac{iS[\varrho, \Psi]}{\hbar}}$$

a normalization constant.

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$$\text{prob. ampl. } [\Psi_{(i)}] = N e^{\frac{iS[g, \Psi]}{\hbar}}$$

□ As usual in quantum theory, the actual or "effective" matter evolution $\langle \Psi_{(i)}(x,t) \rangle$ is close to but not identical to the classical matter

$$\text{prob. ampl. } [\Psi_{(i)}] = N e^{\frac{iS}{\hbar}}$$

- As usual in quantum theory, the actual or "effective" matter evolution $\langle \Psi_{(i)}(x, t) \rangle$ is close to but not identical to the classical matter evolution $\Psi_{(i)}(x, t)$.

Why? Path integral picture: The field evolutions with close-to-extremal actions have very similar values $e^{\frac{iS}{\hbar}}$ because for them $\frac{\delta S}{\delta \phi} \approx 0$ i.e. their prob. amplitudes add up. Other matter evolutions $\Psi_{(j)}(x, t)$ have widely different $e^{\frac{iS}{\hbar}}$, so their probabilities average another

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⇒ Aim: Calculate the "effective action"

$$S_{\text{eff}}[g, \psi]$$

which yields the effective evolution of matter fields when matter quantum effects are taken into account.

□ Problems:

○ These calculations are very difficult.

⇒ Use perturbative methods.

□ Problems:

○ These calculations are very difficult.

⇒ Use perturbative methods.

○ There occur integrals that are divergent at short distances.

⇒ Introduce a cutoff at some minimum length $\underline{l_c}$ (or maximum momentum $\frac{\hbar}{l_c}$).

□ But as always in quantum theory, the effective action will contain terms of all possible forms that

The only question is)

The only question is which prefactors these terms will have. →

□ But as always in quantum theory, the effective action will contain terms of all possible forms that are consistent with the symmetries of the theory i.e. here with general covariance (i.e. that are scalars):

$$S_{\text{eff}}[g, \Psi] = \int_{\mathcal{M}} \left(L^{\text{matter}} + L^{\text{matter}}_{\text{quantum effects}} + c_1 + c_2 R + c_3 \Theta(R^2) \right) \sqrt{g} d^4x$$

quantum "vacuum energy" of matter

this is the local change of the vacuum energy due to curvature deforming the quantum harmonic oscillators of the field modes.

□ The constants c_1, c_2 etc depend on:

□ The constants c_1, c_2 etc depend on:

- the details of the matter action L^{matter} .
(Bosons and Fermions tend to contribute with opposite signs)
- the order of perturbation
- the value of the short-distance cutoff:

(The λ_i are unitless numbers that are roughly of order one, depending on the precise matter Lagrangian)

$$c_1 = \lambda_1 l_c^{-4} \quad (\text{must make up for } [\text{length}]^4 \text{ from } d^4x)$$

$$c_2 = \lambda_2 l_c^{-2} \quad (\text{because } R \text{ has units } [\text{length}]^{-2})$$

$$c_3 = \lambda_3 l_c^0 \quad (\text{terms } R^2 \text{ or } R^{\mu\nu} R_{\mu\nu} \text{ etc have units } [\text{length}]^{-4})$$

$$c_4 = \lambda_4 l_c^2 \quad (\text{higher powers in } R \text{ have prefactors } \sim l_c^{\text{positive power}})$$

□ \Rightarrow For small l we have:

□ Consider the lowest order terms:

$$S_{eff}[g, \psi] = \int_M (L + c_1 + c_2 R) \sqrt{g} d^4x$$

↖ total effective matter Lagrangian
↖ Einstein action
↖ -Λ cosmological constant

and postulate now that the equations of motion for the metric follow from the action principle:

$$\frac{\delta S_{eff}[g, \psi]}{\delta g_{\mu\nu}} = 0$$

□ Einstein had postulated the same action principle!

We note that:

Every (effective) quantum field theory with minimum 14 / 23

The equations of motion for g :

The action principle, $\frac{\delta S_{\text{eff}}}{\delta g_{\mu\nu}} = 0$, yields:

$$\frac{\delta}{\delta g_{\mu\nu}} \int_B (c_1 + c_2 R_{\mu\nu} g^{\mu\nu}) \sqrt{g} d^4x = -\frac{1}{2} \sqrt{g} T^{\mu\nu}$$

in principle, it is
the effective quantum
expectation value

Evaluate the left hand side:

a.)
$$\delta \int_B c_1 \sqrt{g} d^4x = \int_B c_1 \overbrace{\frac{\delta \sqrt{g}}{\delta g_{\mu\nu}}}^{\text{recall: } = \frac{1}{2} g^{\mu\nu} \sqrt{g}} \delta g_{\mu\nu} d^4x$$

$$\frac{\delta}{\delta g_{\mu\nu}} \int_B (c_1 + c_2 R_{\mu\nu} g^{\mu\nu}) \sqrt{g} d^4x = -\frac{1}{2} \sqrt{g} T^{\mu\nu}$$



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Evaluate the left hand side:

$$\begin{aligned} \text{a.)} \quad \delta \int_B c_1 \sqrt{g} d^4x &= \int_B c_1 \frac{\delta \sqrt{g}}{\delta g_{\mu\nu}} \delta g_{\mu\nu} d^4x \\ &= \int_B c_1 \frac{1}{2} g^{\mu\nu} \sqrt{g} \delta g_{\mu\nu} d^4x \end{aligned}$$

recall: $= \frac{1}{2} g^{\mu\nu} \sqrt{g}$

$$\text{b.)} \quad \delta \int c_2 R_{\mu\nu} g^{\mu\nu} \sqrt{g} d^4x$$

Evaluate the left hand side:

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 &= \int_B c_1 \frac{1}{2} g^{\mu\nu} \sqrt{g} \delta g_{\mu\nu} d^4x
 \end{aligned}$$

recall: $= \frac{1}{2} g^{\mu\nu} \sqrt{g}$

$$\begin{aligned}
 \text{b.) } \quad \delta \int_B c_2 R_{\mu\nu} g^{\mu\nu} \sqrt{g} d^4x \\
 = \underbrace{\int_B c_2 (\delta R_{\mu\nu}) g^{\mu\nu} \sqrt{g} d^4x}_{\text{Term I}} + \underbrace{\int_B c_2 R_{\mu\nu} \delta(g^{\mu\nu} \sqrt{g}) d^4x}_{\text{Term II}}
 \end{aligned}$$

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 \end{aligned}$$

Proposition: Term I = 0

Proof: Choose origin of geodesic coordinate system

$$R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu, \lambda} - \Gamma^{\lambda}_{\mu\lambda, \nu} + \underbrace{\Gamma^{\lambda}{}_{\cdot\lambda} \Gamma^{\cdot\cdot}{}_{\cdot\cdot} + \Gamma^{\cdot\cdot}{}_{\cdot\cdot} \Gamma^{\cdot\cdot}{}_{\cdot\cdot}}$$

Proposition: Term I = 0

Proof: Choose origin of geodesic coordinate system

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \underbrace{\Gamma^{\alpha\beta\gamma}\Gamma^{\delta\epsilon\zeta} + \Gamma^{\alpha\beta\gamma}\Gamma^{\delta\epsilon\zeta}}_{\text{vanish at origin because } \Gamma=0}$$

vanish at origin because $\Gamma=0$

Thus:

$$\delta R_{\mu\nu} = (\delta \Gamma^{\alpha}_{\mu\nu})_{,\alpha} - (\delta \Gamma^{\alpha}_{\mu\alpha})_{,\nu}$$

$$\Rightarrow g^{\mu\nu} \delta R_{\mu\nu} = g^{\mu\nu} (\delta \Gamma^{\alpha}_{\mu\nu})_{,\alpha} - g^{\mu\nu} (\delta \Gamma^{\alpha}_{\mu\alpha})_{,\nu}$$

$$= w^{\alpha}_{,\alpha} \text{ for } w^{\alpha} = g^{\mu\nu} \delta \Gamma^{\alpha}_{\mu\nu} - g^{\mu\alpha} \delta \Gamma^{\mu}_{\mu\nu}$$

recall: $g^{\mu\nu}_{,\alpha} = 0$ here.

$$\Rightarrow g^{\mu\nu} \delta R_{\mu\nu} = g^{\mu\nu} (\delta \Gamma^{\lambda}_{\mu\nu})_{;\lambda} - g^{\mu\nu} (\delta \Gamma^{\lambda}_{\mu\lambda})_{;\nu}$$

$$= W^{\lambda}_{;\lambda} \text{ for } W^{\lambda} = g^{\mu\nu} \delta \Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda} \delta \Gamma^{\nu}_{\mu\nu}$$

recall: $g^{\mu\nu}_{;\alpha} = 0$ here.

Thus, in arbitrary coordinate system:

$$g^{\mu\nu} \delta R_{\mu\nu} = W^{\lambda}_{;\lambda}$$

$$\Rightarrow \int_B c_2 g^{\mu\nu} \delta R_{\mu\nu} \sqrt{|g|} d^4x = \int_B W^{\lambda}_{;\lambda} \sqrt{|g|} d^4x$$

$$= \int_B \text{div}_W \Omega$$

$\int_{\text{Grenz}} i_W \Omega$

$= 0$ on ∂B , assuming $\delta g_{\mu\nu}$ and $\delta g_{\mu\nu,s} = 0$ on ∂B .

c.) Evaluate term II:

$$\int_B c_2 R_{\mu\nu} \delta(g^{\mu\nu} \sqrt{g}) d^4x = ?$$

We have:

$$\begin{aligned} \delta(g^{\mu\nu} \sqrt{g}) &= (\delta g^{\mu\nu}) \sqrt{g} + g^{\mu\nu} \frac{\partial \sqrt{g}}{\partial g_{ab}} \delta g_{ab} \\ &= -g^{\mu a} g^{\nu b} \delta g_{ab} \sqrt{g} + g^{\mu\nu} \frac{1}{2} g^{ab} \sqrt{g} \delta g_{ab} \end{aligned}$$

\Rightarrow

$$\int_B c_2 R_{\mu\nu} \delta(g^{\mu\nu} \sqrt{g}) d^4x = - \int_B c_2 \left(+R^{ab} - \frac{1}{2} g^{ab} R \right) \sqrt{g} \delta g_{ab} d^4x$$

recall:

$$= c^{ab}$$

"Einstein"

B

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⇒

$$\int_B c_2 R_{\mu\nu} \delta(g^{\mu\nu} \sqrt{g}) d^4x = - \int_B c_2 \left(+R^{ab} - \frac{1}{2} g^{ab} R \right) \sqrt{g} \delta g_{ab} d^4x$$

recall: $= G^{ab}$ "Einstein tensor"

Bringing together a) + b) + c) ⇒



$$\delta \int_B (c_1 + c_2 R_{\mu\nu} g^{\mu\nu}) \sqrt{g} d^4x$$

$$= \int_B \underbrace{\left(c_1 \frac{1}{2} g^{\mu\nu} - c_2 G^{\mu\nu} \right)}_{\text{symmetric}} \sqrt{g} \delta g_{\mu\nu} d^4x$$

as in the case of the $T^{\mu\nu}$ calculation, one could add an anti-symmetric part here and it would drop from the integrand.

$$\Rightarrow \frac{\delta}{\delta g^{\mu\nu}} \int_B (c_1 + c_2 R_{\mu\nu} g^{\mu\nu}) \sqrt{g} d^4x$$

$$= \left(c_1 \frac{1}{2} g^{\mu\nu} - c_2 G^{\mu\nu} \right) \sqrt{g}$$

(up to anti-sym. components, which we set to zero)

$$\Rightarrow \frac{\delta}{\delta g_{\mu\nu}} \int_B (c_1 + c_2 R_{\mu\nu} g^{\mu\nu}) \sqrt{g} d^4x$$

(up to anti-sym. components, which we set to zero)

$$= (c_1 \frac{1}{2} g^{\mu\nu} - c_2 G^{\mu\nu}) \sqrt{g}$$

Remark: Choosing to add an antisymmetric part would make $G^{\mu\nu}$ and $R^{\mu\nu}$ non-symmetric, in violation of $g_{\mu\nu; \rho} = 0$.

Finally, we conclude:

$\delta S'$

Finally, we conclude:

$$\frac{\delta S'}{\delta g_{\mu\nu}} = 0$$

leads to this equation of motion for g :

$$\left(\frac{1}{2}c_1 g^{\mu\nu} - c_2 G^{\mu\nu}\right) \nabla_{\sigma} g^{\sigma\lambda} = -\frac{1}{2} \nabla_{\sigma} T^{\mu\nu}$$

i.e.:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \frac{c_1}{2c_2} g^{\mu\nu} = \frac{1}{2c_2} T^{\mu\nu}$$

As is well-known, comparison with experiment requires:

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As is well-known, comparison with experiment requires:

$$c_2 = \frac{1}{16\pi G}$$

↑ Newton's constant.

Notice: the "reduced Bianchi identity" $(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)_{;\nu} = 0$ now is equivalent to the statement $T^{\mu\nu}_{;\nu} = 0$

Notice: The symmetry of $R^{\mu\nu}$ (due to $g_{\mu\nu};\rho = 0$) enforces the ...

$16\pi G$

Newton's constant.

Einstein equation:

(Notice: the "reduced Bianchi identity" $(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)_{;\nu} = 0$ now is equivalent to the statement $T^{\mu\nu}_{;\nu} = 0$)

Notice: The symmetry of $R^{\mu\nu}$ (due to $g_{\mu\nu};\rho = 0$) enforces the symmetry of $T^{\mu\nu}$.

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R - \underbrace{8\pi G c_1}_{\Lambda \text{ "cosmol. constant"}} g^{\mu\nu} = 8\pi G T^{\mu\nu}$$

What is Λ ?

Recall: The λ_i are constants prefactor of order one which depends on the matter Lagrangian.

□ First find l_c :

(i.e. also: $G \approx l_c^{-2}$)

Given that $\frac{1}{16\pi G} = c_2 = \lambda_2 l_c^{-2}$



What is Λ ?

Recall: The λ_i are constants prefactor of order one which depends on the matter Lagrangian.

□ First find l_c :

(I.e. also: $G \approx l_c^2$, which we'll need on the next slide)

Given that $\frac{1}{16\pi G} = c_2 = \lambda_2 l_c^{-2}$

we obtain:

$$l_c = \lambda_2 \sqrt{16\pi G} = \sqrt{16\pi \frac{G}{c}}$$

undoing $\hbar=1$ and $c=1$ so that get $1/g_{m,s}$:

$$= \lambda_2 \cdot 4\sqrt{\pi} \cdot 1.616 \times 10^{-35} \text{ m}$$

the "Planck length"

Recall: The λ_i are some numbers of order $O(1)$, depending on details which sort of particles are in the matter action, and their quantum effects

We note, therefore: When quantum field theories are cut off at about the Planck length they induce gravity

□ Now find c_1 :

Given that $l_c \approx 10^{-35}$ m, quantum field theories generate a value of c_1 , i.e., a cosmological constant of about:

$$c_1 = \lambda, l_c^{-4}$$

of \uparrow $\mathcal{O}(1)$

□ Finally, find Λ :

$$\begin{aligned}\Lambda_{\text{theory}} &= -8\pi G c_1 = -8\pi G \lambda, l_c^{-4} \\ &\approx G l_c^{-4} \approx l_c^{-2} \quad (\text{using } \lambda, \approx 1 \text{ and } G \approx l_c^2) \\ &\approx 10^{70} \text{ m}^{-2}\end{aligned}$$

This is the worst physical prediction ever.

Given that $l_c \approx 10^3 \text{ m}$, quantum field theories generate a value of c_1 , i.e., a cosmological constant of about:

$$c_1 = \lambda_1 l_c^{-4}$$

of $\uparrow \mathcal{O}(1)$

□ Finally, find Λ :

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□ Experiment:

Based on cosmic microwave background data and on supernova brightness versus redshift data:

$$\Lambda_{\text{experiment}} \approx 10^{-52} \text{ m}^{-2} \quad \text{i.e.} \quad \frac{\Lambda_{\text{th}}}{\Lambda_{\text{exp}}} \approx 10^{122}$$

→ For some unknown reason, the constant part, c_1 , of the vacuum energy of quantum field theory

□ Experiment:

Based on cosmic microwave background data and on supernova brightness versus redshift data:

$$\Lambda_{\text{experiment}} \approx 10^{-52} \text{ m}^{-2} \quad \text{i.e.} \quad \frac{\Lambda_{\text{th}}}{\Lambda_{\text{exp}}} \approx 10^{122}$$

→ For some unknown reason, the constant part, c_1 , of the vacuum energy of quantum field theories does essentially not gravitate - while its disturbance through curvature, $c_2 R$, is real: it induces regular gravity