Title: General Relativity for Cosmology - Lecture 12

Date: Oct 24, 2013  04:00 PM

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Abstract:

**Status:** Almost ready for GR, i.e., for spacetime dynamics.

But first: Need dynamics of matter fields in curved space, because it will affect the dynamics of spacetime.

**Recall:** On a (pseudo)-Riemannian m/ld, equations are well-defined only if defined independently of any chart.

$\Rightarrow$ Any eqn, including the eqns of motions for matter fields must be eqns among tensors and their covariant derivatives.

$\Rightarrow$ Need a tensor field, $\Psi$, for each species of particle:

$$e^-, \, q, \, gluon, \, \pi^{\pm}_0, \, photon, \, W^{\pm}, \, etc...$$

Notation:

$$\Psi_{(i)}^{\,a...b}{}_{c...d}$$

— contravariant

— covariant

↳ species label

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tensors and their covariant derivatives.

⟹ Need a tensor field, $\Psi$, for each species of particle:

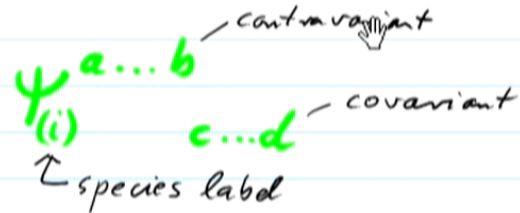$e^-$, $q$, gluon, $\pi^\pm_0$, photon, $W^\pm$, etc...

Notation:

$\Psi^{a...b}_{(i)\ c...d}$

contravariant

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⤷ species label

Note: any spinor equation can also

be expressed as a (complicated) tensor equation

(see e.g. Hawking & Ellis, p. 59)

**Remark:** Could we have also a non-tensor field for some species of matter, namely an additional connection field $\tilde{\Gamma}^{k}_{ij}$?

**Yes, we could:** But, the difference field $Q^{k}_{ij} := \Gamma^{k}_{ij} - \tilde{\Gamma}^{k}_{ij}$ is actually a tensor field!

$$\Gamma^{r}_{ab} \longrightarrow \frac{\partial \bar{x}^{r}}{\partial x^{\mu}} \frac{\partial^{2} x^{j}}{\partial \bar{x}^{a} \partial \bar{x}^{b}} + \frac{\partial \bar{x}^{r}}{\partial x^{\mu}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \Gamma^{k}_{ij}$$

$$\tilde{\Gamma}^{r}_{ab} \longrightarrow \frac{\partial \bar{x}^{r}}{\partial x^{\mu}} \frac{\partial^{2} x^{j}}{\partial \bar{x}^{a} \partial \bar{x}^{b}} + \frac{\partial \bar{x}^{r}}{\partial x^{\mu}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \tilde{\Gamma}^{k}_{ij}$$

$$\Rightarrow \left( \Gamma^{r}_{ab} - \tilde{\Gamma}^{r}_{ab} \right) \longrightarrow \frac{\partial \bar{x}^{r}}{\partial x^{\mu}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \left( \Gamma^{k}_{ij} - \tilde{\Gamma}^{k}_{ij} \right)$$

**Yes, we could:** But, the difference field $Q^{\kappa}{}_{ij} := \Gamma^{\kappa}{}_{ij} - \widetilde{\Gamma}^{\kappa}{}_{ij}$ is actually a tensor field!

$$\Gamma^{r}{}_{ab} \longrightarrow \frac{\partial \bar{x}^{r}}{\partial x^{\kappa}} \frac{\partial^{2} x^{i}}{\partial \bar{x}^{a} \partial \bar{x}^{b}} + \frac{\partial \bar{x}^{r}}{\partial x^{\kappa}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \Gamma^{\kappa}{}_{ij}$$

$$\widetilde{\Gamma}^{r}{}_{ab} \longrightarrow \frac{\partial \bar{x}^{r}}{\partial x^{\kappa}} \frac{\partial^{2} x^{i}}{\partial \bar{x}^{a} \partial \bar{x}^{b}} + \frac{\partial \bar{x}^{r}}{\partial x^{\kappa}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \widetilde{\Gamma}^{\kappa}{}_{ij}$$

$$\Rightarrow \left( \Gamma^{r}{}_{ab} - \widetilde{\Gamma}^{r}{}_{ab} \right) \longrightarrow \frac{\partial \bar{x}^{r}}{\partial x^{\kappa}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \left( \Gamma^{\kappa}{}_{ij} - \widetilde{\Gamma}^{\kappa}{}_{ij} \right)$$

$$\Rightarrow \boxed{ Q^{r}{}_{ab} \longrightarrow \frac{\partial \bar{x}^{r}}{\partial x^{\kappa}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \; Q^{\kappa}{}_{ab} }$$

i.e. $Q^{r}{}_{ab}$ is a tensor due to having the correct transformation property according to the physicist's definition of a tensor.

$\Rightarrow$ Introducing an additional connection $\widetilde{\Gamma}$ is same as introducing simply a new tensor field $Q$

$$\Rightarrow \qquad Q^{\nu}{}_{ab} \longrightarrow \frac{\partial \bar{x}^{\prime}}{\partial x^{r}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \; Q^{\nu}{}_{ab}$$

i.e. $Q^{\nu}{}_{ab}$ is a tensor due to having the correct transformation property according to the physicist's definition of a tensor.

$\Rightarrow$ Introducing an additional connection $\widetilde{\Gamma}$ is same as introducing simply a new tensor field Q.

Notice: Thus, "variations" $\delta\Gamma^{\nu}{}_{ab}$ will behave tensorially!

Equs of motion of matter fields?

Action principle: (As in special relativity)

_Equs of motion of matter fields:_

## Action principle: (As in special relativity)

Any theory of matter fields can be defined by specifying the so-called Lagrangian function, $L$, namely a scalar function of the matter fields $\psi_{(i)}^{a...b}{}_{c...d}$ and their first covariant derivatives, and now also of the metric $g$ :

$$\underset{\text{└ we'll sometimes omit the indices}}{}$$

by specifying the so-called Lagrangian
function, L, namely a scalar function
of the matter fields $\Psi_{(i)}^{a...b}{}_{c...d}$ and their first
covariant derivatives, <u>and now also of the metric $g$:</u>
↳ we'll sometimes omit the indices

$$L(\Psi) = L^{(matter)}\left(\left\{\Psi_{(i)}^{a...b}{}_{c...d}\right\}, \left\{\Psi_{(i)}^{a...b}{}_{c...d;e}\right\}, g\right)$$

◻ Define the action functional:

□ Define the action functional:

$$S[\Psi] := \int_B \underbrace{\overbrace{L(\Psi)}^{\text{scalar}} \overbrace{\sqrt{g}\, d^4x}^{\Omega = \text{volume form}}}_{n-\text{form}} \in \mathbb{R}$$

↖ some bounded
and closed 4-dim
region in M.

Thus, each physical field $\Psi(x,t)$ (as a function of both space *and time*) is mapped into a number $S[\Psi]$.

□ Action principle (or postulate) of classical physics:

☐ Action principle (or postulate) of classical physics:

In nature, physical fields $\Psi$ are such that

$S[\Psi]$ is extremal in the space of all fields $\Psi$.

☐ Thus: The matter fields $\Psi$ obey:

$$\frac{\delta S[\Psi]}{\delta \Psi} = 0 \qquad (\textbf{*})$$

These will be the eqns of motion for the fields $\Psi$

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$$S[\Psi] := \int_B L(\Psi) \sqrt{g} \, d^4x \qquad \in \mathbb{R}$$

$n$-form

$\leftarrow$ some bounded
and closed 4-dim
region in $M$.

Thus, each physical field $\Psi(x,t)$ (as a function of both space _and time_) is mapped into a number $S[\Psi]$.

▢ Action principle (or postulate) of classical physics:

In nature, physical fields $\Psi$ are such that $S[\Psi]$ is extremal in the space of all fields $\Psi$.

**Thus:** The matter fields $\Psi$ obey:

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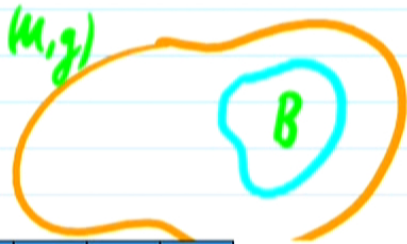These will be the eqns of motion for the fields $\Psi$.

**Definition of $(*)$ ?**

**Def:** A "variation $\delta \Psi$" of the fields $\Psi_{ij}(p)$ $\qquad$ $^{p \in B \subset M}$

in a region $B$ is a one-parameter

☐ <u>Definition of (✗) ?</u>

**Def:** A "variation $\delta \Psi$" of the fields $\Psi_{(i)}(p)$    $p \in B \subset M$

in a region $B$ is a one-parameter

deformation, $\Psi_{(i)}(\lambda, p)$, with $\lambda \in (-\varepsilon, \varepsilon)$,   some finite interval

$\downarrow$ deformation parameter

so that     i.e. $\lambda = 0$ is non-deformation

$(M, g)$

1.) $\Psi_{(i)}(0, p) = \Psi_{(i)}(p)$    $\forall p \in M$

2.) $\Psi_{(i)}(\lambda, p) = \Psi_{(i)}(p)$    $\forall \lambda$, if $p \in M - R$

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## Definition of (✗) ?

$p \in B \subset M$

**Def:** A "variation $\delta \Psi$" of the fields $\Psi_{(i)}(p)$
in a region $B$ is a one-parameter
deformation, $\Psi_{(i)}(\lambda, p)$, with $\lambda \in (-\varepsilon, \varepsilon)$,    $\underbrace{\quad}$ some finite interval

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$(M, g)$

$B$

1.) $\Psi_{(i)}(0, p) = \Psi_{(i)}(p)$    $\forall p \in M$

2.) $\Psi_{(i)}(\lambda, p) = \Psi_{(i)}(p)$    $\forall \lambda$, if $p \in M - B$

**Def:** A variation $\delta I$ of the fields $\psi_{(i)}(p)$ in a region $B$ is a one-parameter deformation, $\psi_{(i)}(\lambda, p)$, with $\lambda \in (-\varepsilon, \varepsilon)$, $\overbrace{\phantom{some finite}}^{\text{some finite}}$ $^{\text{interval}}$

$\hookrightarrow$ deformation parameter

so that $\overset{\text{i.e. } \lambda = 0 \text{ is non-deformation}}{\swarrow}$
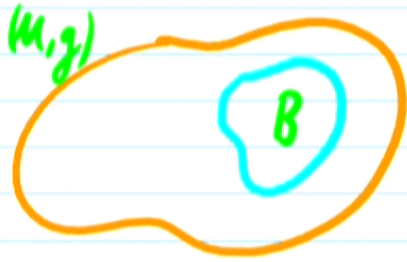
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1.) $\psi_{(i)}(0, p) = \psi_{(i)}(p) \quad \forall p \in M$

2.) $\psi_{(i)}(\lambda, p) = \psi_{(i)}(p) \quad \forall \lambda, \text{ if } p \in M - B$

$\nwarrow$ i.e. no deformation at all outside region $B$.

**Def:** Then, we define:

$(M,g)$



1.) $\quad \Psi_{(i)}(0,\rho) = \Psi_{(i)}(\rho) \qquad \forall \rho \in M$

2.) $\quad \Psi_{(i)}(\lambda,\rho) = \Psi_{(i)}(\rho) \qquad \forall \lambda, \text{ if } \rho \in M-B$

$\quad$ i.e. no deformation at all outside region $B$.

**Def:** Then, we define:

$$\delta \Psi_{(i)}(\rho) := \left. \frac{\partial \Psi_{(i)}(\lambda,\rho)}{\partial \lambda} \right|_{\lambda=0}$$

**Def:** The action principle now reads:

$$0 = \left. \frac{\partial S[\Psi]}{\partial \lambda} \right| \qquad \text{for all variations } \delta\Psi_{(i)}.$$

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2.) $\quad \Psi_{(i)}(\lambda, p) = \Psi_{(i)}(p) \quad \forall \lambda$, if $p \in M - B$

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Def: The action principle now reads:

$$0 = \left. \frac{\partial S'[\Psi]}{\partial \lambda} \right|_{\lambda = 0} \qquad \text{for all variations } \delta \Psi_{(i)}.$$

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## Evaluate:

Term I

$$0 = \frac{\partial S}{\partial \lambda}\Big|_{\lambda=0} = \sum_i \int_B \left[ \frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}} \delta \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d} \right.$$

$$\text{recall:} \quad \equiv \frac{d \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}{d\lambda}\Big|_{\lambda=0}$$

Term II

$$\left. + \frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}} \delta\left(\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\right) \right] \sqrt{g}\, d^4x$$

by assumption,
L depends also on
the 1st cov. derivatives.

$$\boxed{\frac{\delta S[\Psi]}{\delta \Psi} = 0}$$

(✱)

These will be the eqns of motion for the fields $\Psi$.

☐ <u>Definition of (✱) ?</u>

$p \in B \subset M$

Def: A "variation $\delta \Psi$" of the fields $\Psi_{ii}(p)$
in a region $B$ is a one-parameter
deformation. $\Psi_{ii}(\lambda, p)$ with $\lambda \in (-\varepsilon, \varepsilon)$.

some finite
interv.

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□ Define the action functional:

$$S[\Psi] := \int_B \underbrace{\overbrace{L(\Psi)}^{scalar} \underbrace{\sqrt{g}\, d^4 x}_{\substack{\Omega\,=\,volume\ form}}}_{n-form} \quad \in \mathbb{R}$$

↖ some bounded
and closed 4-dim
region in M.

Thus, each physical field $\Psi(x,t)$ (as a function of both space and time) is mapped into a number $S[\Psi]$.

□ Action principle (or postulate) of classical physics:

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*i.e. ...*

so that

$(M, g)$

$B$

1.) $\quad \Psi_{(i)}(0, p) = \Psi_{(i)}(p) \qquad \forall\, p \in M$

2.) $\quad \Psi_{(i)}(\lambda, p) = \Psi_{(i)}(p) \qquad \forall \lambda, \text{ if } p \in M - B$
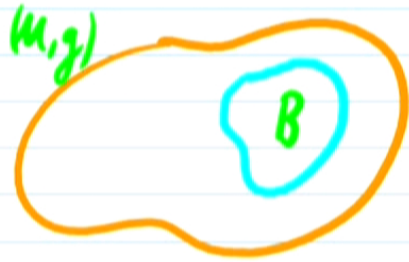
*i.e. no deformation at all outside region B.*

**Def:** Then, we define:

$$\delta \Psi_{(i)}(p) := \left. \frac{\partial \Psi_{(i)}(\lambda, p)}{\partial \lambda} \right|_{\lambda = 0}$$

**Def:** The action principle now reads:

$\partial S[\Psi]$

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i.e. ~~~~ ~~ ~~~~ ~~~~~~~~~~

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**Def:** The action principle now reads:

$$0 = \frac{\partial S'[\Psi]}{\partial \lambda}\bigg|_{\lambda=0} \qquad \text{for all variations } \delta\Psi_{(i)}.$$

**Evaluate:**

$$0 = \frac{\partial S}{\partial \lambda}\bigg|_{\lambda=0} = \sum_i \int_B \left[ \underbrace{\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}}_{Term\ I} \delta \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d} \right.$$

Term I

$$recall: = \frac{d\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}{d\lambda}\bigg|_{\lambda=0}$$

Term II

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## Evaluate:

$$0 = \frac{\partial S}{\partial \lambda}\Big|_{\lambda=0} = \sum_i \int_B \left[ \overbrace{\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}} \, \delta \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}^{Term\ I} \right.$$

recall: $= \dfrac{d \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}{d\lambda}\Big|_{\lambda=0}$

$$\left. + \overbrace{\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}} \, \delta\left(\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\right)}^{Term\ II} \right] \sqrt{g}\, d^4x$$

by assumption,
$L$ depends also on
the 1st cov. derivatives.

# Term II :

**Recall:** At origin of geodesic coordinate system, $\Gamma^k_{ij} = 0$, i.e. $\psi_{;e} = \psi_{,e}$. But then $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial \lambda}$ commute. True in any coordinate system.

▫ **We notice:**

$$\delta\left(\psi_{(i)}{}^{a\cdots b}{}_{c\cdots d;e}\right) = \left(\delta\psi_{(i)}{}^{a\cdots b}{}_{c\cdots d}\right)_{;e}$$

$$\Rightarrow \text{Term II} = \sum_i \int_B \frac{\partial L}{\partial \psi_{(i)}{}^{a\cdots b}{}_{c\cdots d;e}} \left(\delta\psi_{(i)}{}^{a\cdots b}{}_{c\cdots d}\right)_{;e} \sqrt{g}\, d^4x$$

$$= \sum_i \int_B \left[ \underbrace{\left(\frac{\partial L}{\partial \psi_{(i)}{}^{a\cdots b}{}_{c\cdots d;e}} \delta\psi_{(i)}{}^{a\cdots b}{}_{c\cdots d}\right)}_{=: K^e}{}_{;e} \right.$$

$$\left(\begin{array}{l}\text{use Leibniz}\\\text{rule to verify}\end{array}\right)$$

in any coordinate system.

$$\delta\left(\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}\right) = \left(\delta\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\right)_{;e}$$

$$\Rightarrow \quad \text{Term II} = \sum_i \int_B \frac{\partial L}{\partial\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\left(\delta\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\right)_{;e}\sqrt{g}\,d^4x$$

$$=: K^e$$

$$= \sum_i \int_B \left[\left(\frac{\partial L}{\partial\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\delta\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\right)_{;e}\right.$$

$$\left.\quad -\left(\frac{\partial L}{\partial\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\right)_{;e}\delta\Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\right]\sqrt{g}\,d^4x$$

(use Leibnit rule to verify)

One term is a "boundary term":

$$\sum_i \int_B K^e_{;e} \sqrt{g}\, d^4x$$

$$= \sum_i \int_B \mathrm{div}_\Omega K$$

**Exercise:**

show that for all $\xi^\mu$:

$$\xi^\mu_{;\mu} \sqrt{\Omega} = \mathrm{div}_\Omega \xi$$

**Gauß' theorem** $\Rightarrow$

$$= \sum_i \int_{\partial B} i_K \Omega$$

← inner derivation

$$\left( \begin{array}{l} \text{Recall: } \mathrm{div}_\Omega K = L_K \Omega \\ = (i_K \circ d + d \circ i_K)\Omega \\ = d \circ i_K \Omega \end{array} \right)$$

**but:** $K \propto \delta\Psi$ and $\delta\Psi(p) = 0$ if $p \in \partial B$

by property 2) of variations.

$$\{ \text{''}_{in} \ \Omega = div_\Omega \ \} $$

$$= \sum_i \int_B div_\Omega K$$

**Gauß' theorem** $\Rightarrow$

inner derivation

$$= \sum_i \int_{\partial B} i_K \Omega$$

$$\left( \begin{array}{l} \text{Recall:} \ div_\Omega K = L_K \Omega \\ \qquad = (i_K \circ d + d \circ i_K) \Omega \\ \qquad = d \circ i_K \Omega \end{array} \right)$$

**but:** $K \propto \delta \Psi$ and $\delta \Psi(p) = 0$ if $p \in \partial B$

by property 2) of variations.

$\Rightarrow \qquad = 0 \qquad !$

$$\Rightarrow \text{Term II} = \sum_i \int_B \frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}} \left(\delta \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}\right)_{;e} \sqrt{g}\, d^4x$$

$$= \sum_i \int_B \left[ \left( \underbrace{\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}} \delta \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d}}_{=:\, K^e} \right)_{;e} \right.$$

$$\left. - \left(\frac{\partial L}{\partial \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d;e}}\right)_{;e} \delta \Psi_{(i)}{}^{a\ldots b}{}_{c\ldots d} \right] \sqrt{g}\, d^4x$$

$$\left( \begin{array}{l} \text{use Leibnit} \\ \text{rule to verify} \end{array} \right)$$

One term is a "boundary term":

$$\sum_i \int_B K^e{}_{;e} \sqrt{g}\, d^4x$$

**Exercise:**

show that for all $\xi^a$:

Thus, term II simplifies and we obtain:

$$0 = \frac{\partial S}{\partial \lambda}\bigg|_{\lambda=0} = \sum_i \int_B \left[ \underbrace{\frac{\partial L}{\partial \psi_{(i)\ c\ldots d}^{a\ldots b}} \delta \psi_{(i)\ c\ldots d}^{a\ldots b}}_{\text{Term I}} - \overbrace{\left(\frac{\partial L}{\partial \psi_{(i)\ c\ldots d;e}^{a\ldots b}}\right)_{;e} \delta \psi_{(i)\ c\ldots d}^{a\ldots b}}^{\text{Term II}} \right] \sqrt{g}\, d^4x$$

Since must hold for all variations $\delta \psi$

$$\Rightarrow \boxed{\frac{\partial L}{\partial \psi_{(i)\ c\ldots d}^{a\ldots b}} - \left(\frac{\partial L}{\partial \psi_{(i)\ c\ldots d;e}^{a\ldots b}}\right)_{;e} = 0}$$

"Euler-Lagrange equations"

# Example:   A real-valued scalar field $\Psi$ ← real-valued

☐ Such $\Psi$ describe e.g.:

- $\pi^0$ meson (quark + antiquark)
- inflaton

☐ Lagrangian?

- Choose geodesic cds at orb. point and appeal to equiv. principle.
- Obtain from spec. relat. Lagrangian:

$$L = -\frac{1}{2}\left( \Psi_{;a}\, \Psi_{;b}\, g^{ab} + \frac{m^2}{\hbar^2}\Psi^2 \right)$$

☐ Euler-Lagrange equation:   Klein-Gordon equati

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❑ Lagrangian ?

  – Choose geodesic cds at orb. point
    and appeal to equiv. principle.
  – Obtain from spec. relat. Lagrangian:

$$L = -\frac{1}{2}\left( \Psi_{;a} \Psi_{;b} \, g^{ab} + \frac{m^2}{\hbar^2} \Psi^2 \right)$$

❑ Euler - Lagrange equation: Klein-Gordon equation

(Exercise: verify)

$$\boxed{\Psi_{;ab} \, g^{ab} - \frac{m^2}{\hbar^2} \Psi = 0}$$

Example:   The electromagnetic fields

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# Example:   The electromagnetic fields

◻ Assume there are no charges
(i.e. there are only EM waves)

◻ Define the "EM 4-potential" as a
real-number-valued one-form $A$.

◻ Consider the field strength tensor $F$:

$$F := dA$$

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□ The Lagrangian (from equiv. principle):

$$L = -\frac{1}{16\pi} F_{ab} F_{cd} \, g^{ac} g^{bd}$$

(Exercise: write in terms of forms)

□ Varying w. resp. to $A$, the E.L. equations read:

$$F_{ab;c} \, g^{bc} = 0$$

recall: this is $\delta F = 0$

"Maxwell eqns".

□ It is also true that

$$F_{ab;c} + F_{ca;b} + F_{bc;a} = 0$$

but this is not an Euler Lagrange eqn. It

*Example:* A charged scalar field $\psi$, $\longleftarrow$ complex-valued

(such $\psi$ describe, e.g., $\pi^{\pm}$ mesons)

together with electromagnetism.

△ Equiv. principle yields from spec. relativity:

Why $\psi$ complex?
Mixed term is Lorentz force
If $\psi$ was real, it would be
absent:

$-ie A_a \psi^* \psi_{;b} g^{ab}$
$+ie A_b \psi^*_{;a} \psi g^{ab}$

electric charge constant

$$L = -\frac{1}{2}\left(\psi^*_{;a} - ie A_a \psi^*\right)\left(\psi_{;b} + ie A_b \psi\right) g^{ab}$$

$$-\frac{1}{2}\frac{m^2}{\hbar^2}\psi^*\psi - \frac{1}{11-}F_{ab}F_{cd}\,g^{ac}g^{bd}$$

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Why $\Psi$ complex?

Mixed term is Lorentz force
If $\Psi$ was real, it would be absent:

$-ie A_a \Psi^* \Psi_{;b} g^{ab}$
$+ie A_b \Psi^*_{;a} \Psi g^{ab}$
$= ie A_a g^{ab} (\Psi^*_{;a} \Psi - \Psi_{;a} \Psi^*)$
$= 0$ if $\Psi^* = \Psi$

electric charge constant

$$L = -\frac{1}{2}\left(\Psi^*_{;a} - ie A_a \Psi^*\right)\left(\Psi_{;b} + ie A_b \Psi\right) g^{ab}$$

$$-\frac{1}{2}\frac{m^2}{\hbar^2}\Psi^*\Psi - \frac{1}{16\pi} F_{ab} F_{cd}\, g^{ac} g^{bd}$$

▢ **Vary** w. resp. to $\Psi^* \Rightarrow$ E.L. eqn:

$$\Psi_{;ab}\, g^{ab} - \frac{m^2}{\hbar^2}\Psi + ie A_a g^{ab}\left(\Psi_{;b} + ie A_b \Psi\right) + ie A_{a;b}\, g^{ab}\Psi = 0$$

Klein Gordon part          $\Psi$ is affected by A

and varying w. resp. to $\Psi$ yields the complex conj. equation

(Exercise: verify)

$$T_{;ab} g \quad - \frac{m}{t^2} \varphi = 0$$

**Example:**   <u>The electromagnetic fields</u>

- Assume there are no charges (i.e. there are only EM waves)

- Define the "EM 4-potential" as a real-number-valued one-form A.

- Consider the field strength tensor F:

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□ Vary w. resp. to $\Psi^* \Rightarrow$ E.L. eqn:

$$\Psi_{;ab} g^{ab} - \frac{m^2}{\hbar^2} \Psi + ie A_a g^{ab} (\Psi_{;b} + ie A_b \Psi) + ie A_{a;b} g^{ab} \Psi = 0$$

$\underbrace{\phantom{\Psi_{;ab} g^{ab} - \frac{m^2}{\hbar^2} \Psi}}_{\text{Klein Gordon part}}$   $\underbrace{\phantom{+ie A_a g^{ab} (\Psi_{;b} + ie A_b \Psi) + ie A_{a;b} g^{ab} \Psi}}_{\Psi \text{ is affected by } A}$

and varying w. resp. to $\Psi$ yields the compl. conj. equation.

□ Vary w. resp. to $A_a \Rightarrow$ E.L. eqn:

$$\frac{1}{4\pi} F_{ab;c} g^{bc} - ie \Psi (\Psi^*_{;a} - ie A_a \Psi^*) + ie \Psi^* (\Psi_{;a} + ie A_a \Psi) = 0$$

□ Vary w. resp. to $A_a$ ⟹ E.L. eqn:

$$\frac{1}{4\pi} F_{ab;c}\, g^{bc} - ie\, \Psi\left(\Psi^{*}_{;a} - ie A_a \Psi^{*}\right) + ie\, \Psi^{*}\left(\Psi_{;a} + ie A_a \Psi\right) = 0$$

plain Maxwell part

$A$ is affected by $\Psi, \Psi^{*}$.

**Dirac equation:**   (Brief treatment of basics only of Dirac spinors)

**In special relativity:**   (with units such that 🖐 = 1)

# Dirac equation: (Brief treatment of basics only of Dirac spinors)

## In special relativity: (with units such that $\hbar = 1$)

$$\left( i \, \gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \Psi(x) = 0$$

"Dirac equation"

(D)

where $\quad \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} \quad$ is a "Spinor"

↑
describes spin $\frac{1}{2}$ particles
such as electrons and quarks

and the four $4 \times 4$ matrices $\gamma^\mu$ obey:

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$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \eta^{\mu\nu}$$

(*)

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{\mu\nu}$$

17

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \qquad (\text{*})$$

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{\mu\nu}$$

□ <u>Why (*) ?</u> Equation (*) is specifically chosen so that each component of $\psi$ obeys the Klein Gordon equation. Indeed:

$$(D) \implies \left(-i\gamma^\mu \partial_\mu - m\right)\left(i\gamma^\nu \partial_\nu - m\right)\psi = 0$$

$$\implies \left(+\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + i\gamma^\mu \partial_\mu m - im\gamma^\nu \partial_\nu + m^2\right)\psi = 0$$

symmetric under $\mu \leftrightarrow \nu$

17

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17

◻ **Why (✱) ?** Equation (✱) is specifically chosen so that each component of $\Psi$ obeys the Klein Gordon equation. Indeed:

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symmetric under $\mu \leftrightarrow \nu$

$\Rightarrow$ $\left(\frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\partial_\mu\partial_\nu + m^2\right)\Psi = 0$

antisymmetric part not needed, it would drop out.

(✱)

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$$\Rightarrow \left(\frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)\partial_\mu \partial_\nu + m^2\right)\Psi = 0$$

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$$\Rightarrow \mathbb{1}\left(\eta^{\mu\nu}\partial_\mu \partial_\nu + m^2\right)\Psi = 0$$

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(*) $\implies \mathbb{1}\left(\gamma^{\mu\nu}\partial_\mu \partial_\nu + m^2\right)\Psi = 0$

which is the Klein Gordon equation in flat space.

In general relativity:

☐ By choosing an orthonormal tetrad, $\{\Theta^i\}$, we achieve
$$g^{\mu\nu} = \eta^{\mu\nu} \qquad \forall p \in M$$

i.e. one set of matrices $\gamma^\mu$ obeying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ suffices.

☐ This motivates:
$$(i\gamma^\mu \mathbb{P}_\mu - m)\Psi = 0$$

☐ But what is the covariant derivative of a spinor?

# In general relativity:

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- ☐ This motivates:

$$\left(i\gamma^\mu \mathbb{R}_\mu - m\right)\psi = 0$$

- ☐ But what is the covariant derivative of a spinor?

**Recall:** The covariant derivative of a vector yields the infinitesimal Lorentz transformation by which the vector rotates under infinitesimal parallel transport.

**Idea:** The covariant derivative of a __spinor__ should yield the rotation of the spinor by the same infinitesimal Lorentz transformation.

**Recall:** Infinitesimal parallel transport of a vector $e_\varsigma$ in direction $e_\mu$:

$$e_\varsigma \rightarrow e_\varsigma + \nabla_{e_\mu} e_\varsigma = e_\varsigma + \omega^\sigma{}_\varsigma(e_\mu)\, e_\sigma$$

Recall: the curvature 1-form takes values that are infinitesimal Lorentz transformations.

Recall intuition why parallel transport yields Lorentz transformation:

This is an infinitesimal Lorentz transformation $\Lambda_\varsigma{}^\sigma$:

$$e_\varsigma \rightarrow \Lambda_\varsigma{}^\sigma e_\sigma \quad \text{with} \quad \Lambda^\sigma$$

**Recall:** Infinitesimal parallel transport of a vector $e_\zeta$ in direction $e_\mu$:

$$e_\zeta \rightarrow e_\zeta + \nabla_{e_\mu} e_\zeta = e_\zeta + \omega^\sigma_\zeta(e_\mu)\, e_\sigma$$

Recall: the curvature 1-form takes values that are infinitesimal Lorentz transformations.

Recall intuition why parallel transport yields Lorentz transformation: Parallel transport preserves the lengths of vectors, i.e. they can at most "rotate" and in 3+1 dim. this is Lorentz transformations.

This is an infinitesimal Lorentz transformation $\Lambda^\sigma_\zeta$:

$$e_\zeta \rightarrow \Lambda^\sigma_\zeta\, e_\sigma \quad \text{with} \quad \Lambda^\sigma_\zeta = \delta^\sigma_\zeta + \omega^\sigma_\zeta(e_\mu)$$

$$\text{because } \omega^\sigma_\zeta \text{ obeys: } \omega_{\sigma\zeta} = -\omega_{\zeta\sigma}.$$

( Which is the defining equation for infinitesimal Lorentz transformations )

Now that we know the inf. Lorentz transf. for any inf. parallel transport:

$\rightarrow$ **Strategy:** Apply the same inf. Lorentz transformation on spinors

Now that we know the inf. Lorentz transf. for any inf. parallel transpat:

→ **Strategy:** Apply the same inf. Lorentz transformation on spinors for their parallel transpot.

**To this end:** Recall from Special Relativity how an infinitesimal Lorentz transformation acts on a spinor:

▢ Assume $\{s_i\}_{i=1}^{4}$ are ON basis in Spinor space, i.e.

$$\Psi = \Psi^i(x)\, s_i$$

these are Spinor indices: $i = 1, 2, 3, 4$

▢ How do the $s_i$ transform under Lorentz transformations?

□ From special relativity it is known that under infinitesimal Lorentz transformations,

$$\Lambda^\mu_{\ \nu} = \delta^\mu_{\ \nu} + \omega^\mu_{\ \nu}$$

vectors transform as

$$e_\mu \to e_\mu + \omega_\mu^{\ \nu} e_\nu$$

and the Dirac spinors transform as:

$$S_i \to S_i - \frac{1}{4} \omega_\mu^{\ \nu} [\gamma^\mu, \gamma_\nu] S_i$$

Where does $[\gamma^\mu, \gamma^\nu]$ come from?

Recall that e.g. translations in space are generated by momentum operators, $e^{-ia\hat{p}} f(x) e^{ia\hat{p}} = f(x+a)$, if they obey the commutation relations $[x_i, p_j] = i\delta_{ij}$.

Similarly, Lorentz transformations are generated by operators $M^{\mu\nu}$: antisym. antisym.

$$e^{-i\omega_{\mu\nu} M^{\mu\nu}} f \ e^{i\omega_{\mu\nu} M^{\mu\nu}} = \Lambda(f)$$

$$\tilde{\Lambda}^{\tilde{\mu}}_{\nu} = \delta^{\tilde{\mu}}_{\nu} + \omega^{\tilde{\mu}}_{\nu}$$

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$$e^{-i\omega_{\mu\nu} M^{\mu\nu}} f \, e^{i\omega_{\mu\nu} M^{\mu\nu}} = \Lambda(f)$$
if these $M^{\mu\nu}$ obey certain commutation relations. In spinor space, the unique objects that obey these commutation relations are the $M^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$.

$$S_i \rightarrow S_i \; \underbrace{- \frac{1}{4} \omega_\mu^{\tilde{\nu}} [\gamma^\mu, \gamma^\nu] S_i}$$

$\Rightarrow$ under infinitesimal Lorentz transf. the spinor "rotates" by this amount.

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$$S_i \to S_i \; - \tfrac{1}{4} \omega_{\mu\nu} [\gamma^\mu, \gamma^\nu] S_i$$

$\Rightarrow$ under infinitesimal Lorentz transf. the spinor "rotates" by this amount.

# Apply to GR:

If a vector $e_\mu$ is infinitesimally parallel transported in the direction of $e_a$ then it obtains an infinitesimal "rotation", namely the infinitesimal Lorentz transformation

$$\omega^r{}_\nu (e_a)$$

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→ From this one can immediately read off
again the covariant derivative for vectors:

$$\nabla_{e_a} e_\mu = \omega^\nu{}_\mu(e_a) e_\nu$$

□ Now, when a spinor $s_i$ is infinitesimally
parallel transported in the direction of $e_a$
then it too experiences the infinitesimal rotation,
i.e., the infinitesimal Lorentz transformation

$$\omega^r{}_\nu(e_a)$$

which is the value of the connection 1 form. Thus:

❑ *Now, when a spinor* $s_i$ *is infinitesimally, parallel transported in the direction of* $e_a$ *then it too experiences the infinitesimal rotation, i.e., the infinitesimal Lorentz transformation*

$$\omega^{\mu}{}_{\nu}(e_a)$$

*which is the value of the connection 1 form. Thus:*

*local infinitesimal Lorentz transformation, i.e., local value of the connection 1-form.*

$$s_i \rightarrow s_i - \frac{1}{4}\,\overbrace{\omega(e_a)_{\mu}{}^{\nu}}\,[\gamma^{\mu},\gamma_{\nu}]\,s_i$$

❑ *Since, under infinitesimal parallel transport:*

□ *Now, when a spinor* $s_i$ *is infinitesimally, parallel transported in the direction of* $e_a$ *then it too experiences the infinitesimal rotation, i.e., the infinitesimal Lorentz transformation*

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local infinitesimal Lorentz transformation,
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$$S_i \rightarrow S_i - \frac{1}{4} \overbrace{\omega(e_a)_{\mu}}^{} [\gamma^{\nu}, \gamma_{\nu}] S_i$$

☐ Since, under infinitesimal parallel transport:

$$S_i \rightarrow S_i + \nabla_{e_a} S_i$$

⌐ to be determined

⟹ The covariant derivative of the basis vectors $\{S_i\}$ of Dirac spinors is:

$$\nabla_{e_a} s_i = -\frac{1}{4} \omega_\mu^{\ \nu}(e_a) [\gamma^\mu, \gamma_\nu] s_i$$

$\Rightarrow$ For general Dirac spinors $\Psi(x) = \Psi^i(x) s_i$

the Leibniz rule for $\nabla$ yields:

— scalar coefficient functions

$$\nabla_{e_a} \Psi = \nabla_{e_a}\left(\Psi^i(x) s_i\right) = \left(\nabla_{e_a} \Psi^i(x)\right) s_i + \Psi^i(x) \nabla_{e_a} s_i$$

i.e.:

$$\boxed{\nabla_{e_a} \Psi = e_a(\Psi) - \frac{1}{4} \omega(e_a)_\mu^{\ \nu} [\gamma^\mu, \gamma_\nu] \Psi}$$

function

$$e_a(\Psi) = s_i \, e_a(\Psi^i)$$

vector field

# Dirac equation:

The general relativistic Dirac equation

$$(i\gamma^r \nabla_{e_r} - m)\Psi = 0$$

now takes this explicit form:

$$i\gamma^r e_r(\Psi) - i\frac{1}{4}\omega(e_r)^\nu{}_\varsigma [\gamma^\varsigma, \gamma_\nu]\Psi - m\Psi = 0$$

in a chart, this becomes a directional derivative of $\Psi$.

Remark: The relationship between the Dirac operator $D = i\gamma^r \nabla_{e_r}$
and the Laplace or d'Alembert operator $\Box$ also becomes:

$$D = d + S$$

$$\left( i \gamma^r \nabla_{e_r} - m \right) \Psi = 0$$

now takes this explicit form:

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in a chart, this becomes a directional derivative of $\Psi$.

Remark: The relationship between the Dirac operator $D = i \gamma^r \nabla_{e_r}$
and the Laplace or d'Alembert operator $\square$ also becomes:

$$D = d + \delta .$$

To this end, one re-interprets the Grassmann algebra of
differential forms as a so-called Clifford algebra.

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| Go to Page... | Ctrl+G |
| Format... | |

for Cosmology, Achim Kempf, Fall 2013, Lecture 12

s:  Almost ready for GR, i.e., for spacetime dynamics.

But first: Need dynamics of matter fields in curved space, because it will affect the dynamics of spacetime.

Recall:  On a (pseudo)-Riemannian m/ld, equations are well-defined only if defined independently of any chart.

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