Title: General Relativity for Cosmology - Lecture 9

Date: Oct 17, 2013 04:00 PM

URL: http://pirsa.org/13100008

Abstract:

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Questions:

How does ∇ determine "shape"? Through:

Torsion & Curvature!

Recall:

$$\boxed{\bar{\Gamma}^r{}_{ab} = \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^j}{\partial \bar{x}^b}\Gamma^k{}_{ij} + \frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial^2 x^k}{\partial \bar{x}^a \partial \bar{x}^b}}$$

Notice: The antisymmetric part of Γ transforms tensorially!

$$\Gamma^k_{(sym)\,ij} := \frac{1}{2}\left(\Gamma^k{}_{ij} + \Gamma^k{}_{ji}\right)$$

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Recall:

$$\boxed{\bar{\Gamma}^{r}{}_{ab} = \frac{\partial \bar{x}^{r}}{\partial x^{k}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \Gamma^{k}{}_{ij} + \frac{\partial \bar{x}^{r}}{\partial x^{k}} \frac{\partial^2 x^{k}}{\partial \bar{x}^{a} \partial \bar{x}^{b}}}$$

Notice: The antisymmetric part of Γ transforms tensorially!

$$\left.\begin{aligned} \Gamma^{k}_{(sym)\,ij} &:= \tfrac{1}{2}\left(\Gamma^{k}{}_{ij} + \Gamma^{k}{}_{ji}\right) \\ \Gamma^{k}_{(asym)\,ij} &:= \tfrac{1}{2}\left(\Gamma^{k}{}_{ij} - \Gamma^{k}{}_{ji}\right) \end{aligned}\right\} \quad \Gamma^{k}_{ij} = \Gamma^{k}_{(sym)\,ij} + \Gamma^{k}_{(asym)\,ij}$$

$$\Rightarrow \bar{\Gamma}^{r}_{(asym)\,ab} = \frac{\partial \bar{x}^{r}}{\partial x^{k}} \frac{\partial x^{i}}{\partial \bar{x}^{a}} \frac{\partial x^{j}}{\partial \bar{x}^{b}} \Gamma^{k}_{(asym)\,ij} \qquad !$$

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i.e., when $T=0$,

Check: Can we then really always construct a cds so that $\Gamma(p)=0$?

Recall: ξ is autoparallel to a path $\gamma: t \to x(t)$ if

$\xi \in T^1(M)$

$$\boxed{\nabla_{\dot\gamma}\xi = 0}$$

Meaning: ξ is parallel transported along the path γ in M.

Explicitly: $\frac{d\xi^k}{dt} + \Gamma^k{}_{ij}\frac{dx^i}{dt}\xi^j = 0$

Geodesics: A curve $\gamma: t \to x(t)$ is called a geodesic if $\dot\gamma$ is autoparallel along γ, i.e. if

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Geodesics: A curve $\gamma : t \to x(t)$ is called a geodesic if $\dot{\gamma}$ is autoparallel along γ, i.e. if

$$\boxed{\nabla_{\dot{\gamma}} \dot{\gamma} = 0}$$

Meaning: The path γ in M is such that the path's tangent vectors are parallel translates of each other.

- $\Rightarrow$ In charts, geodesics $x^r(t)$ obey:

$$\frac{d^2 x^k}{dt^2} + \Gamma(x)^k{}_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0 \qquad (*)$$

- Theory of ordinary differential equations:

$\Rightarrow$ Given $p = \gamma(0)$, each initial condition

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Meaning: The path γ in $\mathcal{M}$ is such that the path's tangent vectors are parallel translates of each other.

$\Rightarrow$ In charts, geodesics $x^r(t)$ obey:

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$$\frac{d^2x^k}{dt^2} + \Gamma(x)^k{}_{ij}\,\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \qquad (*)$$

- Theory of ordinary differential equations:

$\Rightarrow$ Given $p = \gamma(0)$, each initial condition $\xi = \dot\gamma(0)$ belongs to a unique geodesic γ_ξ of nonzero length.

subscript indicates initial condition vector

- <u>Notice:</u> If $\gamma_\xi(t)$ solves $(*)$ then $\gamma_\xi(\lambda t)$ also solves $(*)$ and for $\lambda \in \mathbb{R}$:

$$\gamma_{\lambda\xi}(t) = \gamma_\xi(\lambda t) \qquad (G)$$

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"Exponential map":

- Consider a fixed point $p \in M$.

The exponential map is defined through:

$$\exp_p : T_p{}'(M) \to M$$

(really from a neighborhood of 0 in $T_p{}'(M)$ to a neighborhood of p in M.)

$$\exp_p : \quad \xi \to \exp_p(\xi) := \gamma_\xi(1)$$

where γ is the geodesic with $\gamma_\xi(0) = p$, $\dot{\gamma}_\xi(0) = \xi$.

- **Observe:**

From (G) we obtain:

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Observe:

From (G) we obtain:

$$\gamma_\xi(\lambda) = \gamma_{\lambda\xi}(1) = \exp_p(\lambda\xi) \qquad (E)$$

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"Geodesic" or "Riemann normal" coordinates:

- $\exp_p$ is a diffeomorphism from a neighborhood of $0 \in T_p'(M) \simeq \mathbb{R}^n$ into a neighborhood of the point $p \in M$.

  ↑ isomorphic

$\Rightarrow$ $\exp_p$ provides a chart around p:

- Choose a basis, say $e_1, e_2, \ldots, e_n$ of $T_p'(M)$, then:

$$\xi = \xi^i e_i$$

- Through $\exp_p$, the ξ^i become the coordinates of points in a neighborhood of $p \in M$:

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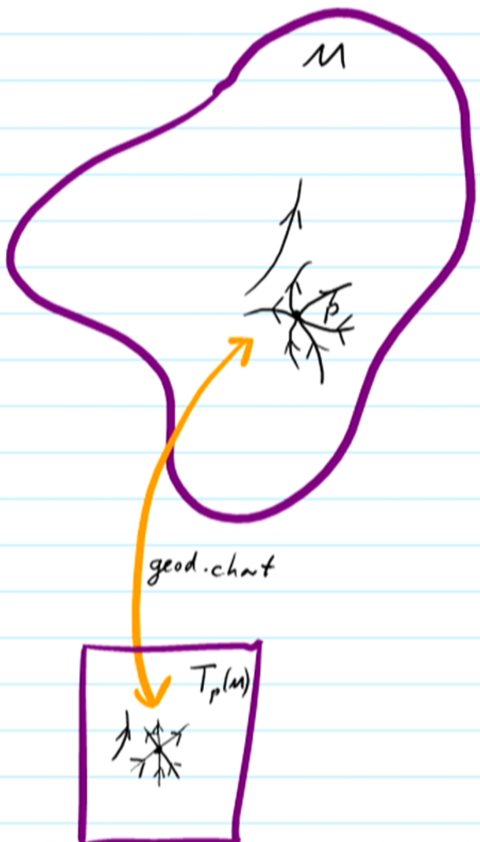
⇒ Geodesics, γ, through p are straight lines in a normal cds about p!



Recall (E):

$$\underbrace{\gamma_\xi(\lambda)}_{\text{for varying } \lambda \text{ one moves along the geodesic in } M} = \exp_p(\lambda\xi)$$

for varying λ one moves along the geodesic in M.

[for varying λ one moves on a straight line in the coordinate system of the ξ^i!

Thus: In geodesic cds, geodesics through p are straight lines of constant velocity ξ.

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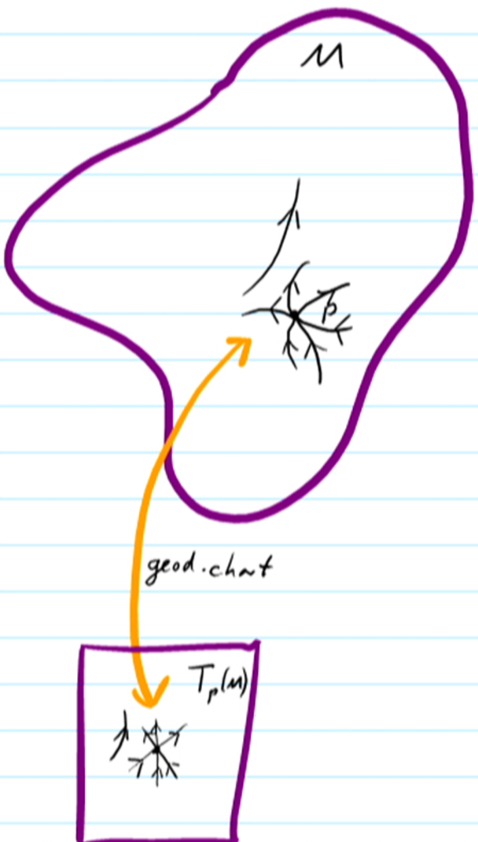
⇒ Geodesics, γ, through p are straight lines in a normal cds about p!



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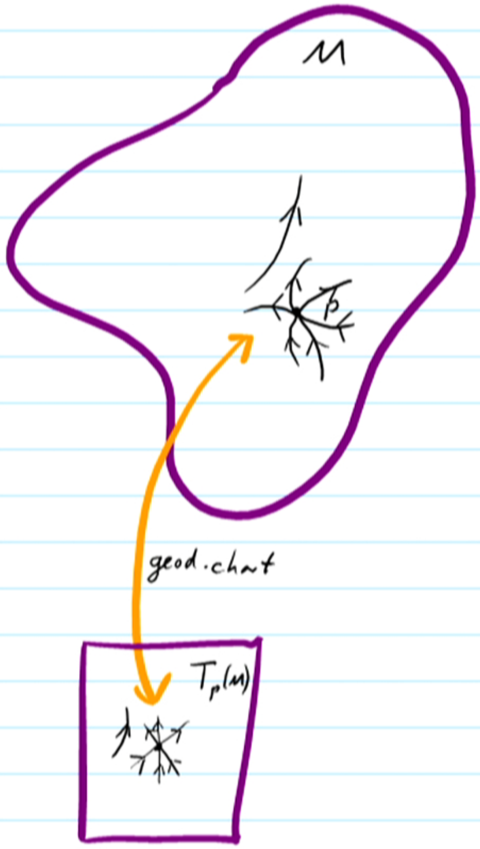

$$\gamma_\xi(\lambda) = \exp_p(\lambda\xi)$$

for varying λ one moves along the geodesic in M.

for varying λ one moves on a straight line in the coordinate system of the ξ^i!

- **Thus:** In geodesic cds, geodesics through p are straight lines of constant velocity ξ.

- Does this mean $\Gamma^k{}_{ij}(p) = 0$? No!

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$\Rightarrow$ **Indeed:** In order to achieve that the entire gravity and pseudo force field vanishes in a suitable chart, $\Gamma^k{}_{ij}(p)=0$,

need: $\Gamma^k_{\text{antisym}\,ij}(p) = \frac{1}{2} T^k{}_{ij}(p) = 0$

$\uparrow$ torsion

Note:
Quantum fluctuations may induce torsion!
So, let's nevertheless ask:

What would torsion mean, geometrically?

Abstract definition of Torsion:

- Assume ξ_1 and ξ_2 are tangent vectors at $p \in M$:

Then, the **Torsion map** is defined as:

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Compare with prior definition:

- Choose canonical bases $\frac{\partial}{\partial x^i}$, dx^i:

- $$J^k{}_{ij} := dx^k\left(J\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right)$$

$$= \left\langle dx^k, J\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right\rangle$$ (more convenient notation)

$$= \left\langle dx^k, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} - \underbrace{\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]}\right\rangle$$

Recall:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k{}_{ij} \frac{\partial}{\partial x^k}$$

$$\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\right) f = 0 \quad \forall f$$

$$= \left\langle dx^k, \Gamma^r{}_{ij} \frac{\partial}{\partial x^r} - \Gamma^r{}_{ji} \frac{\partial}{\partial x^r}\right\rangle = \Gamma^r{}_{ij}\,\delta^k{}_r - \Gamma^r{}_{ji}\,\delta^k{}_r$$

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$$T^k{}_{ij} := dx^k\left(T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right)$$

$$= \left\langle dx^k, T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right\rangle$$ (more convenient notation)

$$= \left\langle dx^k, \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i} - \underbrace{\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]}\right\rangle$$

Recall:

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \Gamma^k{}_{ij}\frac{\partial}{\partial x^k}$$

$$\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\right) f = 0 \quad \forall f$$

$$= \left\langle dx^k, \Gamma^r{}_{ij}\frac{\partial}{\partial x^r} - \Gamma^r{}_{ji}\frac{\partial}{\partial x^r}\right\rangle = \Gamma^r{}_{ij}\delta^k_r - \Gamma^r{}_{ji}\delta^k_r$$

$$\Rightarrow \boxed{T^k{}_{ij} = \Gamma^k{}_{ij} - \Gamma^k{}_{ji}}$$

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$$\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\right) f = 0 \quad \forall f$$

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$$\Rightarrow \boxed{T^k{}_{ij} = \Gamma^k{}_{ij} - \Gamma^k{}_{ji}}$$

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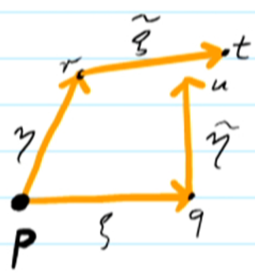
Geometric meaning of torsion? Parallelograms would not close!

Travel from p infinitesimally in ξ and then η direction, and compare with the reverse:



$$\xi, \eta \in T_p^1$$
$$\tilde{\xi} \in T_r^1$$
$$\tilde{\eta} \in T_q^1$$

Recall parallel transport: $\nabla_{\dot{x}} \xi = 0$

$$\frac{d\xi^k}{dt} + \Gamma^k_{ij} \frac{dx^i}{dt} \xi^j = 0$$

$\tilde{\xi}(r) = ?$

$$\tilde{\xi}^k(x^i + \eta^i) \approx \xi^k(x^i) + \frac{d\xi^k}{dt}(x^i)$$
$$= \xi^k(x^i) - \Gamma(x)^k_{ij} \eta^i \xi^j$$

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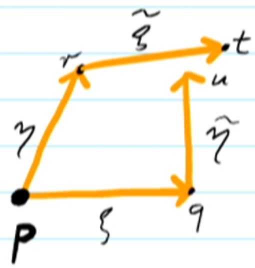
compare with the reverse:



$\xi, \eta \in T_p^1$

$\tilde\xi \in T_r^1$

$\tilde\eta \in T_q^1$

Recall parallel transport: $\nabla_{\dot\gamma}\xi = 0$

$$\frac{d\xi^k}{dt} + \Gamma^k_{ij}\frac{dx^i}{dt}\xi^j = 0$$

$\tilde\xi(r) = ?$

$$\tilde\xi^k(x^i+\eta^i) \approx \xi^k(x^i) + \frac{d\xi^k}{dt}(x^i)$$

$$= \xi^k(x^i) - \Gamma(x)^k{}_{ij}\,\eta^i\xi^j$$

$\Rightarrow$ Cds. of t: $x^a + \eta^a + \xi^a - \Gamma(x)^a{}_{ij}\,\eta^i\xi^j$

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Comment: We had:

$$\tilde{\xi}^k(x^i+\eta^i) \approx \xi^k(x^i) + \frac{d\xi^k}{dt}(x^i) = \xi^k(x^i) - \Gamma(x)^k{}_{ij}\,\eta^i\xi^j$$

this is also:

$$= \xi^k(x^i) - \left(\eta^i\xi^k{}_{,i} + \Gamma(x)^k{}_{ij}\,\eta^i\xi^j\right) + \eta^i\xi^k{}_{,i}$$

$$= \xi^k(x^i) - \eta^i\xi^k{}_{;i} + \eta^i\xi^k{}_{,i}$$

Thus: cd distance from u to t is:

$$\cancel{x^a} + \cancel{\eta^a} + \cancel{\xi^a} - \eta^i\xi^k{}_{;i} + \eta^i\xi^k{}_{,i} - \cancel{x^a} - \cancel{\xi^a} - \cancel{\eta^a} + \xi^i\eta^k{}_{;i} - \eta^i\xi^k{}_{,i} = T^a{}_{ji}\,\eta^i\xi^j$$

Recall that indeed: $T: \eta,\xi \rightarrow T(\eta,\xi) = \nabla_\eta\xi - \nabla_\xi\eta - [\eta,\xi]$

Curvature:

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Analogously obtain: Cds. of u: $x^a+\xi^a+\eta^a-\Gamma(x)^a{}_{ij}\,\xi^i\eta^j$

Torsion!

$\Rightarrow$ Cd. distance from u to t is: $\left(\Gamma(x)^a{}_{ji}-\Gamma(x)^a{}_{ij}\right)\eta^i\xi^j = T^a_{ji}\,\eta^i\xi^j$ ✓

Comment: We had:

$$\tilde{\xi}^k(x^i+\eta^i) \approx \xi^k(x^i)+\frac{d\xi^k}{dt}(x^i) = \xi^k(x^i)-\Gamma(x)^k{}_{ij}\,\eta^i\xi^j$$

this is also:

$$= \xi^k(x^i)-\left(\eta^i\xi^k{}_{;i}+\Gamma(x)^k{}_{ij}\,\eta^i\xi^j\right)+\eta^i\xi^k{}_{,i}$$

$$= \xi^k(x^i)-\eta^i\xi^k{}_{;i}+\eta^i\xi^k{}_{,i}$$

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$$T: T_p^1(M) \times T_p^1(M) \to T_p^1(M)$$

This will be the amount by which an infinitesimal parallelogram spanned by ξ_1 and ξ_2 does not close.

$$T: \xi_1, \xi_2 \longrightarrow T(\xi_1, \xi_2) := \overbrace{\nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1 - [\xi_1, \xi_2]}$$

for proof it's a tensor, see Straumann

- It is used to define the Torsion tensor, $\mathcal{T}$,

$$\mathcal{T} \in T_{p\,2}^{\,1}(M)$$

we could also write: $= \omega(T(\xi_1, \xi_2))$

through:

contraction yields a number

feeding 1 covector & 2 vectors to a (1,2) tensor yields a number $\longrightarrow$ $$\mathcal{T}(\omega, \xi_1, \xi_2) := \overbrace{\langle \underbrace{\omega}_{\in T_{p1}(M)}, \underbrace{T(\xi_1, \xi_2)}_{\in T_p^1(M)} \rangle} \in \mathbb{R}$$

Compare with prior definition:

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Curvature:

Assume ξ_1, ξ_2 and ξ_3 are tangent vectors at $p \in M$.

- The **curvature map**, R, is defined through:

$$R: \xi_1, \xi_2, \xi_3 \rightarrow \underbrace{R(\xi_1, \xi_2)}_{\text{an operator, or map, acting on } \xi_3} \xi_3 = \overbrace{\left(\nabla_{\xi_1} \nabla_{\xi_2} - \nabla_{\xi_2} \nabla_{\xi_1} - \nabla_{[\xi_1, \xi_2]}\right) \xi_3}^{\in T_p{}'(M)}$$

- It defines the curvature tensor, R,

can be fed one covector and 3 vectors to yield a number

$$R \in T_p{}^1{}_3(M)$$

through:

$\omega(R(\xi_1, \xi_2)\xi_3)$ //

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through:

$$R(\omega, \xi_1, \xi_2, \xi_3) := \overbrace{\langle \omega, R(\xi_1, \xi_2)\xi_3 \rangle}^{= \omega(R(\xi_1, \xi_2)\xi_3)} \in \mathbb{R}$$

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through:

$$R(\omega,\xi_1,\xi_2,\xi_3) := \langle \omega, \overbrace{R(\xi_1,\xi_2)\xi_3}^{=\ \omega(R(\xi_1,\xi_2)\xi_3)} \rangle \in \mathbb{R}$$

In a chart:

$$R^i{}_{jkl} = \langle dx^i, R\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\right)\frac{\partial}{\partial x^j} \rangle$$

$$= \langle dx^i, \left(\nabla_{\frac{\partial}{\partial x^k}}\nabla_{\frac{\partial}{\partial x^l}} - \nabla_{\frac{\partial}{\partial x^l}}\nabla_{\frac{\partial}{\partial x^k}} - \nabla_{\underbrace{\left[\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\right]}_{=0}}\right)\frac{\partial}{\partial x^j} \rangle$$

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In a chart:

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$$= \left\langle dx^i, \nabla_{\frac{\partial}{\partial x^k}} \Gamma^s{}_{lj} \frac{\partial}{\partial x^s} - \nabla_{\frac{\partial}{\partial x^l}} \Gamma^s{}_{kj} \frac{\partial}{\partial x^s} \right\rangle$$

$$= \left\langle dx^i, \left(\Gamma^s{}_{lj,k} + \Gamma^r{}_{lj}\Gamma^s{}_{kr} - \Gamma^s{}_{kj,l} - \Gamma^r{}_{kj}\Gamma^s{}_{lr}\right)\frac{\partial}{\partial x^s} \right\rangle$$

$$= \Gamma^i{}_{lj,k} - \Gamma^i{}_{kj,l} + \Gamma^s{}_{lj}\Gamma^i{}_{ks} - \Gamma^s{}_{kj}\Gamma^i{}_{ls}$$

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$$= \left\langle dx^i, \left(\nabla_{\frac{\partial}{\partial x^k}} \nabla_{\frac{\partial}{\partial x^l}} - \nabla_{\frac{\partial}{\partial x^l}} \nabla_{\frac{\partial}{\partial x^k}} - \underbrace{\nabla_{\left[\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right]}}_{=0}\right) \frac{\partial}{\partial x^j} \right\rangle$$

$$= \left\langle dx^i, \nabla_{\frac{\partial}{\partial x^k}} \Gamma^s_{lj} \frac{\partial}{\partial x^s} - \nabla_{\frac{\partial}{\partial x^l}} \Gamma^s_{kj} \frac{\partial}{\partial x^s} \right\rangle$$

$$= \left\langle dx^i, \left(\Gamma^s_{lj,k} + \Gamma^r_{lj} \Gamma^s_{kr} - \Gamma^s_{kj,l} - \Gamma^r_{kj} \Gamma^s_{lr}\right) \frac{\partial}{\partial x^s} \right\rangle$$

$$= \Gamma^i_{lj,k} - \Gamma^i_{kj,l} + \underbrace{\Gamma^s_{lj} \Gamma^i_{ks} - \Gamma^s_{kj} \Gamma^i_{ls}}$$

(at origin of geodesic cds they vanish.)

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(at origin of geodesic cds they vanish,

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$$= \langle dx^i, \frac{\partial \Gamma^s_{\ell j}}{\partial x^k} \frac{\partial}{\partial x^s} - \frac{\partial \Gamma^s_{kj}}{\partial x^\ell} \frac{\partial}{\partial x^s} \rangle$$

$$= \langle dx^i, \left(\Gamma^s_{\ell j,k} + \Gamma^r_{\ell j}\Gamma^s_{kr} - \Gamma^s_{kj,\ell} - \Gamma^r_{kj}\Gamma^s_{\ell r}\right)\frac{\partial}{\partial x^s} \rangle$$

$$= \Gamma^i_{\ell j,k} - \Gamma^i_{kj,\ell} + \underbrace{\Gamma^s_{\ell j}\Gamma^i_{ks} - \Gamma^s_{kj}\Gamma^i_{\ell s}}_{\text{(at origin of geodesic cds they vanish.)}}$$

Curvature tensor's meaning?

Intuition:

- Contains derivatives of Γ $\Rightarrow$

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Curvature tensor's meaning?

Intuition:

- Contains derivatives of Γ $\Rightarrow$
- expresses variation in pseudo and gravitational forces $\Rightarrow$
- expresses the strength and direction of "tidal forces".

Geometry:

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Geometry:

Curvature expresses noncommutativity of two parallel transports, namely:

Proposition: (Ricci Identity)

Assume the torsion vanishes and that ξ is a vector field. Then:

$$\boxed{\xi^a{}_{;cd} - \xi^a{}_{;dc} = R^a{}_{cdb}\, \xi^b}$$

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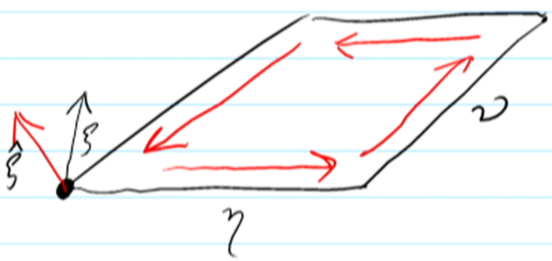
that ξ is a vector field. Then:

$$\boxed{\xi^a{}_{;cd} - \xi^a{}_{;dc} = R^a{}_{cdb}\,\xi^b}$$

(here: $\xi^a{}_{;cd} := \xi^a{}_{;c;d}$ etc.)

Remark: (a bit messy to derive because need Taylor expansion, see, e.g., text by Stewart or Einstein)

It implies that for parallel transport along infinitesimal parallelogram:



$$\boxed{(\hat\xi - \xi)^a \approx \eta^b \nu^c R^a{}_{bcd}\,\xi^d}$$

Proof of Ricci identity:

- Assume ξ, η, v are vector fields.

- Then, $R(\xi,\eta)v := \nabla_\xi(\nabla_\eta v) - \nabla_\eta(\nabla_\xi v) - \nabla_{[\xi,\eta]} v$ reads

in basis:

$$R^a{}_{bcd}\,\xi^b \eta^c v^d = \left(v^a{}_{;d}\,\eta^d\right)_{;c}\xi^c - \left(v^a{}_{;d}\,\xi^d\right)_{;c}\eta^c$$
$$- v^a{}_{;d}\underbrace{\left(\eta^d{}_{;c}\,\xi^c - \xi^d{}_{;c}\,\eta^c\right)}$$

used Torsion $= T(\xi_1,\xi_2) := \nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1 - [\xi_1,\xi_2] = 0$

i.e.: $[\xi,\eta] = \nabla_\xi \eta - \nabla_\eta \xi$

Terms cancel:

$$\Longrightarrow R^a{}_{bcd}\,\xi^b \eta^c v^d = \left(v^a{}_{;d;c} - v^a{}_{;c;d}\right)\xi^c \eta^d$$

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in basis: $$R^a{}_{bcd}\,\xi^b\eta^c v^d = \left(v^a{}_{;d}\,\eta^d\right)_{;c}\xi^c - \left(v^a{}_{;d}\,\xi^d\right)_{;c}\eta^c - v^a{}_{;d}\underbrace{\left(\eta^d{}_{;c}\,\xi^c - \xi^d{}_{;c}\,\eta^c\right)}$$

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True $\forall \xi,\eta$ $$\Longrightarrow R^a{}_{bcd}\,v^d = v^a{}_{;cb} - v^a{}_{;bc} \quad \checkmark$$

The "Bianchi Identities":

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The "Bianchi Identities":

- They are automatic relations among torsion and curvature, by construction.

- Preparation: $\boxed{\nabla \text{ for maps!}}$

Consider an arbitrary $\mathcal{F}(M)$-linear map:

(e.g. Torsion or Curvature map)

$$K: \underbrace{\xi_1 \times \xi_2 \times \ldots \times \xi_r}_{\text{tangent vectors}} \longrightarrow \underbrace{K(\xi_1, \ldots, \xi_r)}_{\text{tangent vector}}$$

i.e. at each $p \in M$:

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Preparation: $\boxed{\nabla \text{ for maps!}}$

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i.e. at each $p \in M$:

$$K : T_p(M)^r \longrightarrow T_p(M)'$$

We can view K as a tensor $\tilde{K} \in T_p(M)^1_r$,

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- We can view K as a tensor $\tilde{K} \in T_p(M)^1_r$,

(as we did for R and J)

namely:

$$\tilde{K}(\omega, \xi_1, \ldots, \xi_r) := \langle \omega, K(\xi_1, \ldots, \xi_r) \rangle$$

- Now let the usual derivative of the tensor $\tilde{K}$ define the derivative of the map K:

$$\langle \omega, (\underbrace{\nabla_\xi K}_{\substack{\text{new concept:} \\ \text{covariant derivative} \\ \text{of a map } K: T_p(M)^r \to T_p(M)^1}})(\xi_1, \ldots, \xi_r) \rangle := \underbrace{\nabla_\xi \tilde{K}(\omega, \xi_1, \ldots \xi_r)}_{\substack{\text{usual cov. derivative} \\ \text{of a } (1, r) \text{ tensor}}}$$

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1st Bianchi Identity:

$$\sum_{\text{cyclic}} R(\xi,\eta)v = \sum_{\text{cyclic}} \Big(\mathcal{T}(\mathcal{T}(\xi,\eta),v) + (\nabla_\xi \mathcal{T})(\eta,v) \Big)$$

2nd Bianchi Identity:

$$\sum_{\text{cyclic}} \Big((\nabla_\xi R)(\eta,v) + R(\mathcal{T}(\xi,\eta),v) \Big) = 0$$

with obvious simplification in case $\mathcal{T} = 0$.

... or like the homogeneous Maxwell equations

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1st Bianchi Identity:

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or like the homogeneous Maxwell eqn

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Proof of 1st Bianchi: (assuming no torsion)

$$\underbrace{\sum_{\text{cyclic}} R(\xi,\eta)\nu}_{\parallel} = 0$$

Indeed:

$$\overbrace{(\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi)\nu - \nabla_{[\xi,\eta]}\nu + \text{cyclic}}$$

$$= \nabla_\xi(\nabla_\eta \nu - \nabla_\nu \eta) - \nabla_{[\eta,\nu]}\xi + \text{cyclic}$$

Exercise: Prove that: (without torsion) $\boxed{\nabla_\eta \nu - \nabla_\nu \eta = [\eta,\nu]}$

$$= \nabla_\xi [\eta,\nu] - \nabla_{[\eta,\nu]}\xi + \text{cyclic}$$

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Indeed: $\overbrace{(\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi)\nu - \nabla_{[\xi,\eta]}\nu + \text{cyclic}}$

$$= \nabla_\xi(\nabla_\eta \nu - \nabla_\nu \eta) - \nabla_{[\eta,\nu]}\xi + \text{cyclic}$$

Exercise: Prove that: (without torsion) $\boxed{\nabla_\eta \nu - \nabla_\nu \eta = [\eta,\nu]}$

$$= \underbrace{\nabla_\xi [\eta,\nu] - \nabla_{[\eta,\nu]}\xi}_{\text{// because again } \nabla_a b - \nabla_b a = [a,b]} + \text{cyclic}$$

$$= \overbrace{[\xi,[\eta,\nu]]} + \text{cyclic}$$

$= 0$ by Jacobi identity for all lin. maps.

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Recall:

Assume A, B, C are linear maps $V \to V$

Then: $[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0$

i.e., the Jacobi identity holds.

Proof: Simply spell out the commutators.

Remark: This means that, e.g., in quantum mechanics, all objects that that need to be representable as operators on the Hilbert space must obey the Jacobi identity, e.g., generators of symmetries.

This is why the Jacobi identity is one of the axioms of Lie Algebras.

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Recall:

Assume A, B, C are linear maps $V \to V$

Then: $[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0$

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