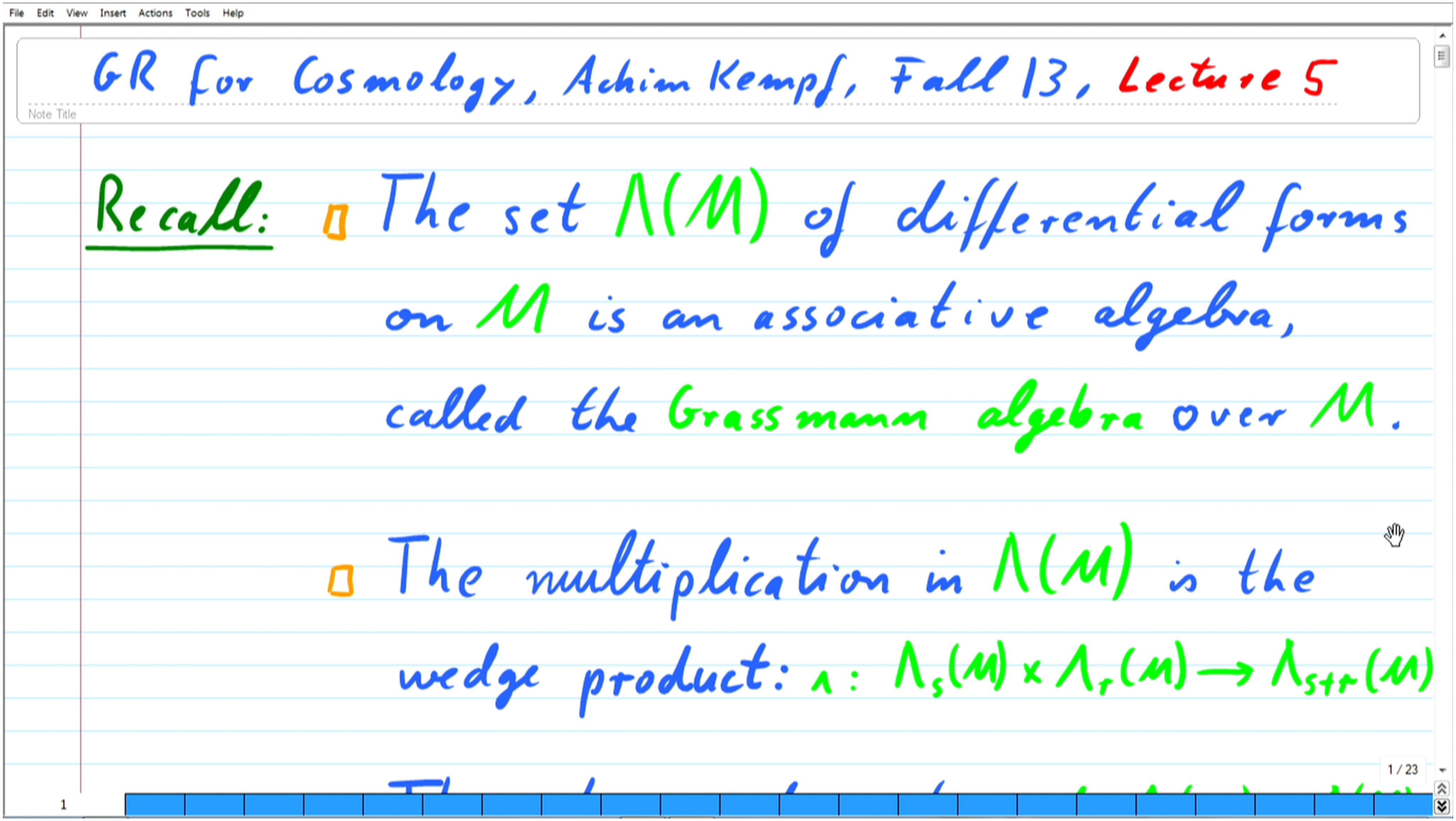
Title: General Relativity for Cosmology - Lecture 5

Date: Oct 10, 2013 04:00 PM

URL: http://pirsa.org/13100007

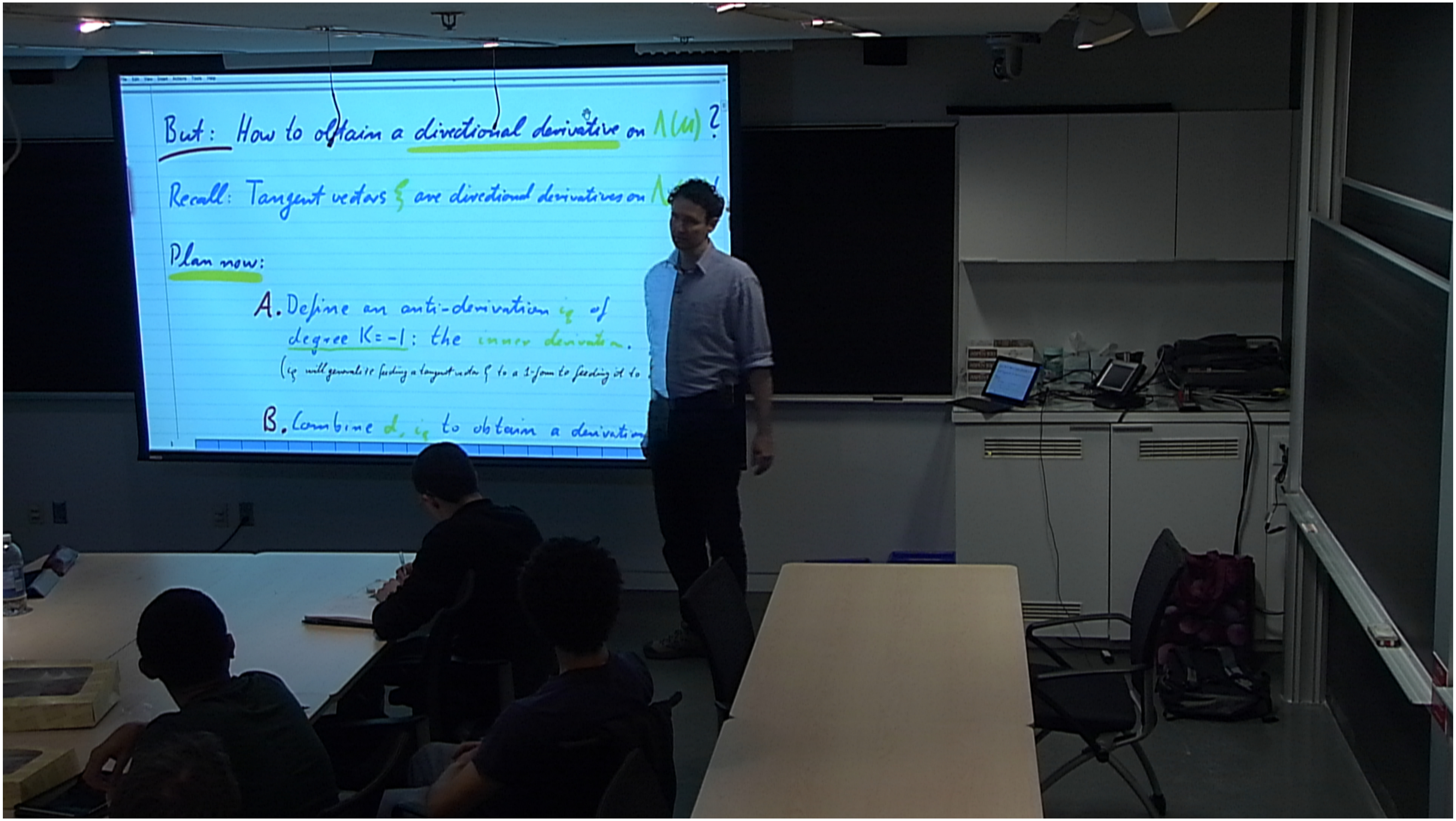
Abstract:



GR for Cosmology, Achim Kempf, Fall 13, Lecture 5

Recall:

- The set $\Lambda(M)$ of differential forms on M is an associative algebra, called the Grassmann algebra over M.
- The multiplication in $\Lambda(M)$ is the wedge product: $\wedge : \Lambda_s(M) \times \Lambda_r(M) \rightarrow \Lambda_{s+r}(M)$



But: How to obtain a directional derivative on $\Lambda(M)$?

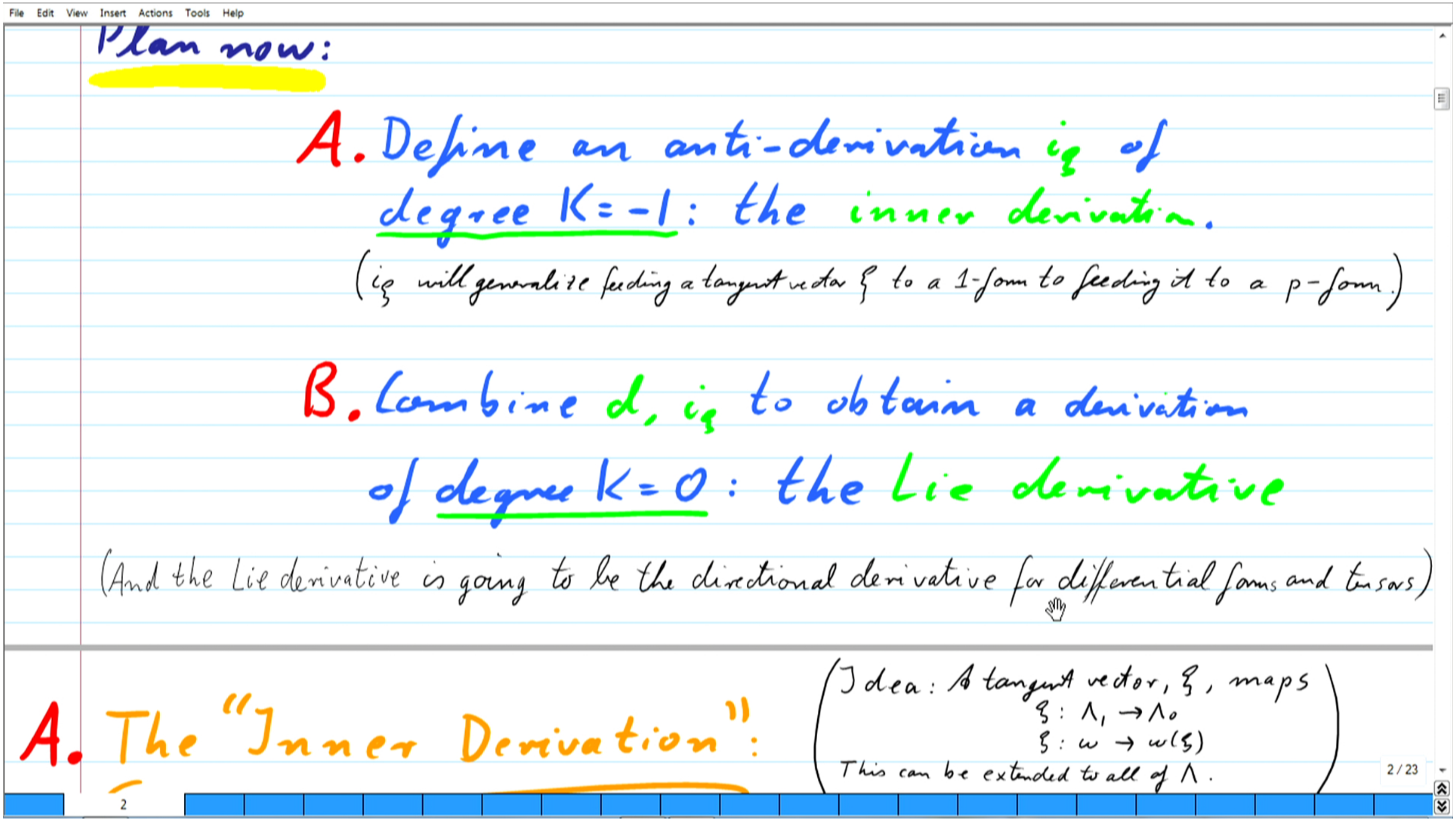
Recall: Tangent vectors ξ are directional derivatives on Λ

Plan now:

A. Define an anti-derivation i_ξ of degree $K=-1$: the inner derivation.

(i_ξ will generalize feeding a tangent vector ξ to a 1-form to feeding it to

B. Combine d, i_ξ to obtain a derivation



Plan now:

A. Define an anti-derivation i_ξ of degree $K=-1$: the inner derivation.

(i_ξ will generalize feeding a tangent vector ξ to a 1-form to feeding it to a p-form.)

B. Combine d, i_ξ to obtain a derivation of degree $K=0$: the Lie derivative

(And the Lie derivative is going to be the directional derivative for differential forms and tensors)

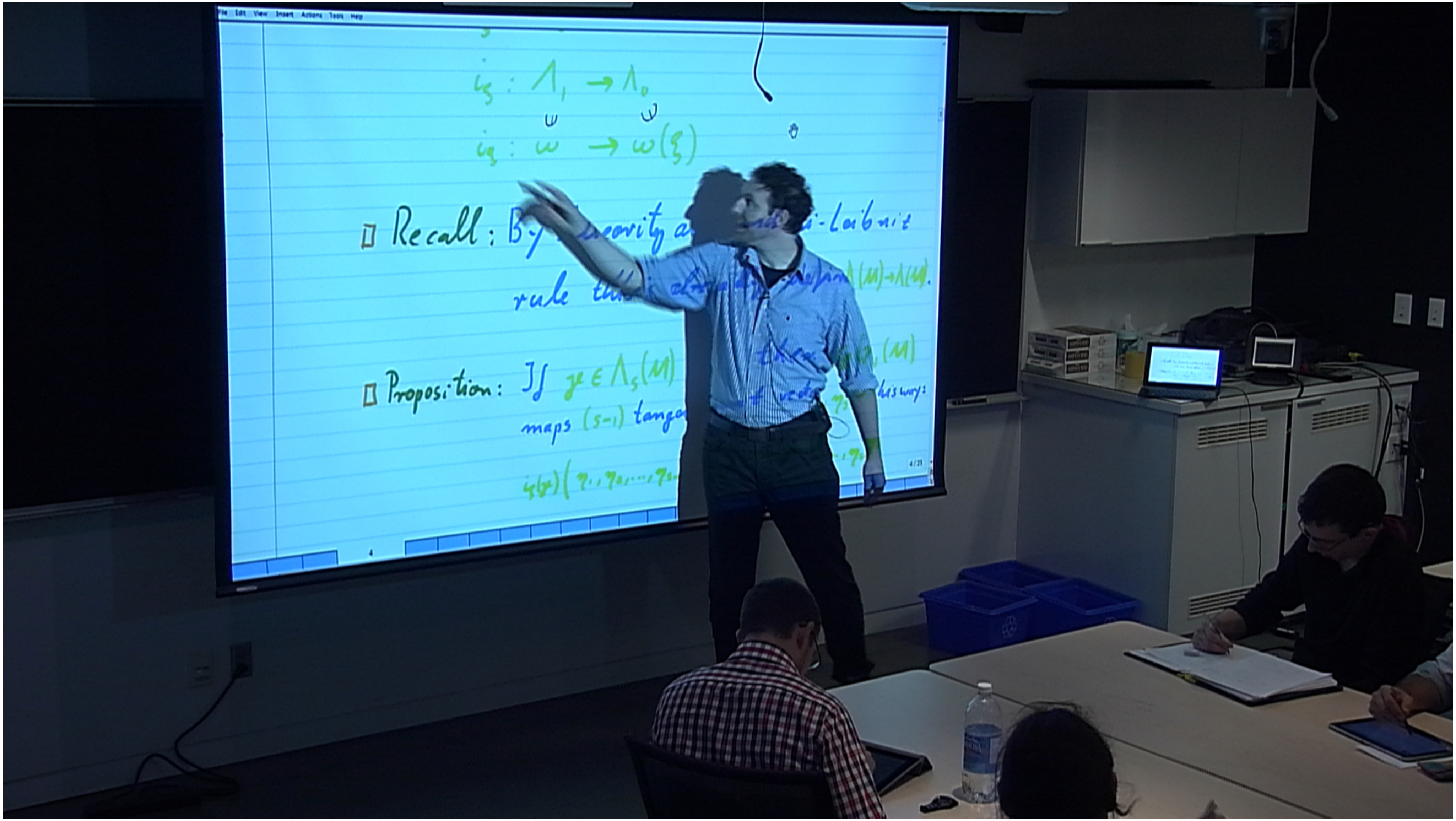
A. The "Inner Derivation":

(Idea: A tangent vector, ξ, maps

$$\xi : \Lambda_1 \to \Lambda_0$$

$$\xi : \omega \to \omega(\xi)$$

This can be extended to all of Λ.)

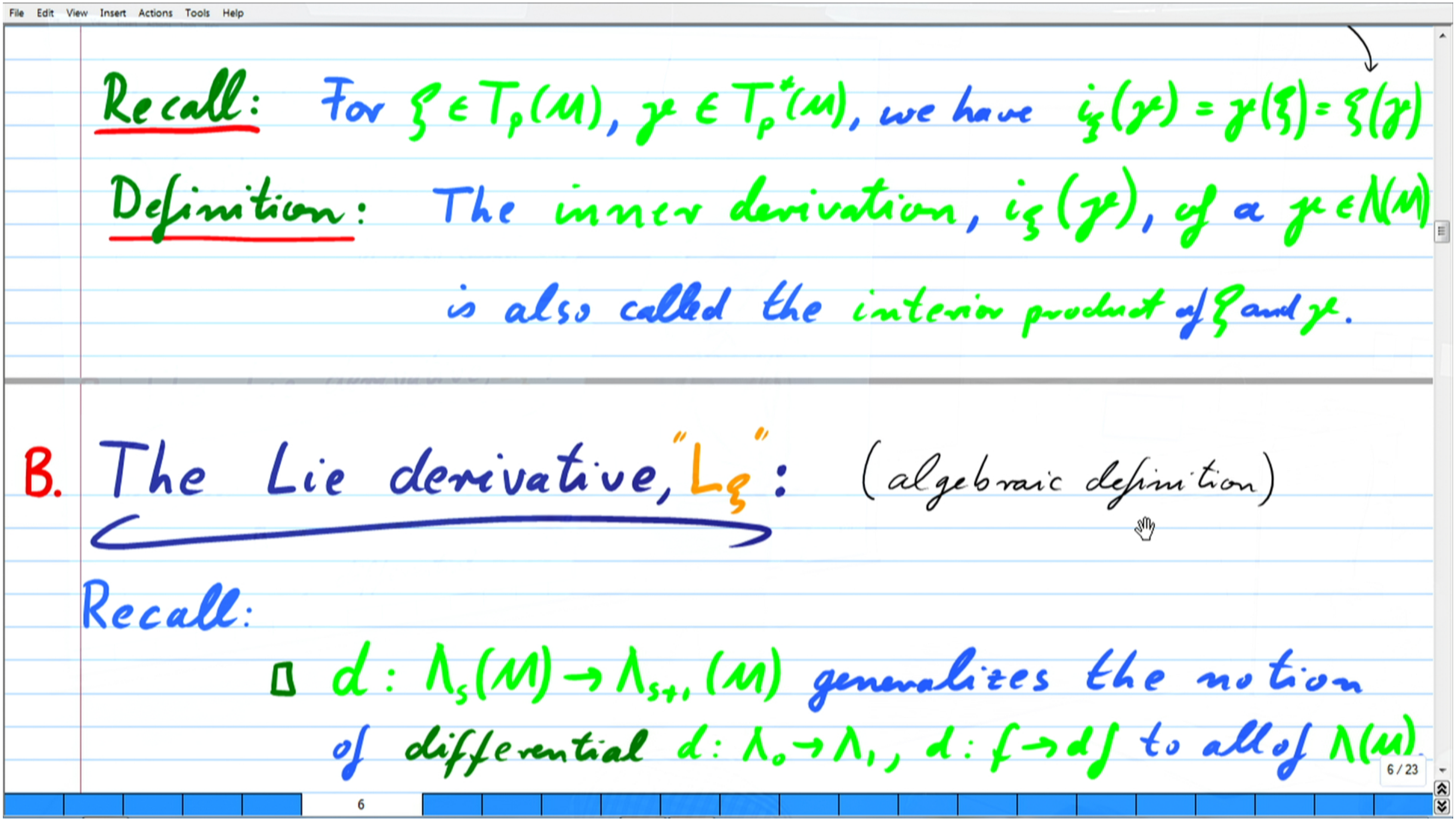


$$\begin{array}{cccc} i_\xi : & \Lambda_1 & \to & \Lambda_0 \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ i_\xi : & \omega & \to & \omega(\xi) \end{array}$$

☐ Recall: By [illegible] linearity a[illegible] -Leibniz rule th[illegible] $\Lambda(M) \to \Lambda(M)$.

☐ Proposition: If $\gamma \in \Lambda_s(M)$ [illegible] then [illegible] $\Lambda_{s-1}(M)$ [illegible] this way: maps $(s-1)$ tang[illegible]

$i_\xi(\gamma)(\eta_1, \eta_2, \ldots, \eta_{s-1}$ [illegible]



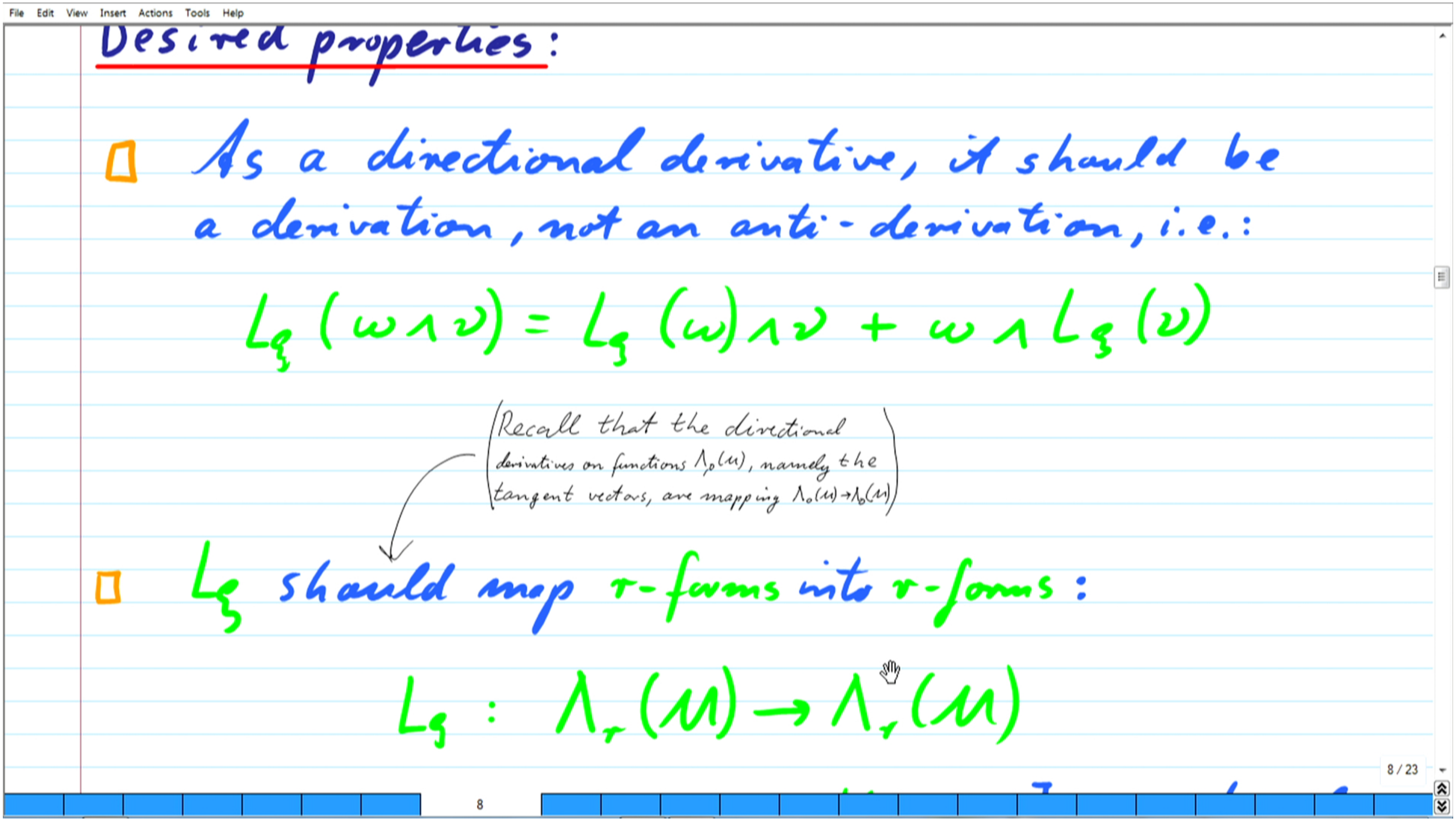
Recall: For $\xi \in T_p(M)$, $\omega \in T_p^*(M)$, we have $i_\xi(\omega) = \omega(\xi) = \xi(\omega)$

Definition: The inner derivation, $i_\xi(\omega)$, of a $\omega \in \Lambda(M)$ is also called the interior product of ξ and ω.

B. The Lie derivative, "L_ξ": (algebraic definition)

Recall:

- $d: \Lambda_s(M) \to \Lambda_{s+1}(M)$ generalizes the notion of differential $d: \Lambda_0 \to \Lambda_1$, $d: f \to df$ to all of $\Lambda(M)$.



Desired properties:

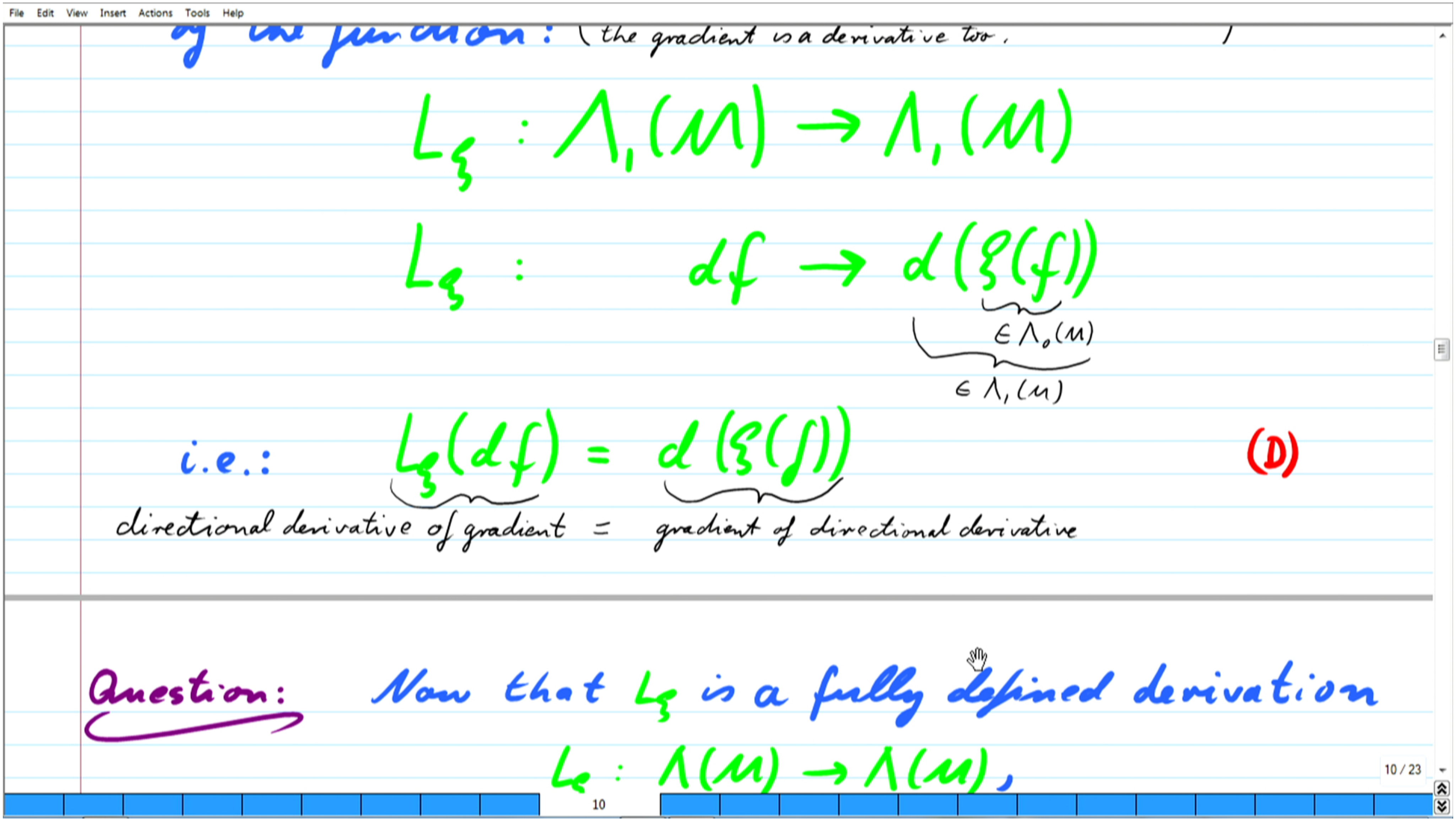
☐ As a directional derivative, it should be a derivation, not an anti-derivation, i.e.:

$$L_\xi(\omega \wedge \nu) = L_\xi(\omega) \wedge \nu + \omega \wedge L_\xi(\nu)$$

(Recall that the directional derivatives on functions $\Lambda_0(M)$, namely the tangent vectors, are mapping $\Lambda_0(M) \to \Lambda_0(M)$)

☐ L_ξ should map r-forms into r-forms:

$$L_\xi : \Lambda_r(M) \to \Lambda_r(M)$$



of the function: (the gradient is a derivative too,)

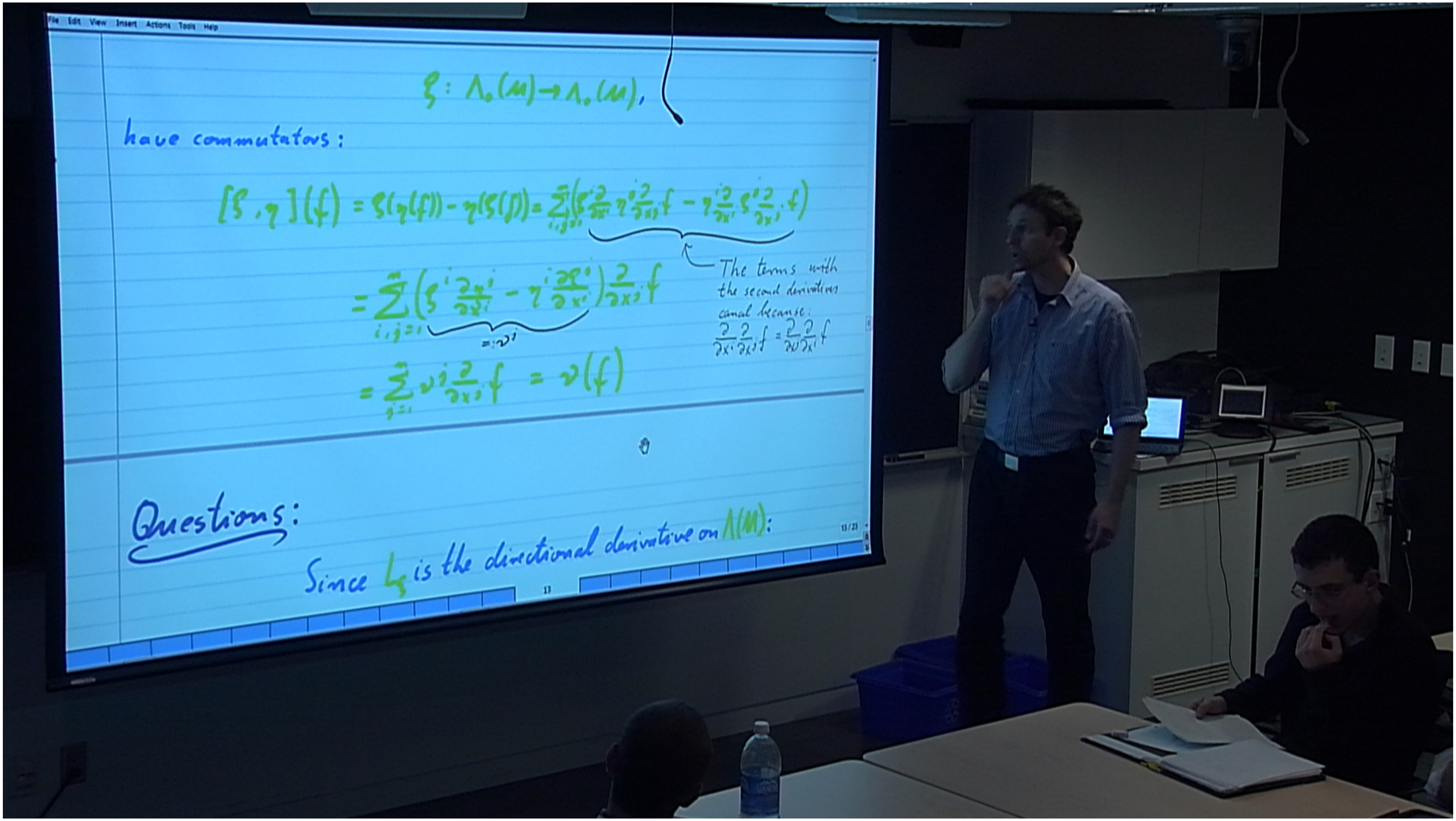
$$L_\xi : \Lambda_1(M) \to \Lambda_1(M)$$

$$L_\xi : \quad df \to \underbrace{d(\underbrace{\xi(f)}_{\in \Lambda_0(M)})}_{\in \Lambda_1(M)}$$

i.e.: $$\underbrace{L_\xi(df)}_{\text{directional derivative of gradient}} = \underbrace{d(\xi(f))}_{\text{gradient of directional derivative}} \qquad \textbf{(D)}$$

Question: Now that L_ξ is a fully defined derivation

$$L_\xi : \Lambda(M) \to \Lambda(M),$$



$$\xi : \Lambda_0(M) \to \Lambda_0(M),$$

have commutators:

$$[\xi,\eta](f) = \xi(\eta(f)) - \eta(\xi(f)) = \sum_{i,j=1}^{m}\underbrace{\left(\xi^i \frac{\partial}{\partial x^i}\eta^j\frac{\partial}{\partial x^j}f - \eta^i\frac{\partial}{\partial x^i}\xi^j\frac{\partial}{\partial x^j}f\right)}$$

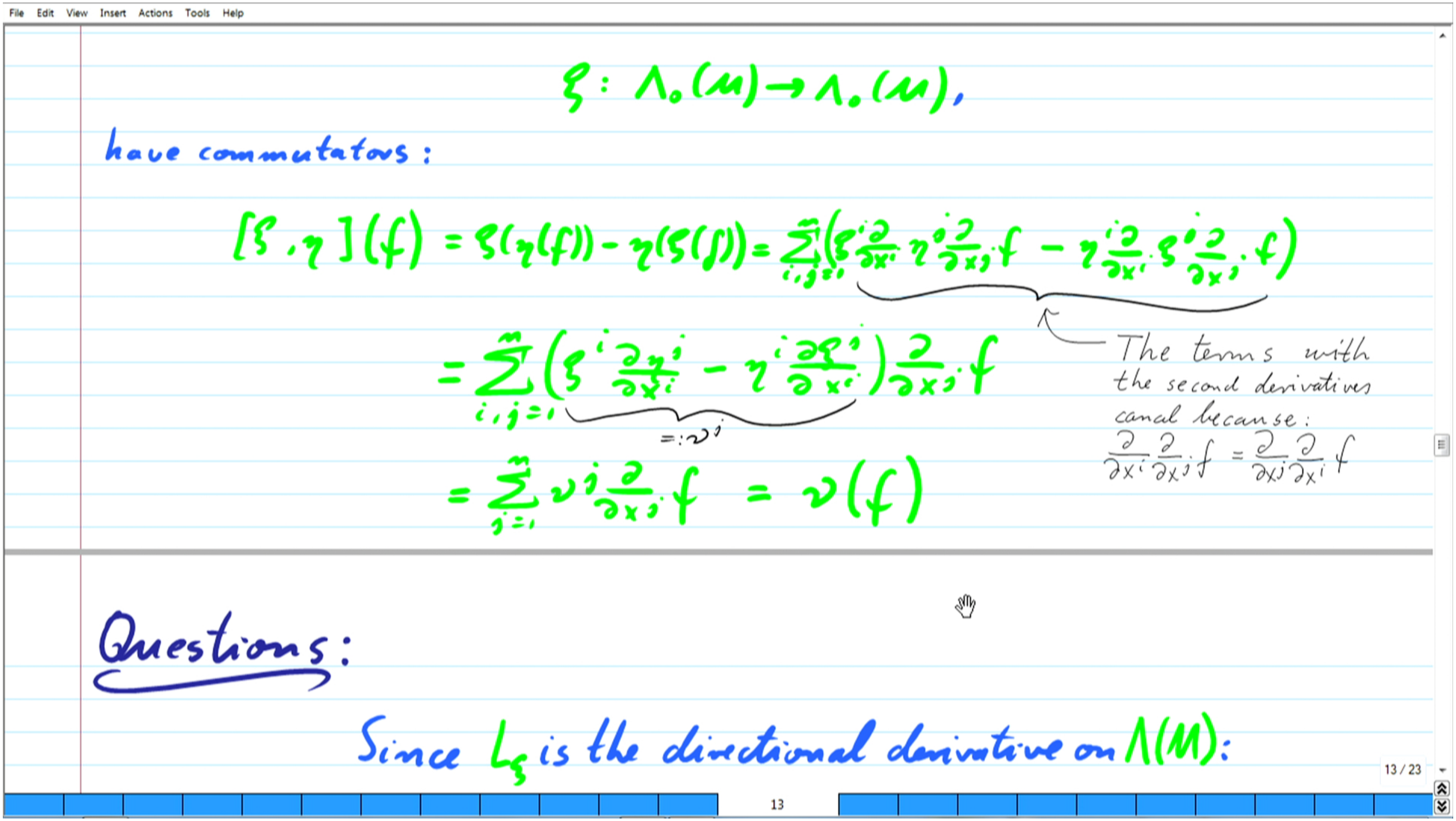
The terms with the second derivatives cancel because: $\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}f = \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}f$

$$= \sum_{i,j=1}^{m}\underbrace{\left(\xi^i\frac{\partial \eta^j}{\partial x^i} - \eta^i\frac{\partial \xi^j}{\partial x^i}\right)}_{=:\nu^j}\frac{\partial}{\partial x^j}f$$

$$= \sum_{j=1}^{m}\nu^j\frac{\partial}{\partial x^j}f = \nu(f)$$

Questions:

Since L_ξ is the directional derivative on $\Lambda(M)$:



$$\xi : \Lambda_0(M) \rightarrow \Lambda_0(M),$$

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$$[\xi, \eta](f) = \xi(\eta(f)) - \eta(\xi(f)) = \sum_{i,j=1}^{n} \underbrace{\left(\xi^i \frac{\partial}{\partial x^i} \eta^j \frac{\partial}{\partial x^j} f - \eta^i \frac{\partial}{\partial x^i} \xi^j \frac{\partial}{\partial x^j} f \right)}$$

The terms with the second derivatives cancel because:

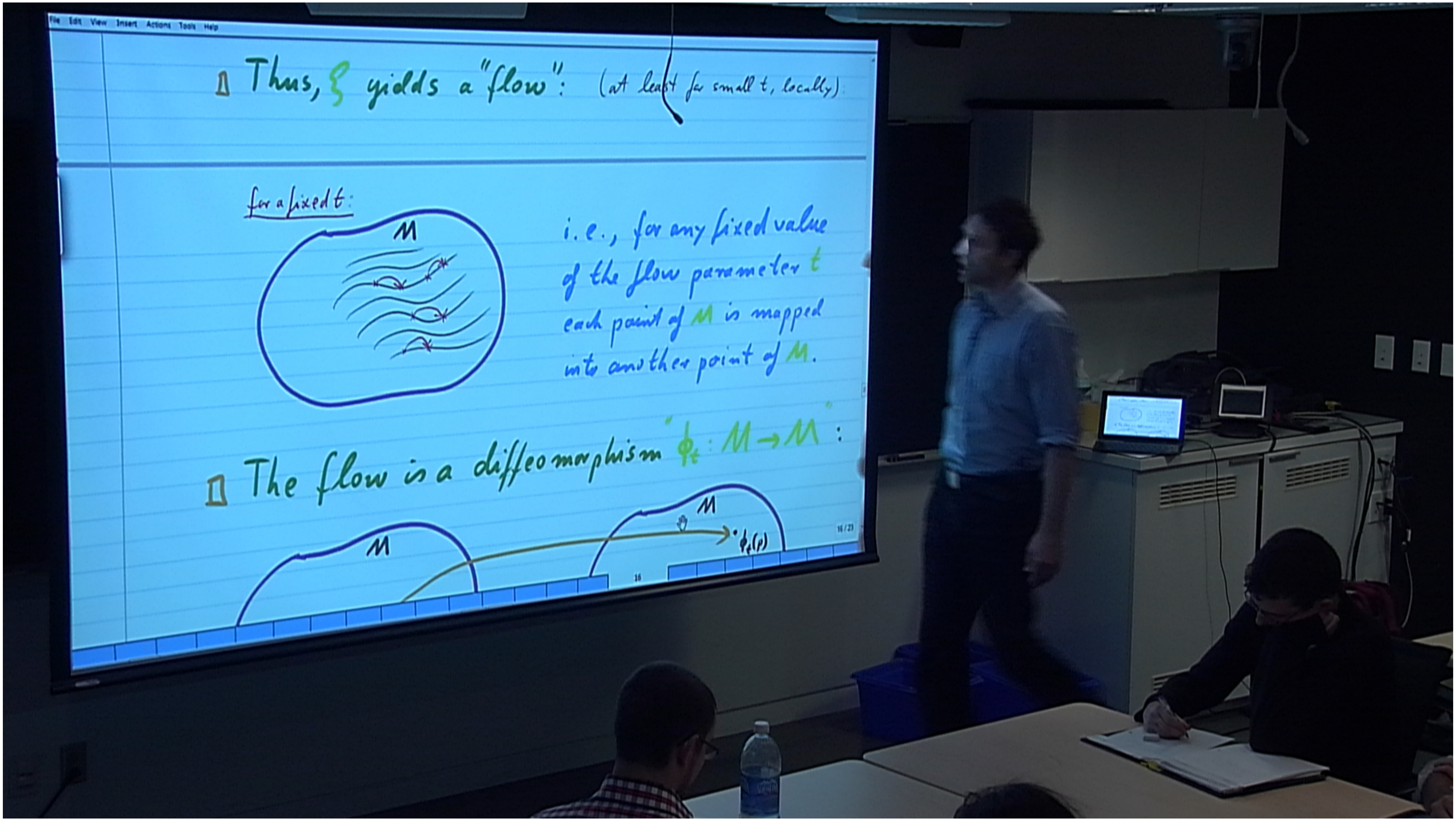
$$\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f = \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} f$$

$$= \sum_{i,j=1}^{n} \underbrace{\left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i} \right)}_{=: v^j} \frac{\partial}{\partial x^j} f$$

$$= \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j} f = v(f)$$

Questions:

Since L_ξ is the directional derivative on $\Lambda(M)$:

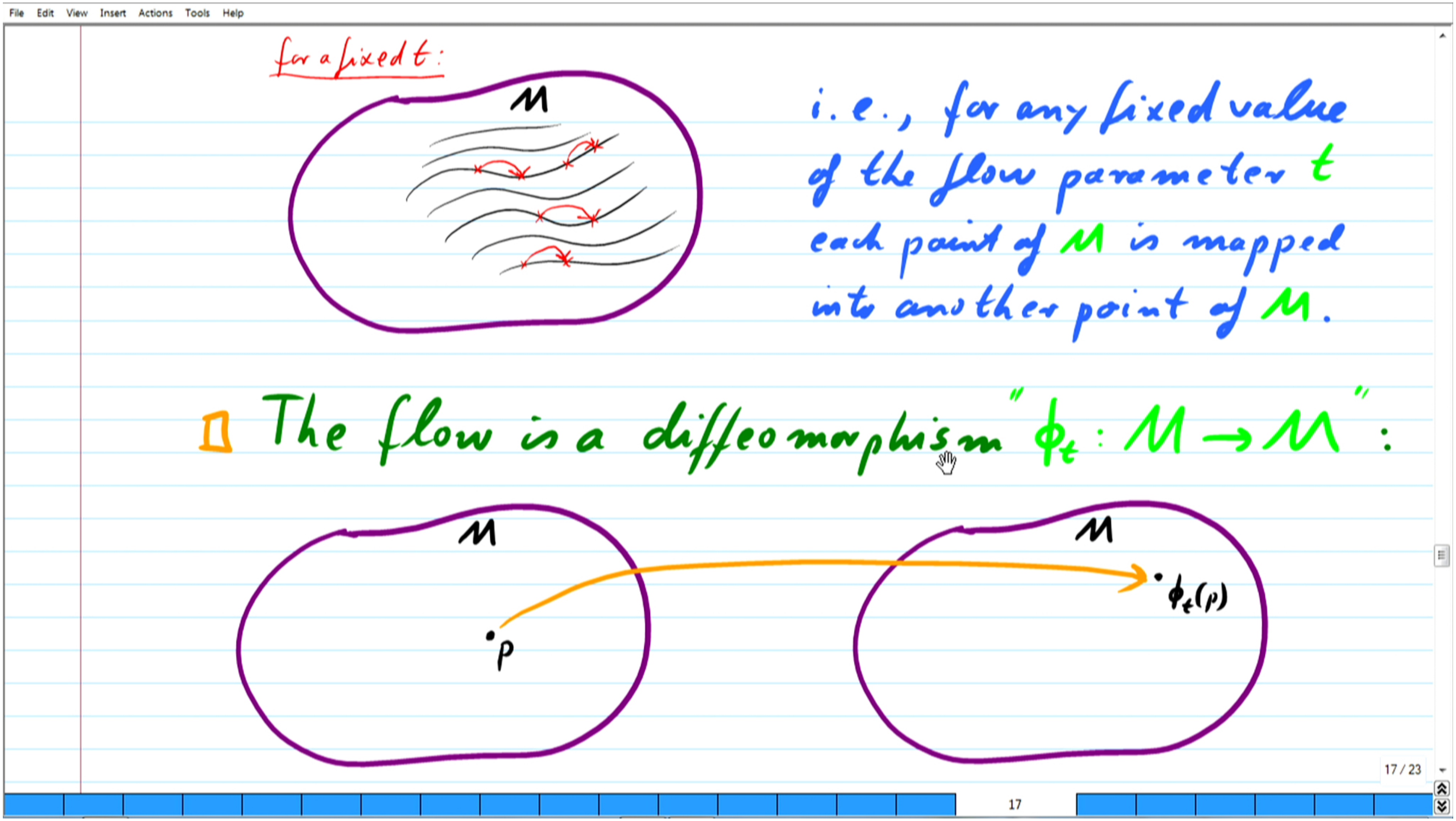


Thus, ξ yields a "flow": (at least for small t, locally)

for a fixed t:

i.e., for any fixed value of the flow parameter t each point of M is mapped into another point of M.

The flow is a diffeomorphism "$\phi_t : M \to M$":



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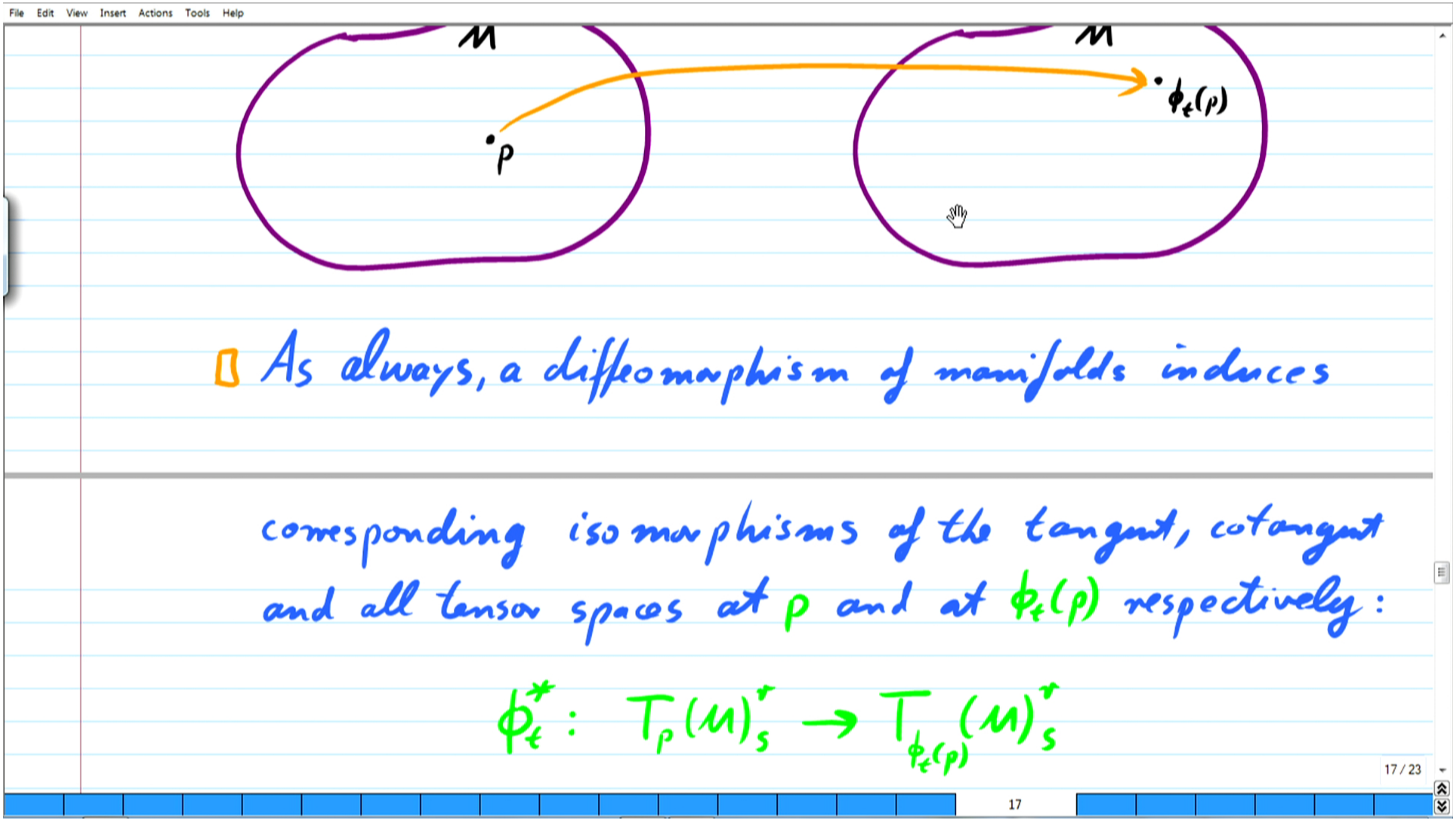
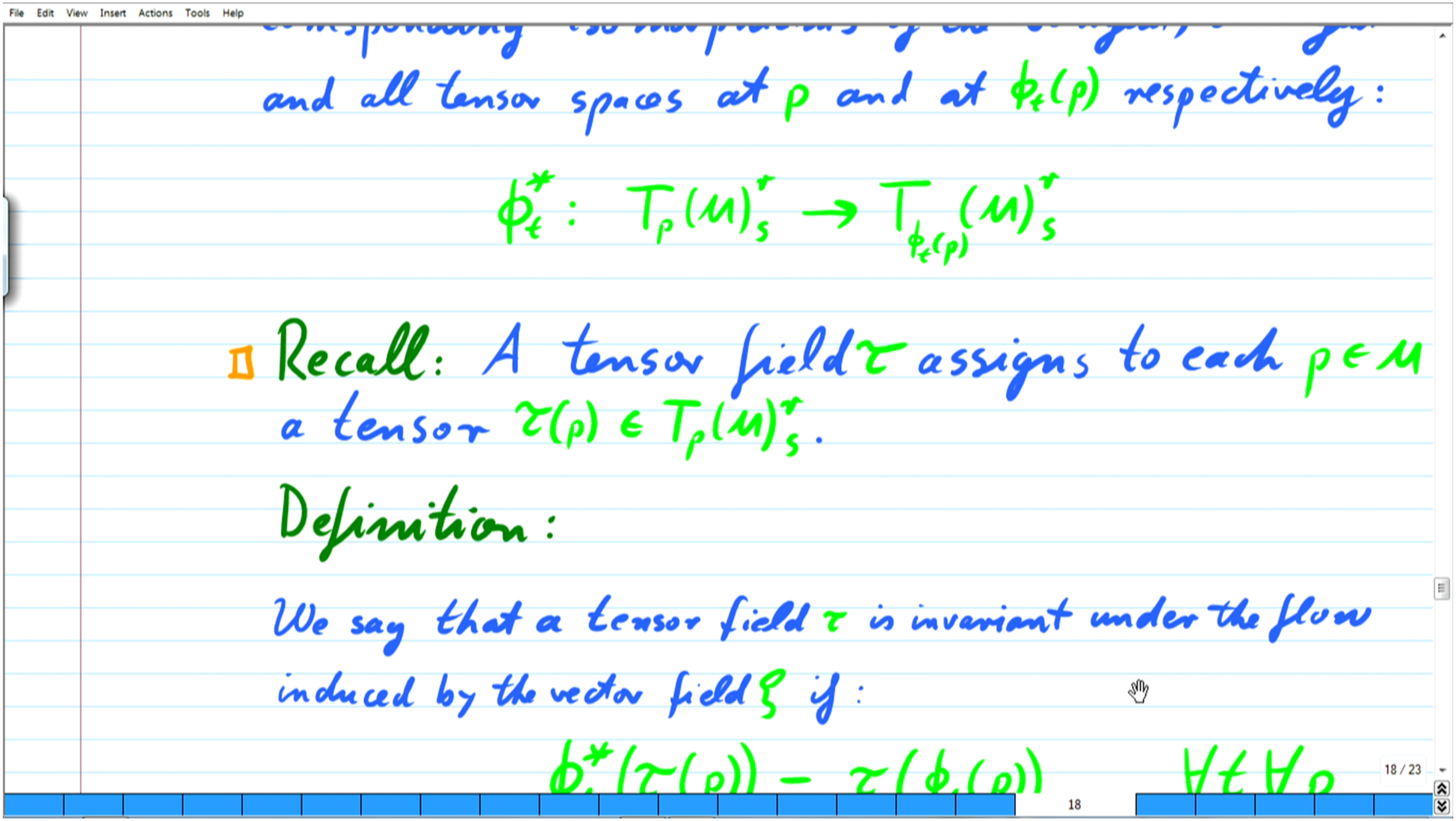


As always, a diffeomorphism of manifolds induces corresponding isomorphisms of the tangent, cotangent and all tensor spaces at p and at $\phi_t(p)$ respectively:

$$\phi_t^*: T_p(M)^r_s \to T_{\phi_t(p)}(M)^r_s$$



and all tensor spaces at p and at $\phi_t(p)$ respectively:

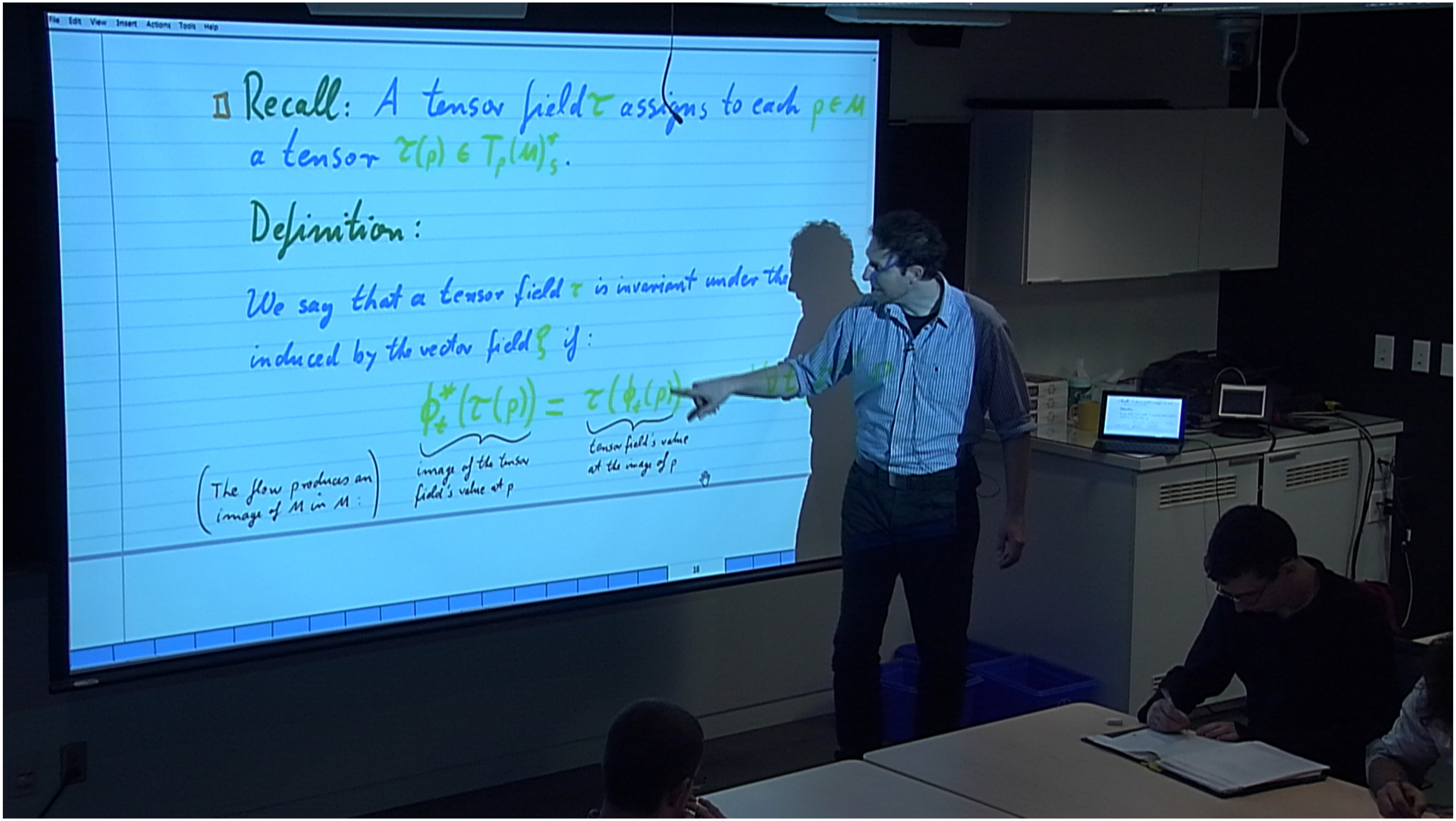
$$\phi_t^*: T_p(M)^r_s \to T_{\phi_t(p)}(M)^r_s$$

□ **Recall:** A tensor field τ assigns to each $p \in M$ a tensor $\tau(p) \in T_p(M)^r_s$.

Definition:

We say that a tensor field τ is invariant under the flow induced by the vector field ξ if:

$$\phi_t^*(\tau(p)) = \tau(\phi_t(p)) \qquad \forall t \, \forall p$$



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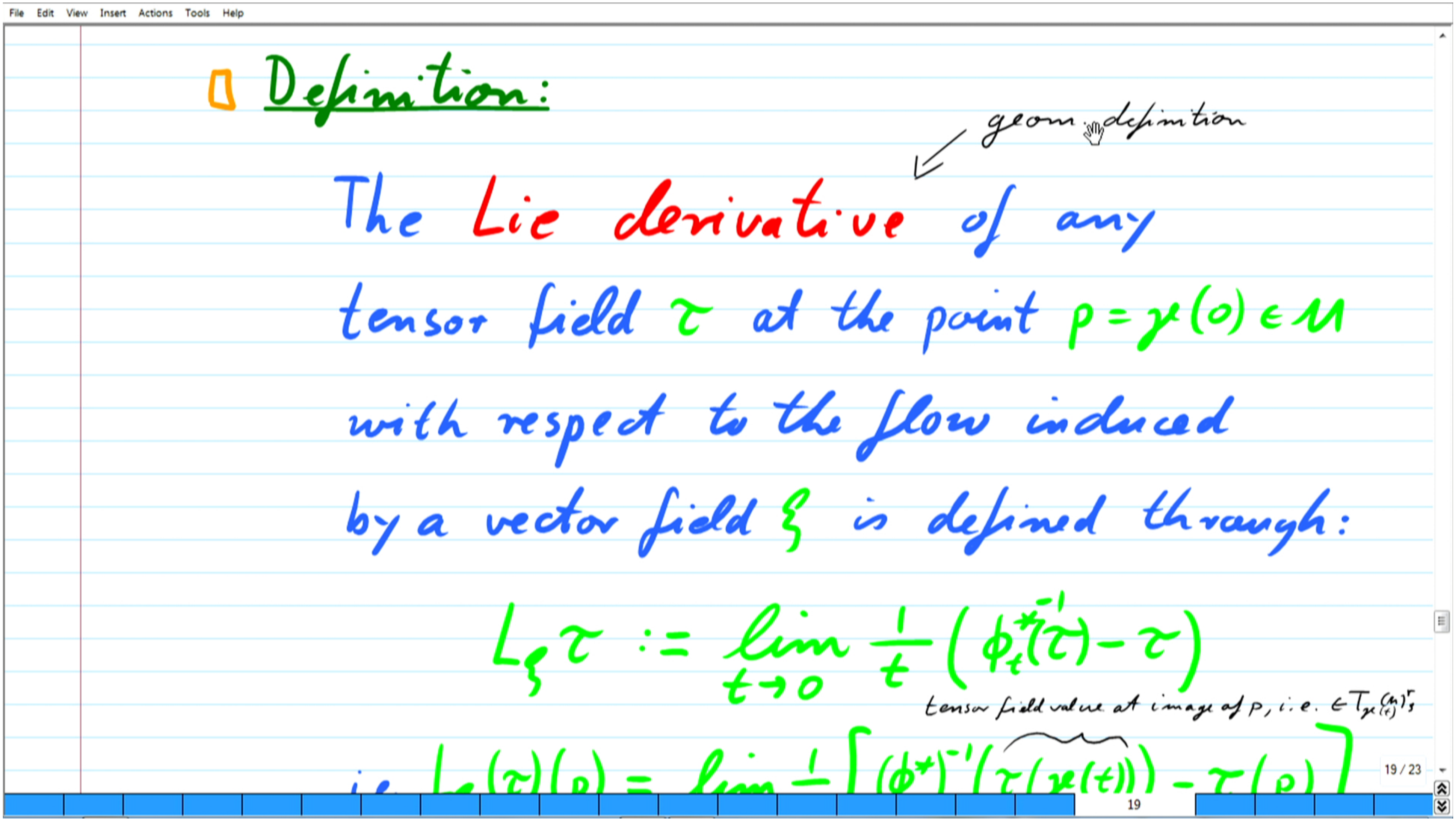
Definition:

We say that a tensor field τ is invariant under the

induced by the vector field ξ if:

$$\underbrace{\phi_t^*(\tau(p))}_{\text{image of the tensor field's value at } p} = \underbrace{\tau(\phi_t(p))}_{\text{tensor field's value at the image of } p}$$

(The flow produces an image of M in M:)



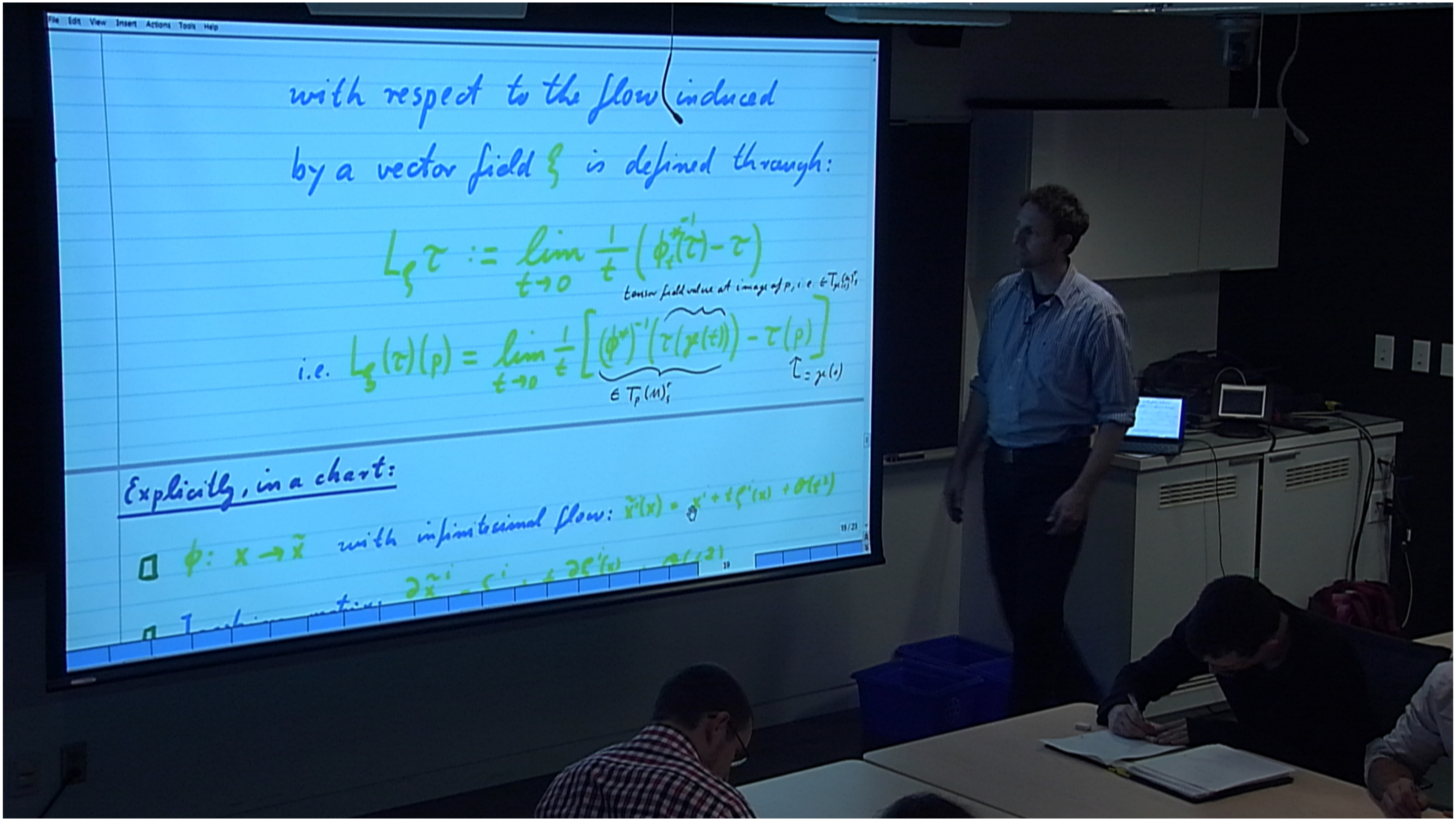
Definition:

geom. definition

The Lie derivative of any tensor field τ at the point $p = \gamma(0) \in M$ with respect to the flow induced by a vector field ξ is defined through:

$$L_\xi \tau := \lim_{t \to 0} \frac{1}{t}\left(\phi_t^{*-1}(\tau) - \tau\right)$$

i.e. $L(\tau)(p) = \lim \frac{1}{t}\left[(\phi^*)^{-1}\big(\overbrace{\tau(\gamma(t))}^{\text{tensor field value at image of } p \text{, i.e. } \in T_{\gamma(t)}(M)^r_s}\big) - \tau(p)\right]$



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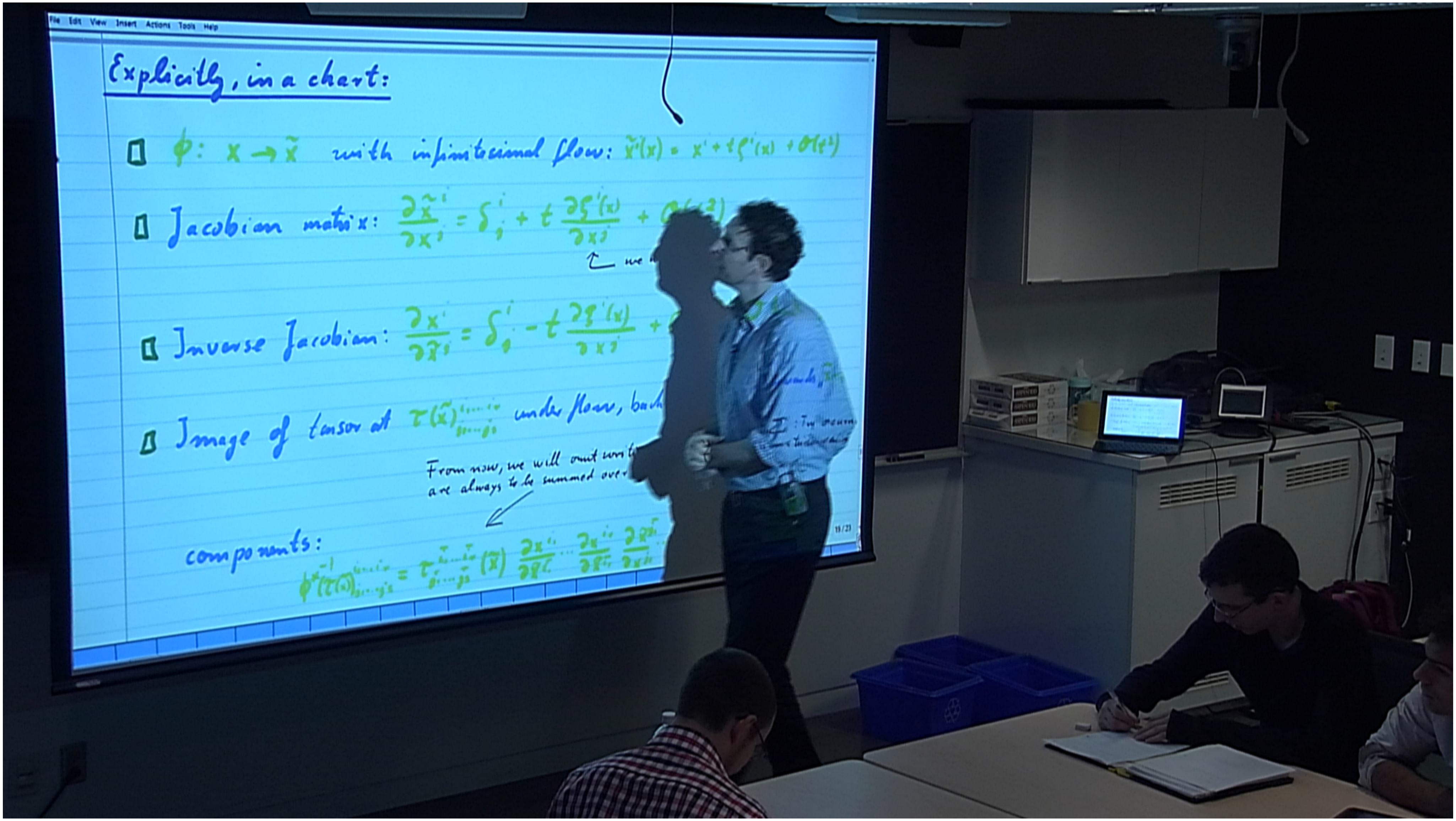
Explicitly, in a chart:

☐ $\phi: x \to \tilde{x}$ with infinitesimal flow: $\tilde{x}^i(x) = x^i + t\xi^i(x) + O(t^2)$



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↰ we

☐ Inverse Jacobian: $\frac{\partial x^i}{\partial \tilde{x}^j} = \delta^i_j - t\frac{\partial \xi^i(x)}{\partial x^j} +$

☐ Image of tensor at $\tau(\tilde{x})^{i_1 \dots i_r}_{j_1 \dots j_s}$ under flow, back

From now, we will omit writ
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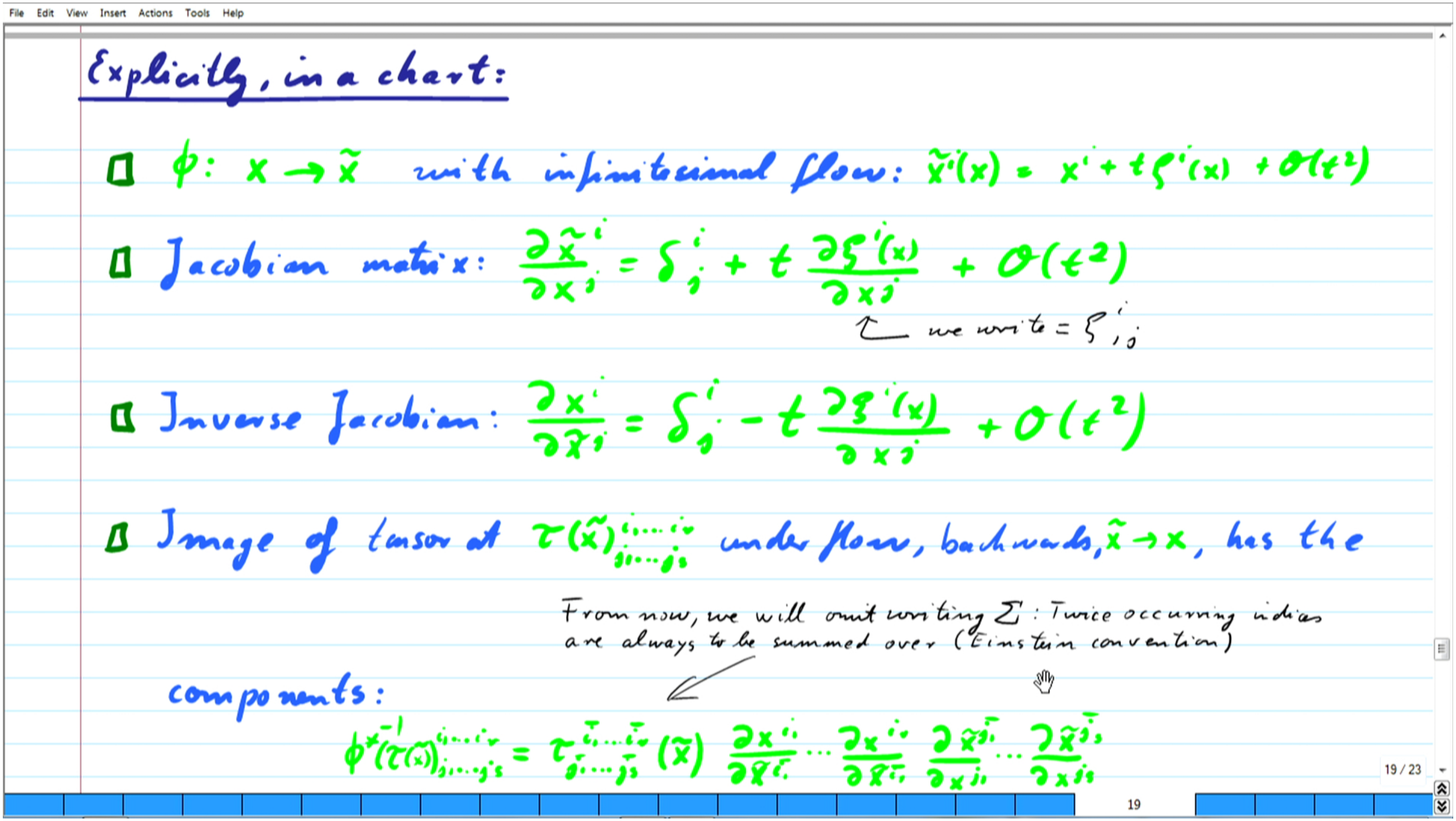
components:

$$\phi^{*-1}(\tau(x))^{i_1 \dots i_r}_{j_1 \dots j_s} = \tau^{\tilde{i}_1 \dots \tilde{i}_r}_{\tilde{j}_1 \dots \tilde{j}_s}(\tilde{x}) \frac{\partial x^{i_1}}{\partial \tilde{x}^{\tilde{i}_1}} \cdots \frac{\partial x^{i_r}}{\partial \tilde{x}^{\tilde{i}_r}} \frac{\partial \tilde{x}^{\tilde{j}_1}}{\partial x^{j_1}} \cdots$$



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- ☐ $\phi: x \to \tilde{x}$ with infinitesimal flow: $\tilde{x}^i(x) = x^i + t\xi^i(x) + \mathcal{O}(t^2)$
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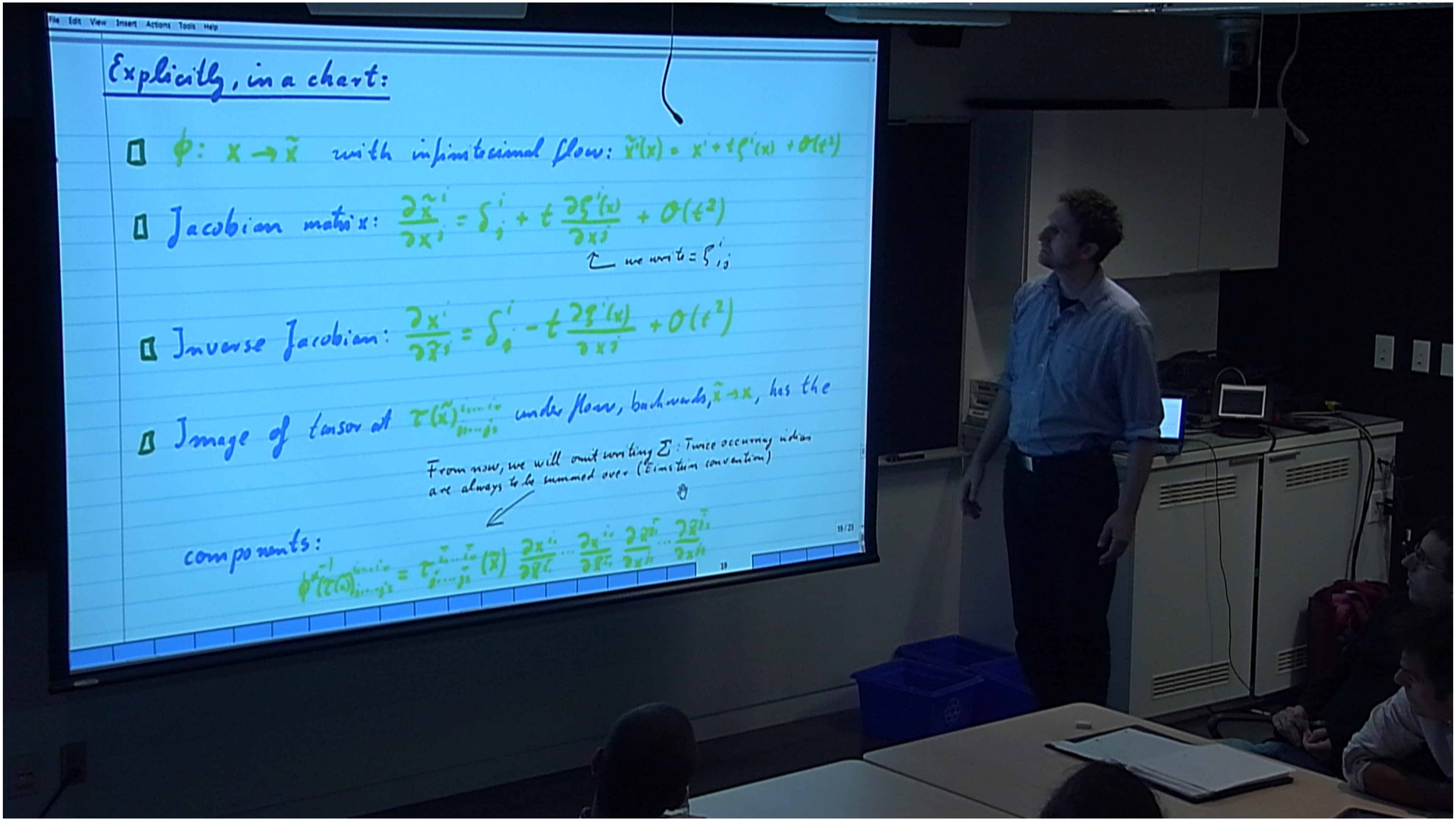
  $\hookleftarrow$ we write $= \xi^i_{,j}$
- ☐ Inverse Jacobian: $\frac{\partial x^i}{\partial \tilde{x}^j} = \delta^i_j - t\,\frac{\partial \xi^i(x)}{\partial x^j} + \mathcal{O}(t^2)$
- ☐ Image of tensor at $\tau(\tilde{x})^{i_1\dots i_r}_{j_1\dots j_s}$ under flow, backwards, $\tilde{x} \to x$, has the components:

From now, we will omit writing Σ: Twice occurring indices are always to be summed over (Einstein convention)

$$\phi^{*-1}(\tau(\tilde{x}))^{i_1\dots i_r}_{j_1\dots j_s} = \tau^{\bar{i}_1\dots\bar{i}_r}_{\bar{j}_1\dots\bar{j}_s}(\tilde{x})\,\frac{\partial x^{i_1}}{\partial \tilde{x}^{\bar{i}_1}}\cdots\frac{\partial x^{i_r}}{\partial \tilde{x}^{\bar{i}_r}}\,\frac{\partial \tilde{x}^{\bar{j}_1}}{\partial x^{j_1}}\cdots\frac{\partial \tilde{x}^{\bar{j}_s}}{\partial x^{j_s}}$$



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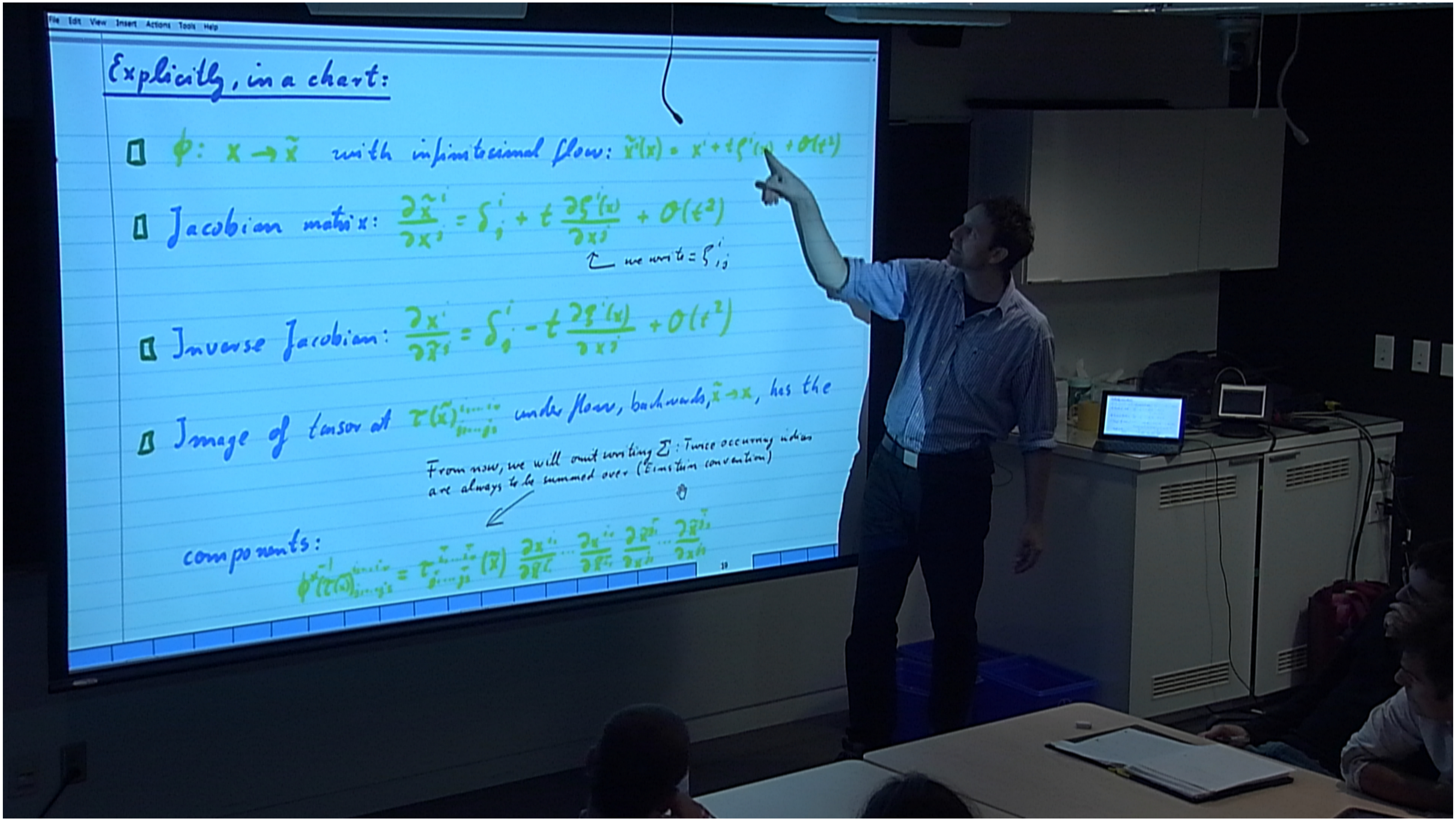

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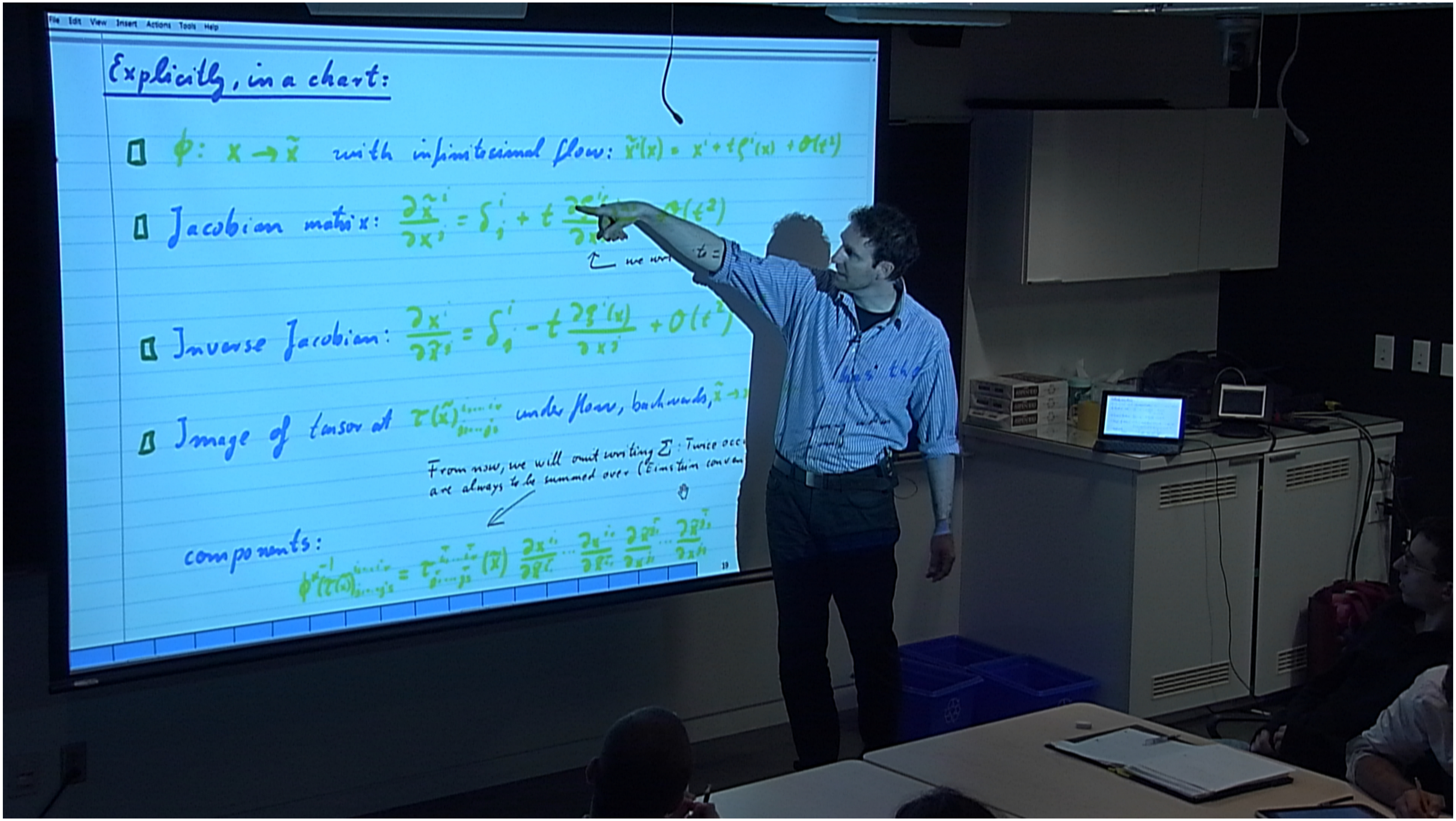

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  - ↰ we wr… $t =$
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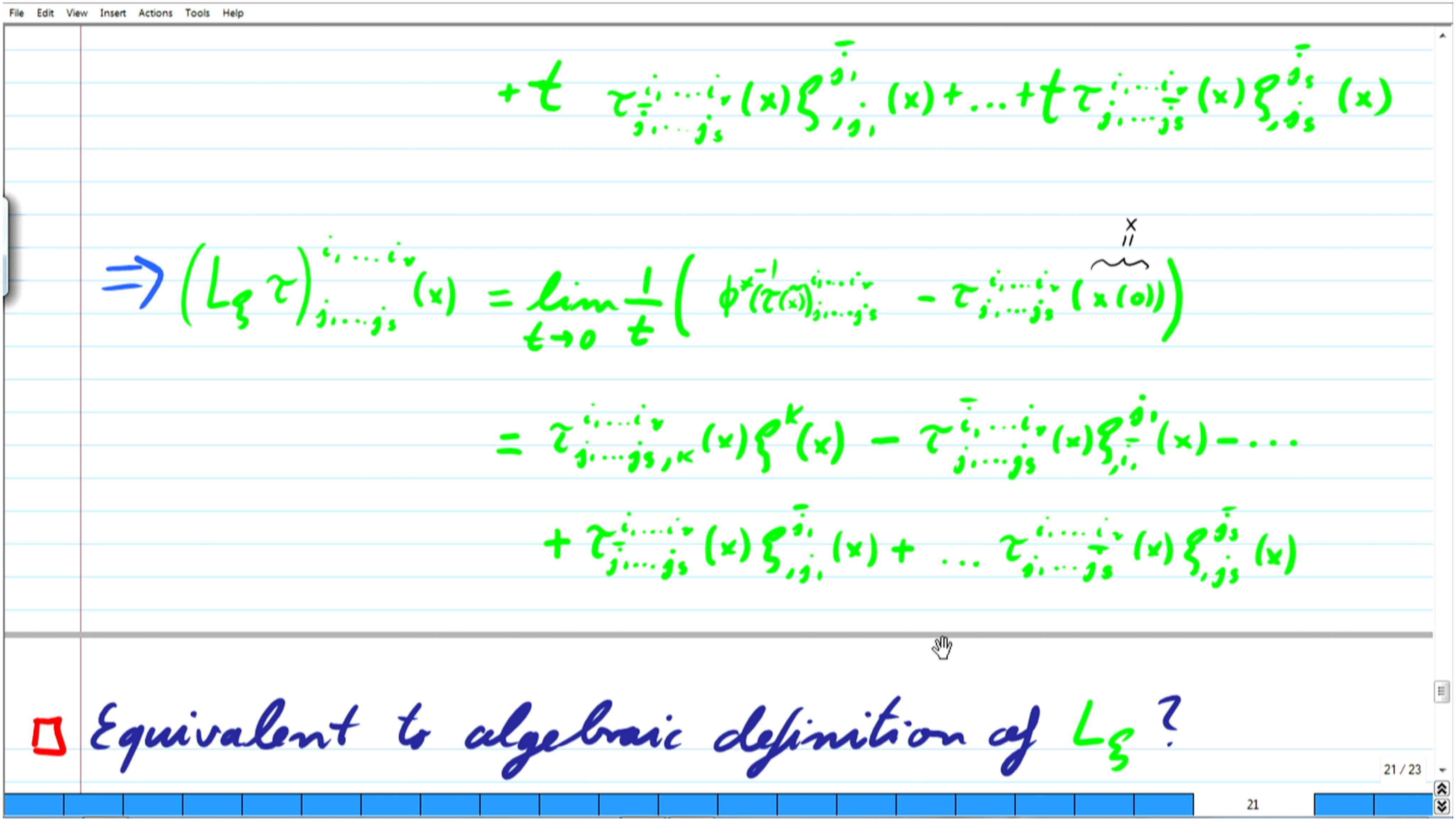
components:

$$\phi^{*-1}(\tau(x))^{i_1\dots i_r}_{j_1\dots j_s} = \tau^{\tilde{k}_1\dots\tilde{k}_r}_{\tilde{l}_1\dots\tilde{l}_s}(\tilde{x})\,\frac{\partial x^{i_1}}{\partial \tilde{x}^{k_1}}\cdots\frac{\partial x^{i_r}}{\partial \tilde{x}^{k_r}}\,\frac{\partial \tilde{x}^{l_1}}{\partial x^{j_1}}\cdots\frac{\partial \tilde{x}^{l_s}}{\partial x^{j_s}}$$



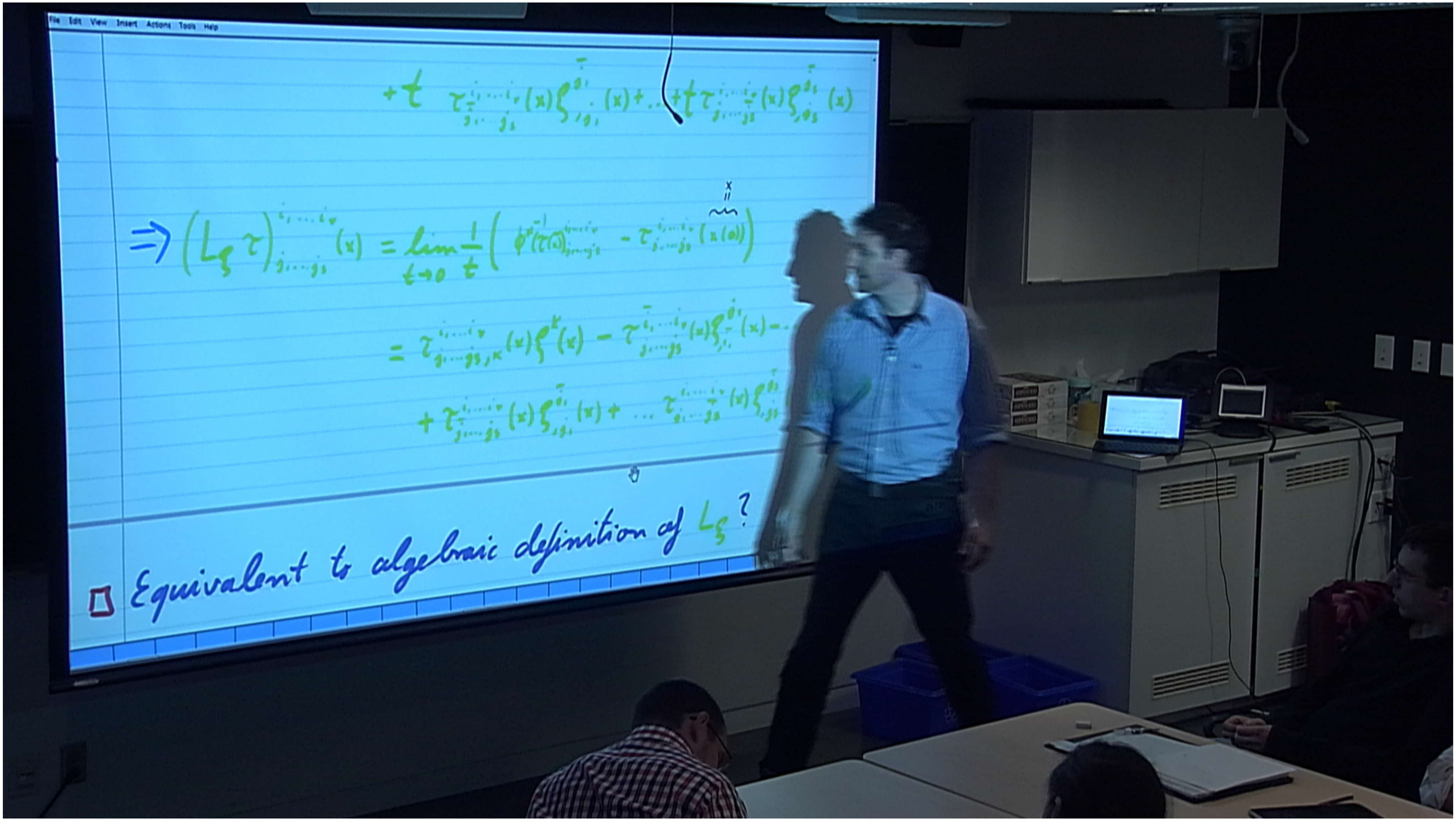
$$+t\,\tau^{i_1\dots i_r}_{\bar{j}_1\dots j_s}(x)\,\xi^{\bar{j}_1}_{,j_1}(x)+\dots+t\,\tau^{i_1\dots i_r}_{j_1\dots \bar{j}_s}(x)\,\xi^{\bar{j}_s}_{,j_s}(x)$$

$$\Rightarrow (L_\xi\tau)^{i_1\dots i_r}_{j_1\dots j_s}(x)=\lim_{t\to 0}\frac{1}{t}\Big(\phi^{*-1}(\tau(x))^{i_1\dots i_r}_{j_1\dots j_s}-\tau^{i_1\dots i_r}_{j_1\dots j_s}(\overset{x}{\overset{\shortparallel}{\overbrace{x(0)}}})\Big)$$

$$=\tau^{i_1\dots i_r}_{j_1\dots j_s,k}(x)\,\xi^k(x)-\tau^{\bar{i}_1\dots i_r}_{j_1\dots j_s}(x)\,\xi^{i_1}_{,\bar{i}_1}(x)-\dots$$

$$+\tau^{i_1\dots i_r}_{\bar{j}_1\dots j_s}(x)\,\xi^{\bar{j}_1}_{,j_1}(x)+\dots\tau^{i_1\dots i_r}_{j_1\dots \bar{j}_s}(x)\,\xi^{\bar{j}_s}_{,j_s}(x)$$

☐ Equivalent to algebraic definition of L_ξ?



$$+t\,\tau^{i_1\cdots i_r}_{\cdot\cdots j_s}(x)\,\xi^{\cdot}_{,j_1}(x)+\cdots+t\,\tau^{i_1\cdots i_r}_{j_1\cdots \cdot}(x)\,\xi^{\cdot}_{,j_s}(x)$$

$$\Rightarrow \left(L_\xi \tau\right)^{i_1\cdots i_r}_{j_1\cdots j_s}(x)=\lim_{t\to 0}\frac{1}{t}\left(\phi^{*-1}(\tau(x))^{i_1\cdots i_r}_{j_1\cdots j_s}-\tau^{i_1\cdots i_r}_{j_1\cdots j_s}(\overset{x}{\overset{\shortparallel}{\widetilde{x(0)}}})\right)$$

$$=\tau^{i_1\cdots i_r}_{j_1\cdots j_s,k}(x)\,\xi^{k}(x)-\tau^{\cdot\cdots i_r}_{j_1\cdots j_s}(x)\,\xi^{i_1}_{,\cdot}(x)-\cdots$$

$$+\tau^{i_1\cdots i_r}_{\cdot\cdots j_s}(x)\,\xi^{\cdot}_{,j_1}(x)+\cdots\tau^{i_1\cdots i_r}_{j_1\cdots \cdot}(x)\,\xi^{\cdot}_{,j_s}$$

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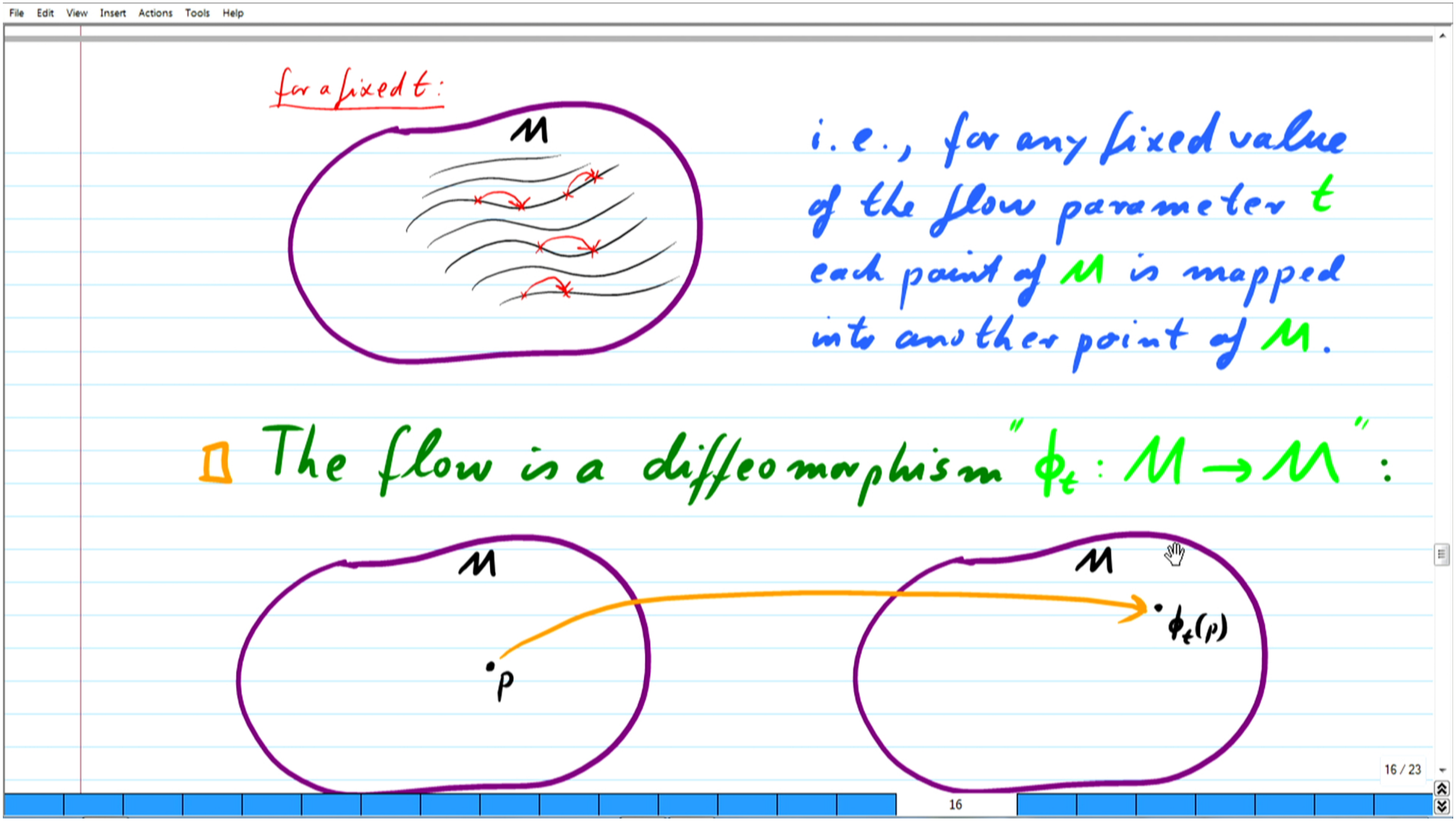
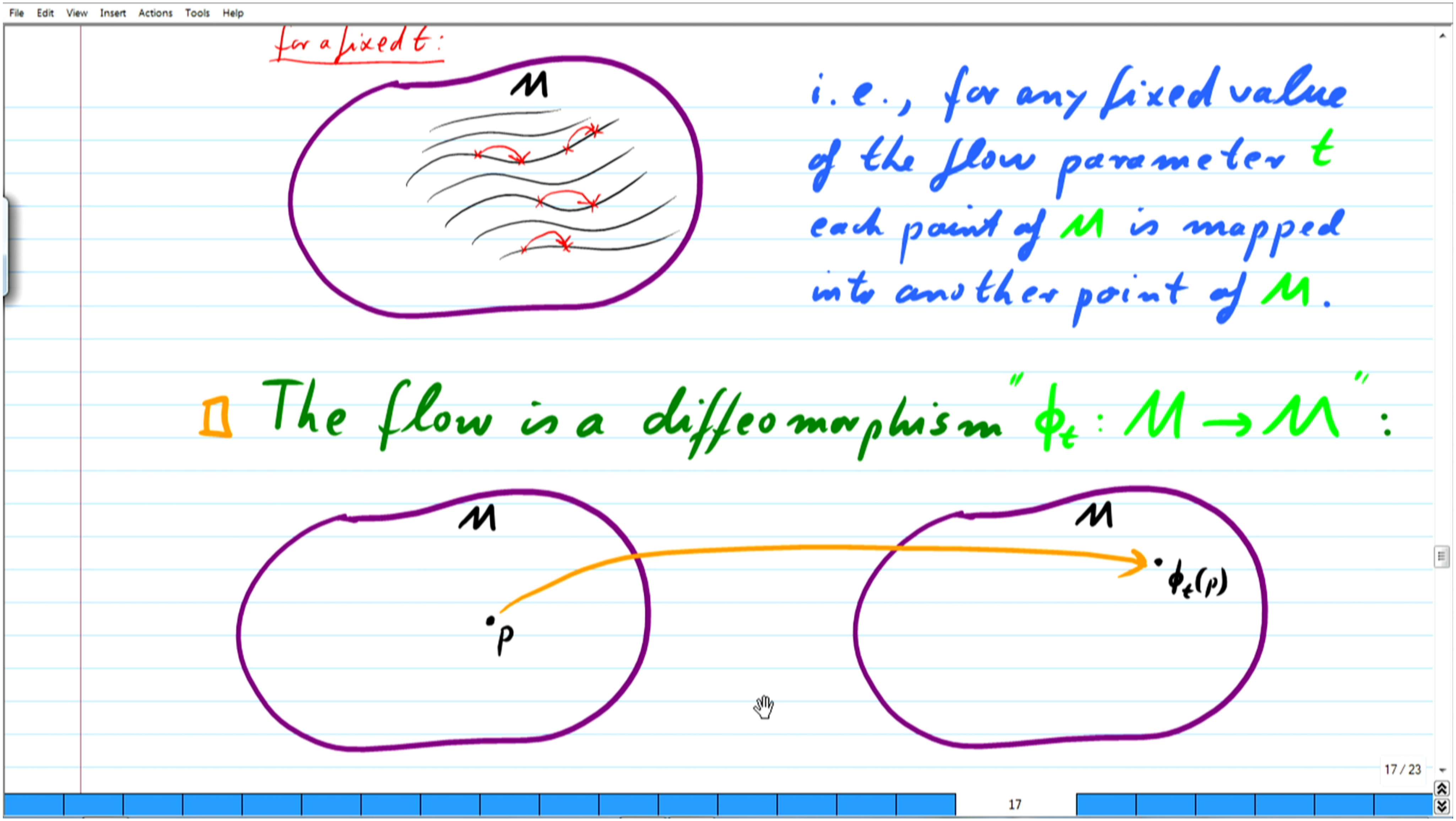


i.e., for any fixed value of the flow parameter t each point of M is mapped into another point of M.

☐ The flow is a diffeomorphism "$\phi_t : M \to M$":



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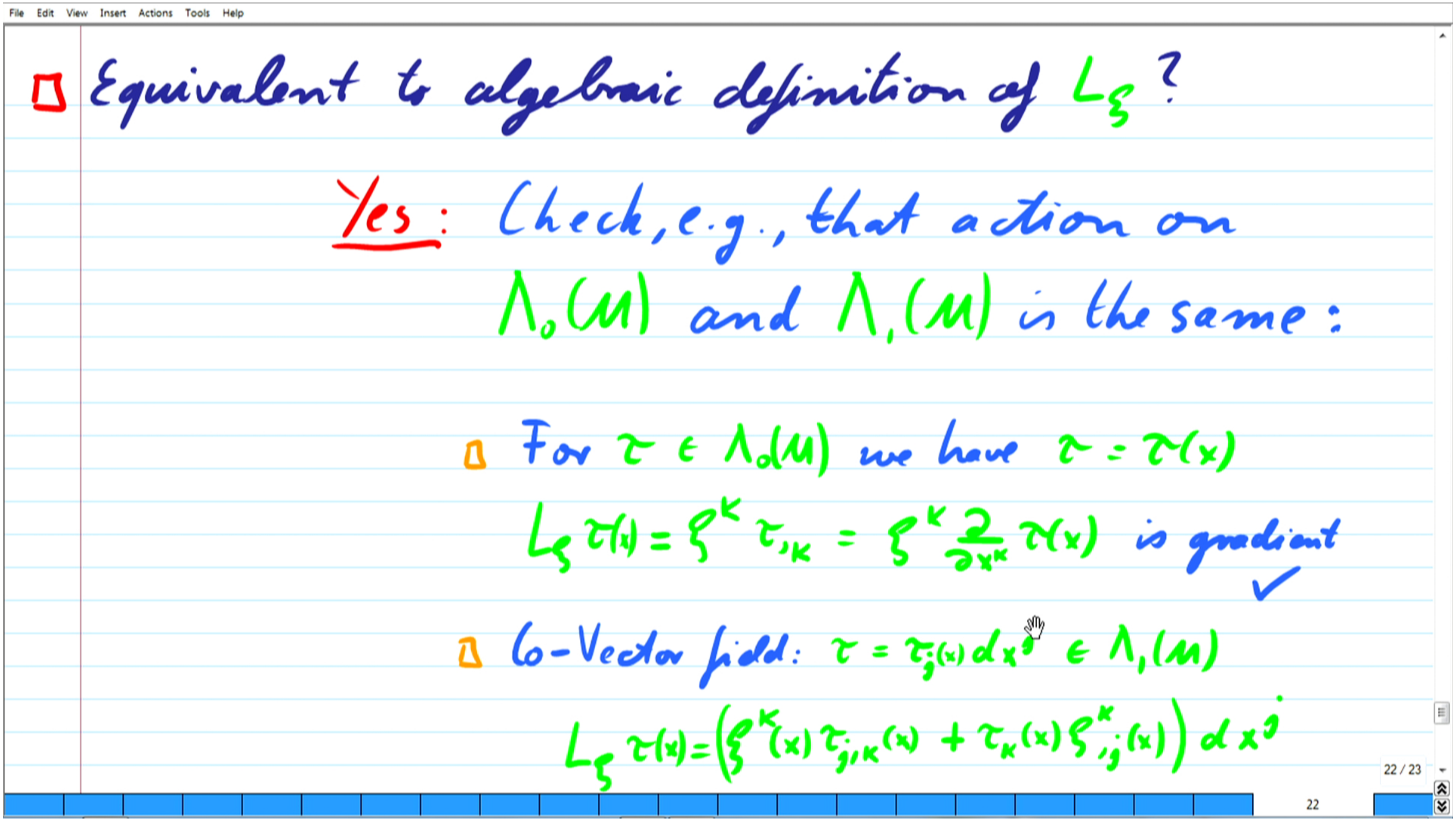
i.e., for any fixed value of the flow parameter t each point of M is mapped into another point of M.

□ The flow is a diffeomorphism "$\phi_t : M \to M$":



☐ Equivalent to algebraic definition of L_ξ?

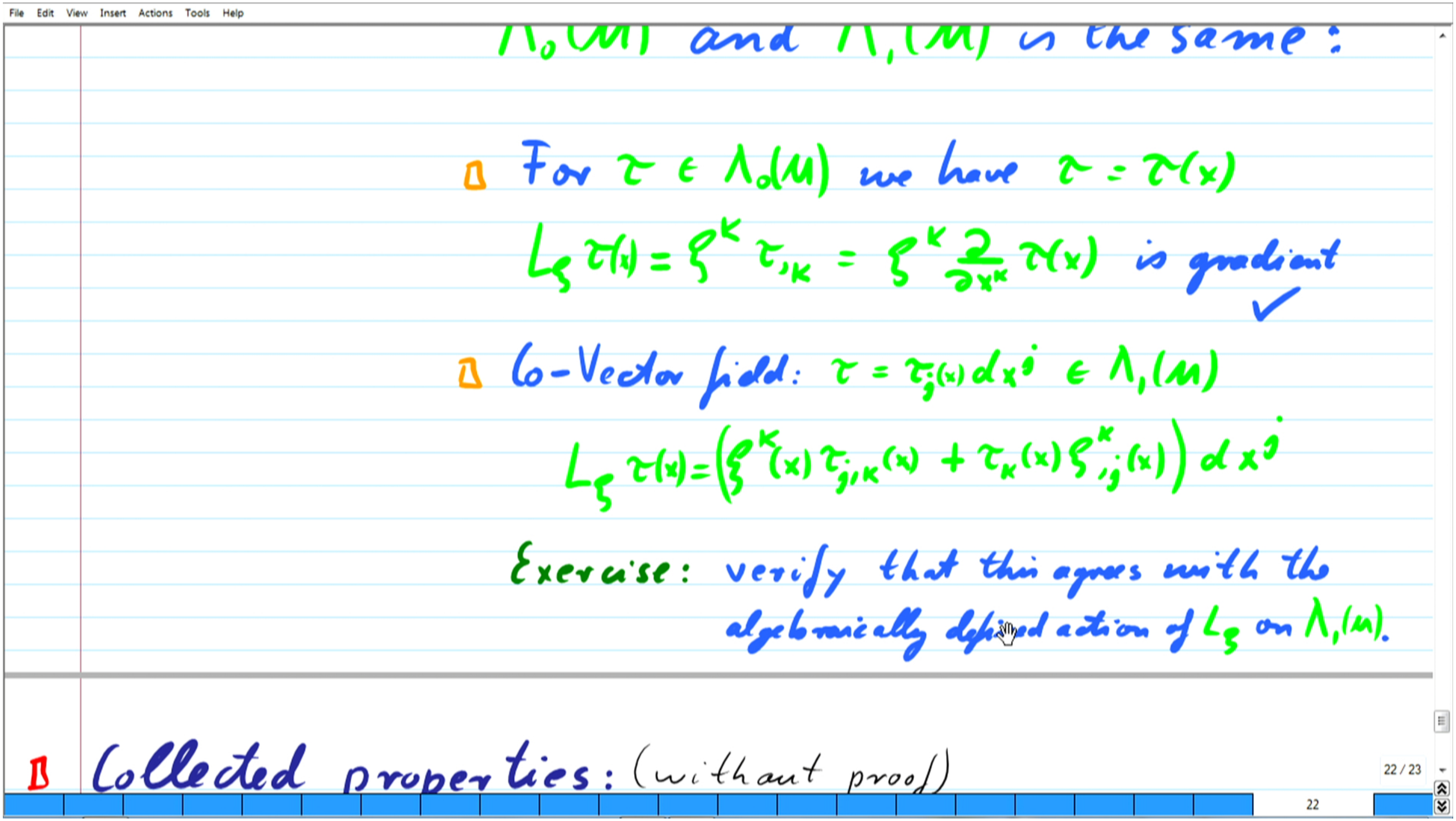
Yes: Check, e.g., that action on $\Lambda_0(M)$ and $\Lambda_1(M)$ is the same:

☐ For $\tau \in \Lambda_0(M)$ we have $\tau = \tau(x)$

$$L_\xi \tau(x) = \xi^k \tau_{,k} = \xi^k \frac{\partial}{\partial x^k} \tau(x)$$ is gradient ✓

☐ Co-Vector field: $\tau = \tau_j(x)\, dx^j \in \Lambda_1(M)$

$$L_\xi \tau(x) = \left(\xi^k(x)\, \tau_{j,k}(x) + \tau_k(x)\, \xi^k_{,j}(x) \right) dx^j$$



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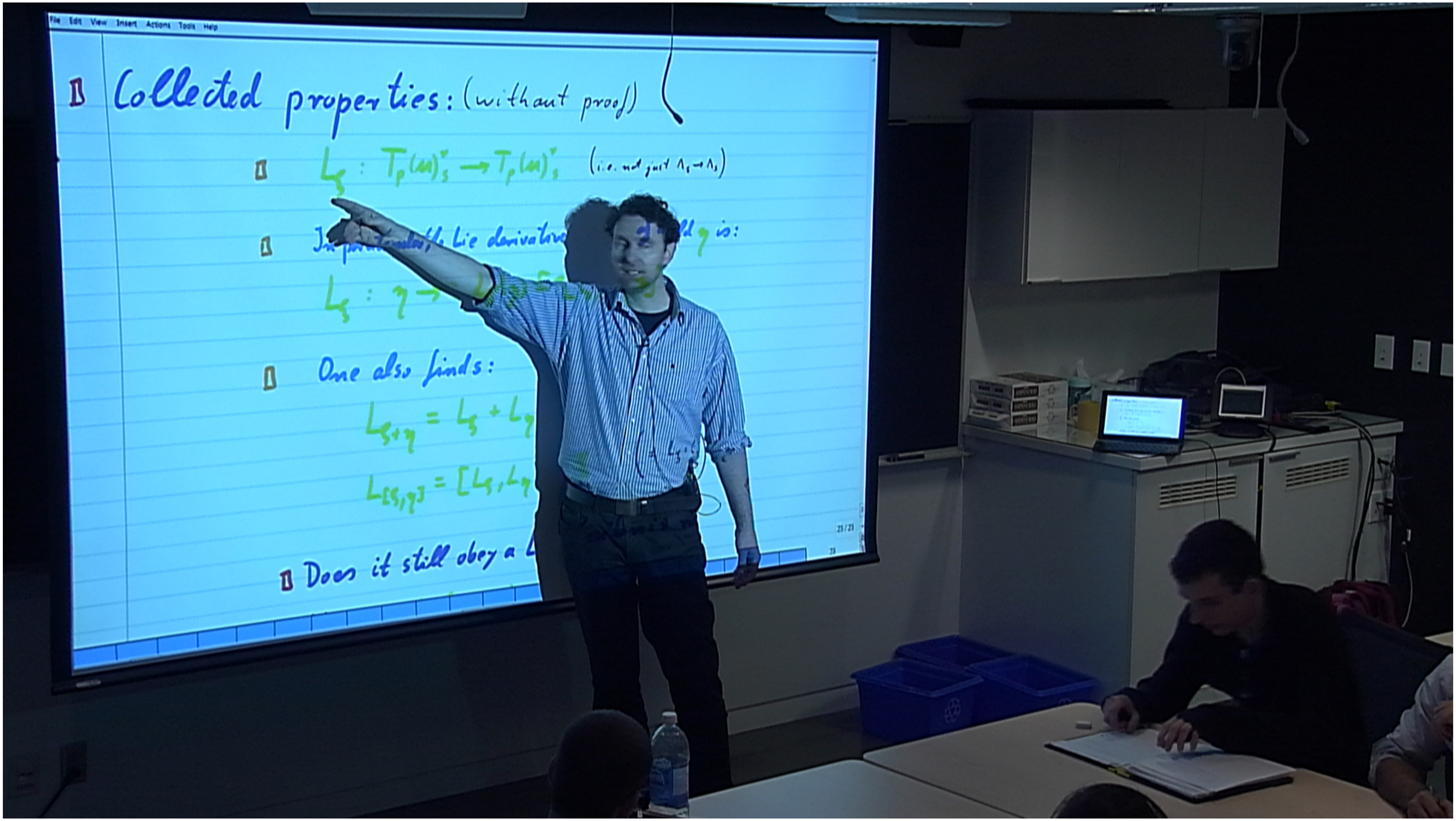
Exercise: verify that this agrees with the algebraically defined action of L_ξ on $\Lambda_1(M)$.

- Collected properties: (without proof)



Collected properties: (without proof)

- $L_\xi : T_p(M)^r_s \to T_p(M)^r_s$ (i.e. not just $\Lambda_s \to \Lambda_s$)
- In particular, the Lie derivative [illegible] field η is:

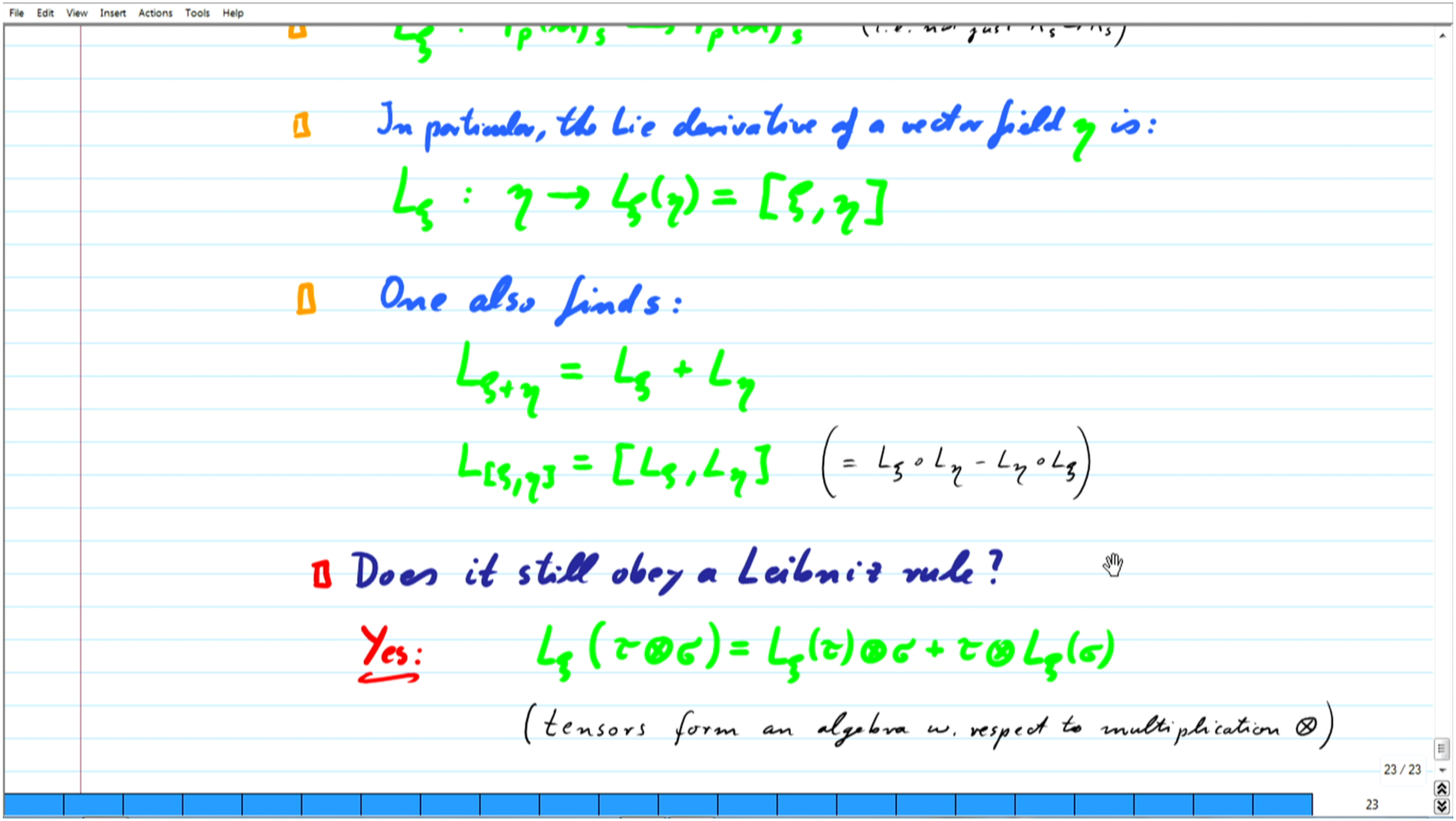
  $L_\xi : \eta \to$ [illegible]

- One also finds:

  $L_{\xi+\eta} = L_\xi + L_\eta$

  $L_{[\xi,\eta]} = [L_\xi, L_\eta$

- Does it still obey a L



- $L_\xi : T_p(r,s)_S \to T_p(r,s)_S$ (i.e. not just $K_S \to K_S$)

- In particular, the Lie derivative of a vector field η is:

$$L_\xi : \eta \to L_\xi(\eta) = [\xi, \eta]$$

- One also finds:

$$L_{\xi+\eta} = L_\xi + L_\eta$$

$$L_{[\xi,\eta]} = [L_\xi, L_\eta] \quad \left(= L_\xi \circ L_\eta - L_\eta \circ L_\xi \right)$$

- Does it still obey a Leibniz rule?

Yes:

$$L_\xi(\tau \otimes \sigma) = L_\xi(\tau) \otimes \sigma + \tau \otimes L_\xi(\sigma)$$

(tensors form an algebra w. respect to multiplication $\otimes$)