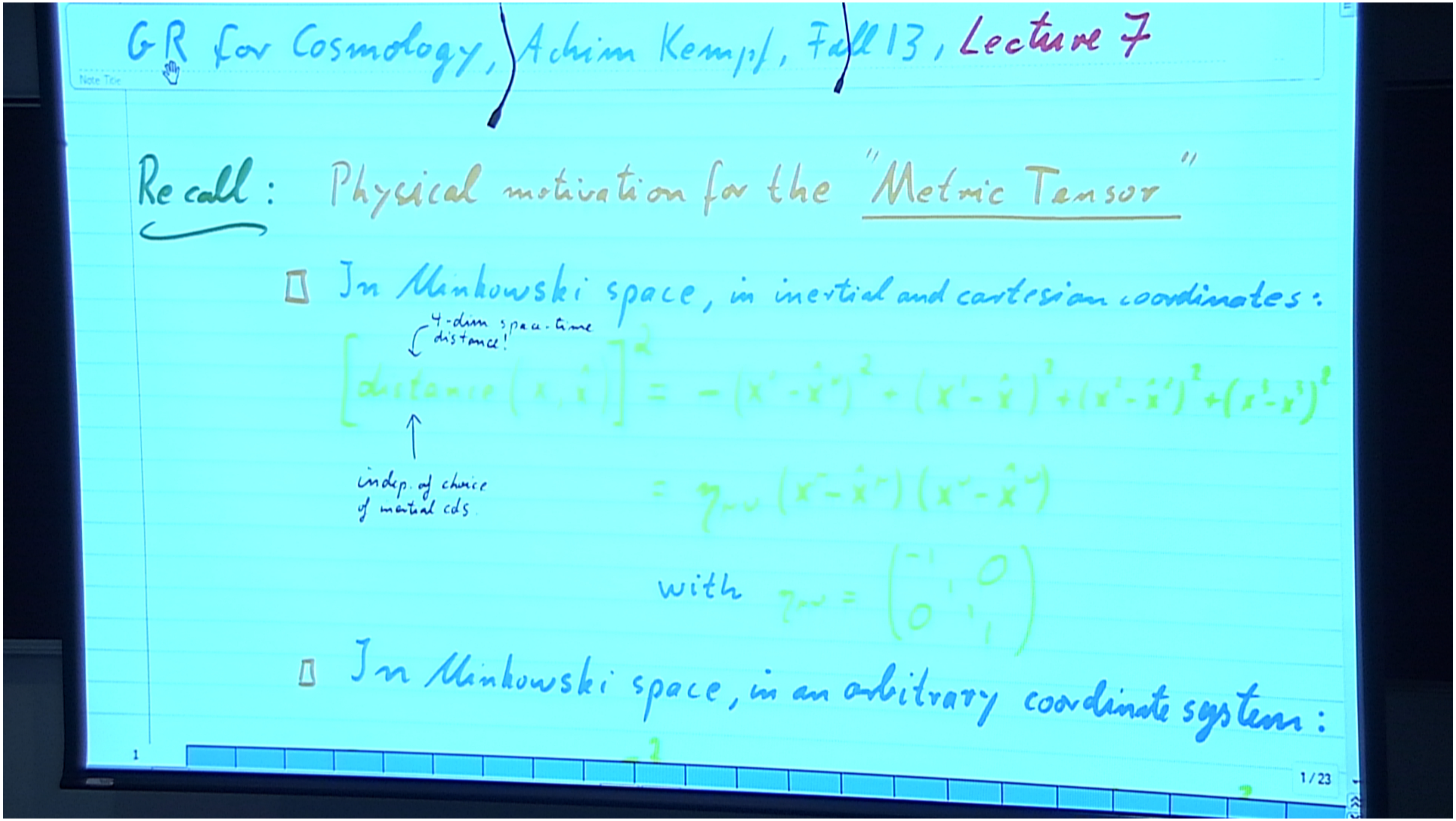
Title: General Relativity for Cosmology - Lecture 7

Date: Oct 15, 2013 04:00 PM

URL: http://pirsa.org/13100003

Abstract:



GR for Cosmology, Achim Kempf, Fall 13, Lecture 7

Recall: Physical motivation for the "Metric Tensor"

- ☐ In Minkowski space, in inertial and cartesian coordinates:

4-dim space-time distance!

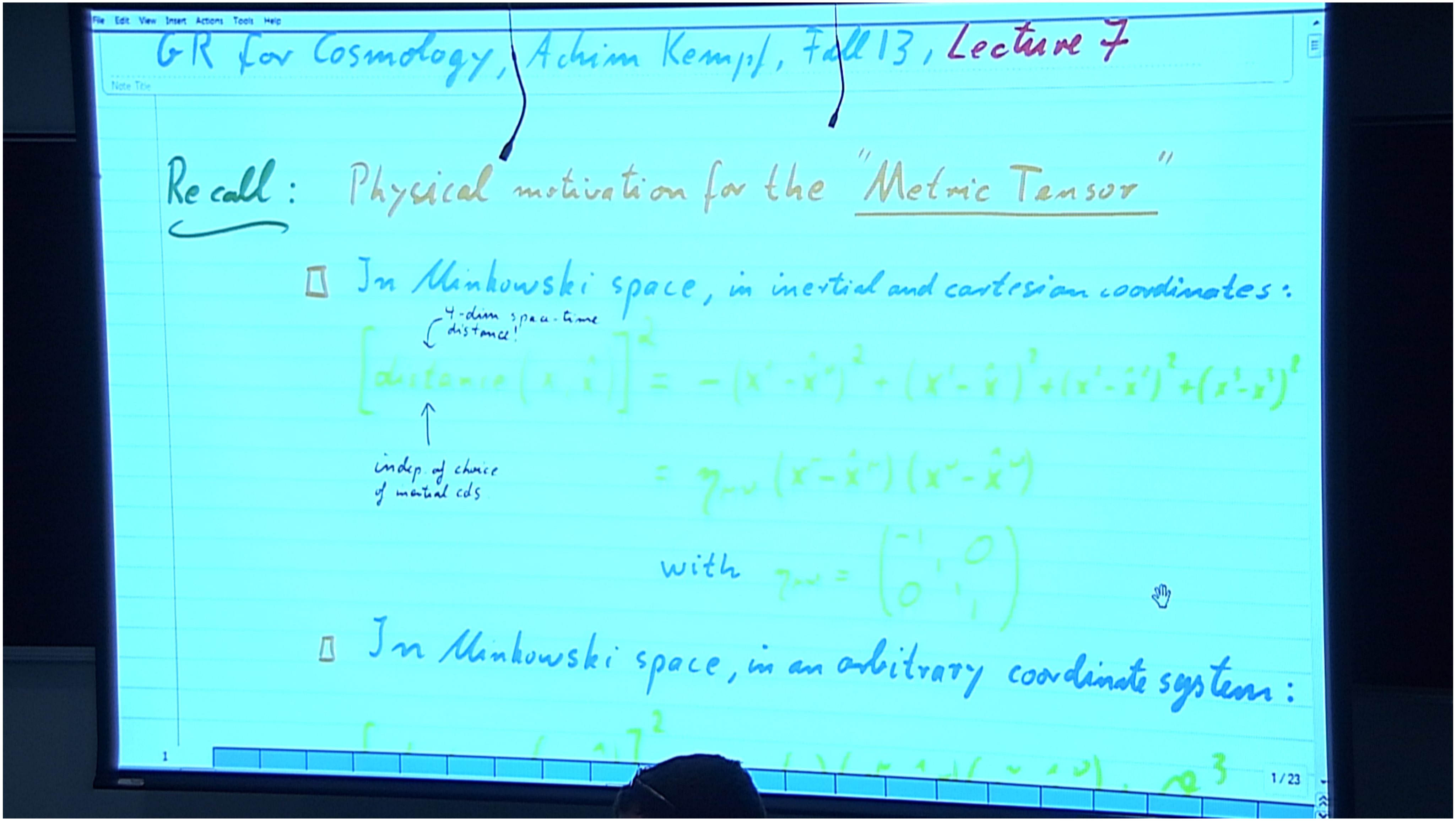
$$\left[\text{distance}(x,\hat{x})\right]^2 = -(x^0-\hat{x}^0)^2 + (x^1-\hat{x}^1)^2 + (x^2-\hat{x}^2)^2 + (x^3-x^3)^2$$

indep. of choice of inertial cds

$$= \eta_{\mu\nu}(x^\mu-\hat{x}^\mu)(x^\nu-\hat{x}^\nu)$$

with $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

- ☐ In Minkowski space, in an arbitrary coordinate system:



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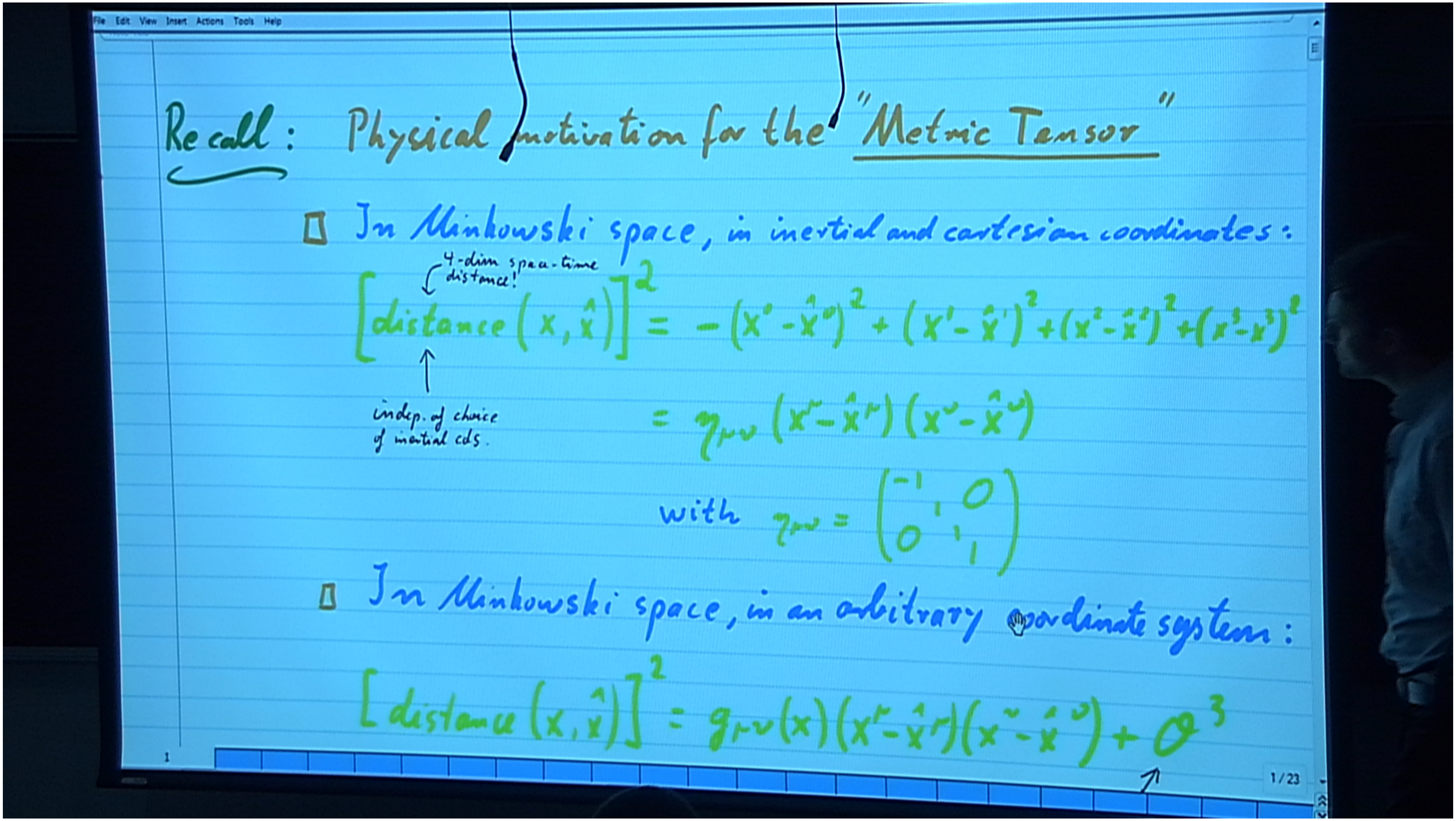

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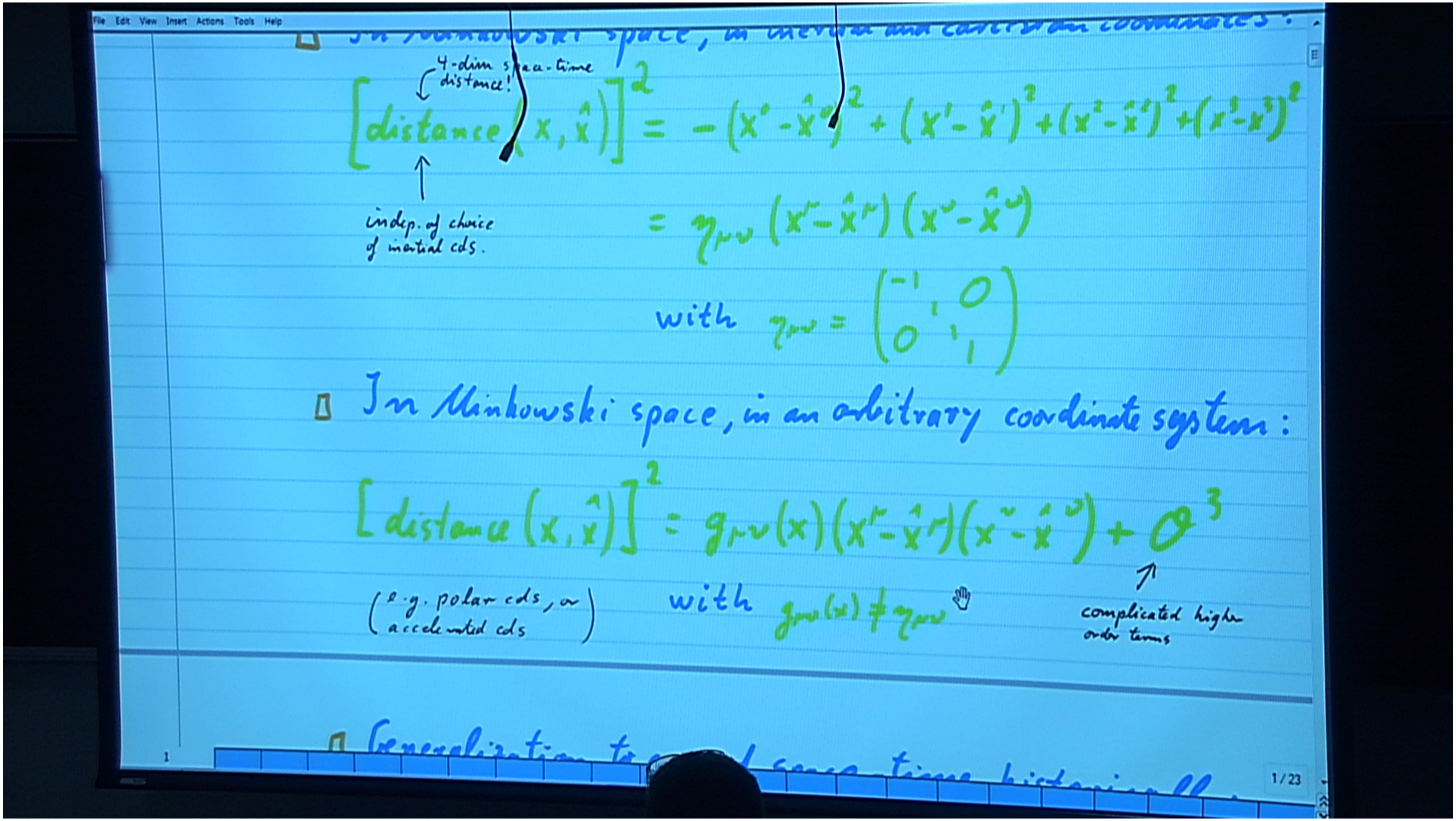

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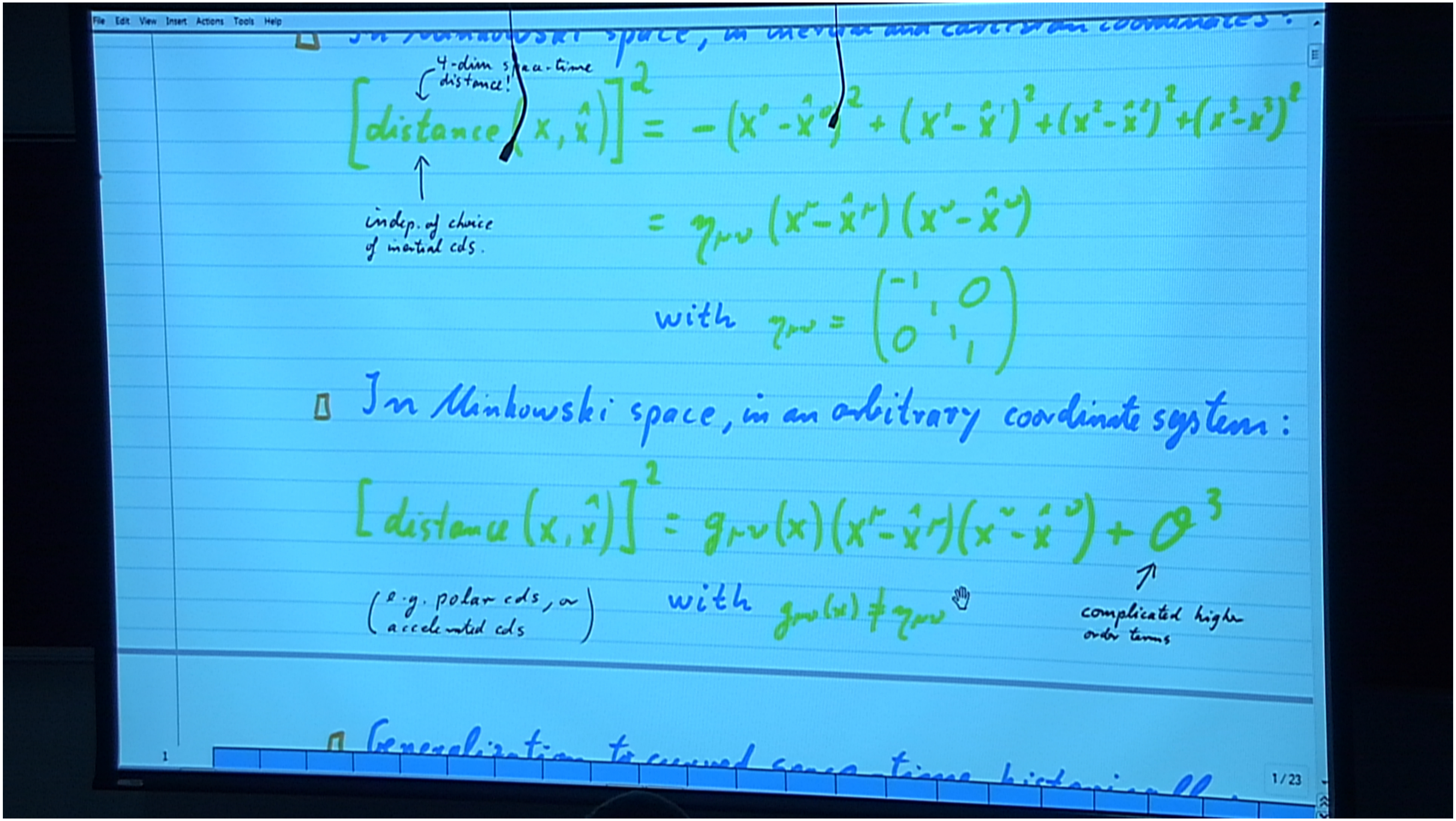
(e.g. polar cds, or accelerated cds)

$$\text{with } g_{\mu\nu}(x) \neq \eta_{\mu\nu}$$

complicated higher order terms

☐ Generalization to curved space-time historically:



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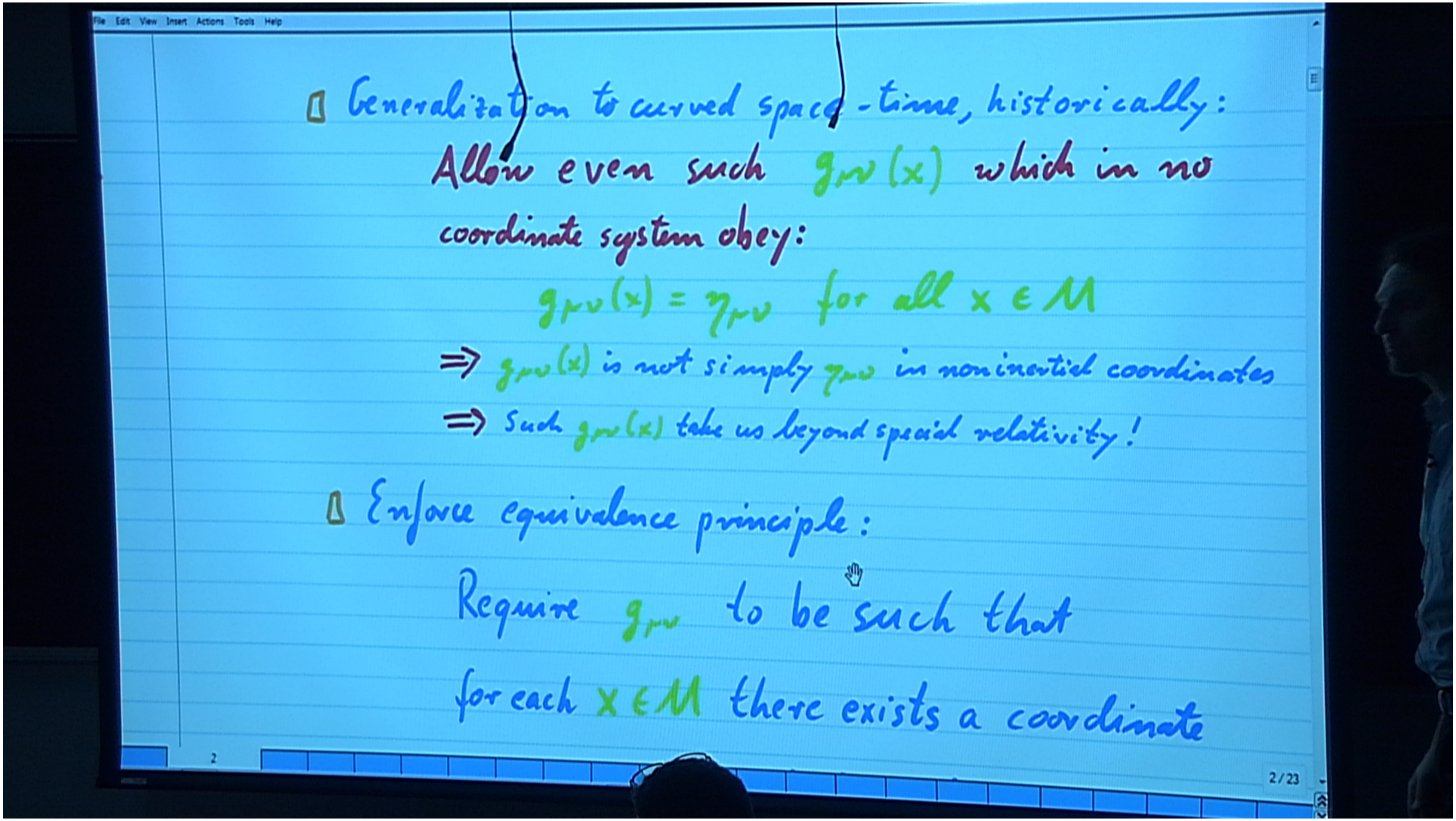


- Generalization to curved space-time, historically:

  Allow even such $g_{\mu\nu}(x)$ which in no coordinate system obey:

  $$g_{\mu\nu}(x) = \eta_{\mu\nu} \quad \text{for all } x \in M$$

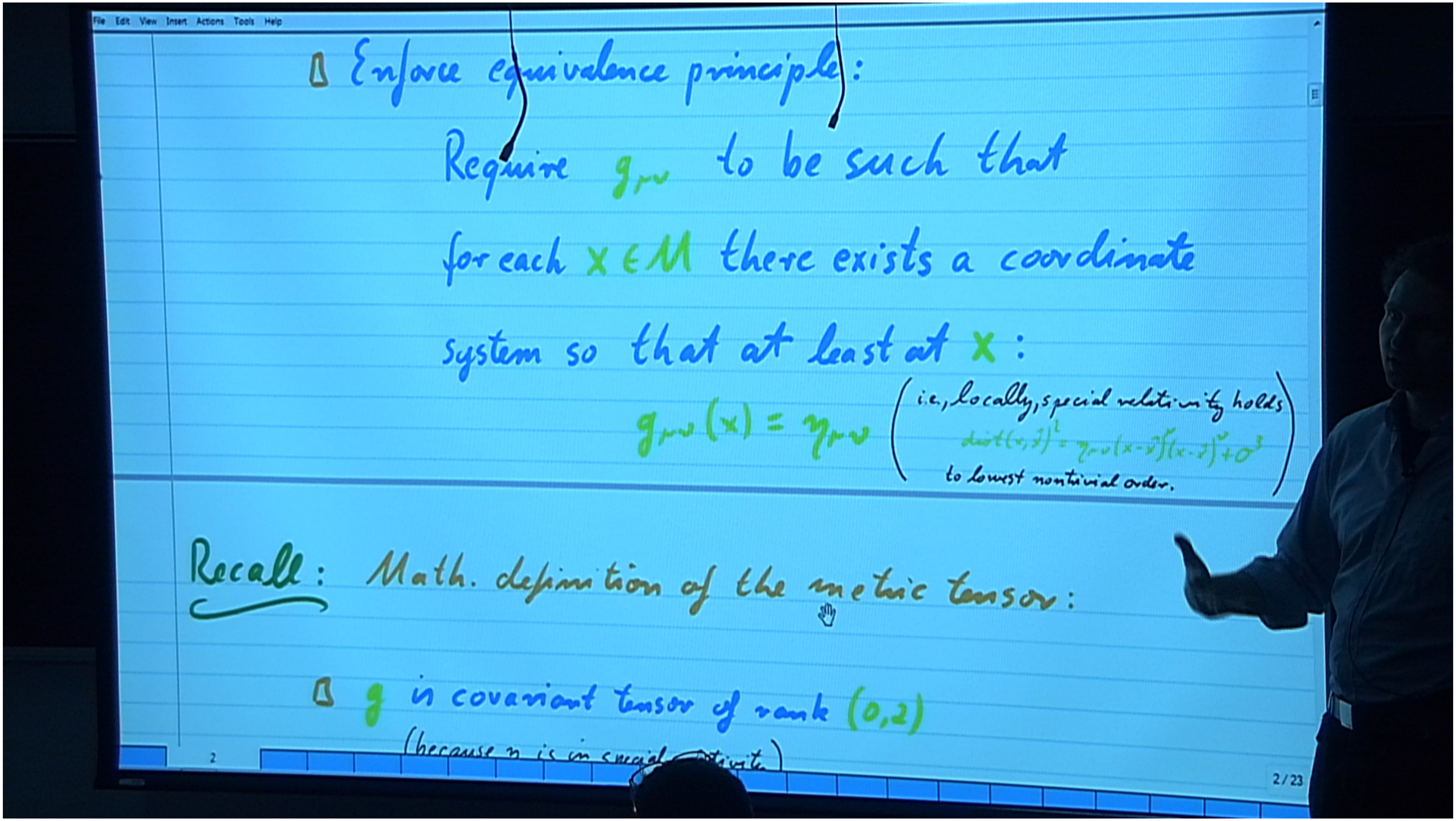
  ⇒ $g_{\mu\nu}(x)$ is not simply $\eta_{\mu\nu}$ in noninertial coordinates

  ⇒ Such $g_{\mu\nu}(x)$ take us beyond special relativity!

- Enforce equivalence principle:

  Require $g_{\mu\nu}$ to be such that

  for each $x \in M$ there exists a coordinate



- Enforce equivalence principle:

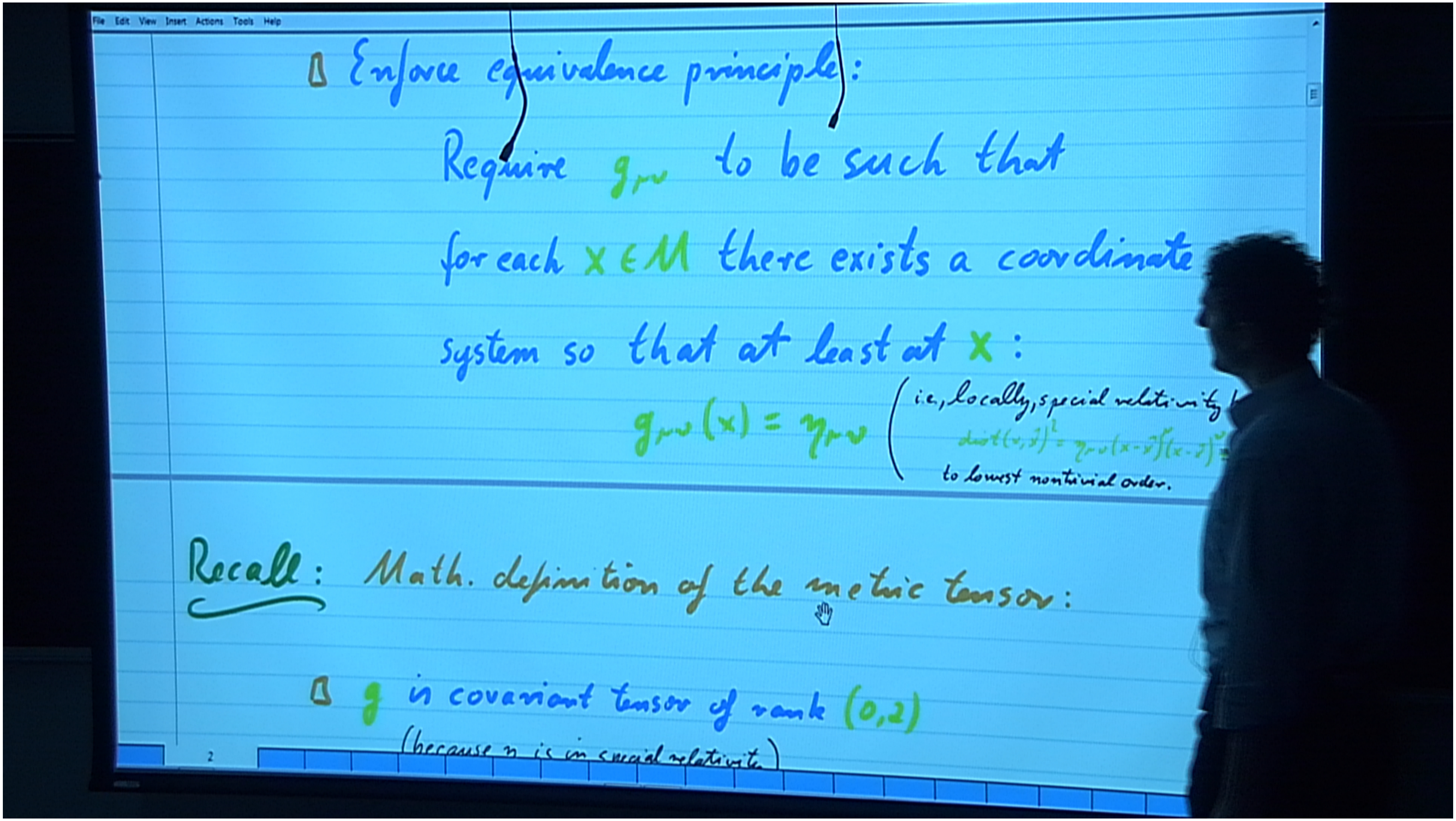
Require $g_{\mu\nu}$ to be such that for each $x \in M$ there exists a coordinate system so that at least at x:

$$g_{\mu\nu}(x) = \eta_{\mu\nu}$$

(i.e., locally, special relativity holds $dist(x,\tilde{x})^2 = \eta_{\mu\nu}(x-\tilde{x})^\mu(x-\tilde{x})^\nu + O^3$ to lowest nontrivial order.)

Recall: Math. definition of the metric tensor:

- g is covariant tensor of rank $(0,2)$
(because η is in special [illegible]tivity)



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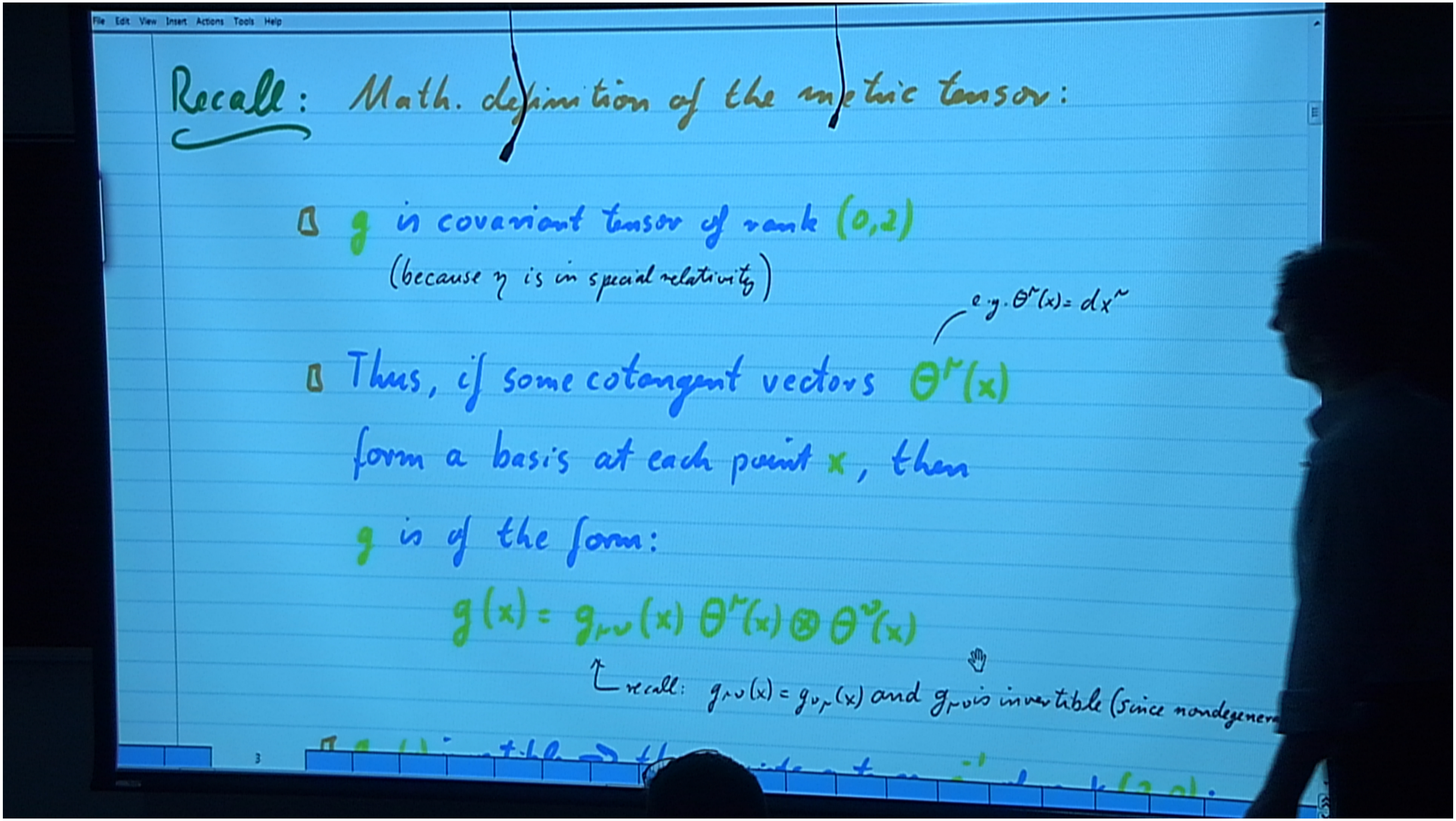

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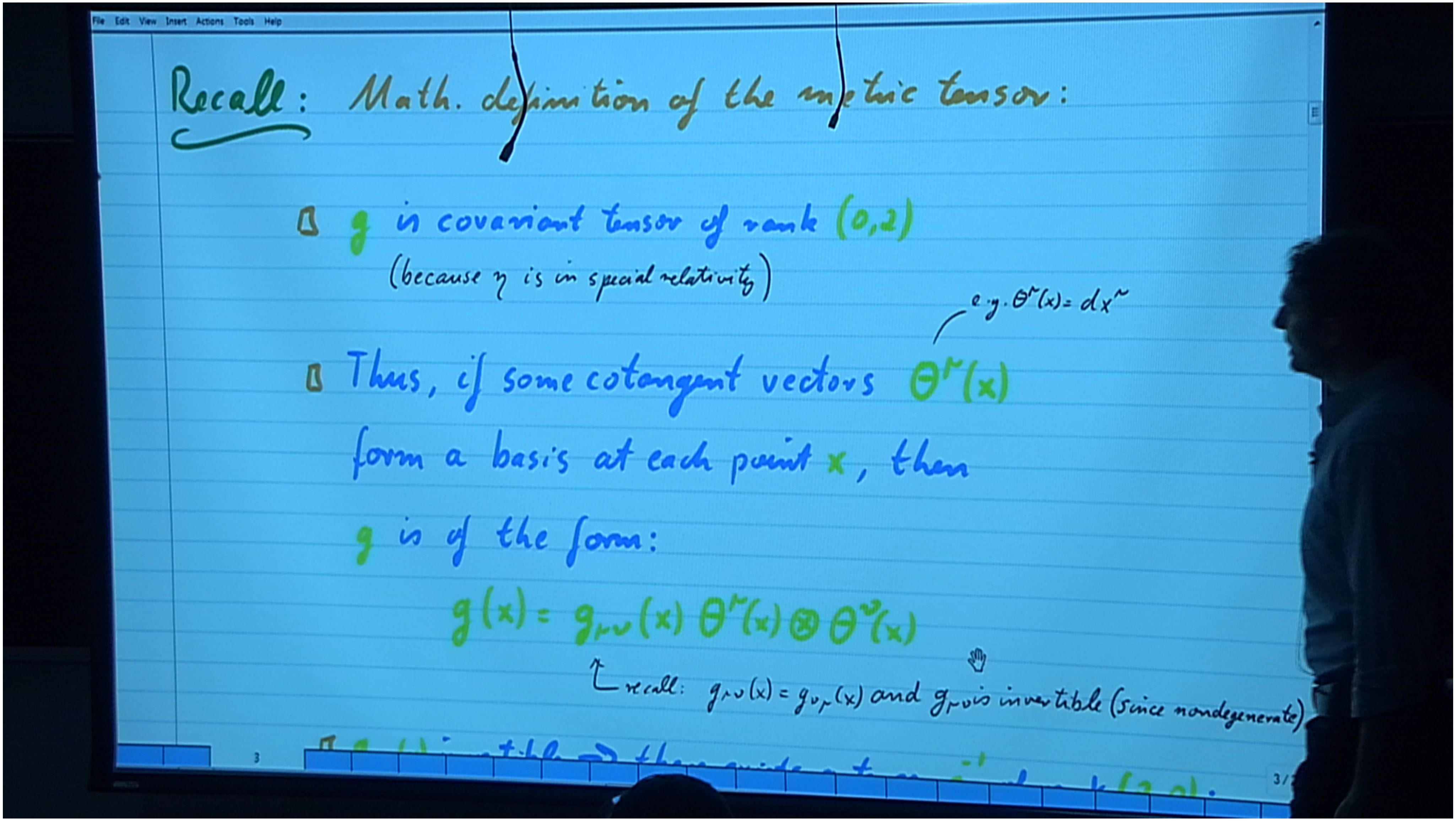
- Thus, if some cotangent vectors $\theta^\mu(x)$ form a basis at each point x, then g is of the form:

e.g. $\theta^\mu(x) = dx^\mu$

$$g(x) = g_{\mu\nu}(x)\,\theta^\mu(x) \otimes \theta^\nu(x)$$

recall: $g_{\mu\nu}(x) = g_{\nu\mu}(x)$ and $g_{\mu\nu}$ is invertible (since nondegenera



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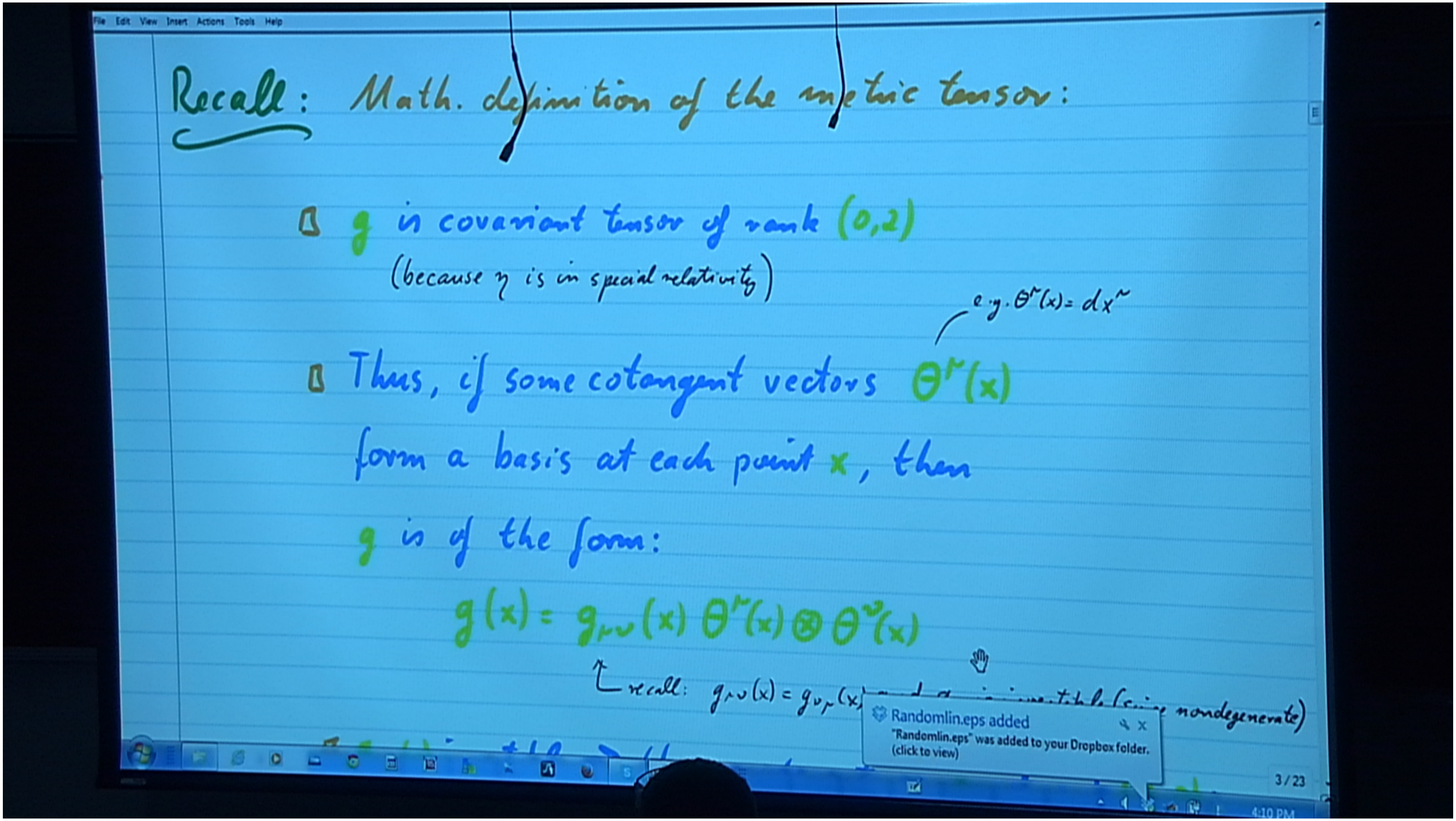

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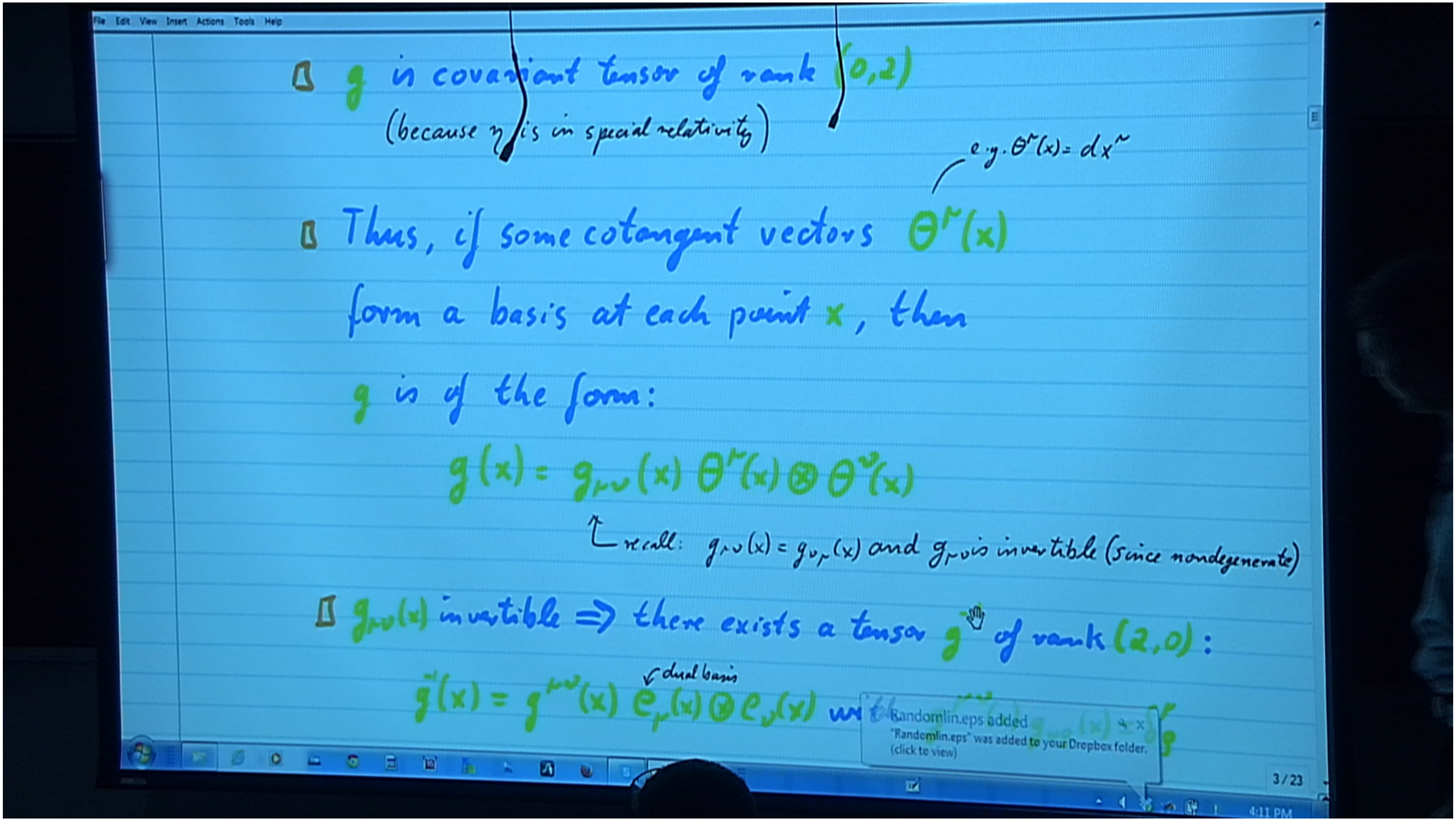

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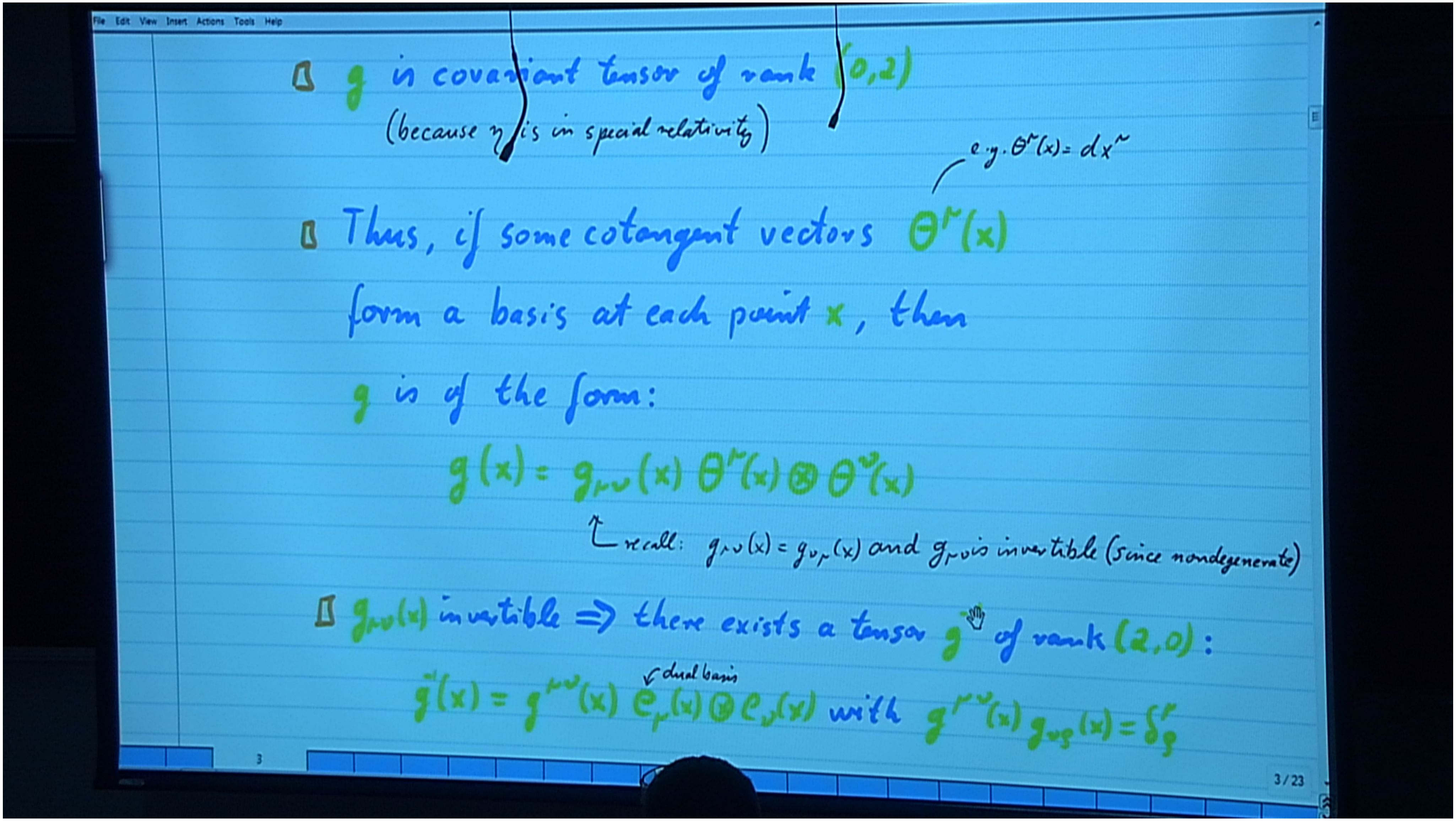

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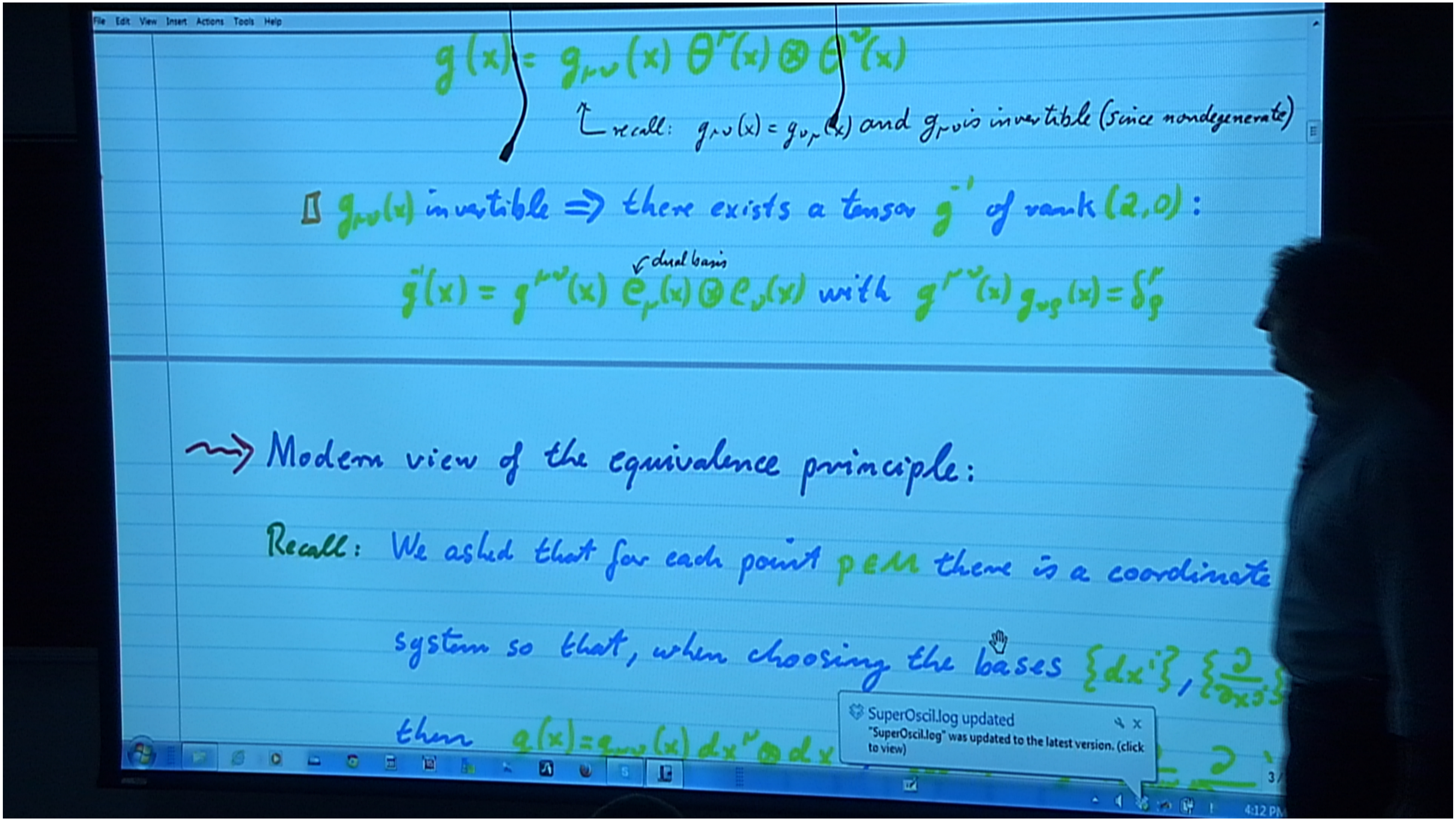

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⟿ Modern view of the equivalence principle:

Recall: We asked that for each point $p \in M$ there is a coordinate system so that, when choosing the bases $\{dx^i\}, \{\frac{\partial}{\partial x^j}\}$

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SuperOscil.log updated

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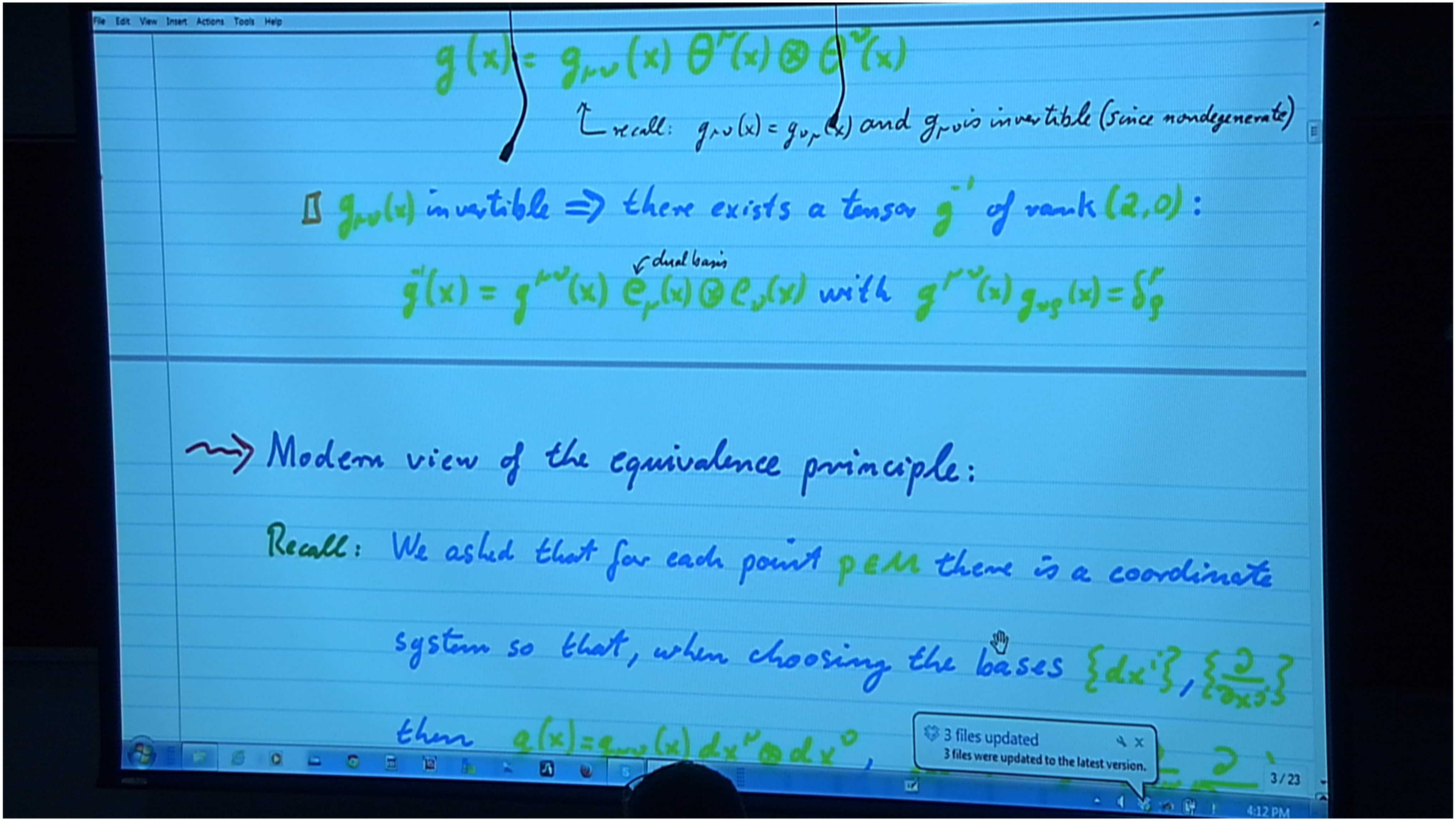

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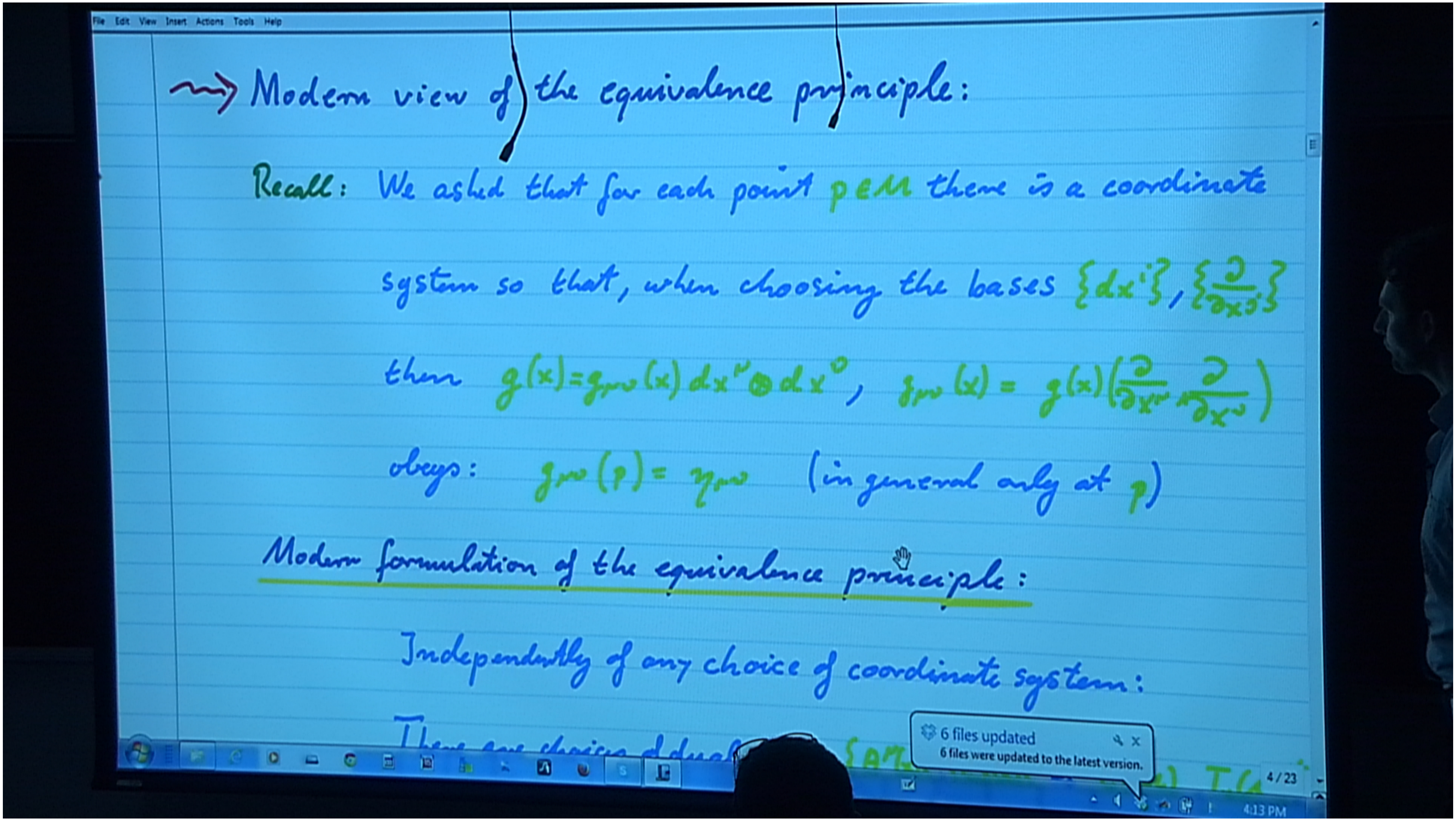

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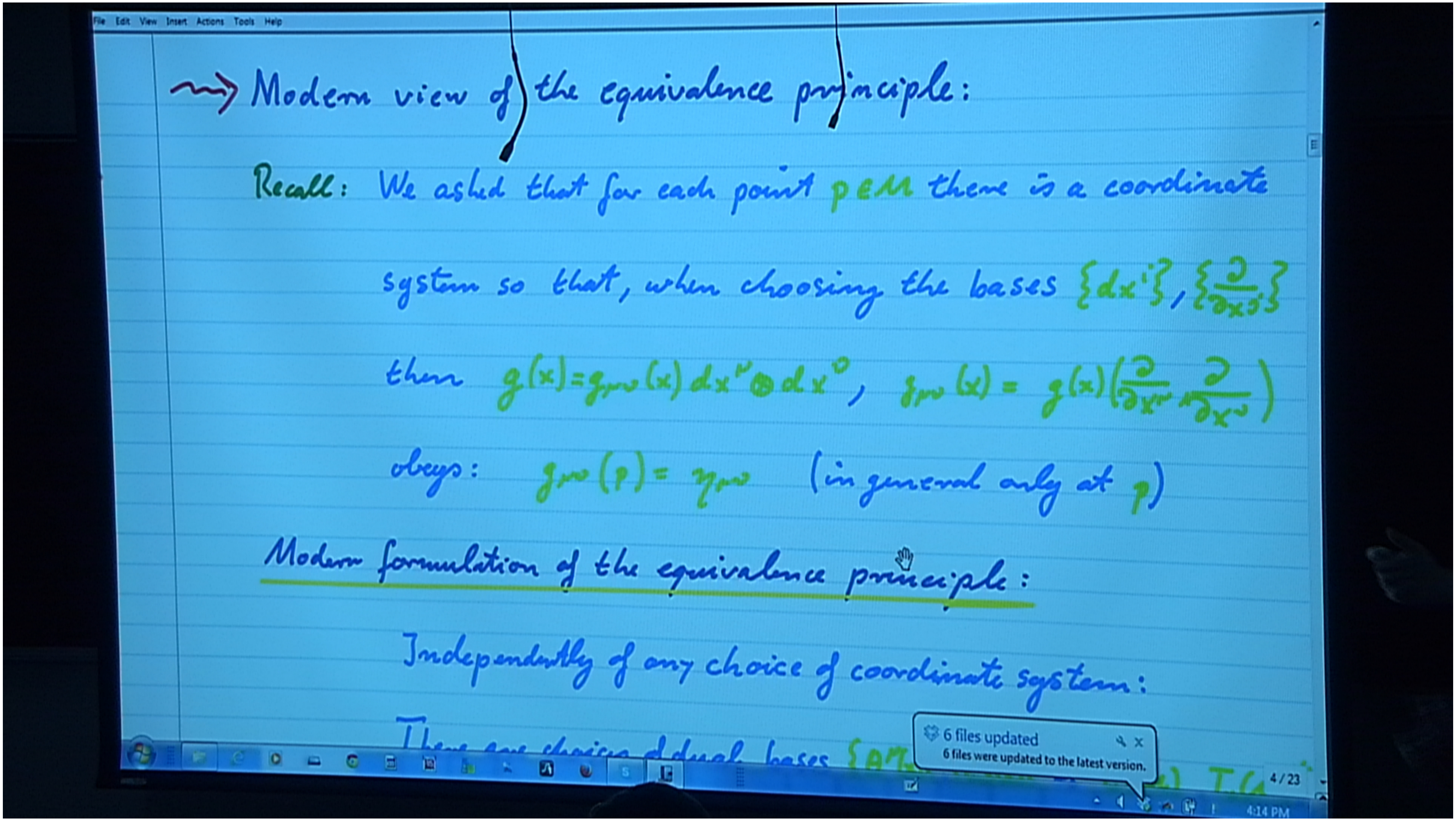
Modern formulation of the equivalence principle:

Independently of any choice of coordinate system:

There are choices of dual



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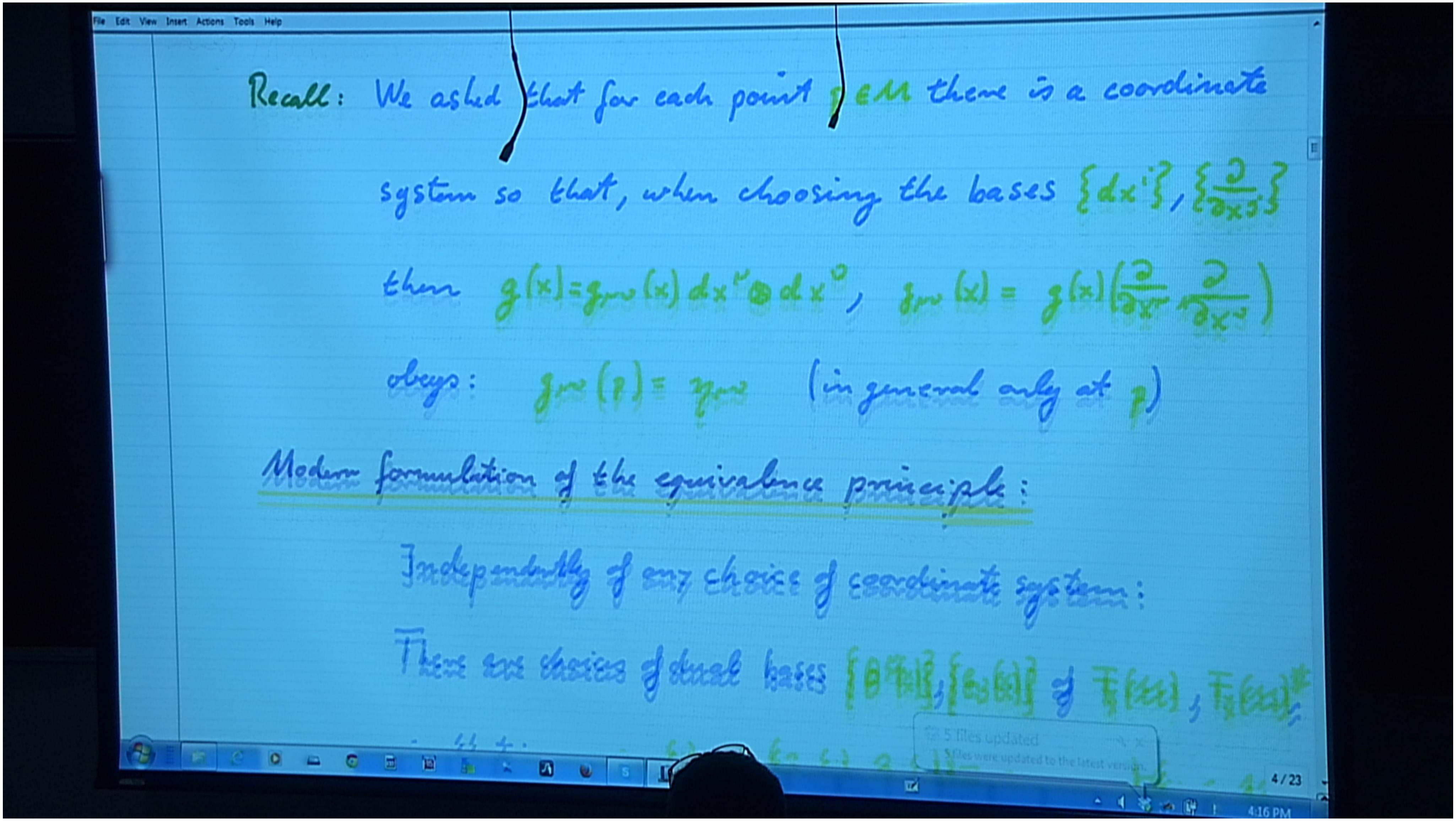

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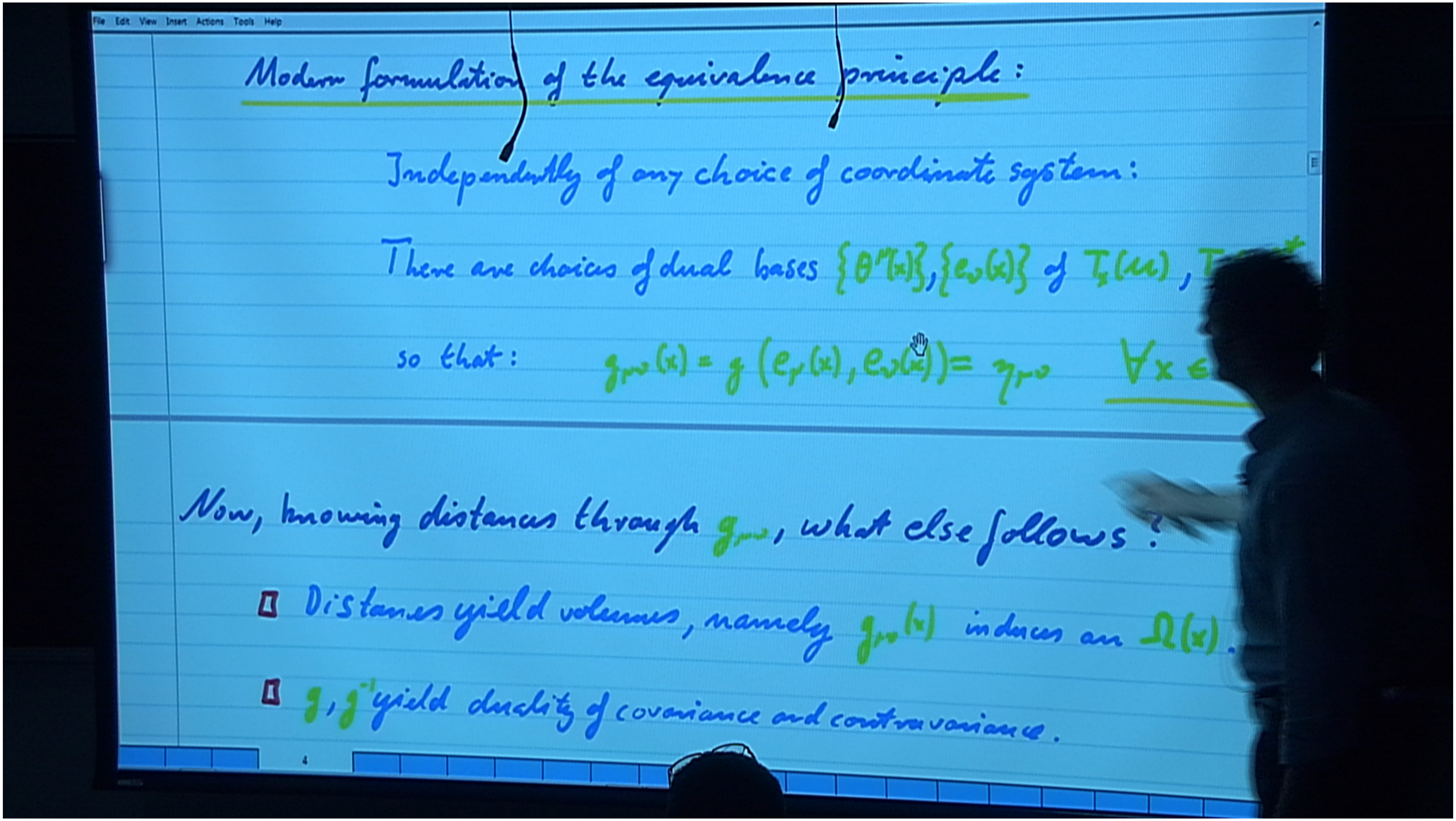

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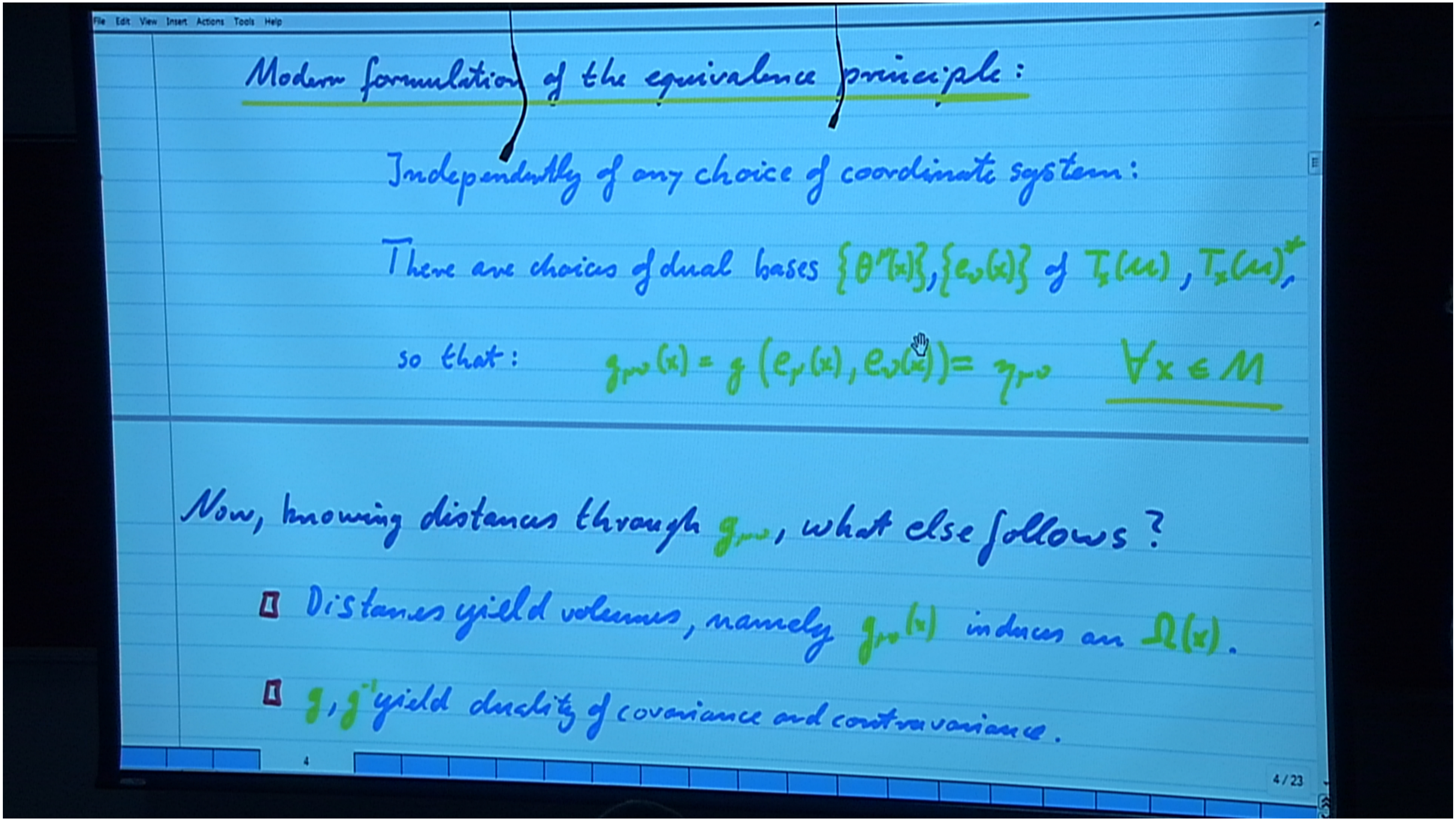
☐ g, g^{-1} yield duality of covariance and contravariance.



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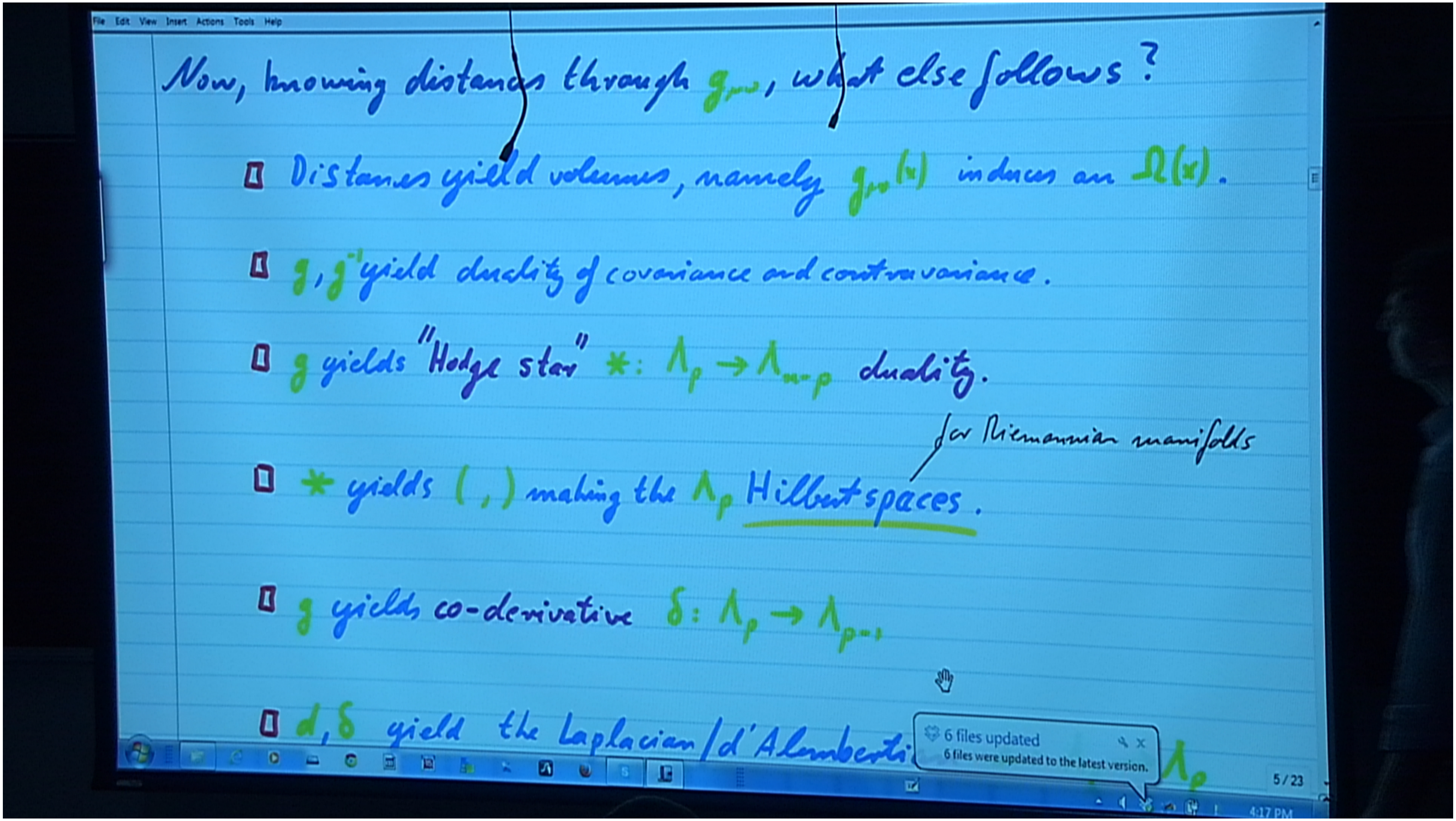

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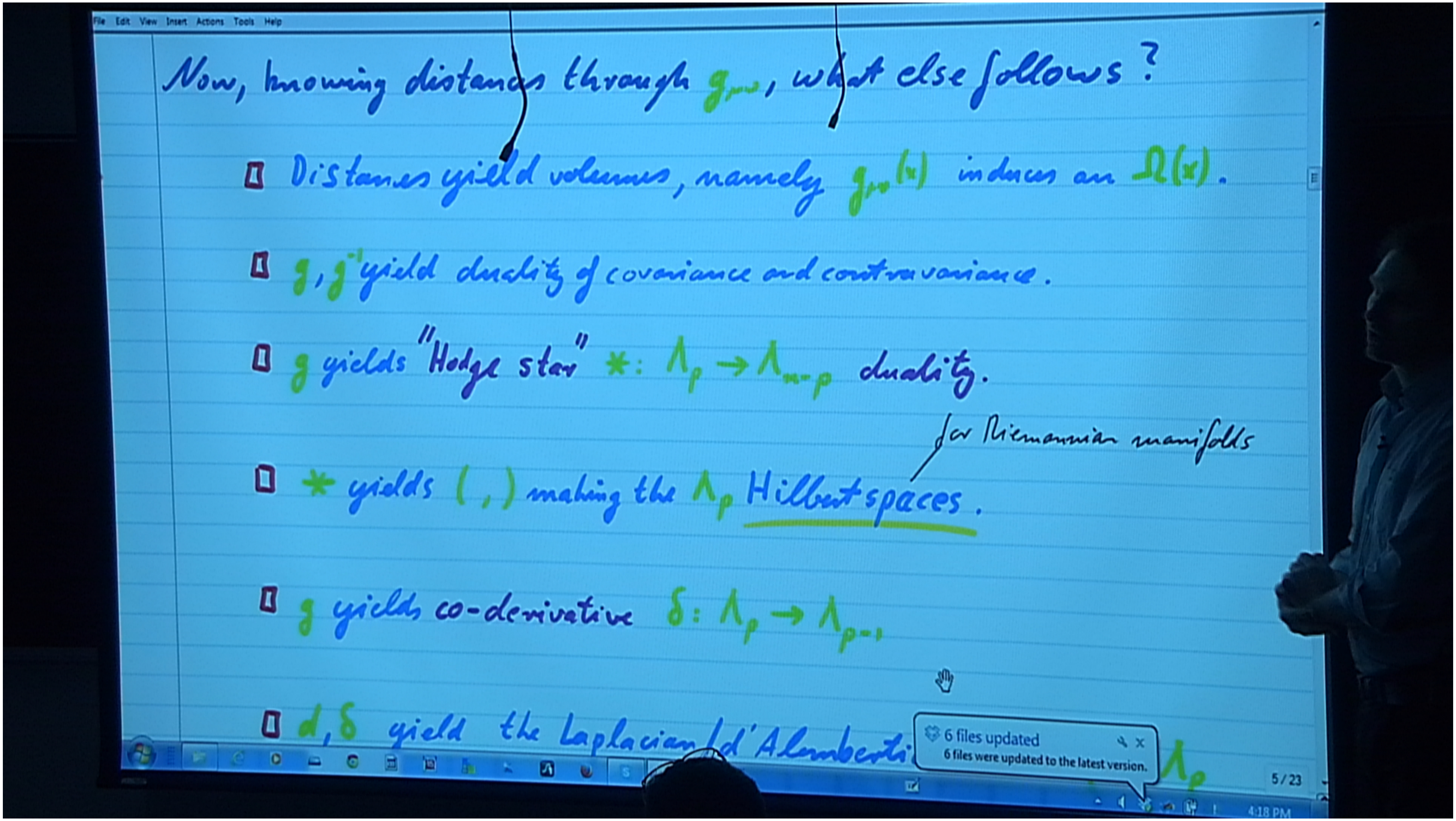
- ☐ $*$ yields $(\,,\,)$ making the Λ_p Hilbert spaces. for Riemannian manifolds
- ☐ g yields co-derivative $\delta: \Lambda_p \to \Lambda_{p-1}$
- ☐ d, δ yield the Laplacian/d'Alemberti… Λ_p



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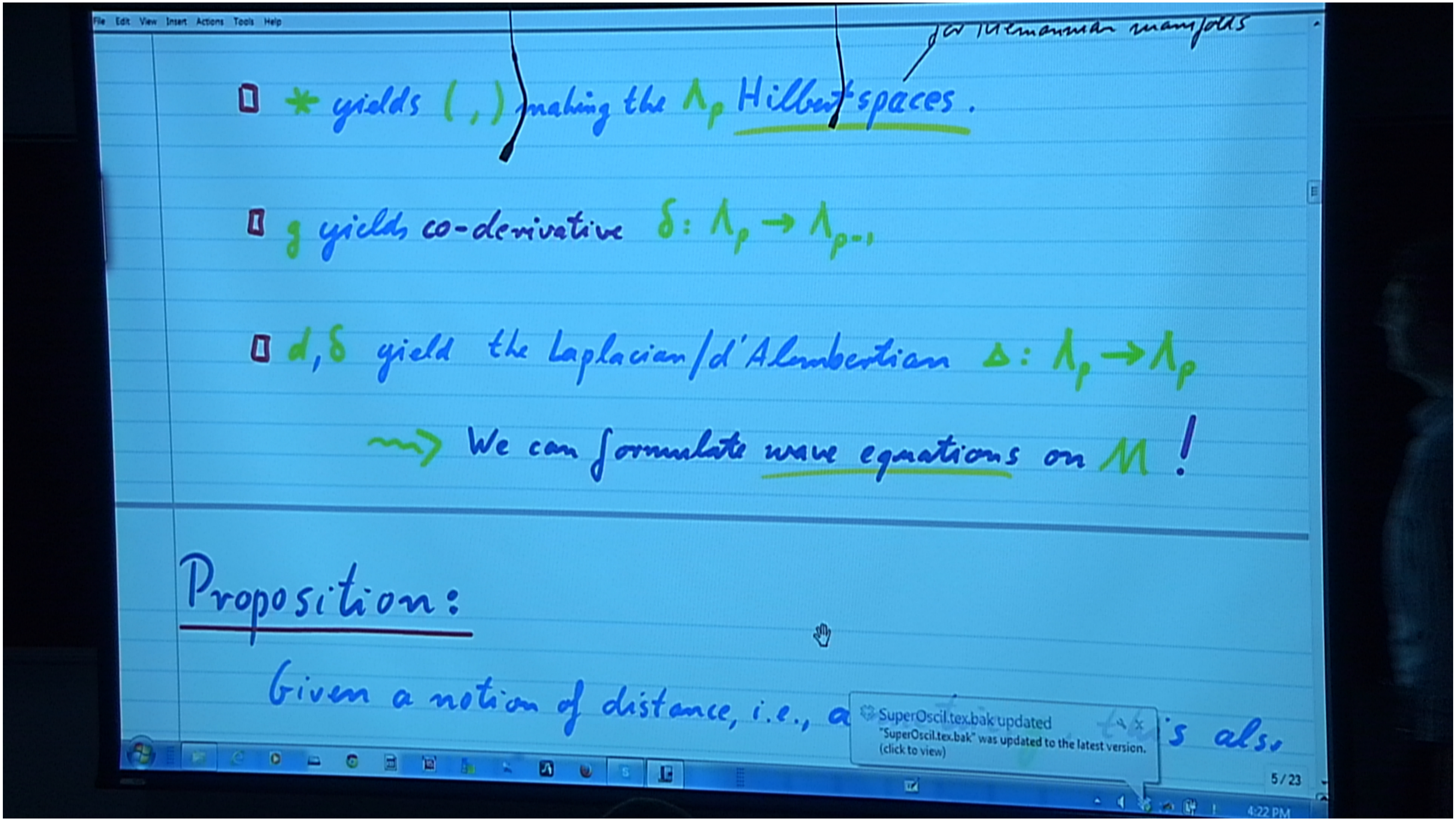

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$\rightsquigarrow$ We can formulate wave equations on M !

Proposition:

Given a notion of distance, i.e., a … 's also

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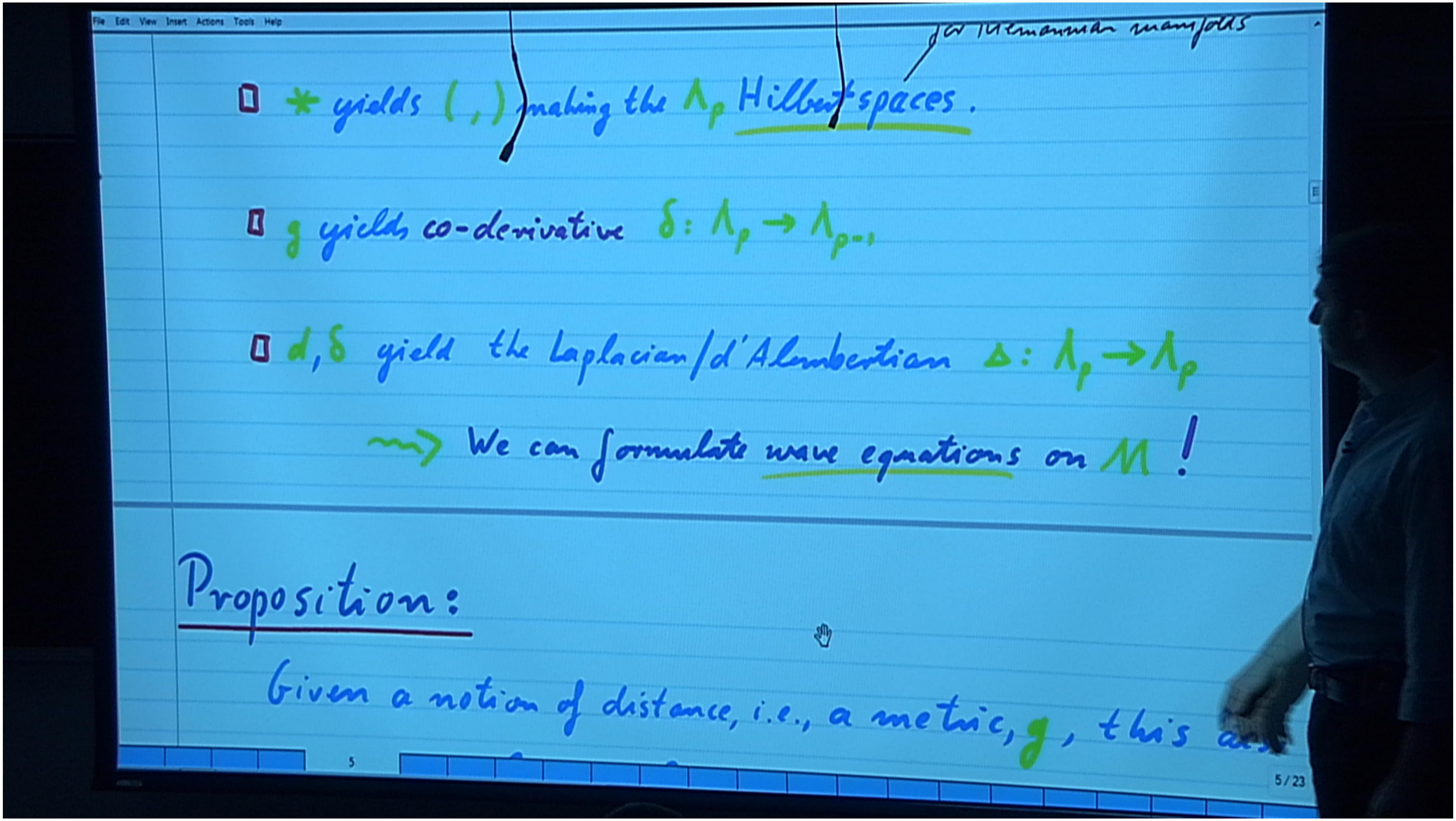

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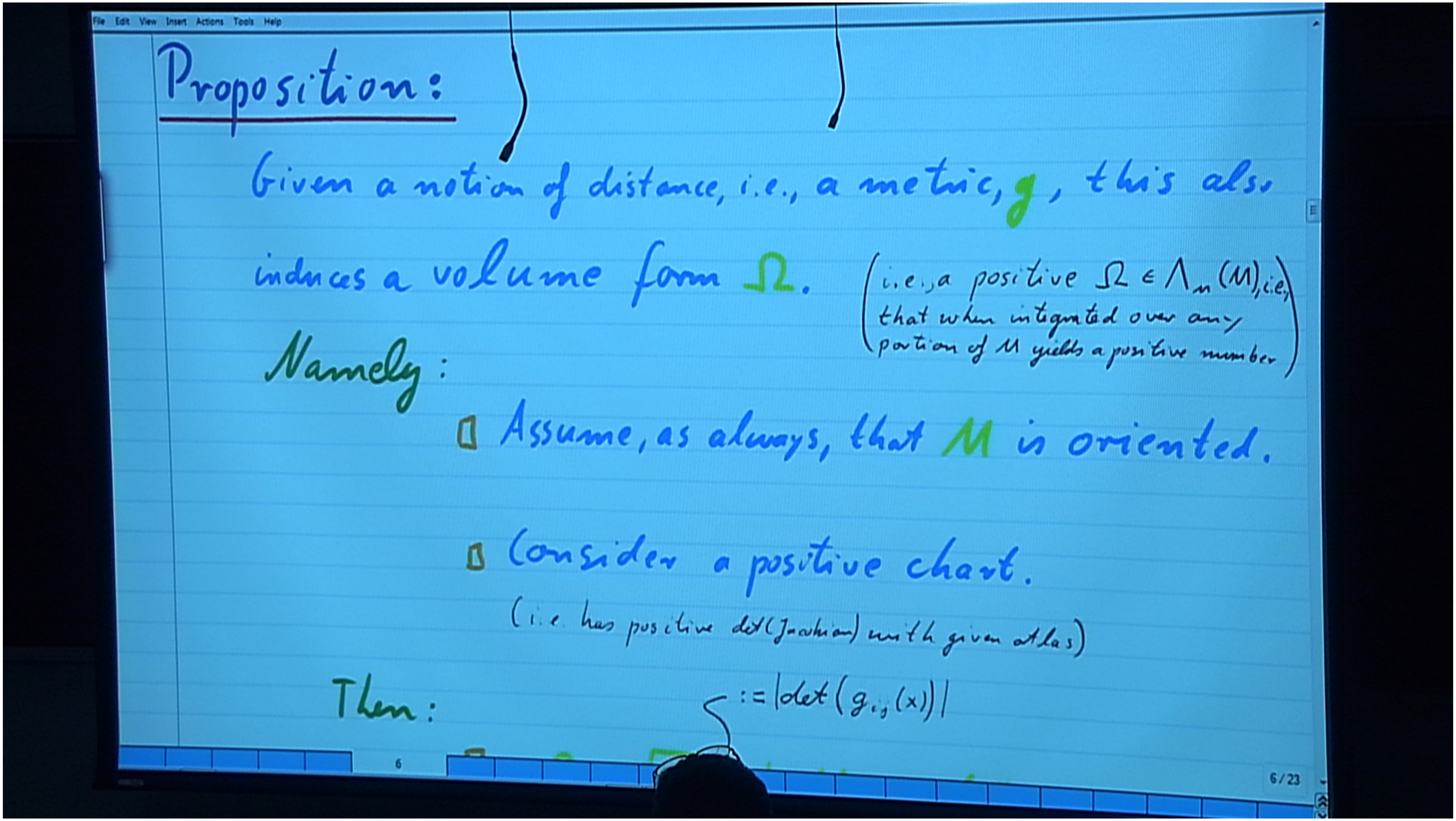


Proposition:

Given a notion of distance, i.e., a metric, g, this also induces a volume form Ω.

(i.e., a positive $\Omega \in \Lambda_n(M)$, i.e., that when integrated over any portion of M yields a positive number)

Namely:

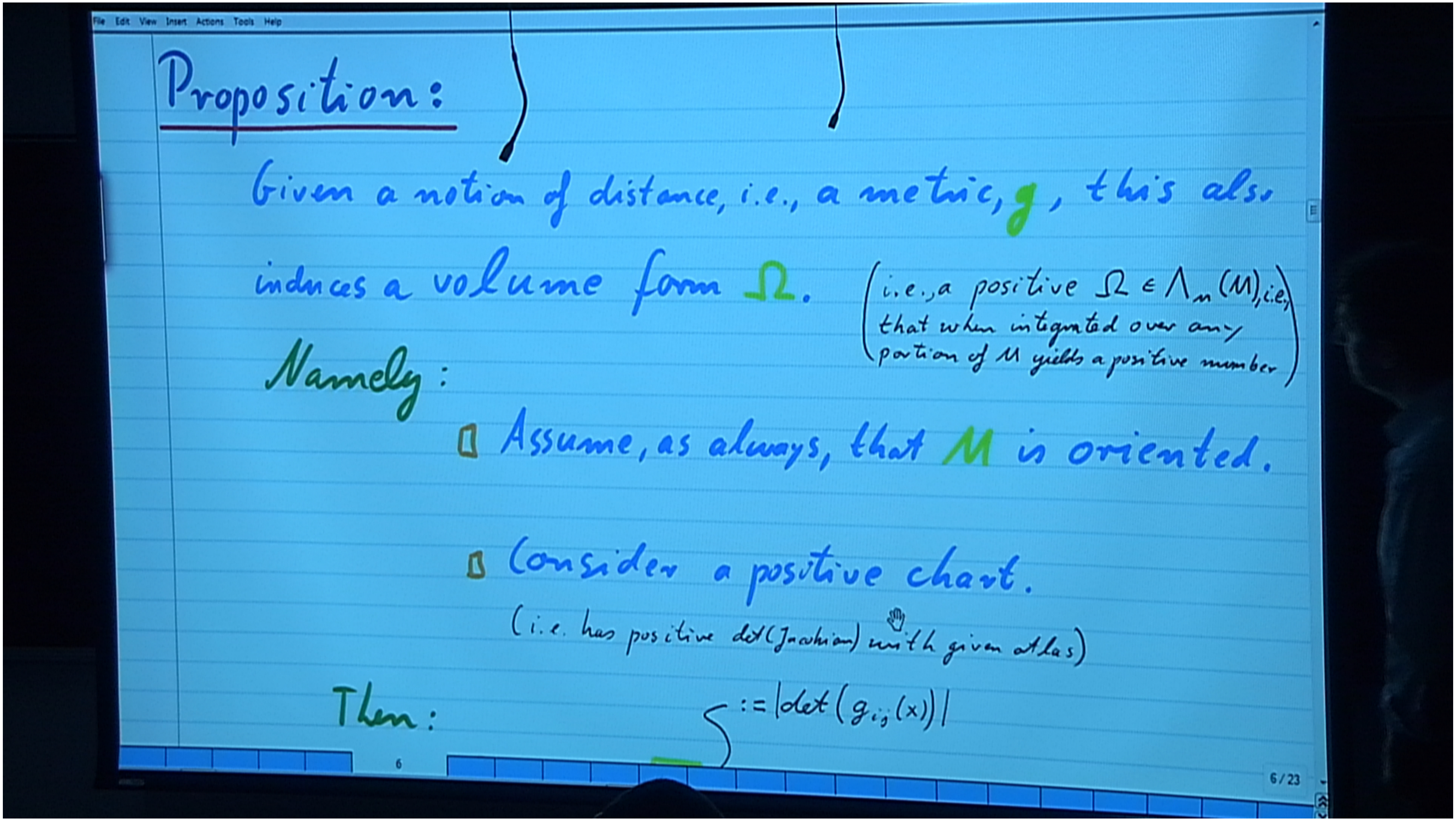
☐ Assume, as always, that M is oriented.

☐ Consider a positive chart.

(i.e. has positive det(Jacobian) with given atlas)

Then:

$:= |\det(g_{ij}(x))|$



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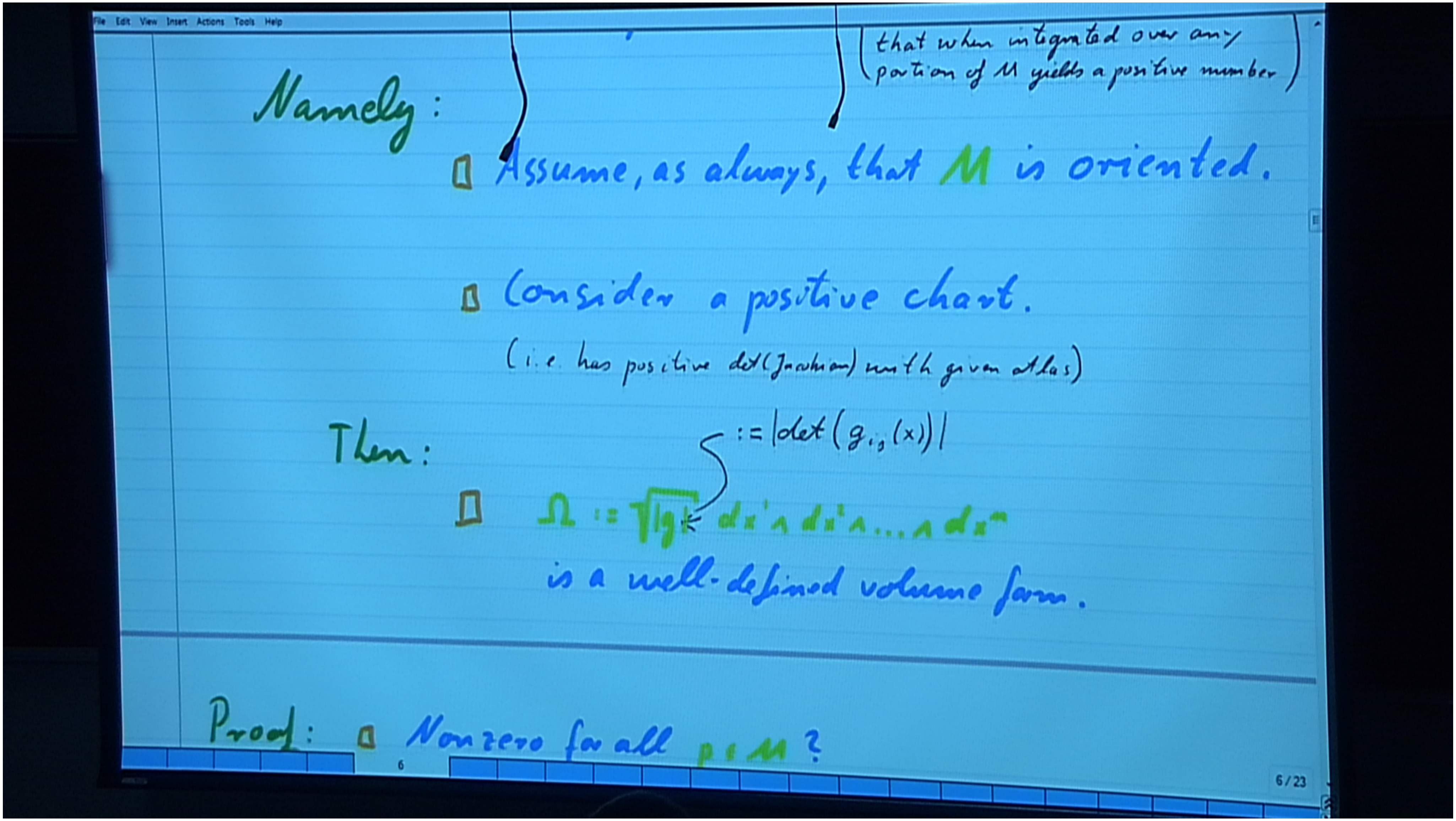

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$$:= |\det(g_{ij}(x))|$$

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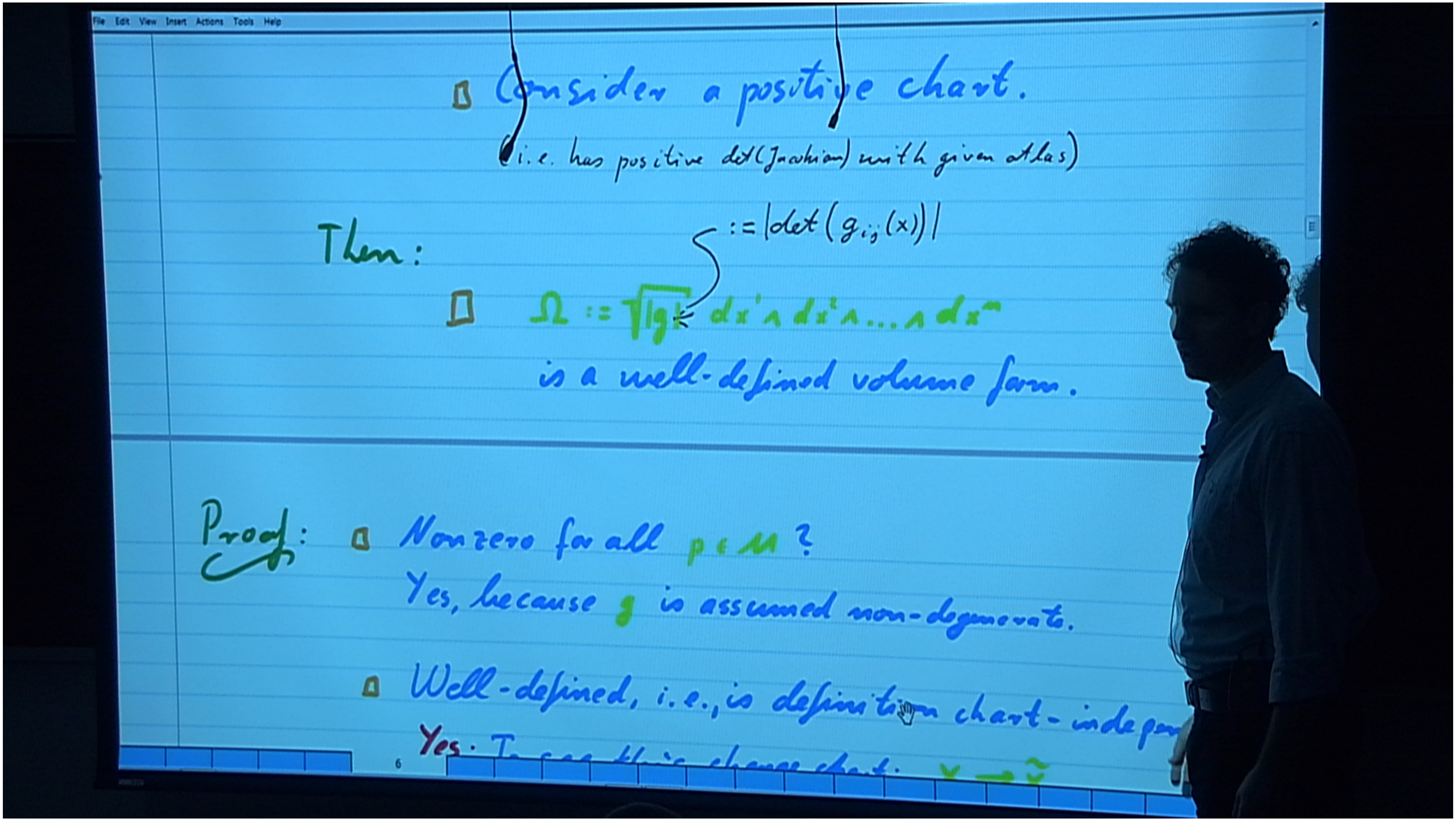
is a well-defined volume form.

Proof: ☐ Nonzero for all $p \in M$?



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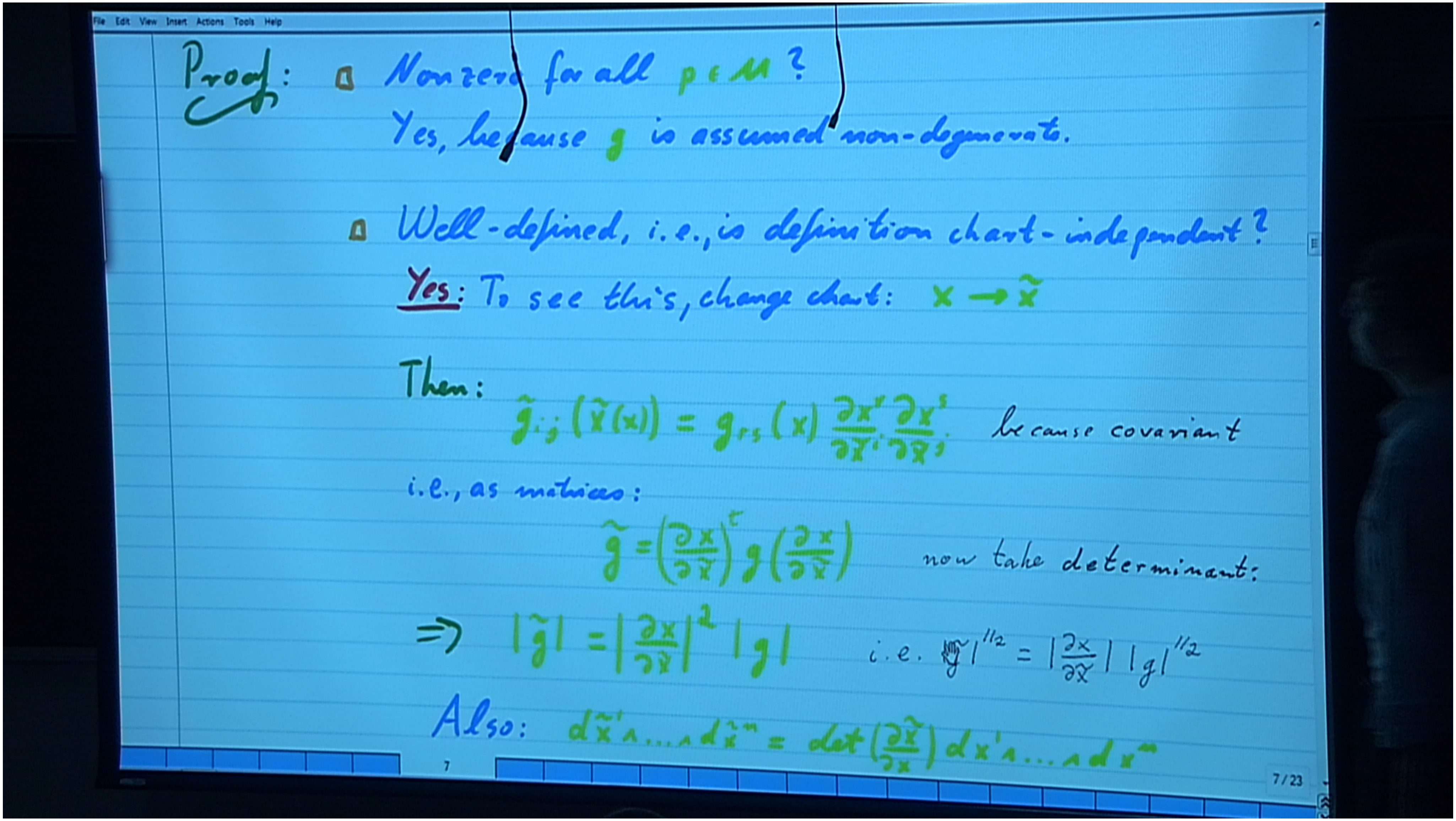

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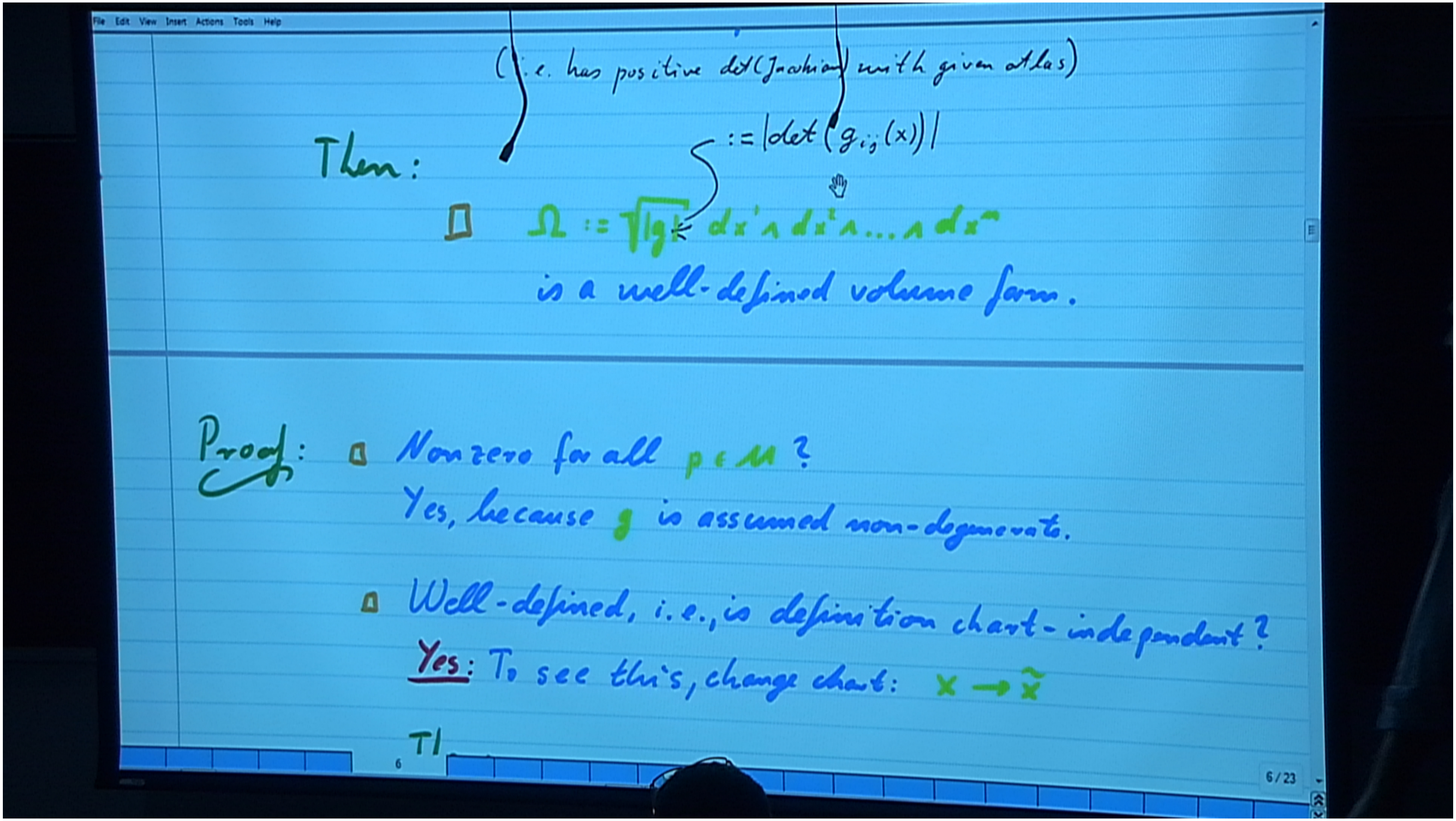

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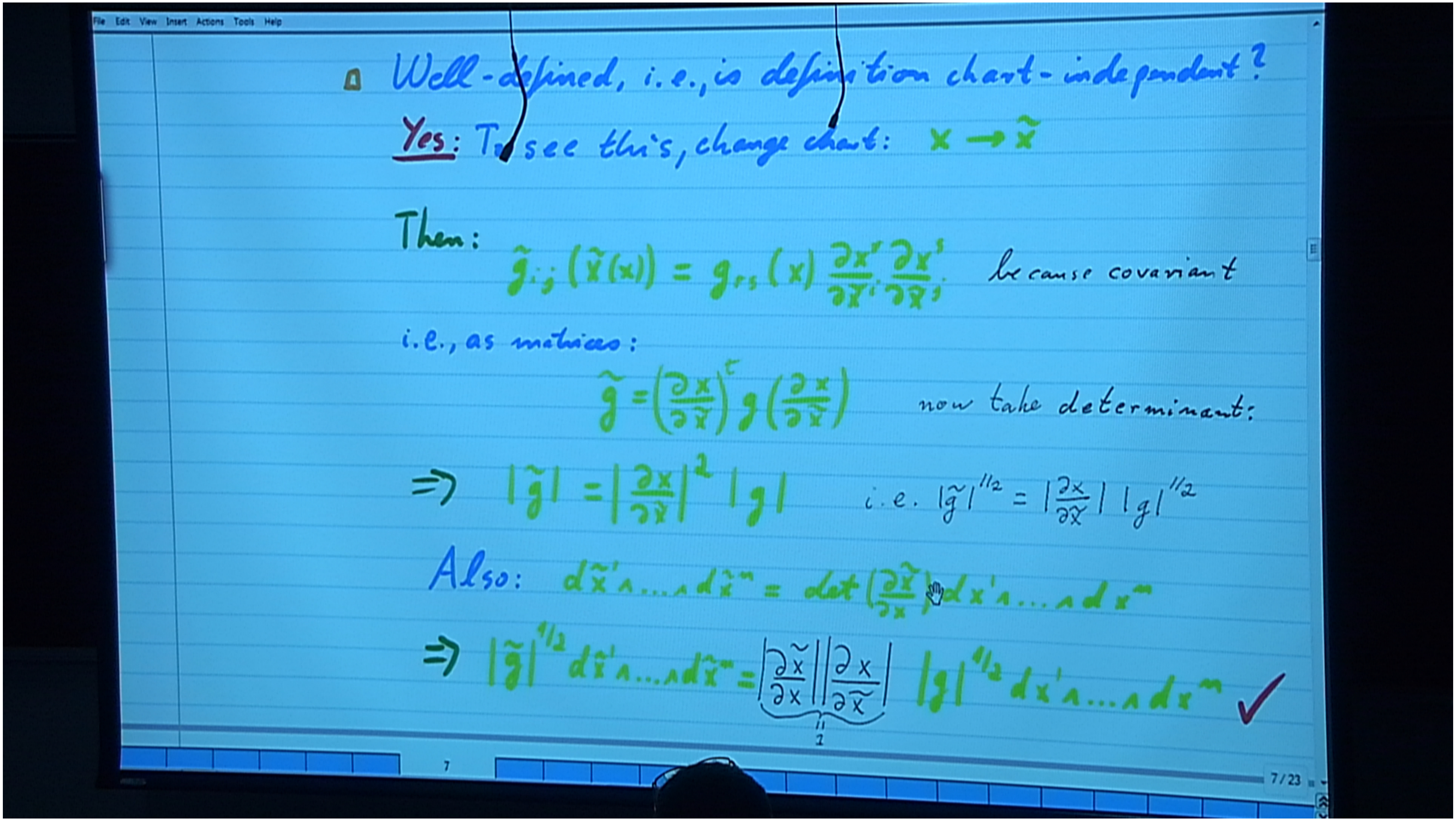
Tl



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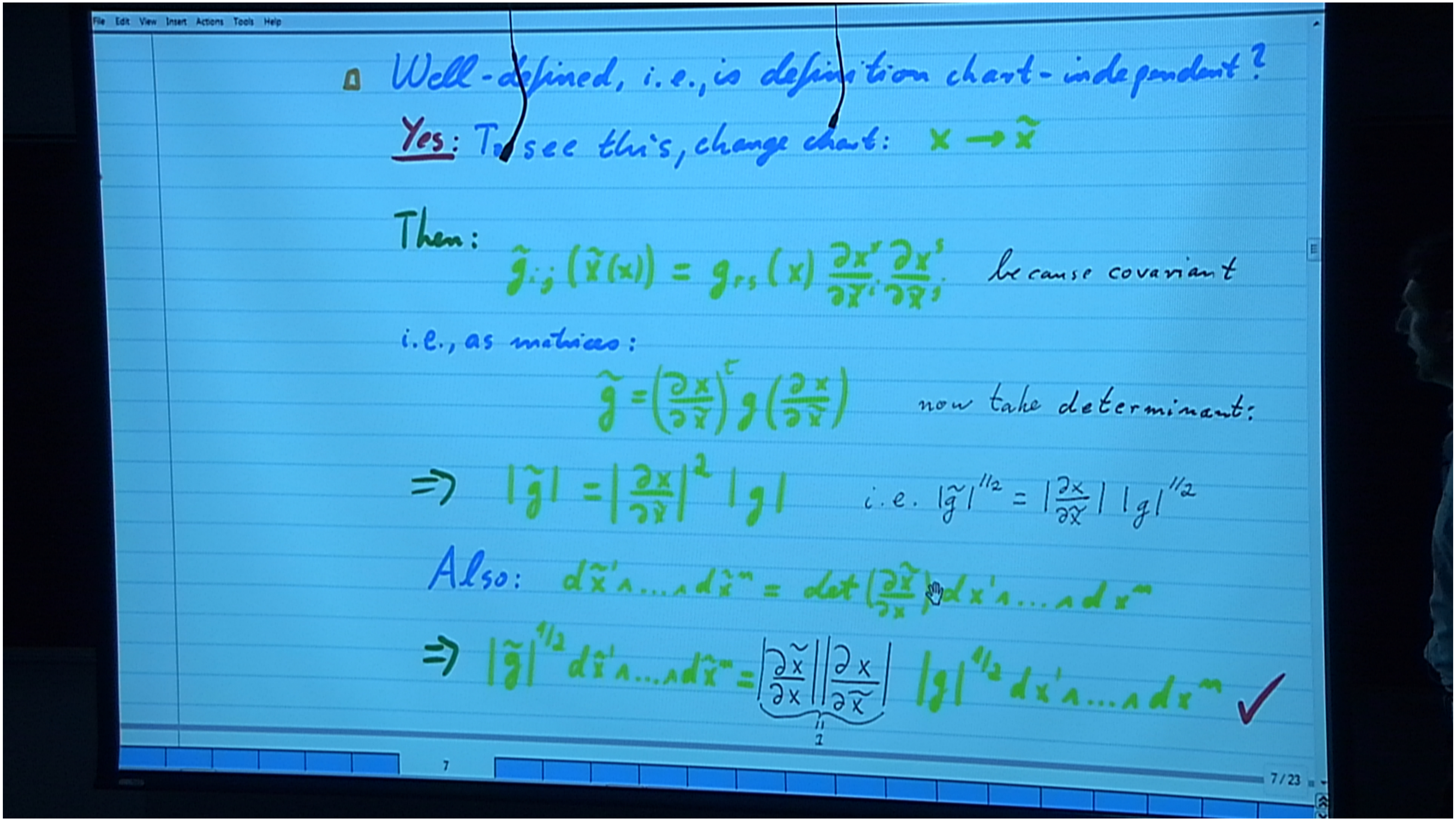

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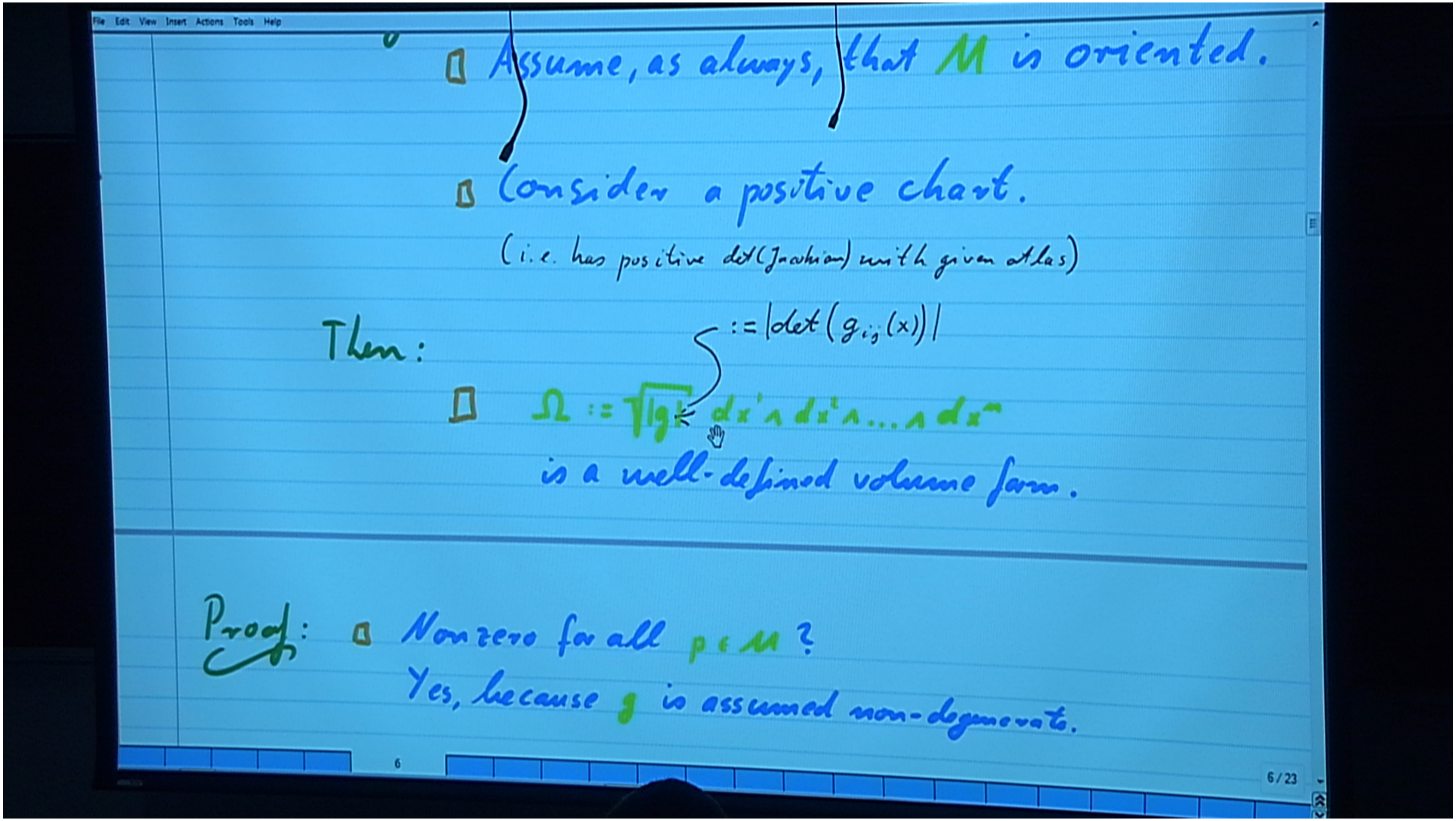

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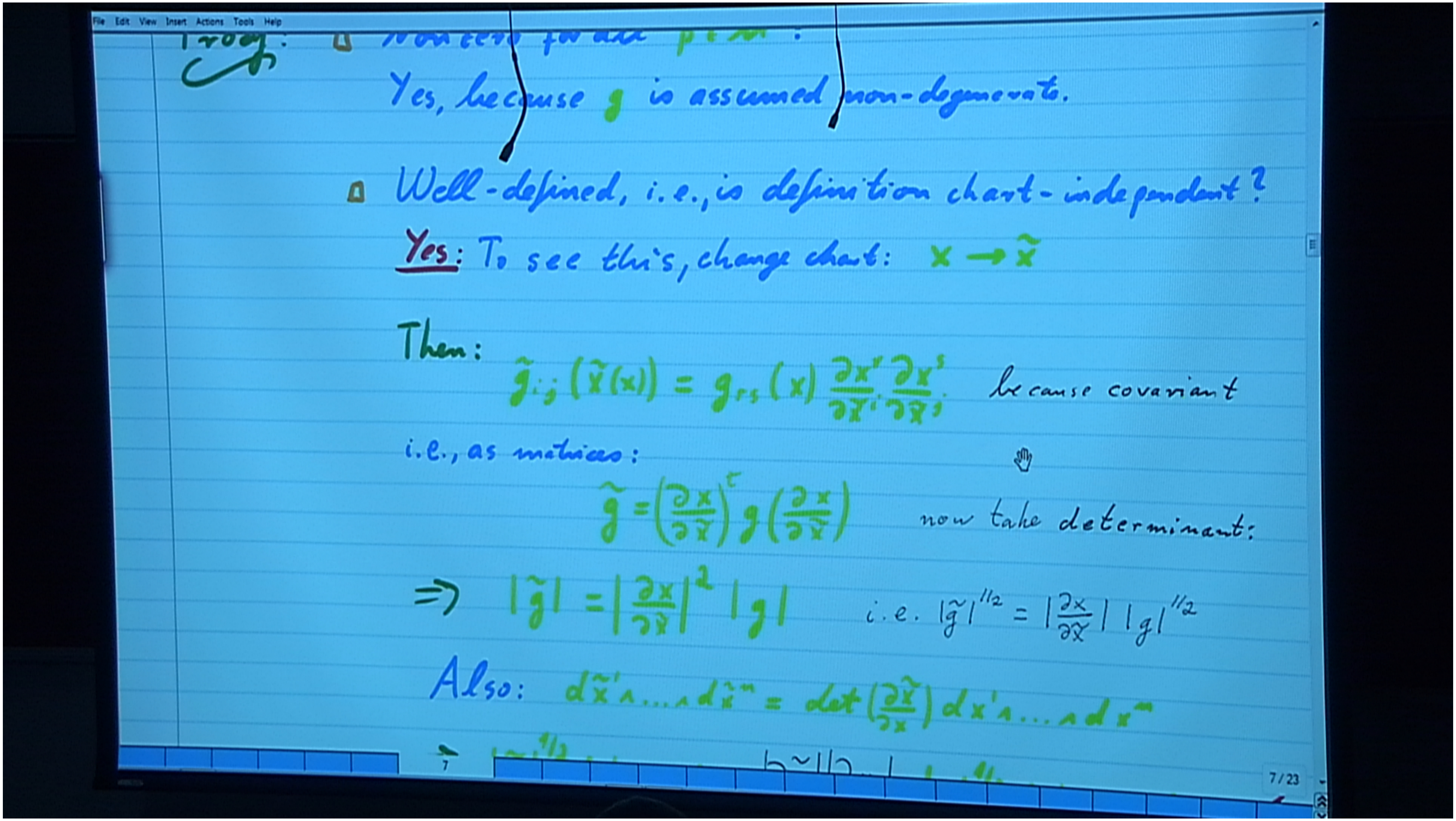

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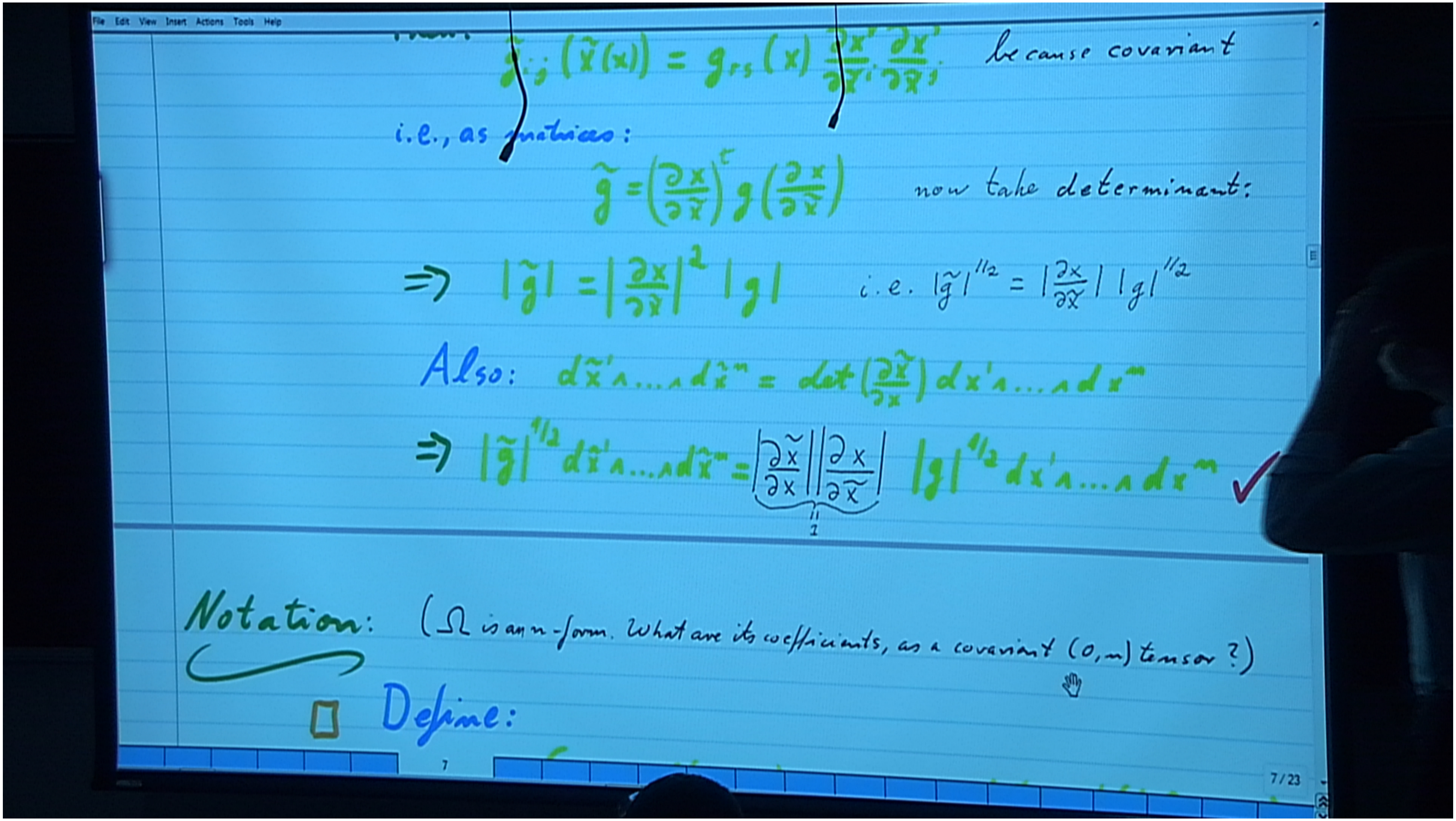

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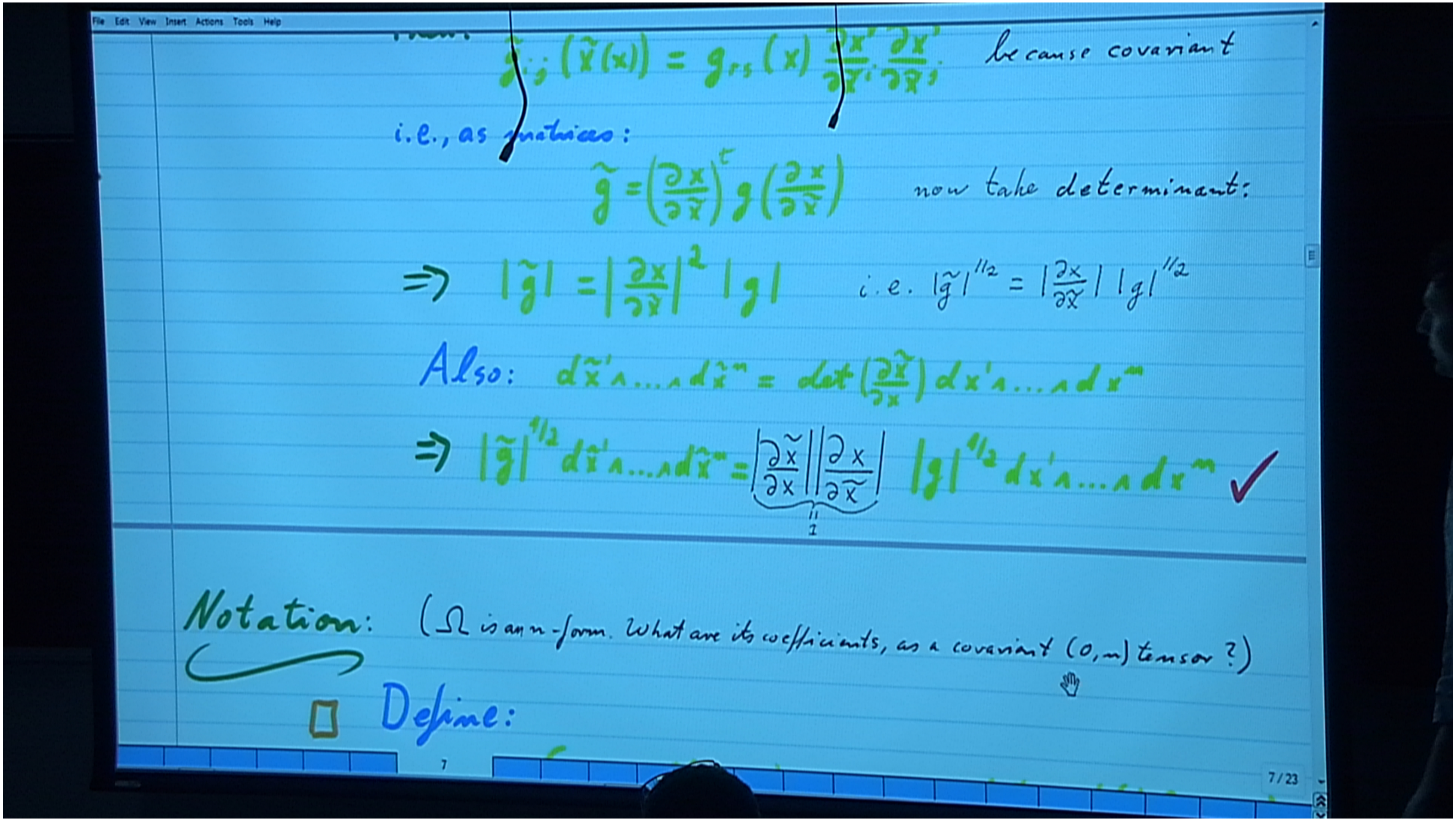
Notation: (Ω is an n-form. What are its coefficients, as a covariant $(0,n)$ tensor?)

☐ Define:



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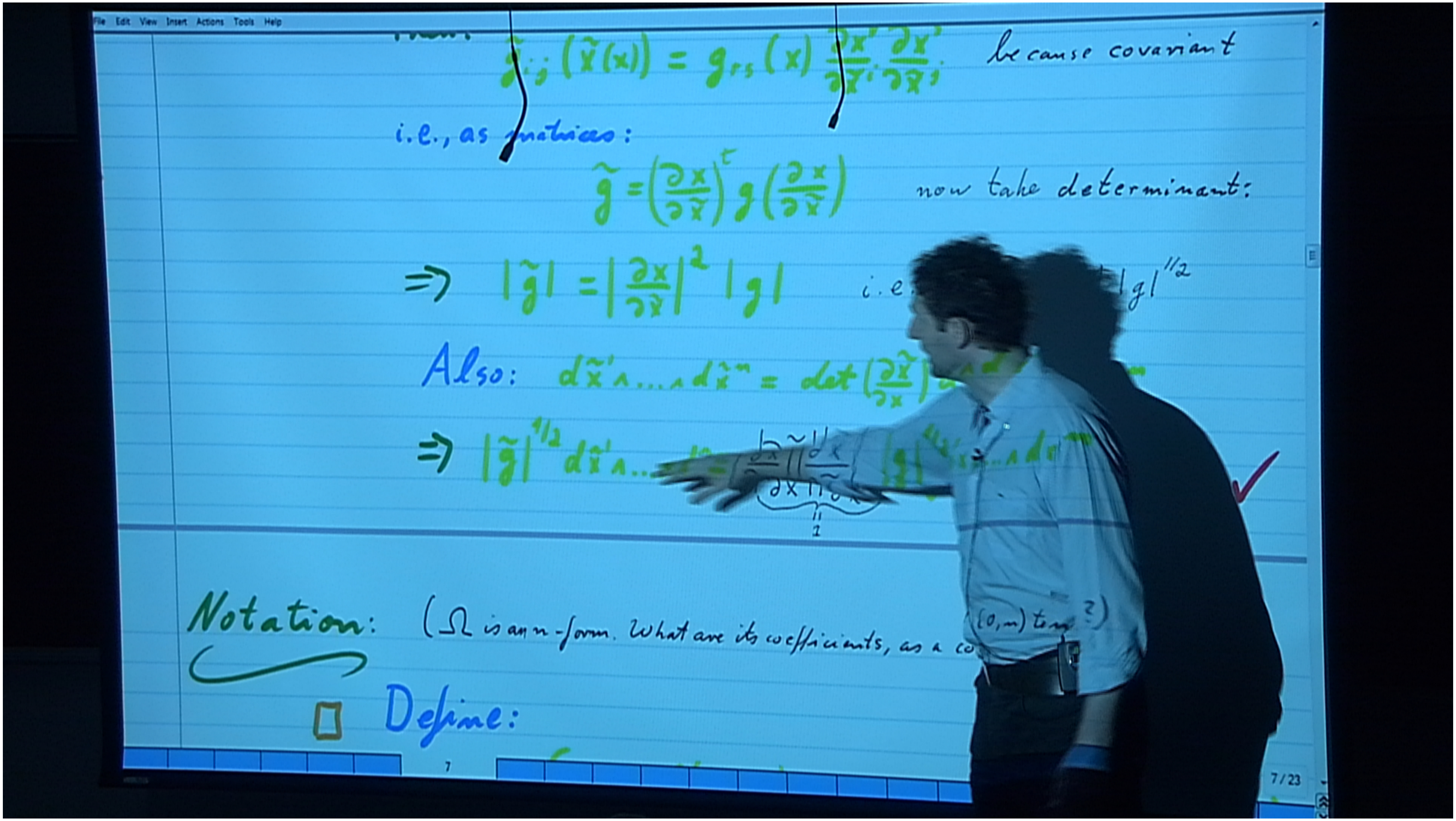

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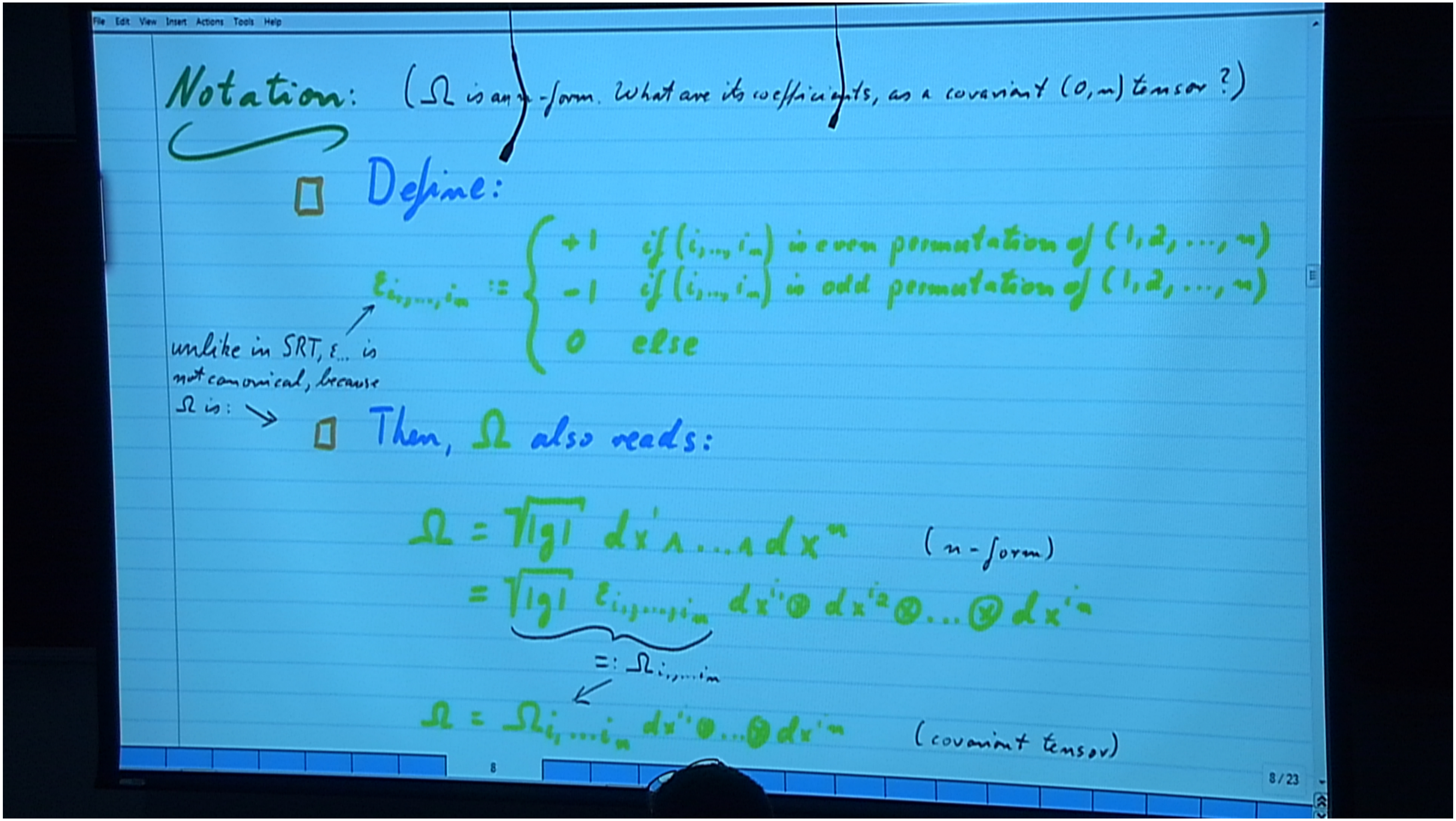

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$$\varepsilon_{i_1,\dots,i_n} := \begin{cases} +1 & \text{if } (i_1,\dots,i_n) \text{ is even permutation of } (1,2,\dots,n) \\ -1 & \text{if } (i_1,\dots,i_n) \text{ is odd permutation of } (1,2,\dots,n) \\ 0 & \text{else} \end{cases}$$

unlike in SRT, $\varepsilon_{\dots}$ is not canonical, because Ω is: →

☐ Then, Ω also reads:

$$\Omega = \sqrt{|g|}\, dx^1 \wedge \dots \wedge dx^n \qquad (n\text{-form})$$

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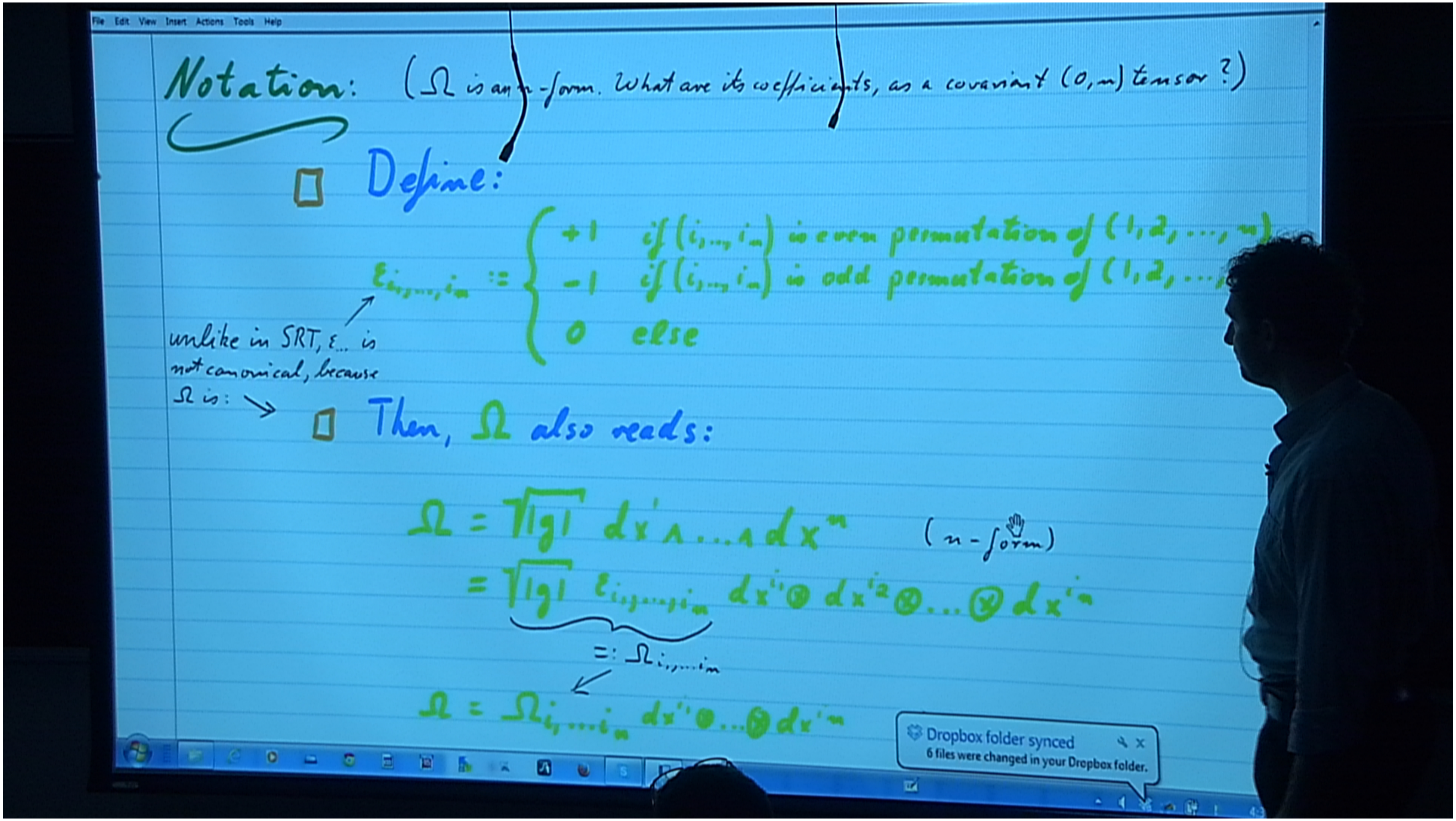

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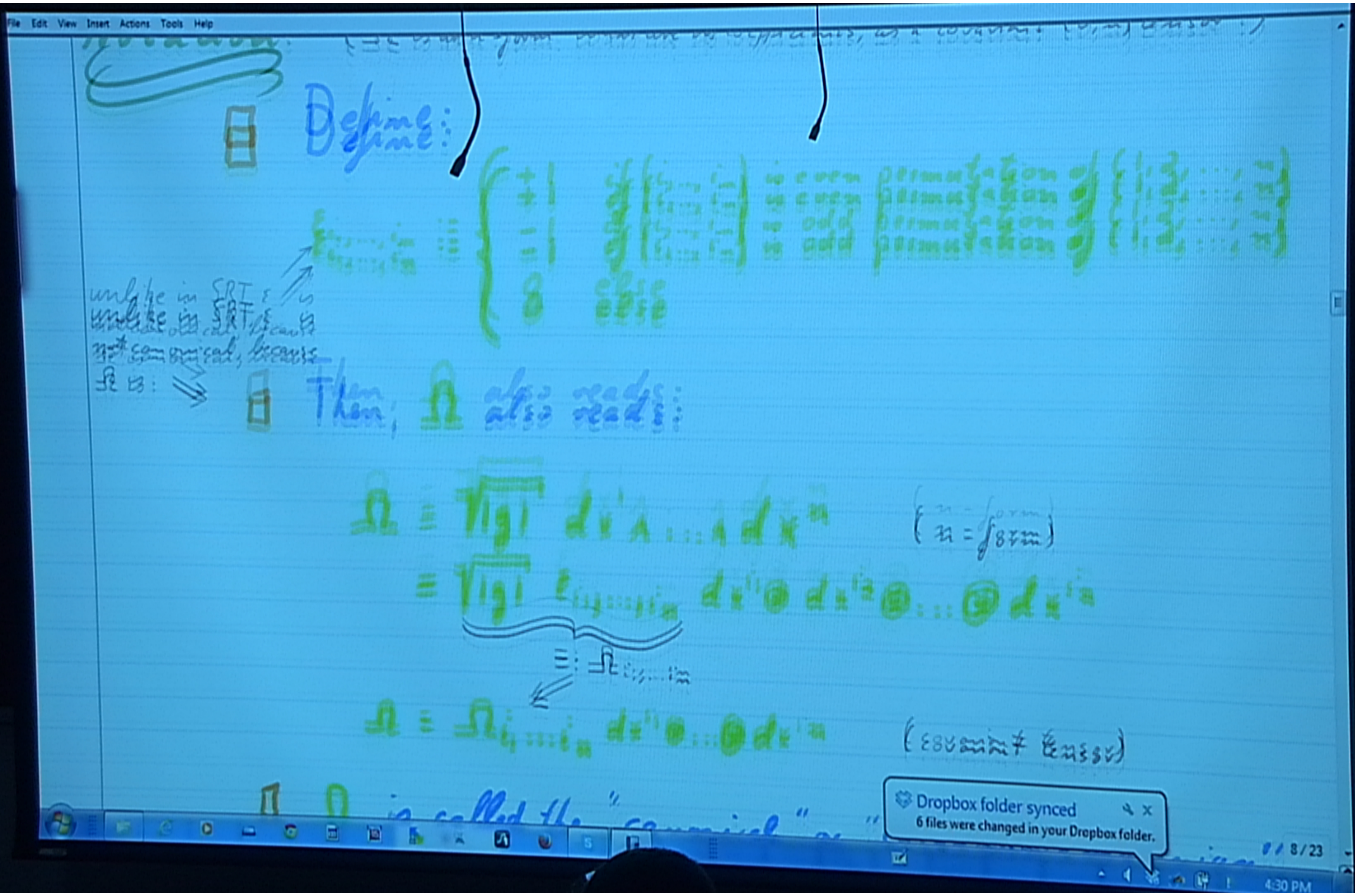

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Ω is called the "canonical" or "

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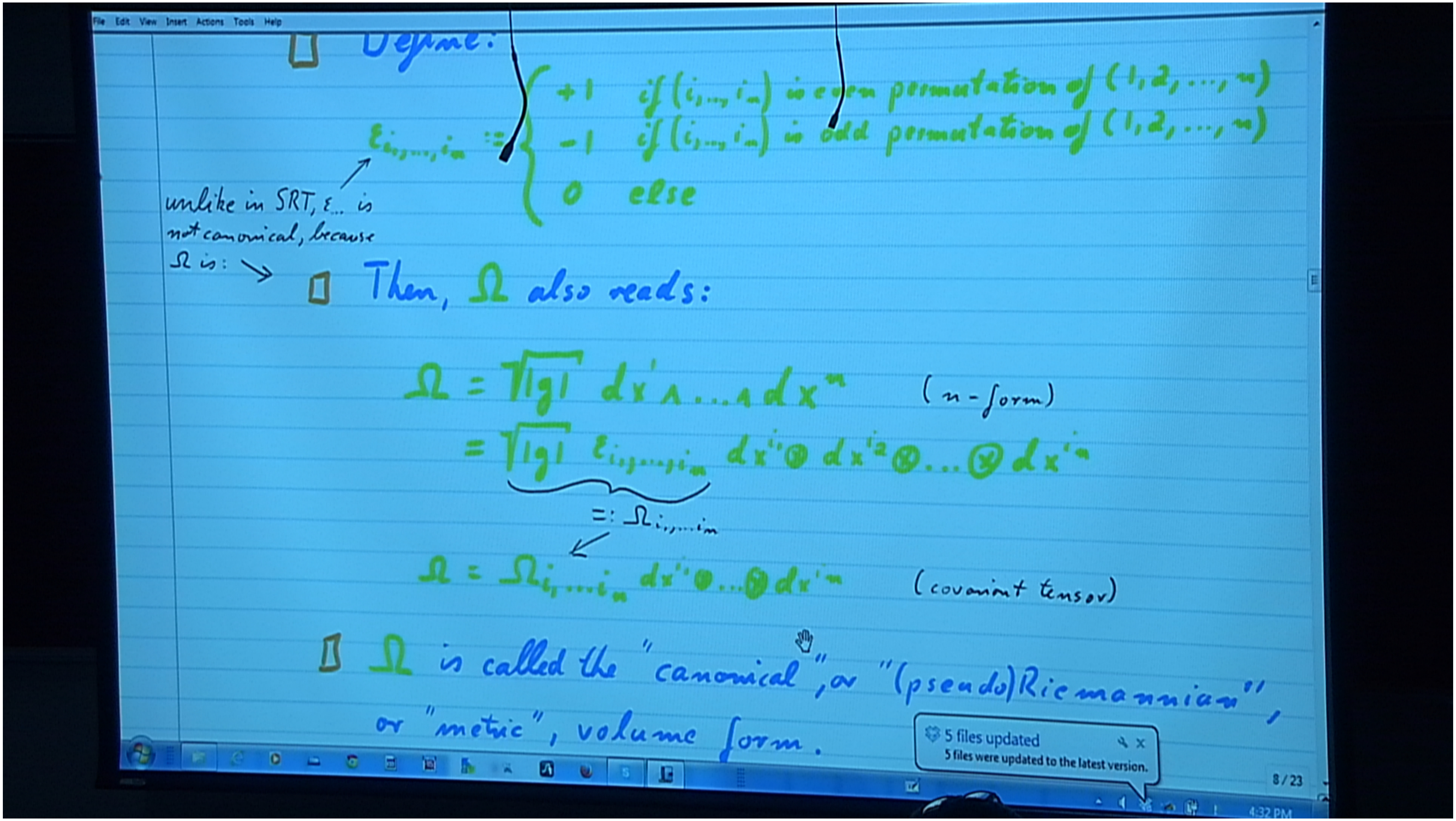

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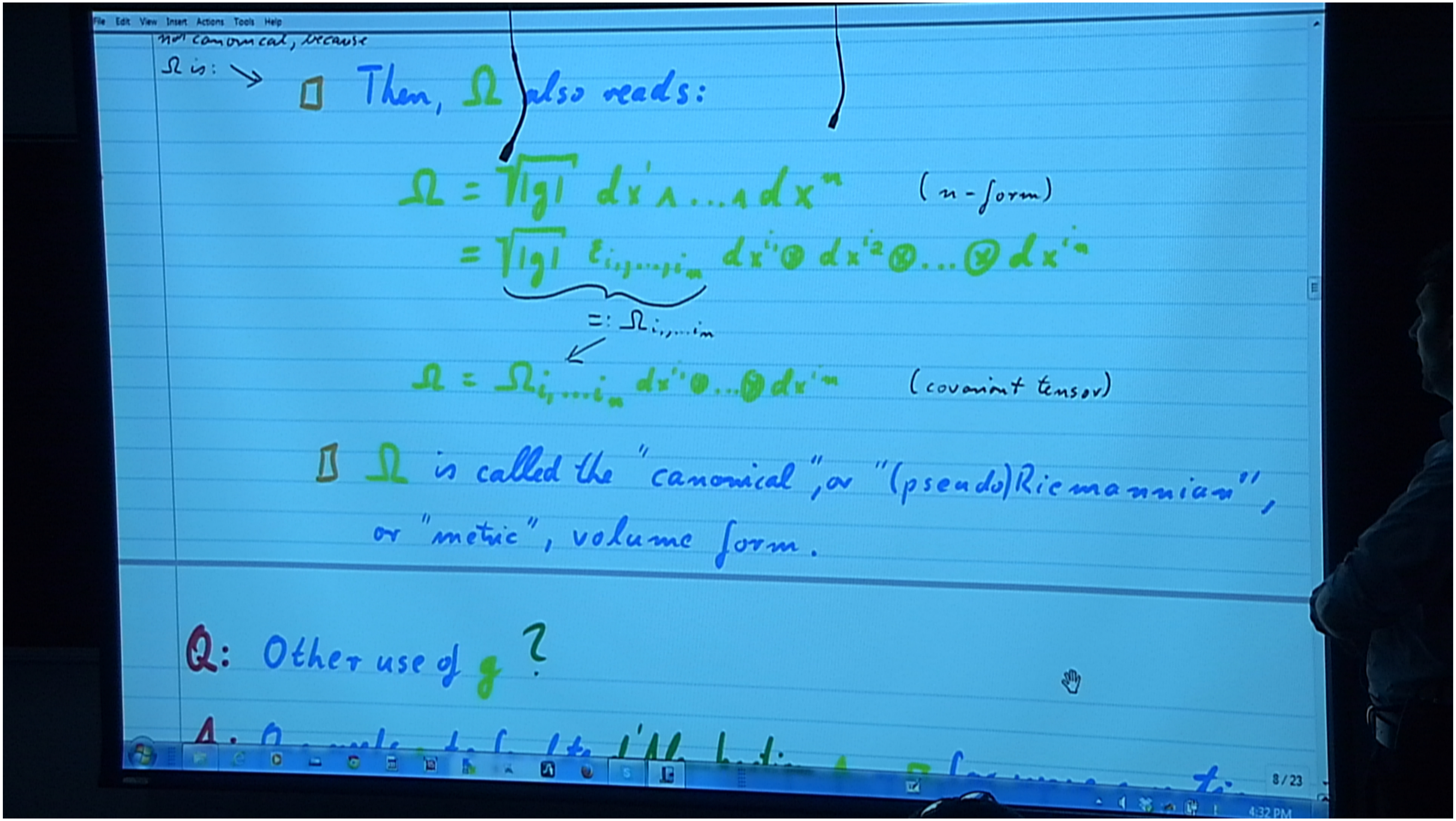

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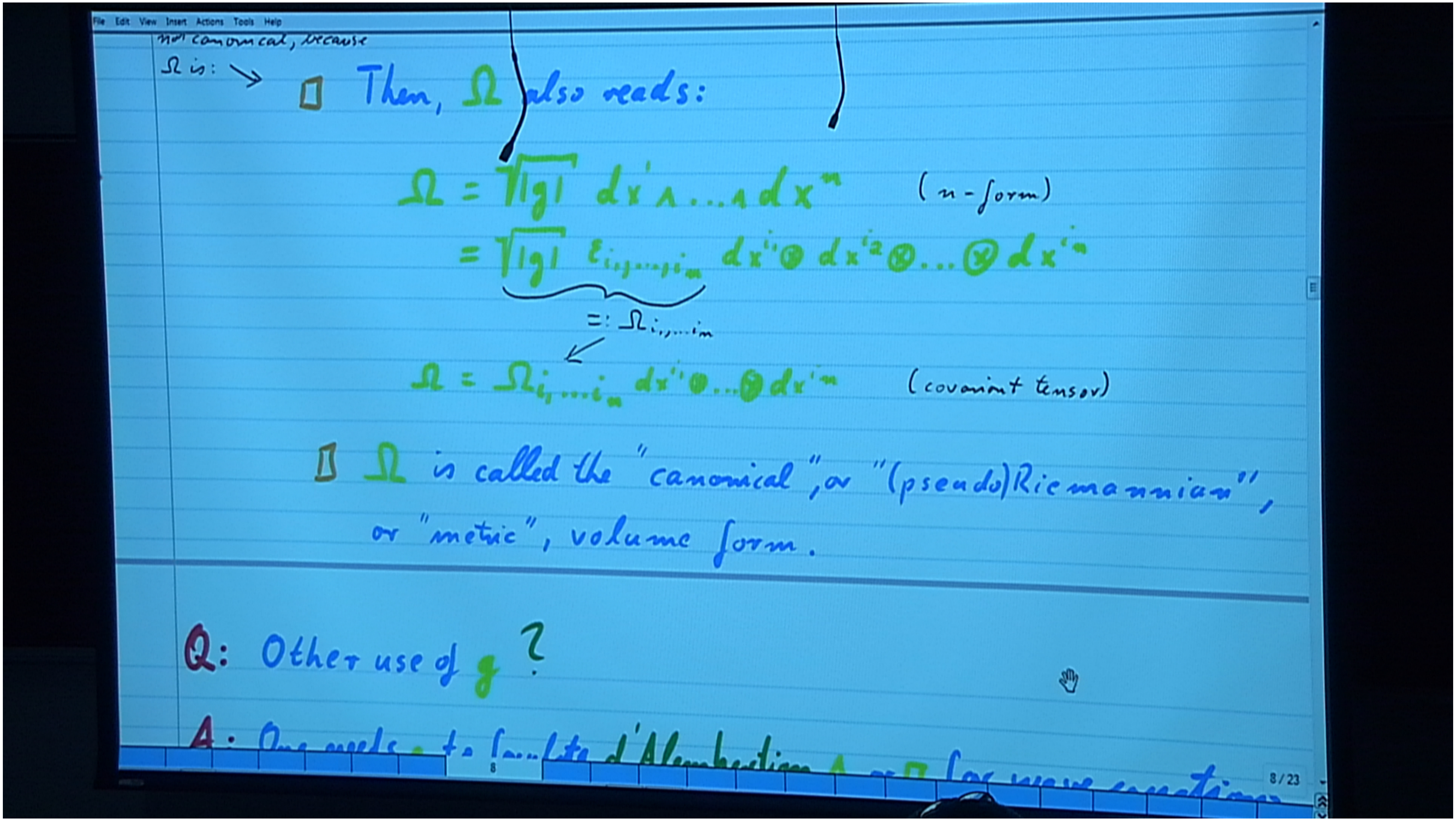
Q: Other use of g ?

A: [illegible]



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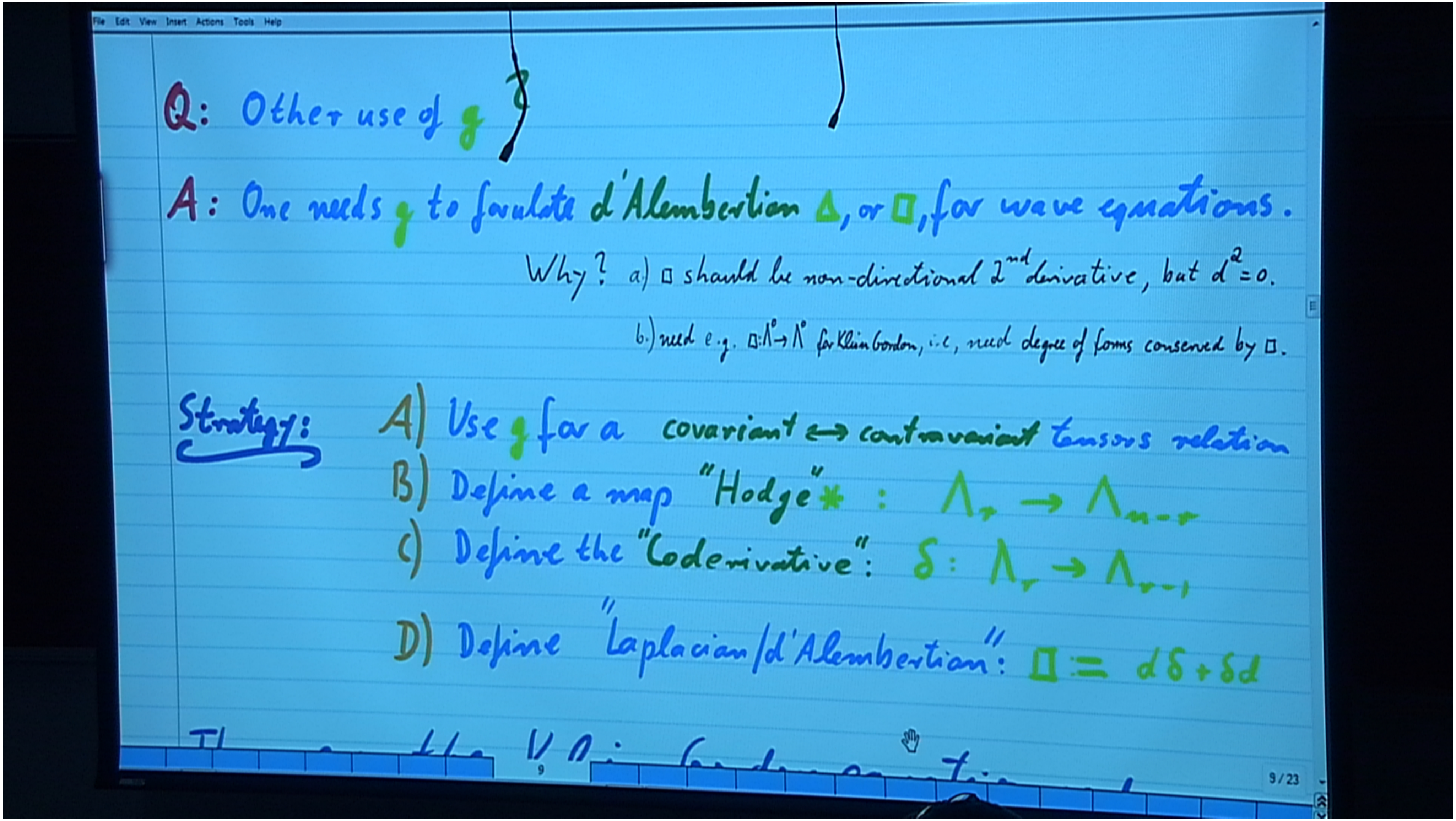

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Why? a) $\Box$ should be non-directional 2^{nd} derivative, but $d^2 = 0$.

b.) need e.g. $\Box : \Lambda^0 \to \Lambda^0$ for Klein Gordon, i.e, need degree of forms conserved by $\Box$.

Strategy:

A) Use g for a covariant $\leftrightarrow$ contravariant tensors relation

B) Define a map "Hodge" $*$: $\Lambda_r \to \Lambda_{n-r}$

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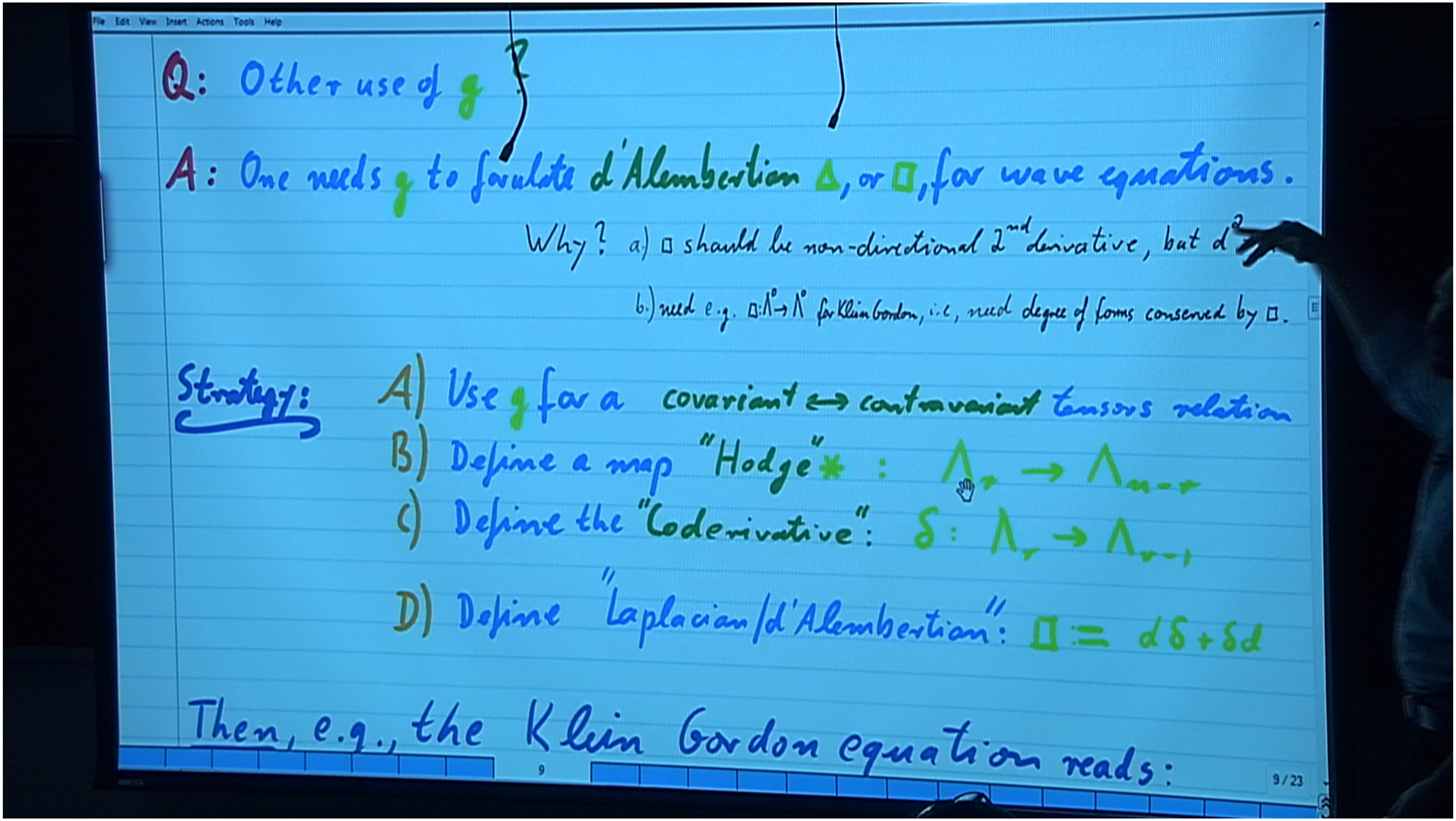

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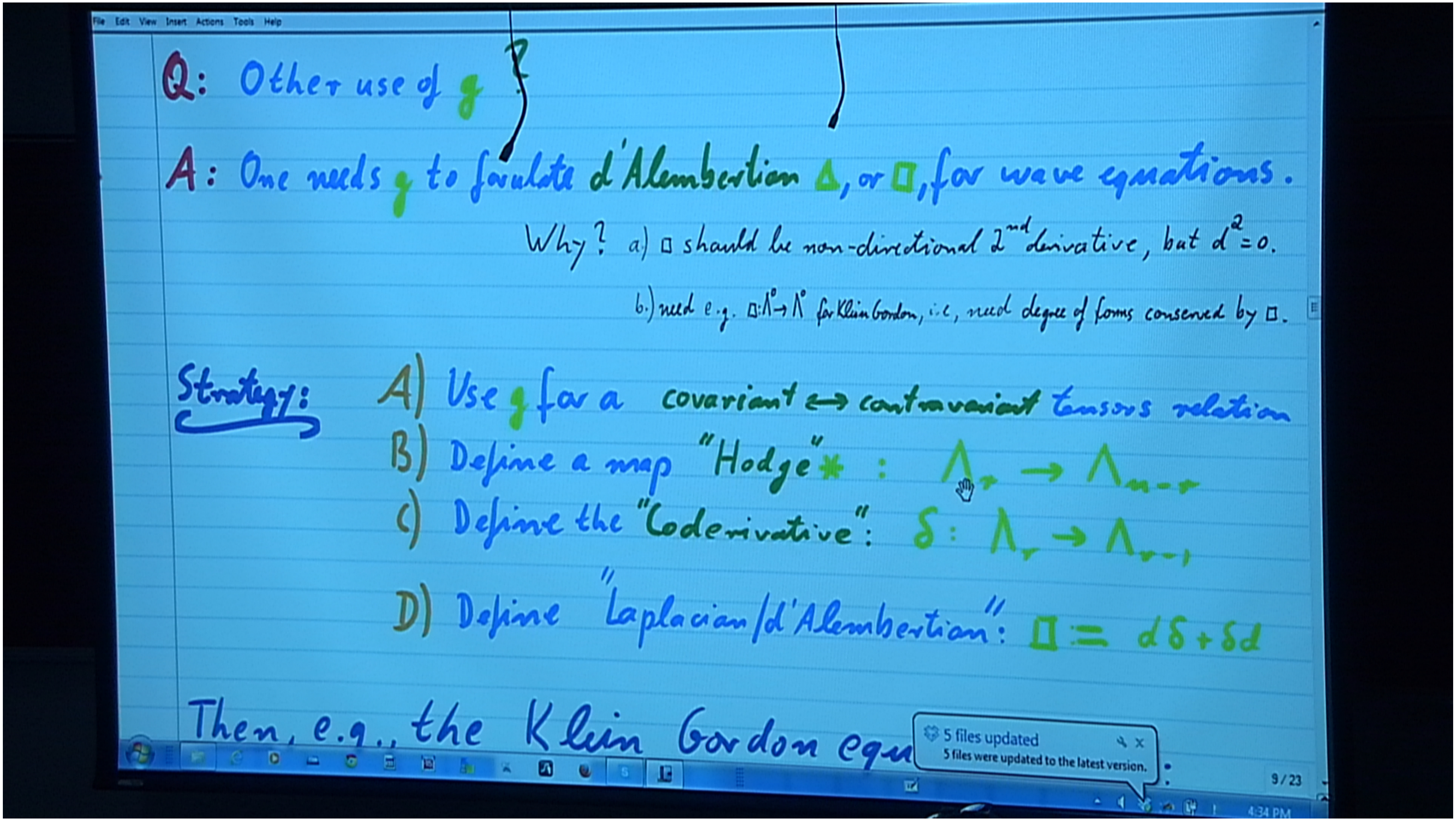

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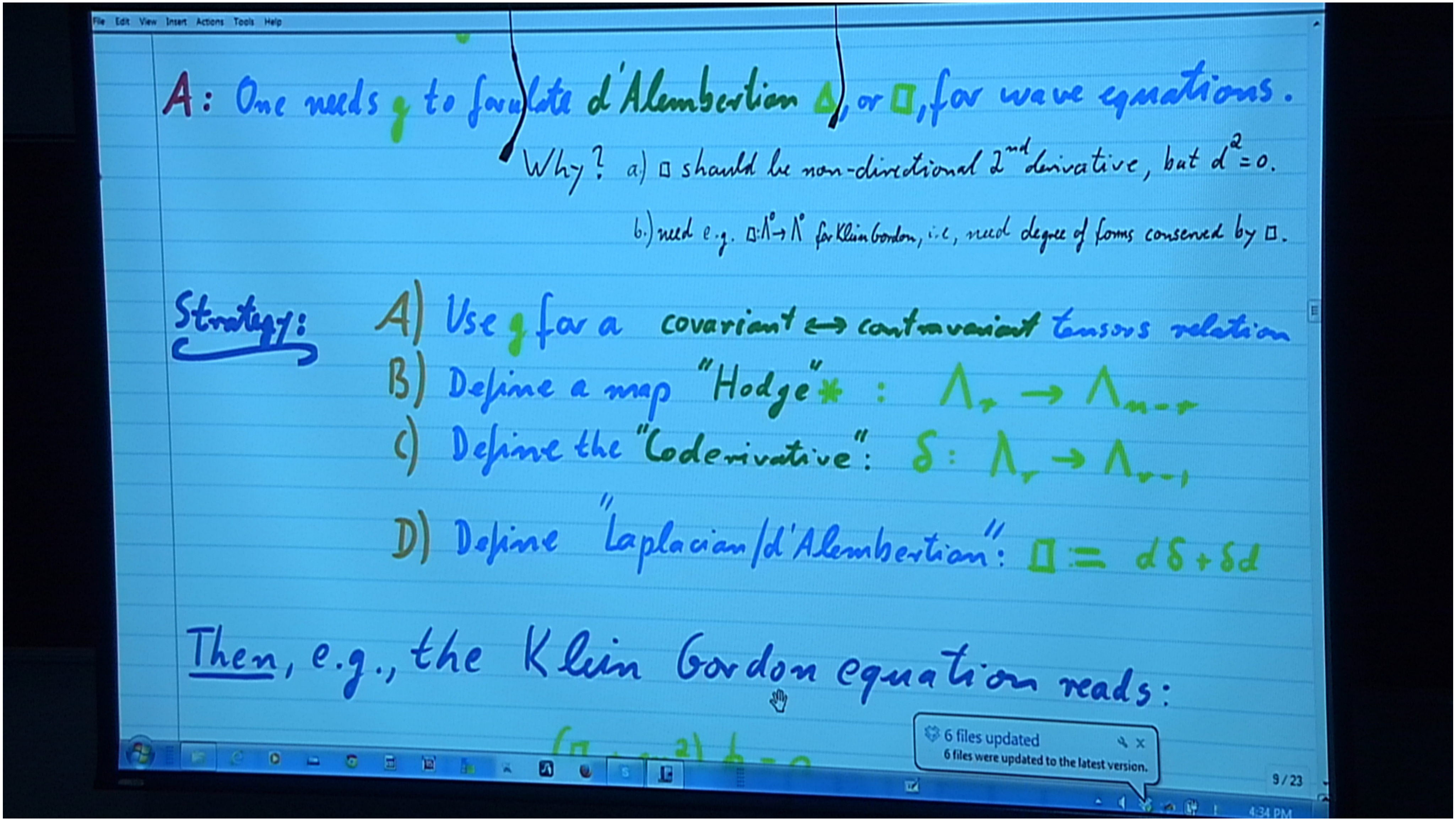

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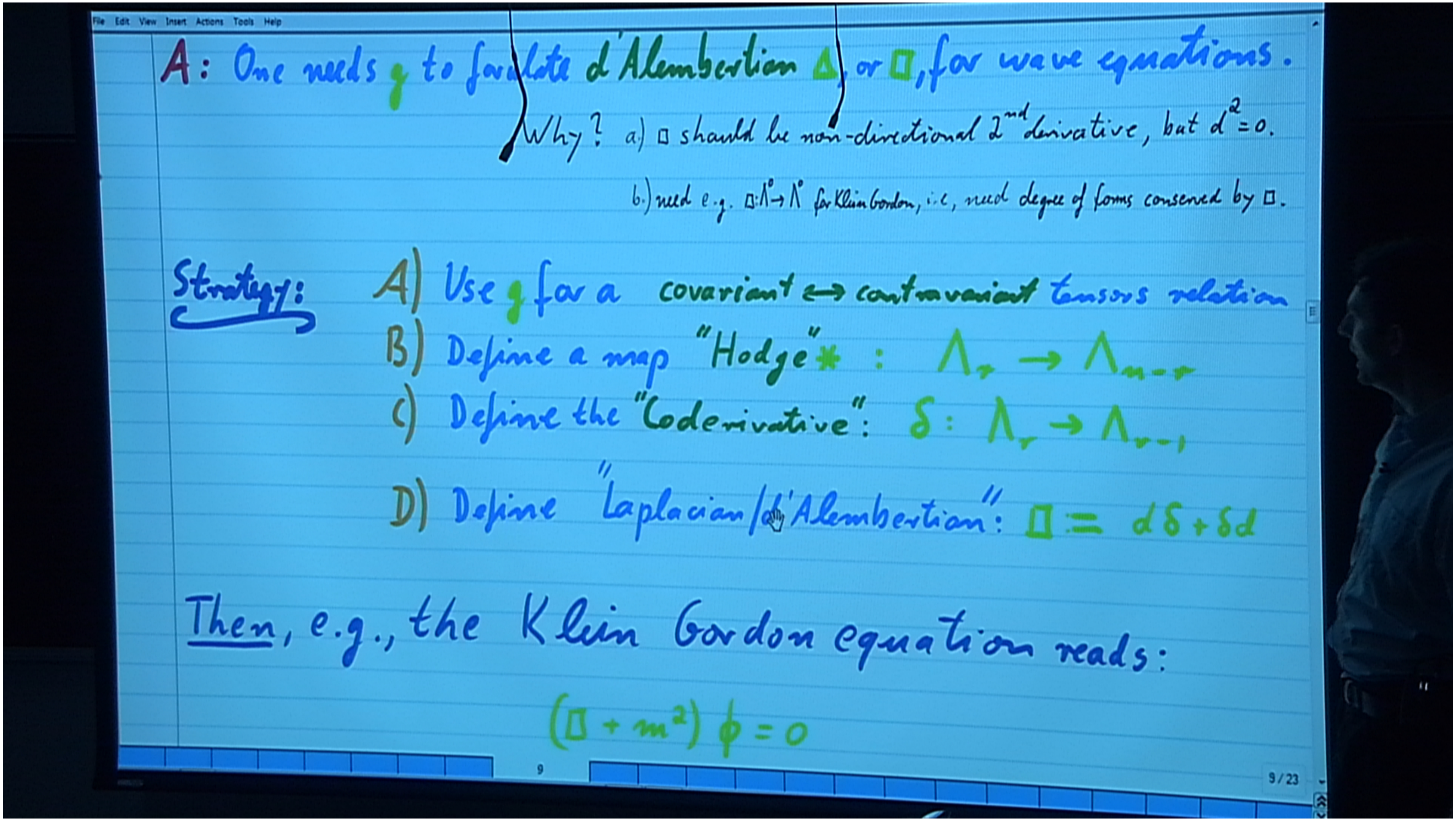

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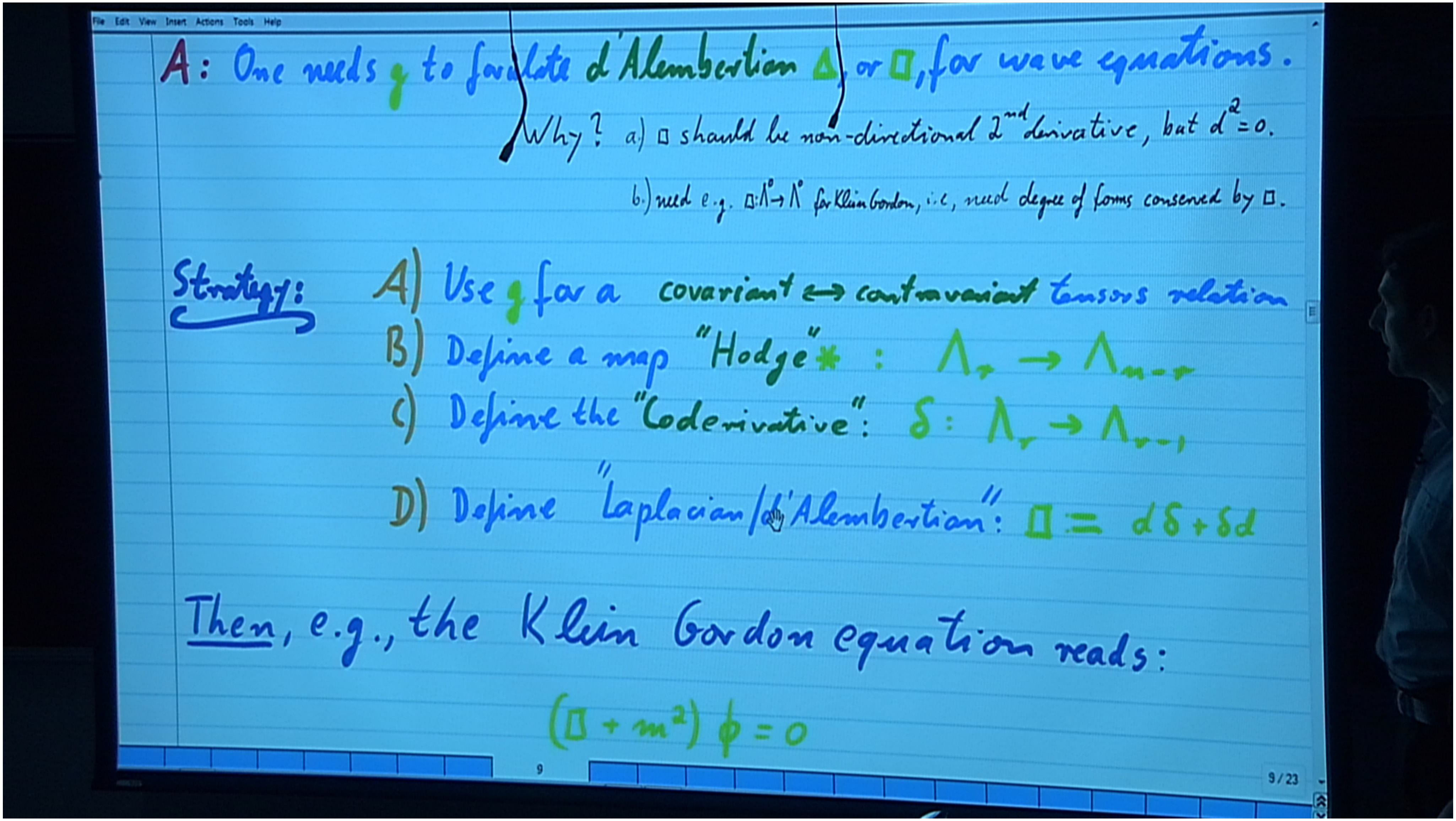


A: One needs g to formulate d'Alembertian Δ, or $\square$, for wave equations.

Why? a.) $\square$ should be non-directional 2nd derivative, but $d^2 = 0$.

b.) need e.g. $\square : \Lambda^0 \to \Lambda^0$ for Klein Gordon, i.e., need degree of forms conserved by $\square$.

Strategy:

A) Use g for a covariant ⟷ contravariant tensors relation

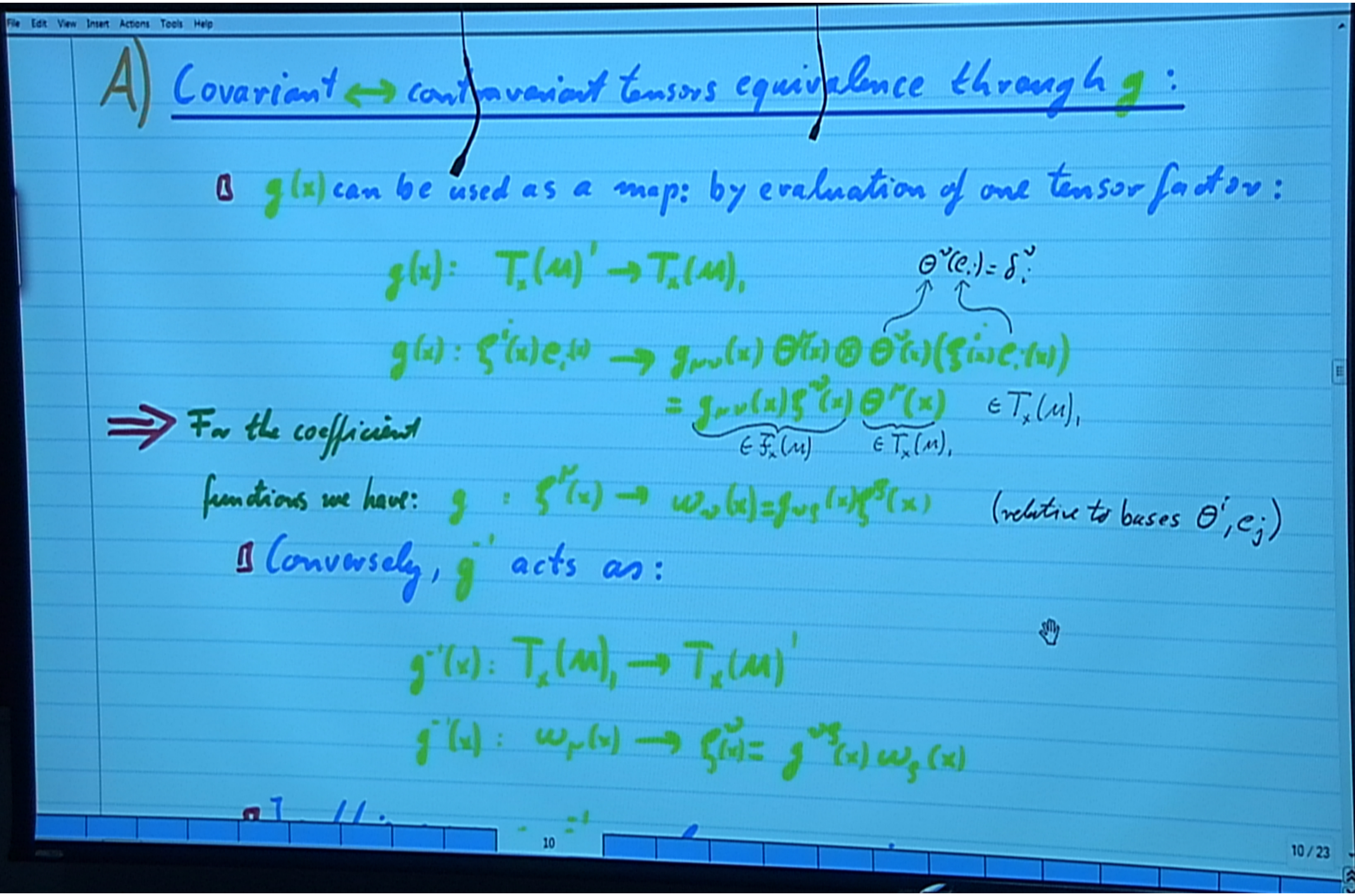
B) Define a map "Hodge" $*$: $\Lambda_r \to \Lambda_{n-r}$

C) Define the "Coderivative": $\delta : \Lambda_r \to \Lambda_{r-1}$

D) Define "Laplacian/d'Alembertian": $\square = d\delta + \delta d$

Then, e.g., the Klein Gordon equation reads:

$$(\square + m^2)\phi = 0$$



A) Covariant ↔ contravariant tensors equivalence through g:

- $g(x)$ can be used as a map: by evaluation of one tensor factor:

$$g(x):\ T_x(M)' \to T_x(M),$$

$$\theta^\nu(e_i) = \delta^\nu_i$$

$$g(x):\ \xi^i(x)e_i(x) \to g_{\mu\nu}(x)\,\theta^\mu(x)\otimes\theta^\nu(x)\big(\xi^i(x)e_i(x)\big)$$

$$= \underbrace{g_{\mu\nu}(x)\xi^\nu(x)}_{\in \mathcal{F}_x(M)}\ \underbrace{\theta^\mu(x)}_{\in T_x(M),} \quad \in T_x(M),$$

⟹ For the coefficient functions we have: $g\ :\ \xi^\nu(x) \to \omega_\nu(x) = g_{\nu\sigma}(x)\xi^\sigma(x)$ (relative to bases θ^i, e_j)

- Conversely, g^{-1} acts as:

$$g^{-1}(x):\ T_x(M), \to T_x(M)'$$

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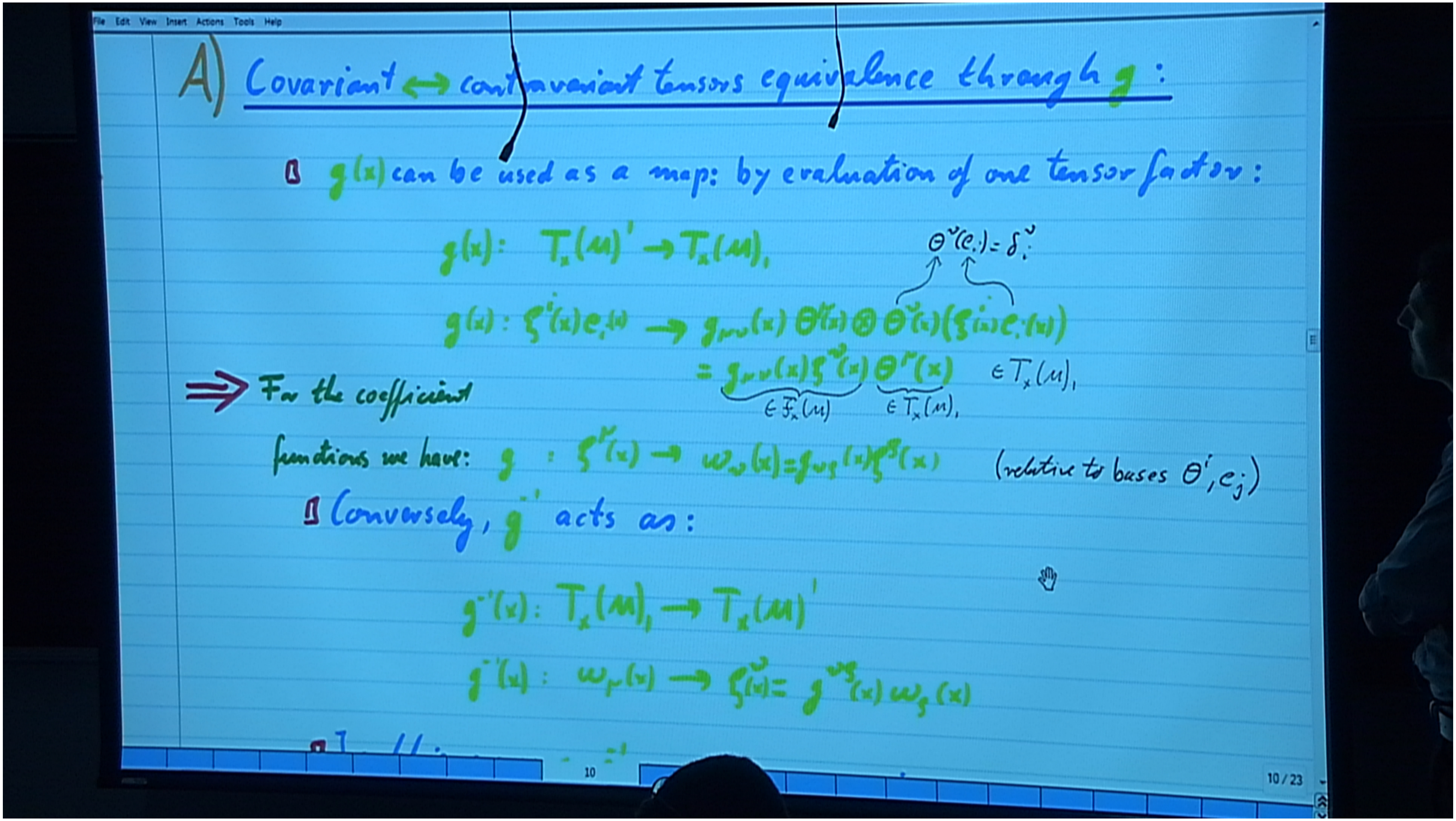

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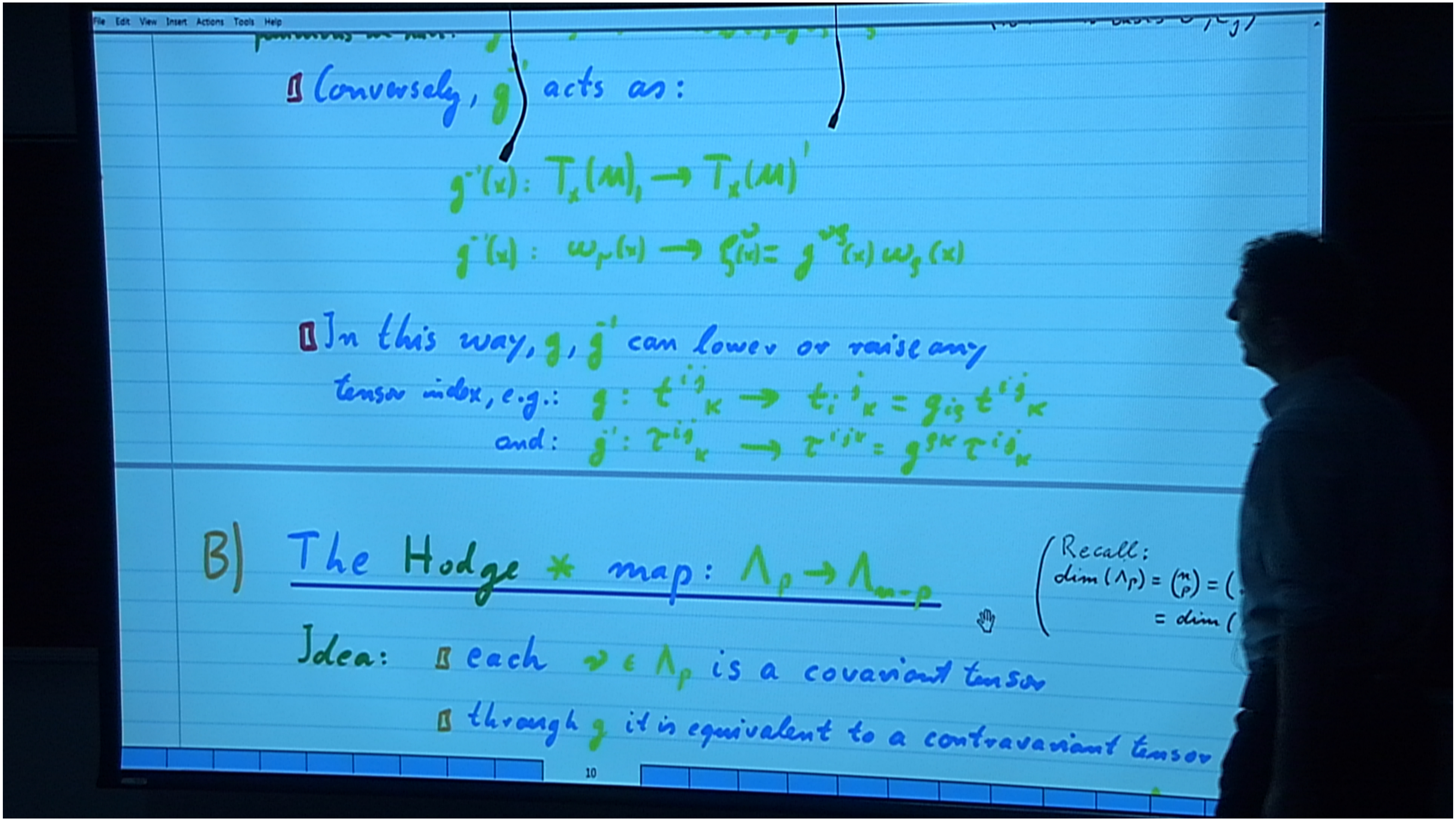

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- In this way, g, g^{-1} can lower or raise any tensor index, e.g.: $g: t^{ij}{}_k \to t_i{}^j{}_k = g_{ij}\,t^{ij}{}_k$

  and: $g^{-1}: \tau^{ij}{}_k \to \tau^{ijk} = g^{jk}\,\tau^{ij}{}_k$

B) The Hodge $*$ map: $\Lambda_p \to \Lambda_{n-p}$

(Recall:
$\dim(\Lambda_p) = \binom{n}{p} = ($
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Idea:
- each $\nu \in \Lambda_p$ is a covariant tensor
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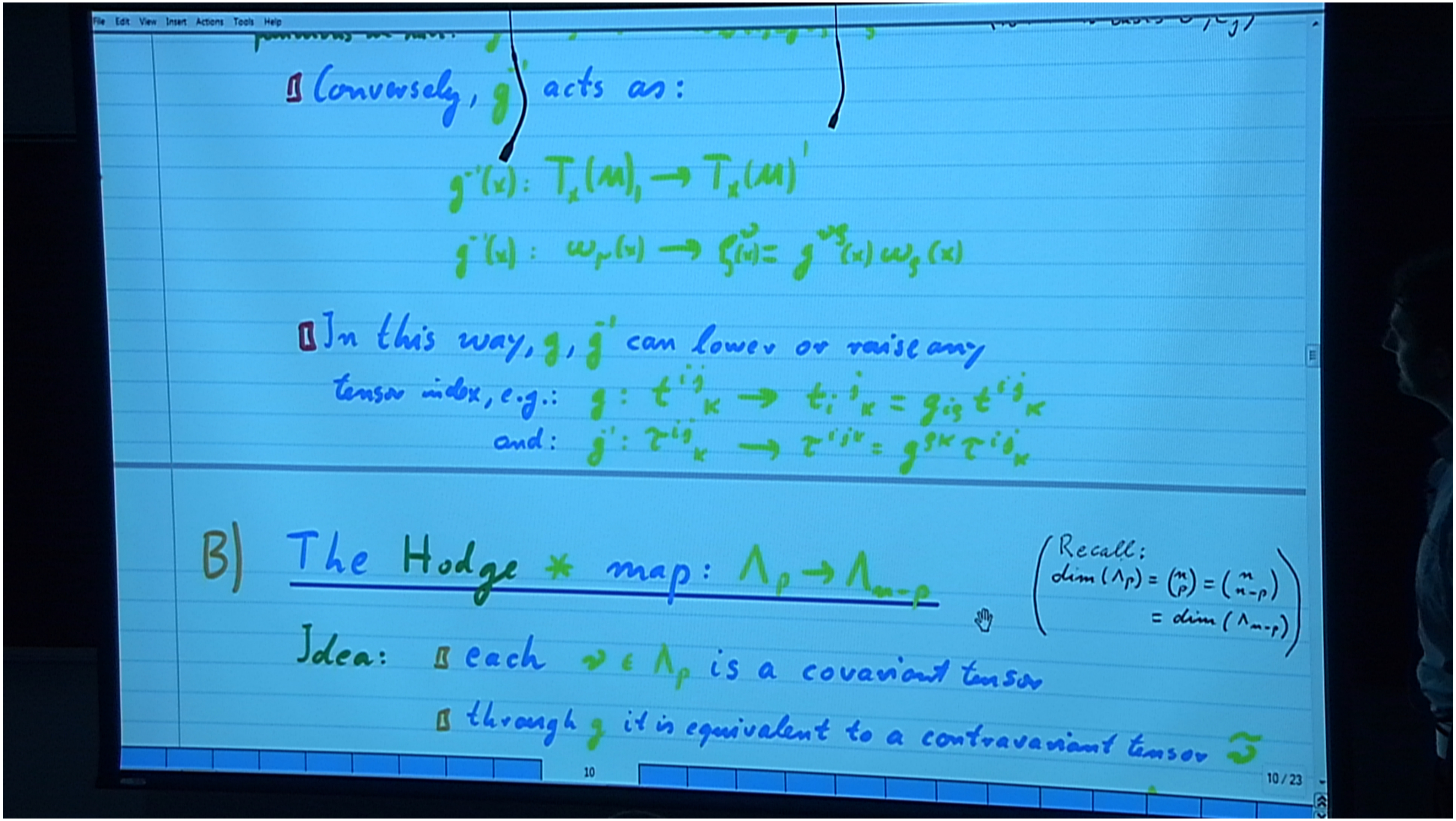

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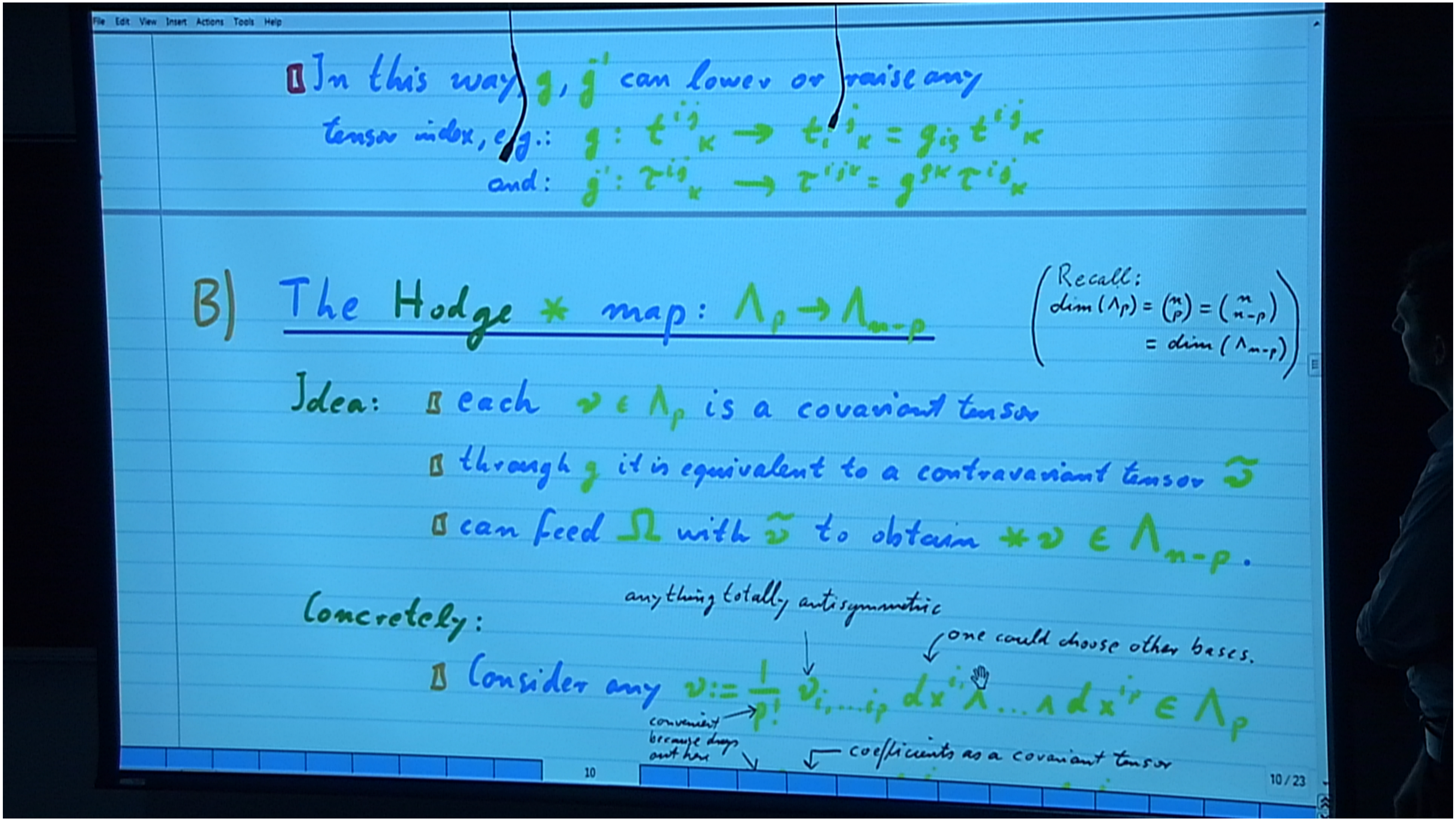

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Concretely:

anything totally antisymmetric

one could choose other bases.

- Consider any $\nu := \frac{1}{p!} \nu_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p} \in \Lambda_p$

convenient because drops out here

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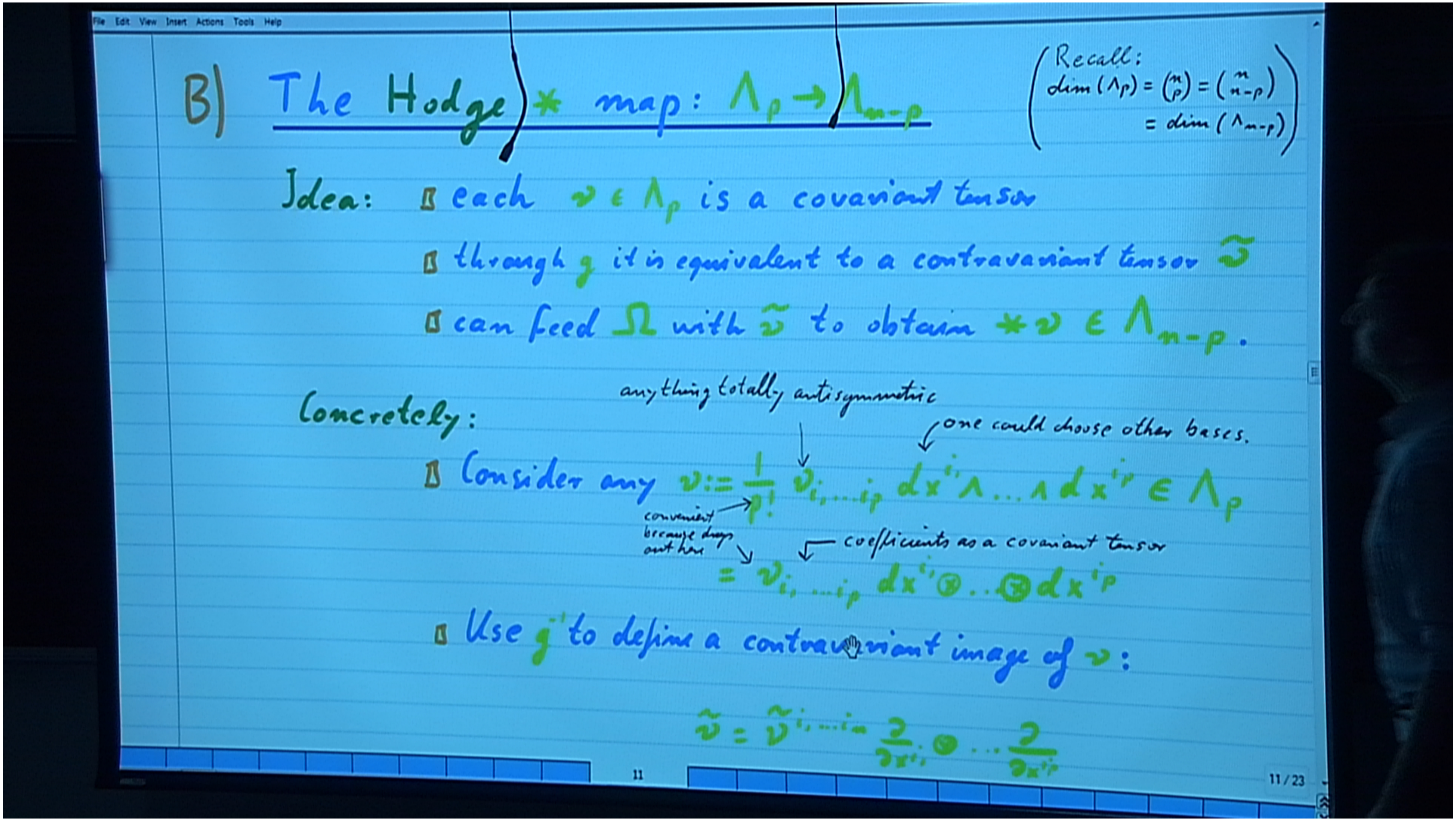

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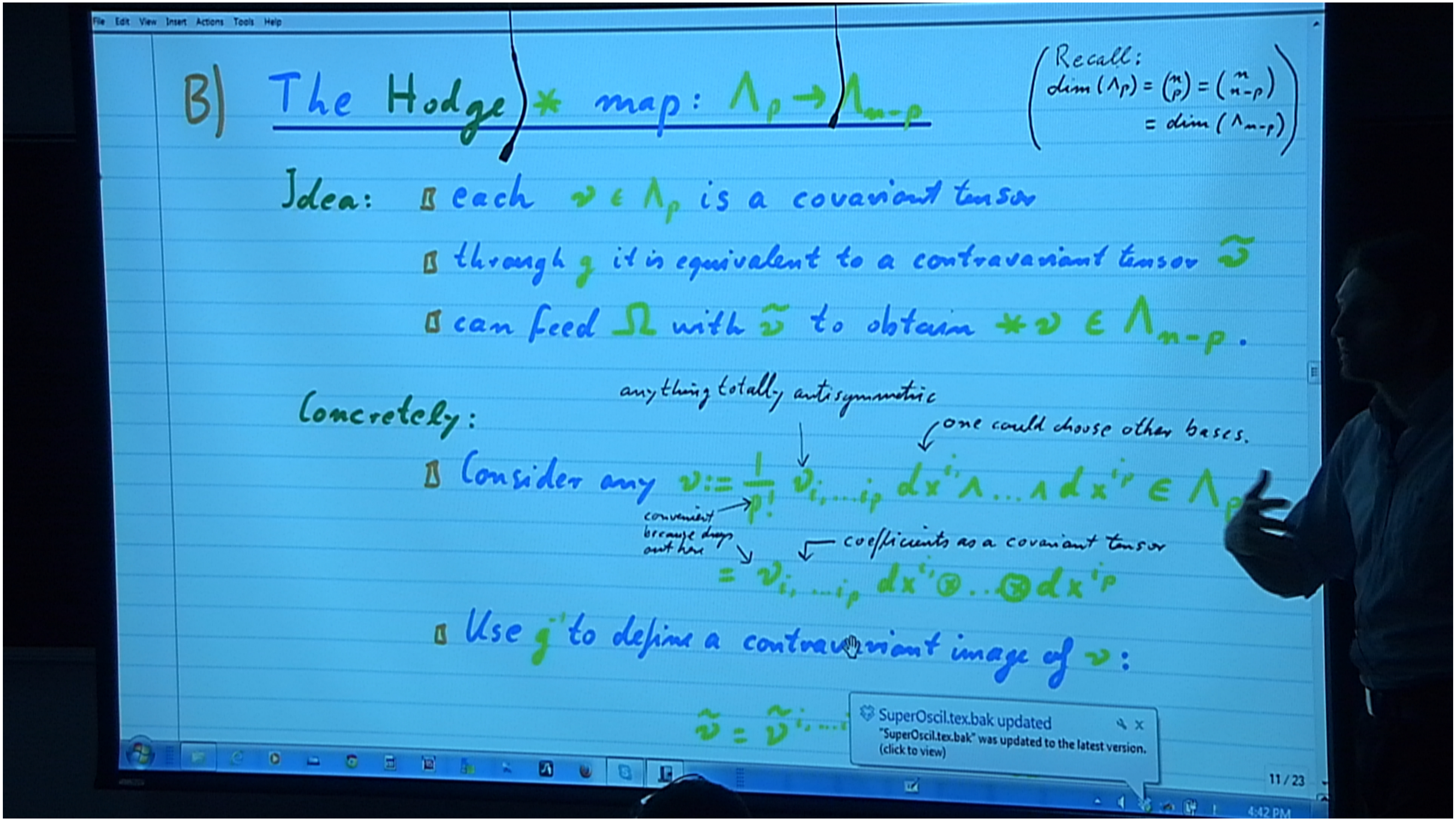

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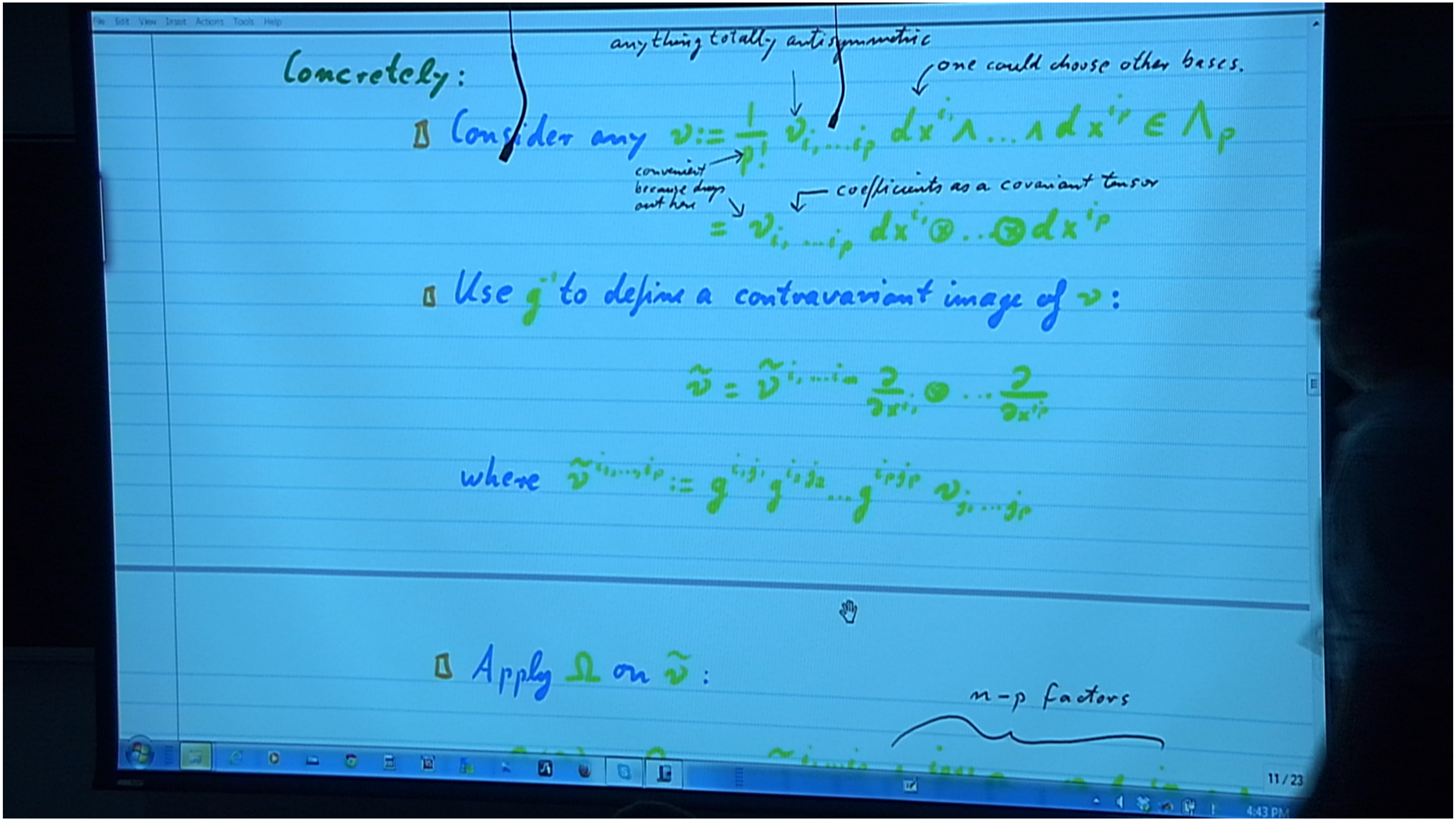
SuperOscil.tex.bak updated

"SuperOscil.tex.bak" was updated to the latest version.
(click to view)



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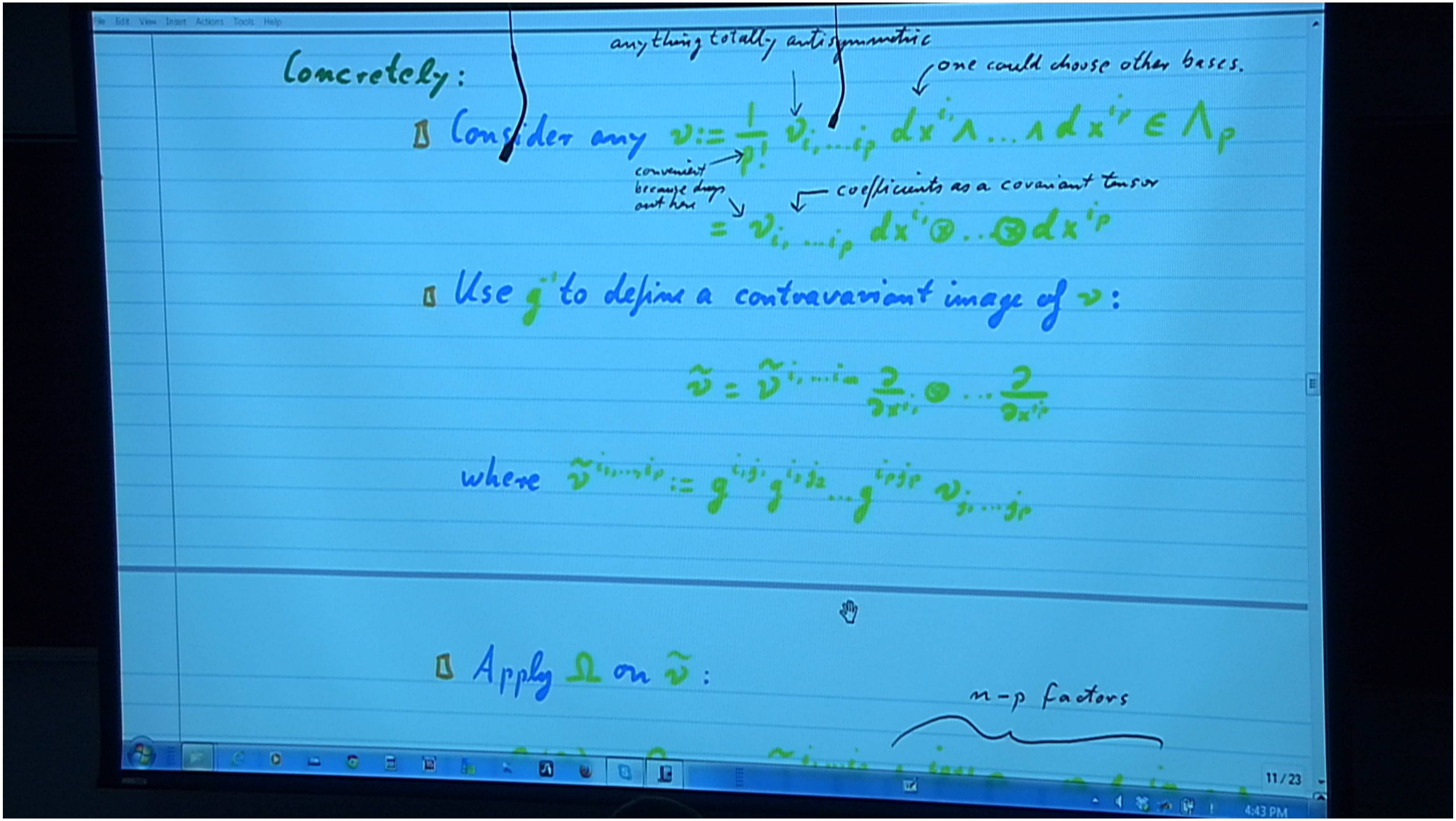
▯ Apply Ω on $\tilde{v}$:

n-p factors



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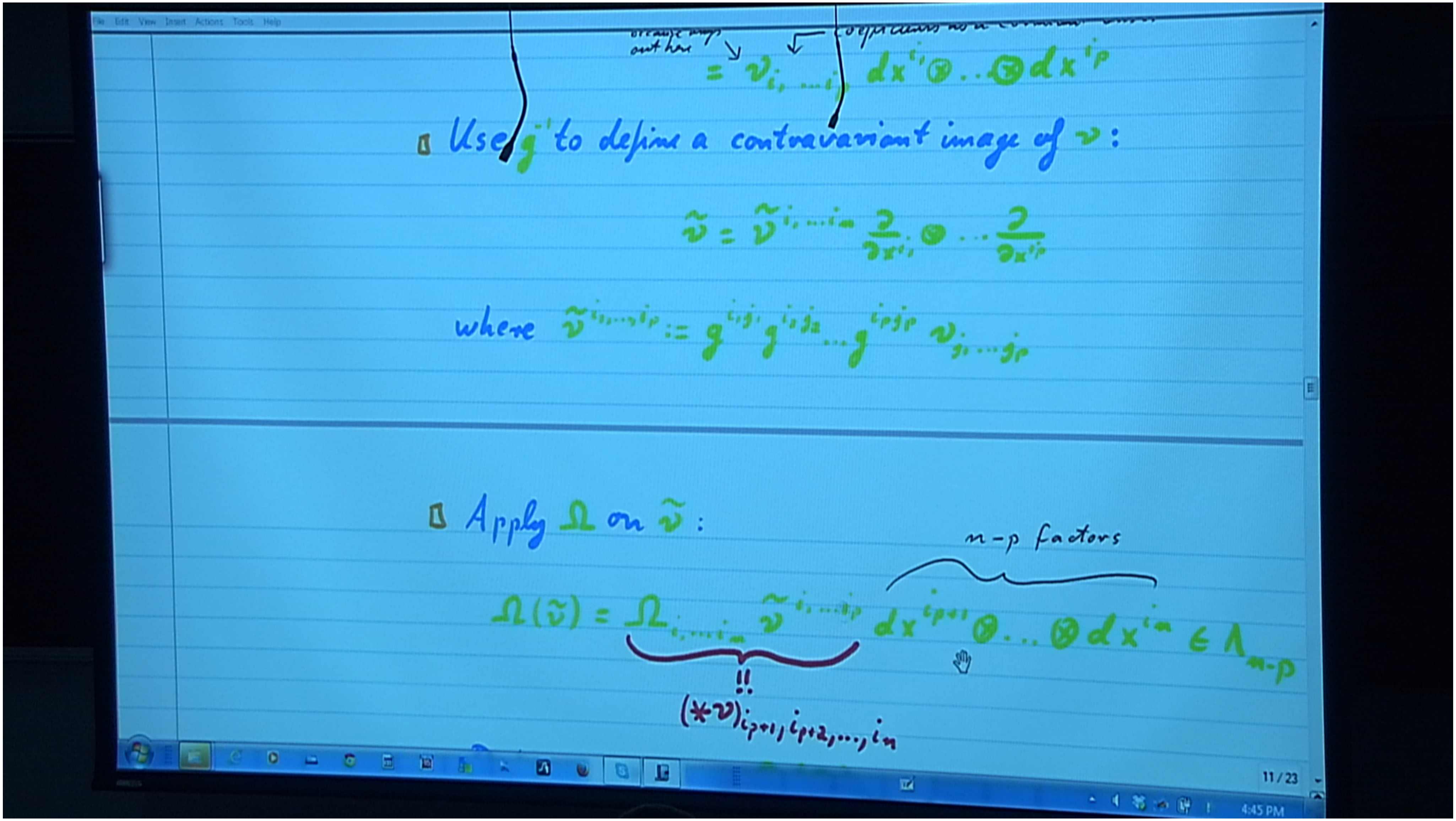

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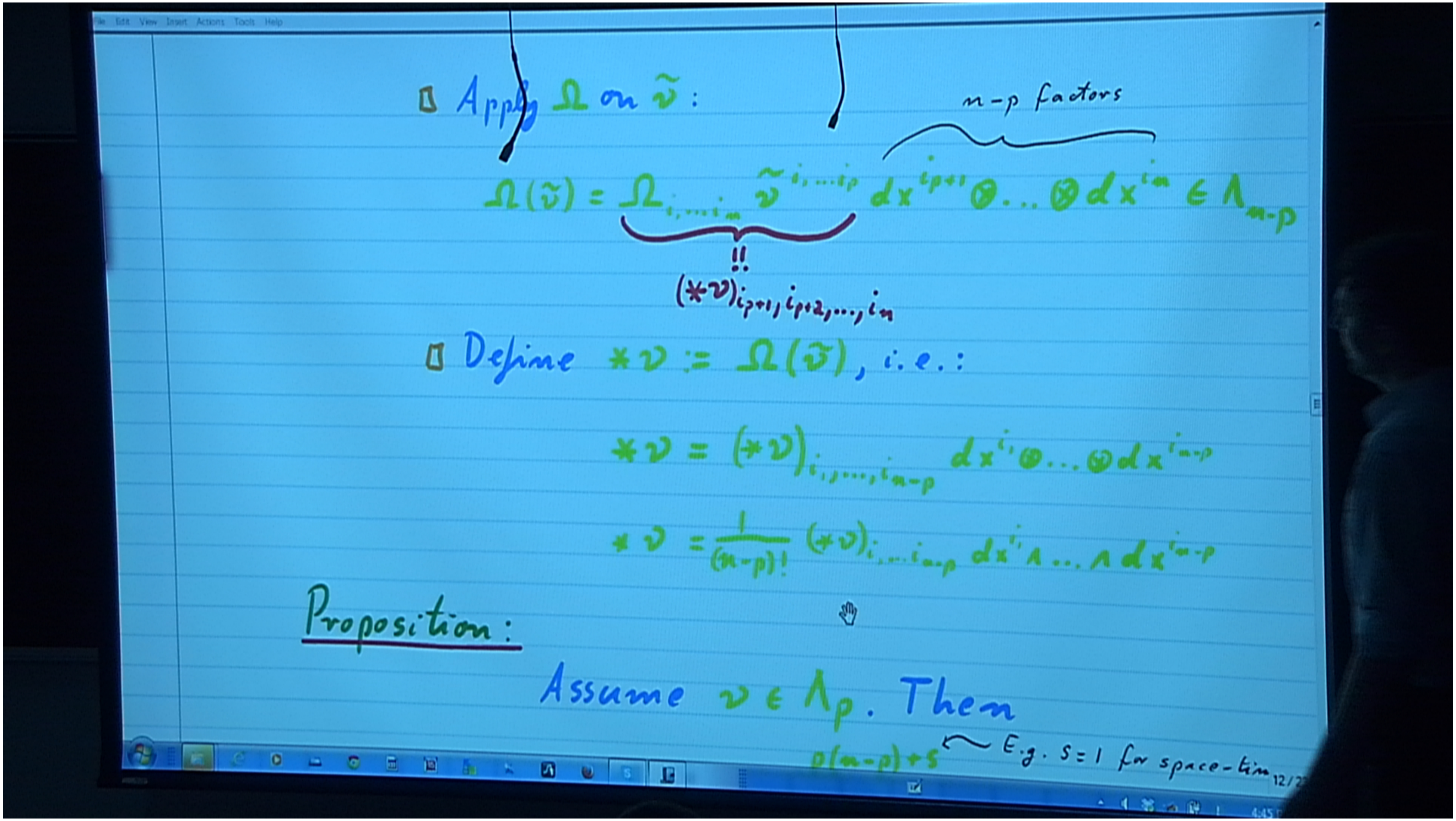

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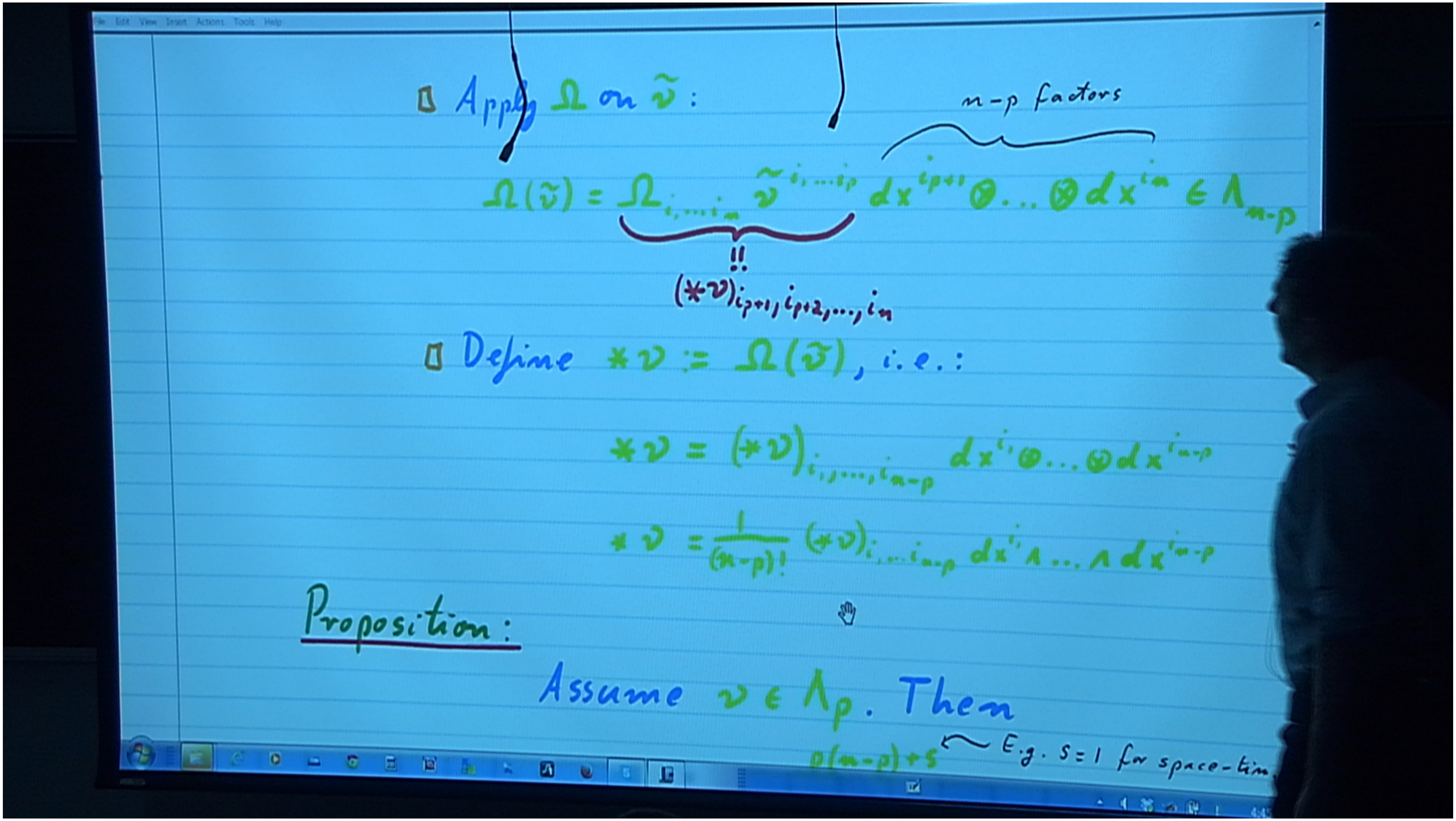
Assume $v \in \Lambda_p$. Then

$p(n-p)+s \leftarrow$ E.g. $s=1$ for space-tim



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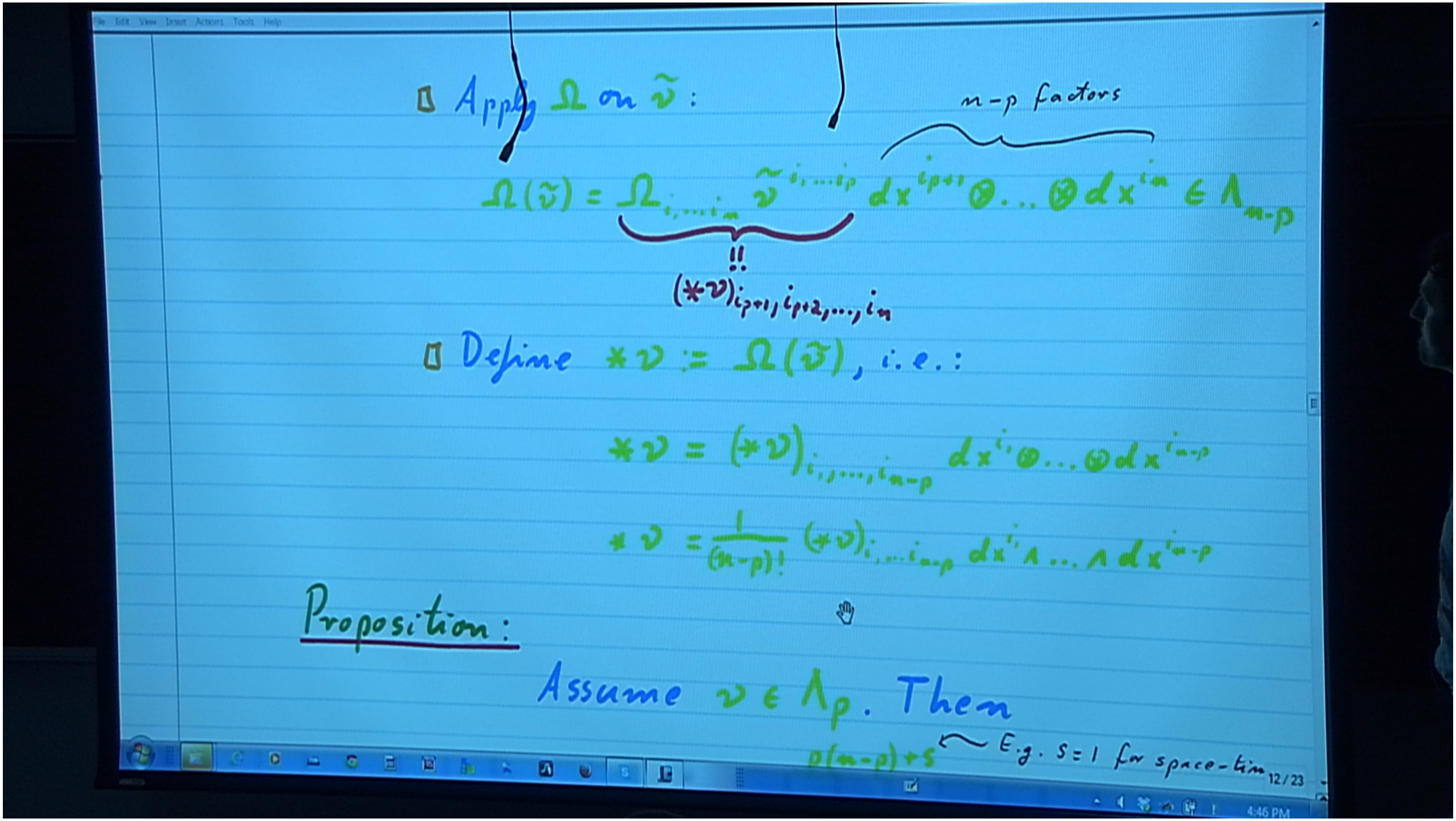

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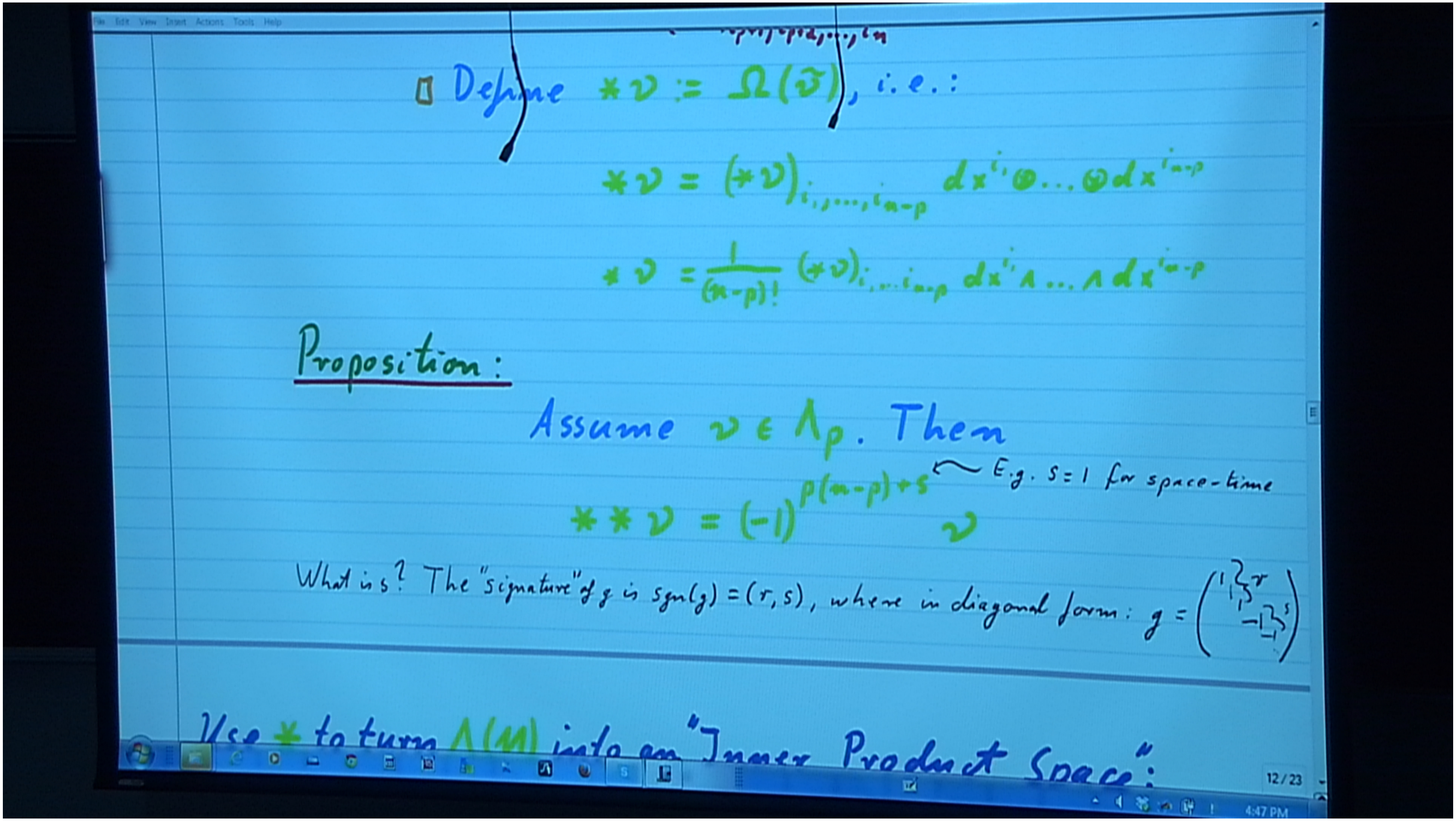

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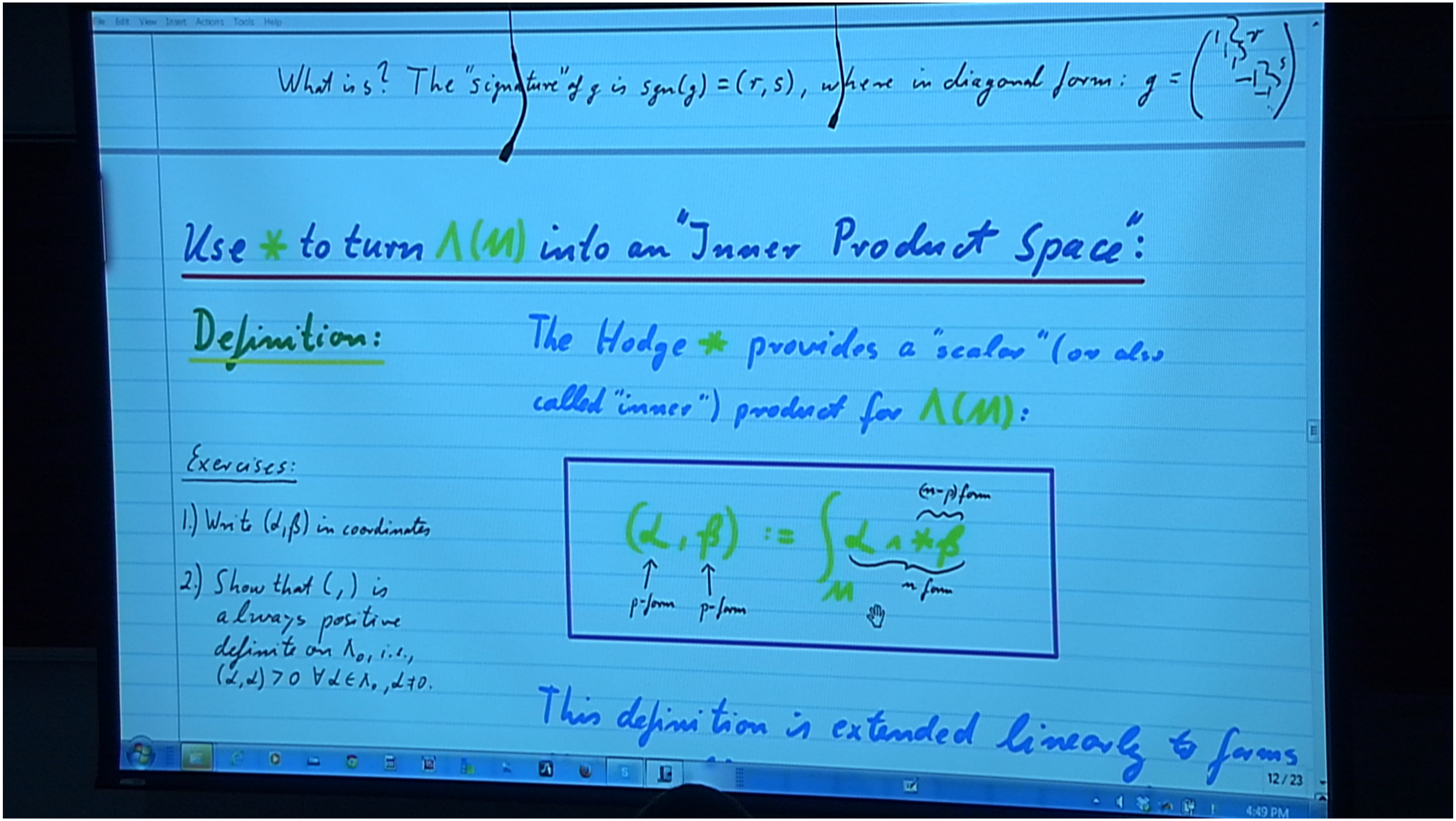
$$* * v = (-1)^{p(n-p)+s} v$$

← E.g. $s = 1$ for space-time

What is s? The "signature" of g is $\text{sgn}(g) = (r, s)$, where in diagonal form: $g = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ $\left.\right\}r$ $\left.\right\}s$

Use $*$ to turn $\Lambda(M)$ into an "Inner Product Space":



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Definition: The Hodge $*$ provides a "scalar" (or also called "inner") product for $\Lambda(M)$:

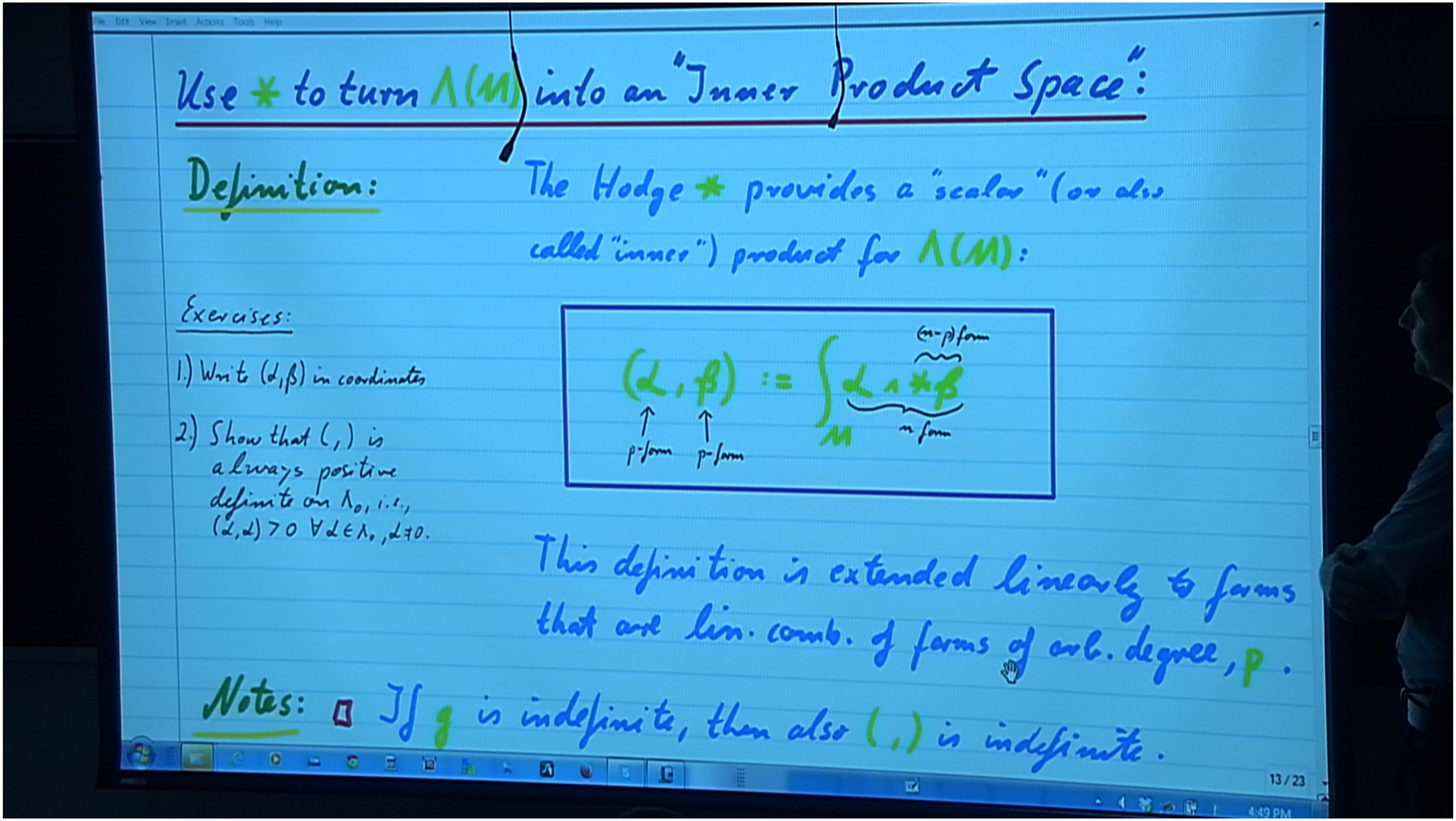
$$\boxed{\underset{\text{p-form}}{\underset{\uparrow}{(\alpha}}, \underset{\text{p-form}}{\underset{\uparrow}{\beta)}} := \int_M \underbrace{\alpha \wedge \overbrace{*\beta}^{(n-p)\,\text{form}}}_{n\text{ form}}}$$

Exercises:

1.) Write (α, β) in coordinates

2.) Show that $(,)$ is always positive definite on Λ_0, i.e., $(\alpha, \alpha) > 0 \ \forall \alpha \in \Lambda_0, \alpha \neq 0$.

This definition is extended linearly to forms



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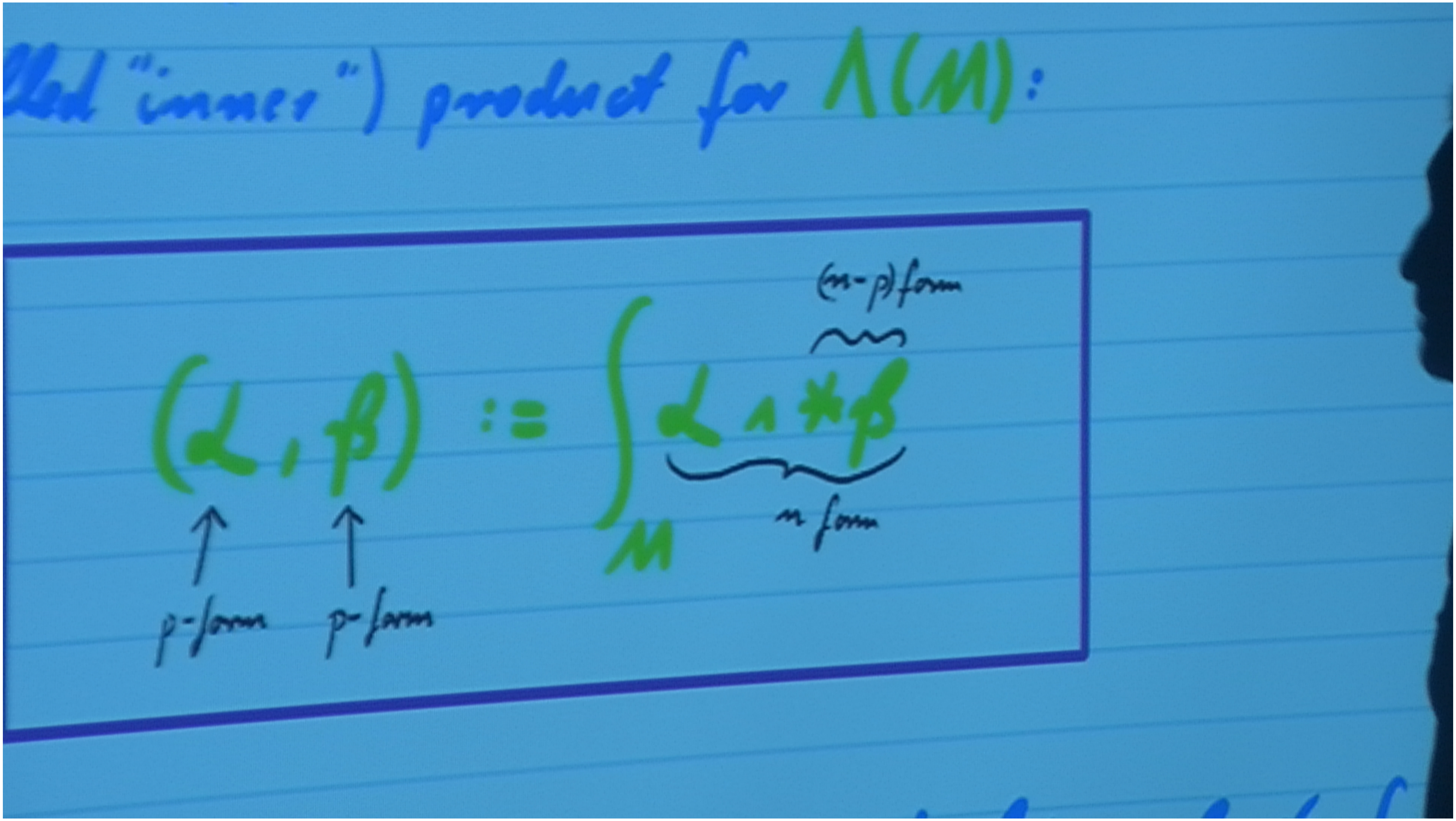
This definition is extended linearly to forms that are lin. comb. of forms of arb. degree, p.

Notes: ☐ If g is indefinite, then also $(\,,\,)$ is indefinite.



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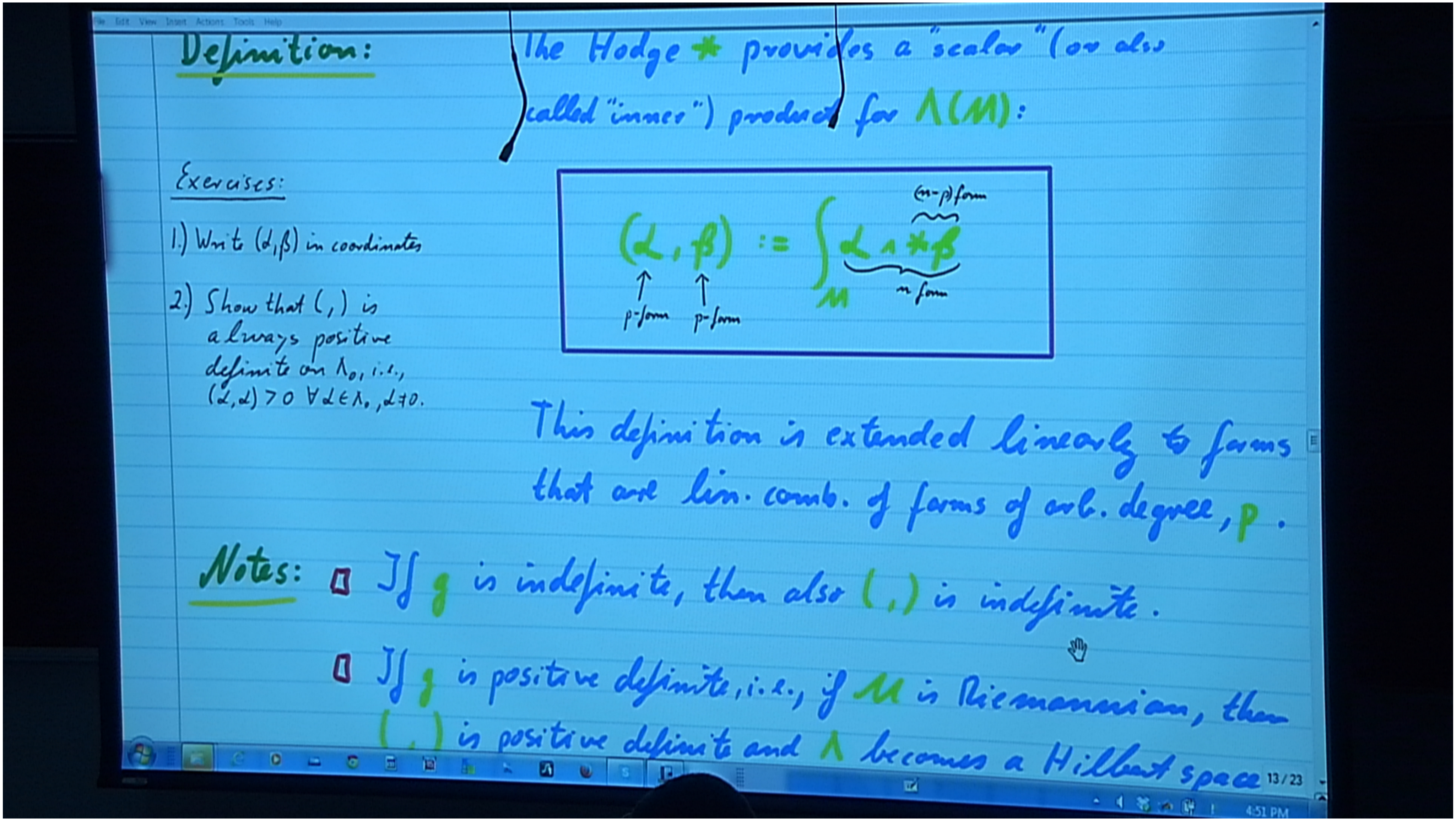

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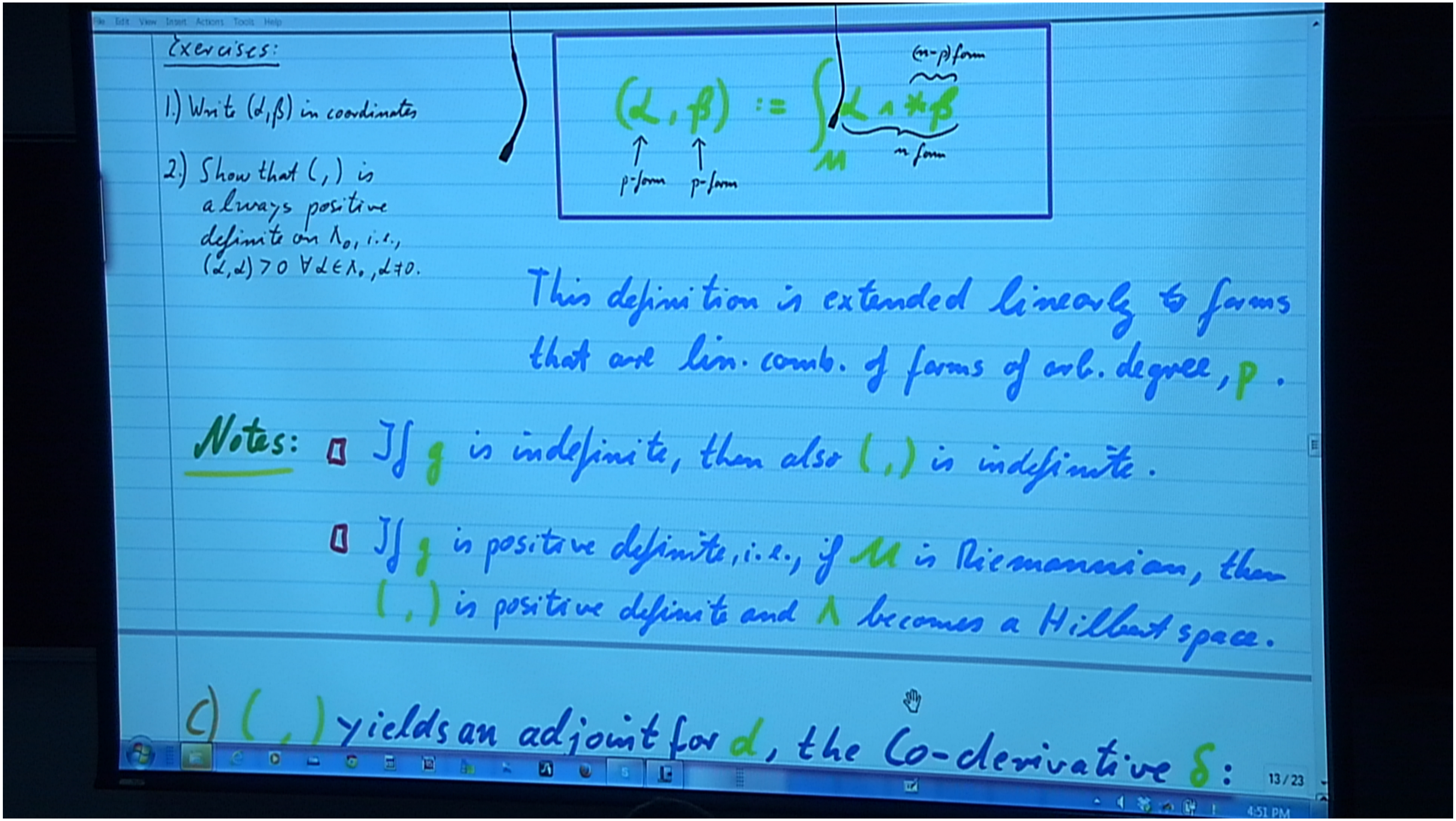

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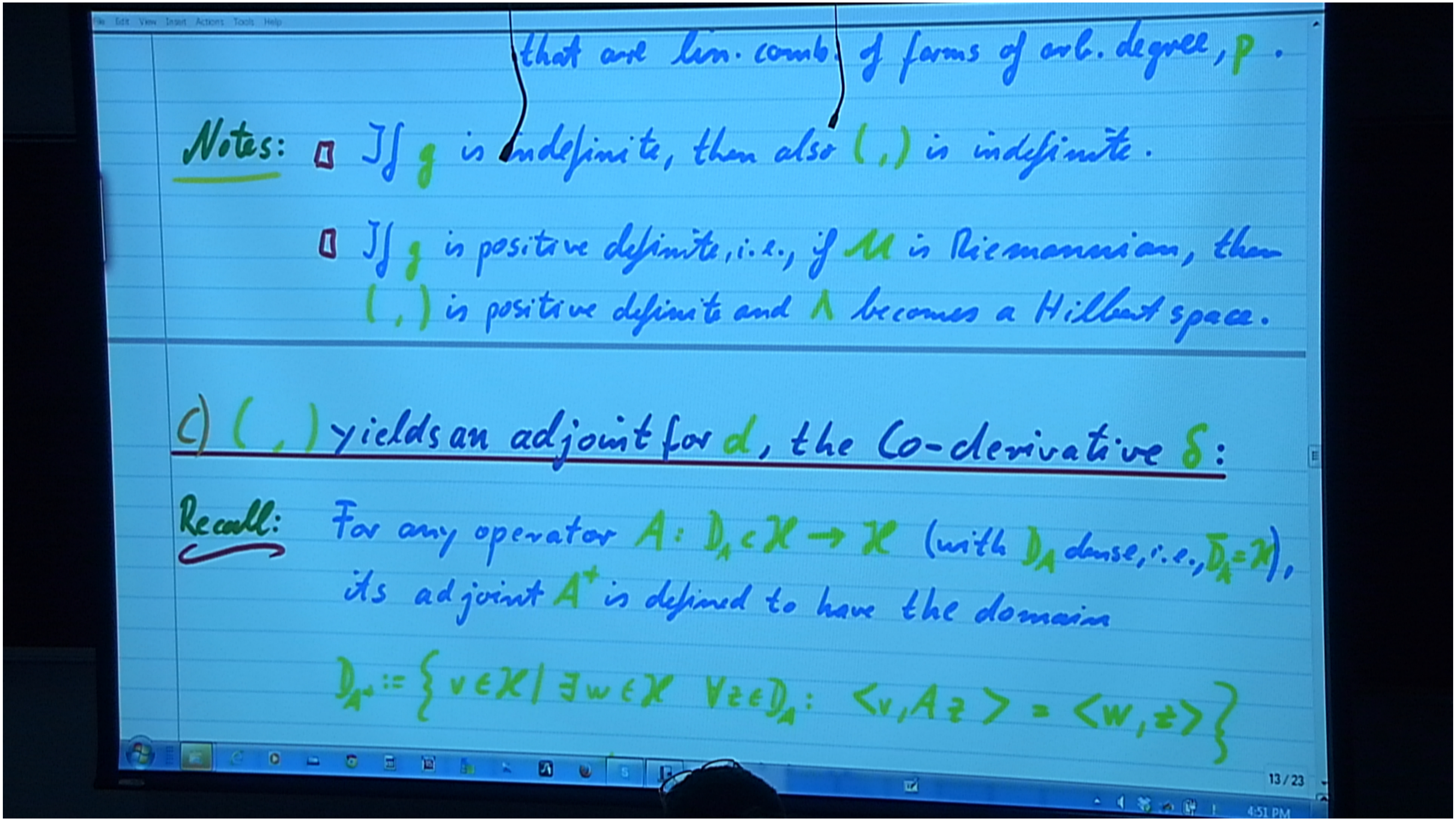

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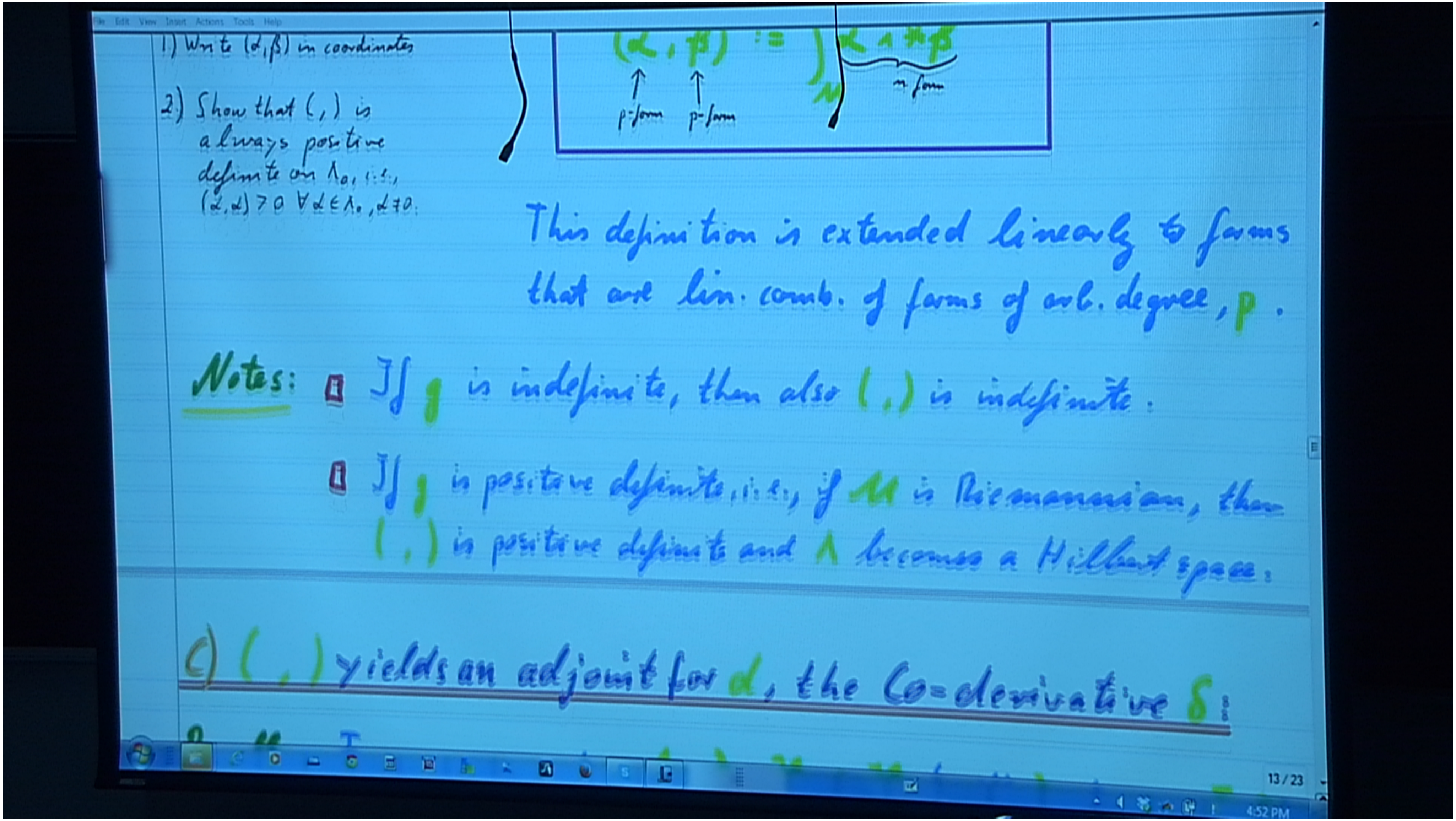
Recall: For any operator $A: D_A \subset \mathcal{H} \to \mathcal{H}$ (with D_A dense, i.e., $\overline{D_A} = \mathcal{H}$), its adjoint A^+ is defined to have the domain

$$D_{A^+} := \{ v \in \mathcal{H} \mid \exists w \in \mathcal{H} \ \forall z \in D_A : \langle v, Az \rangle = \langle w, z \rangle \}$$



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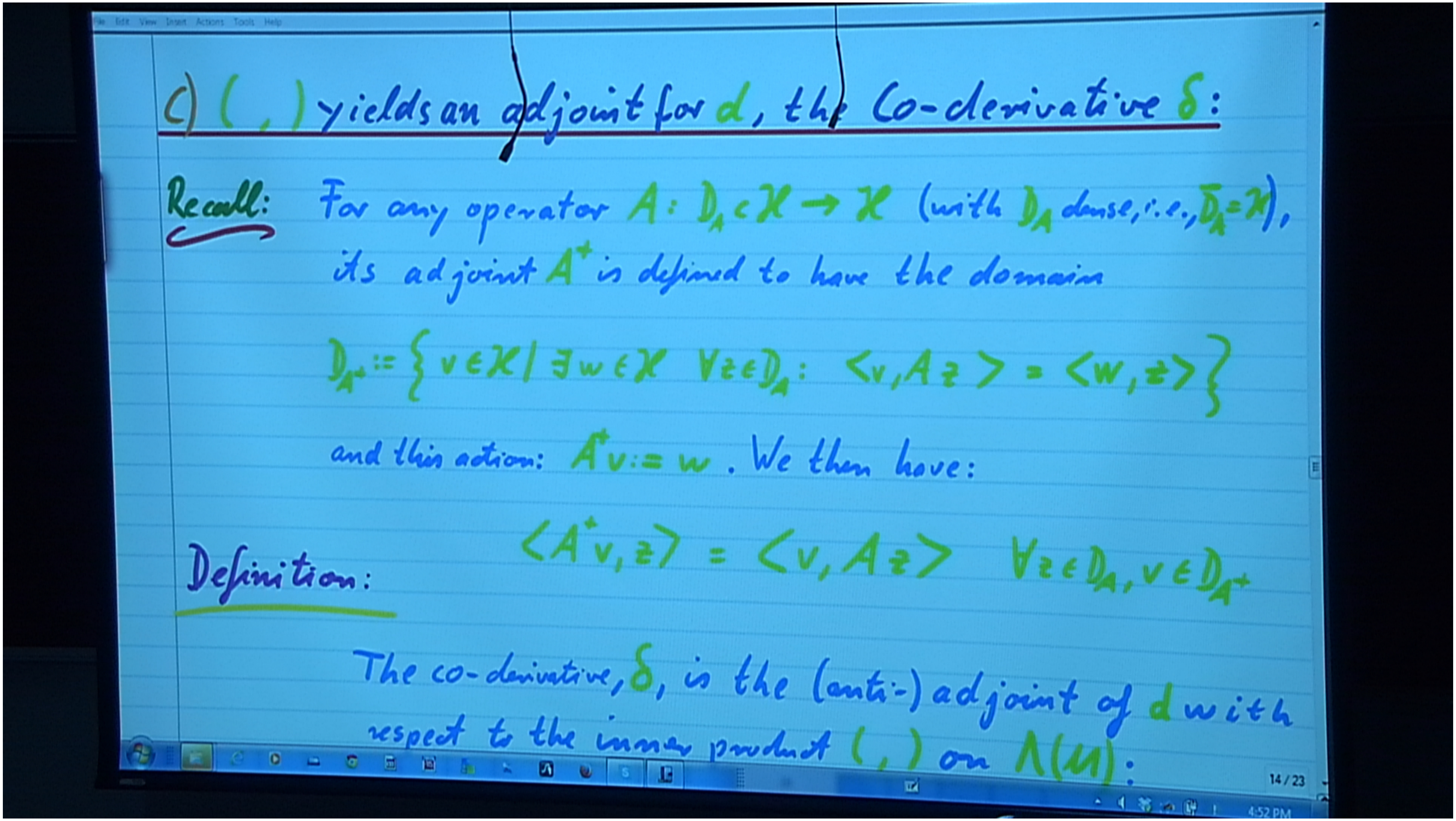

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Recall: For any operator $A : D_A \subset \mathcal{H} \to \mathcal{H}$ (with D_A dense, i.e., $\overline{D_A} = \mathcal{H}$), its adjoint A^+ is defined to have the domain

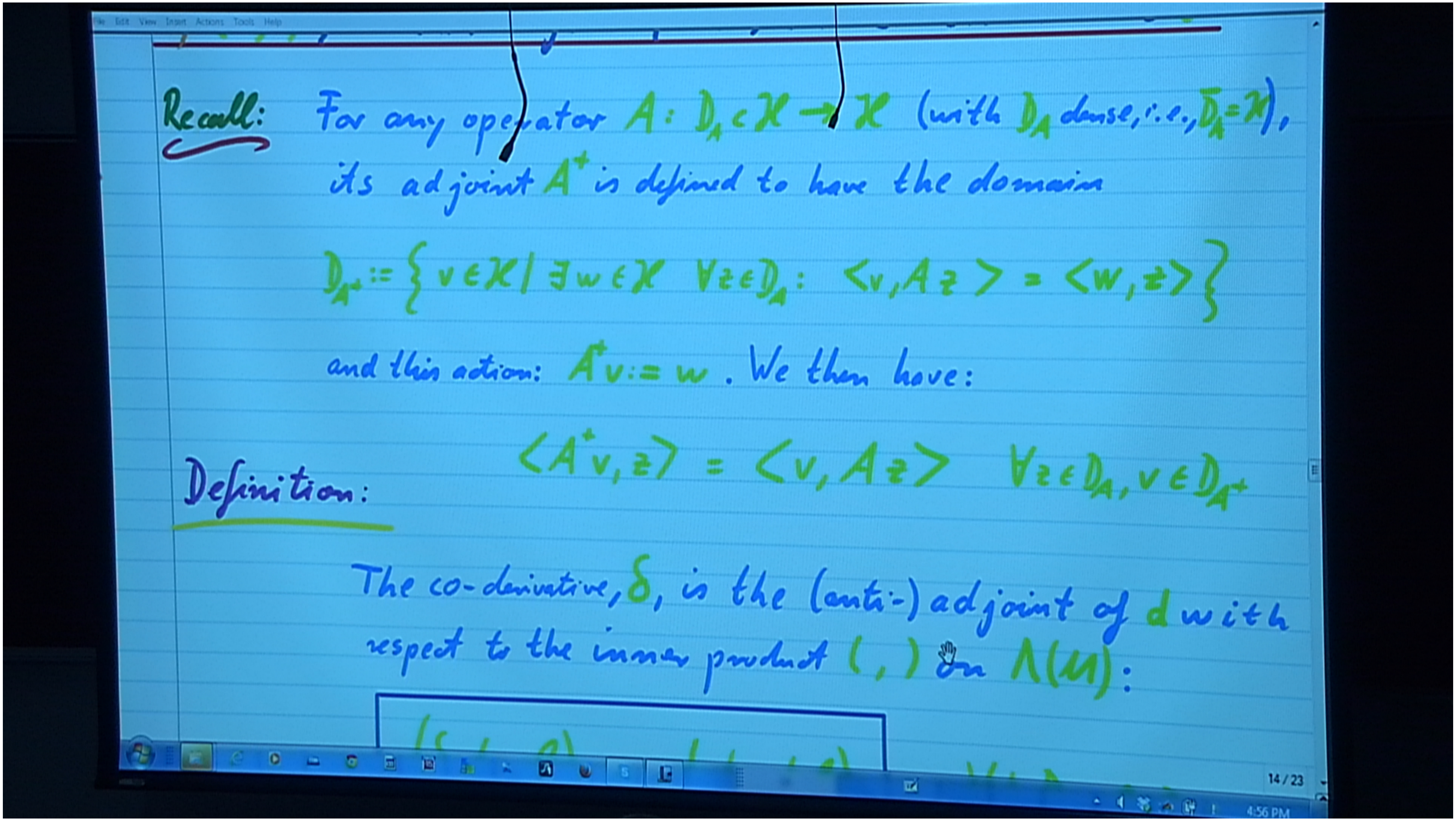
$$D_{A^+} := \{ v \in \mathcal{H} \mid \exists w \in \mathcal{H} \ \forall z \in D_A : \ \langle v, Az \rangle = \langle w, z \rangle \}$$

and this action: $A^+ v := w$. We then have:

$$\langle A^+ v, z \rangle = \langle v, Az \rangle \quad \forall z \in D_A, v \in D_{A^+}$$

Definition:

The co-derivative, δ, is the (anti-) adjoint of d with respect to the inner product (,) on $\Lambda(M)$:



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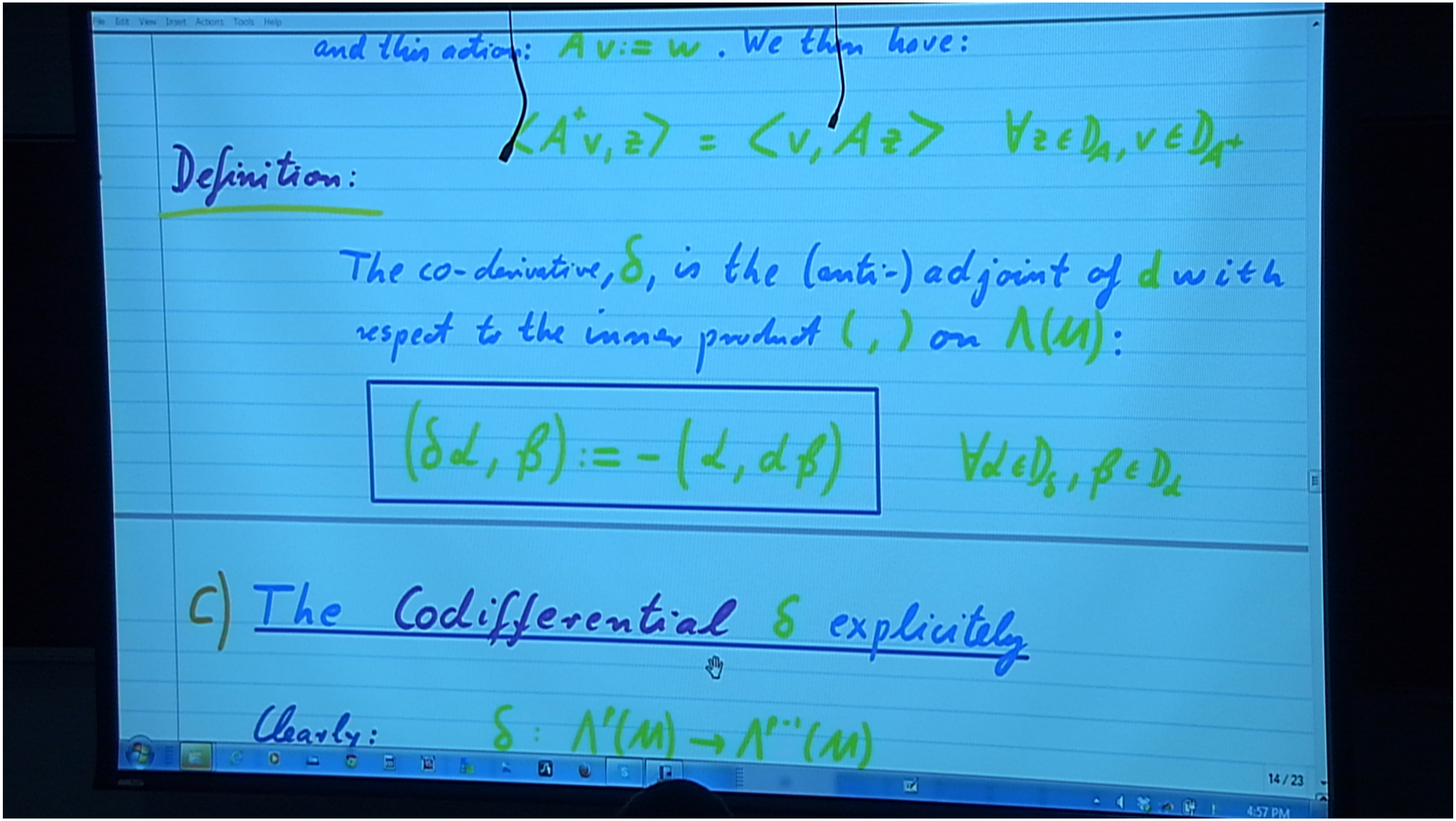

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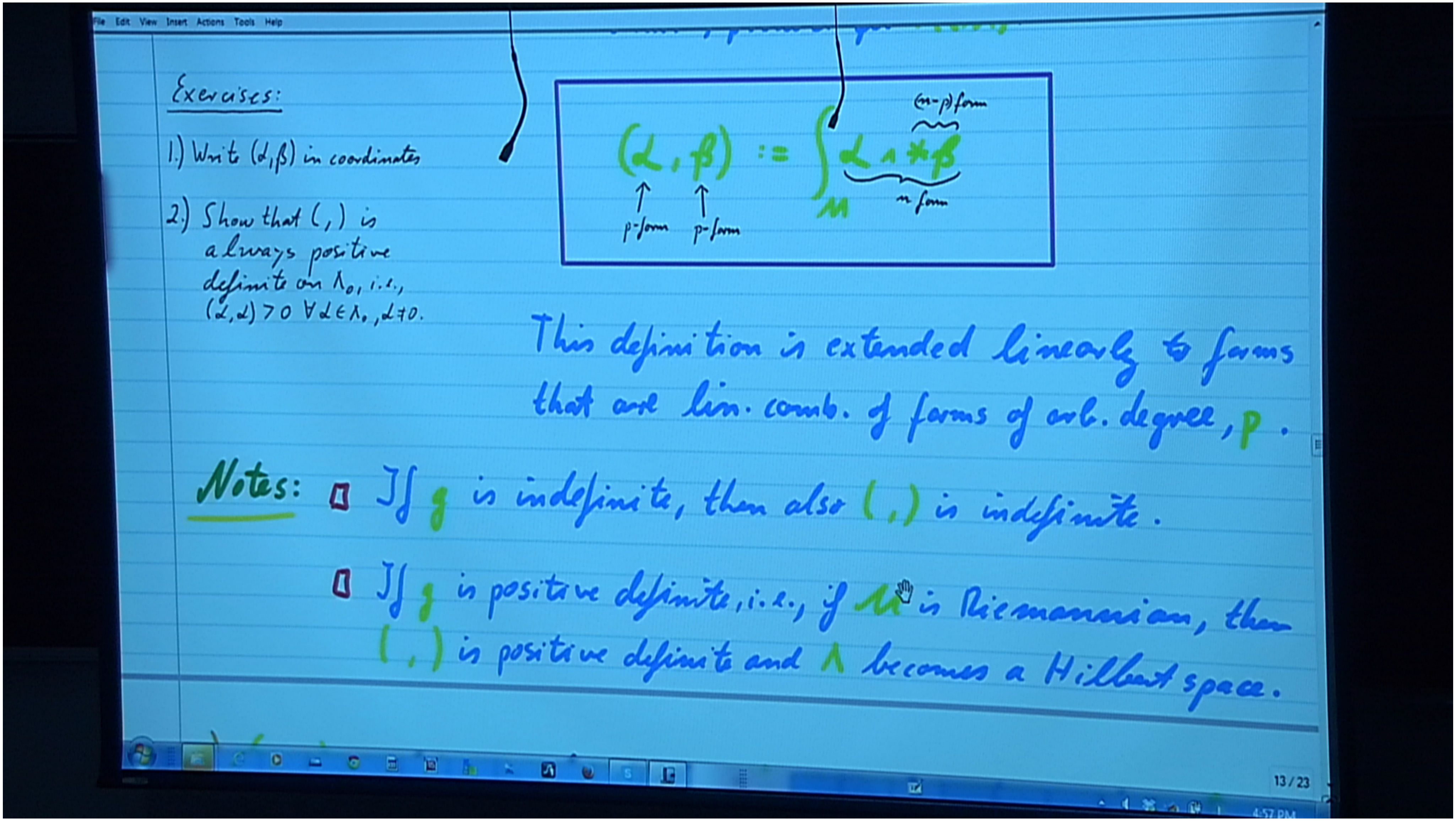
$$\boxed{(\delta\alpha, \beta) := -(\alpha, d\beta)} \quad \forall \alpha \in D_\delta, \beta \in D_d$$

c) The Codifferential δ explicitly

Clearly: $\delta : \Lambda^{r}(M) \to \Lambda^{r-1}(M)$



Exercises:

1.) Write (α,β) in coordinates

2.) Show that $(,)$ is always positive definite on Λ_0, i.e., $(\alpha,\alpha) > 0 \ \forall \alpha \in \Lambda_0, \alpha \neq 0$.

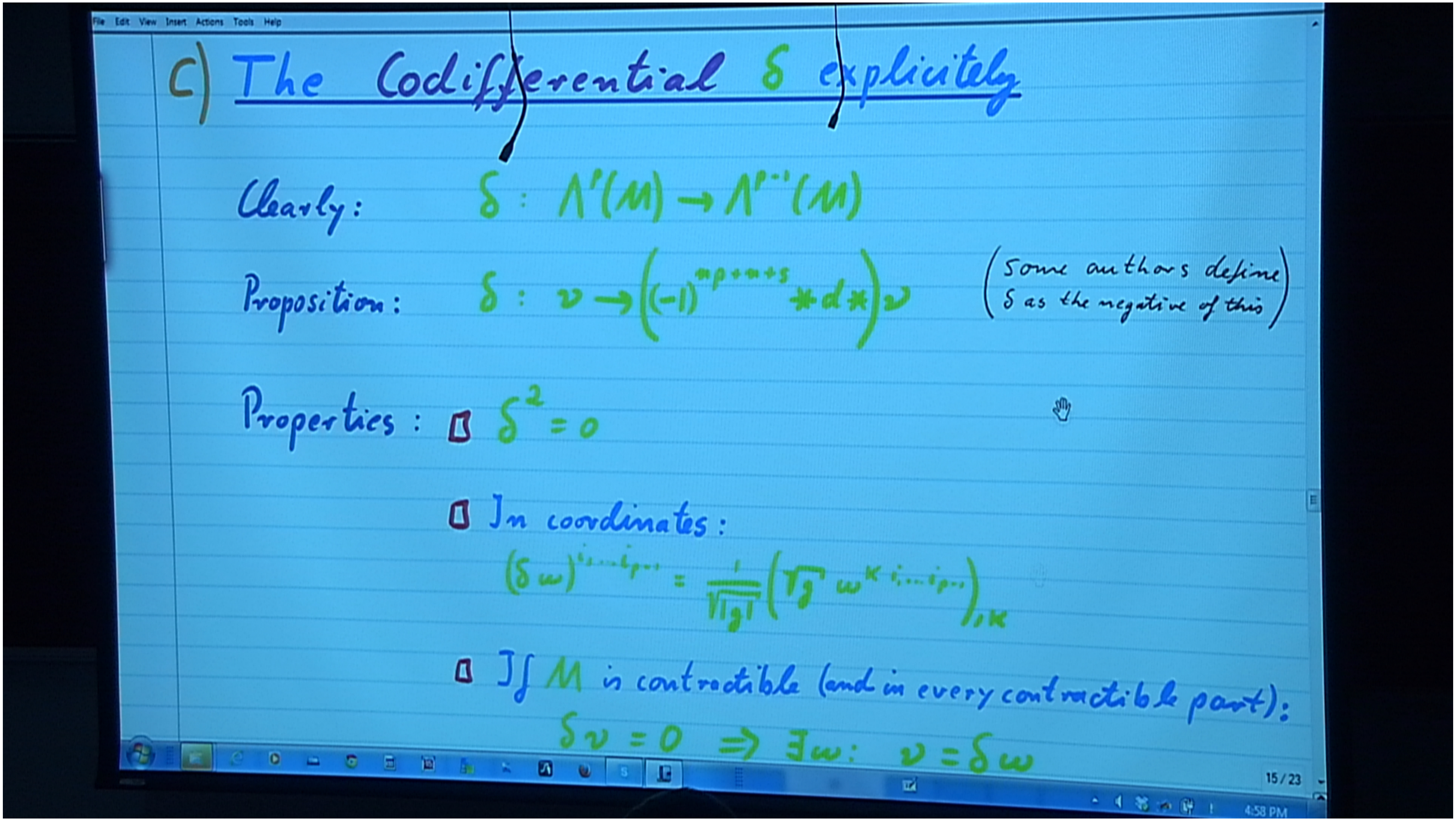
$$\boxed{(\underset{\substack{\uparrow\\ p\text{-form}}}{\alpha},\underset{\substack{\uparrow\\ p\text{-form}}}{\beta}) := \int_M \underbrace{\alpha\wedge\overbrace{*\beta}^{(n-p)\text{ form}}}_{n\text{ form}}}$$

This definition is extended linearly to forms that are lin. comb. of forms of arb. degree, p.

Notes:

☐ If g is indefinite, then also $(,)$ is indefinite.

☐ If g is positive definite, i.e., if M is Riemannian, then $(,)$ is positive definite and Λ becomes a Hilbert space.



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Clearly: $\delta : \Lambda^p(M) \to \Lambda^{p-1}(M)$

Proposition: $\delta : v \to \left((-1)^{mp+m+s} * d *\right) v$

(Some authors define δ as the negative of this)

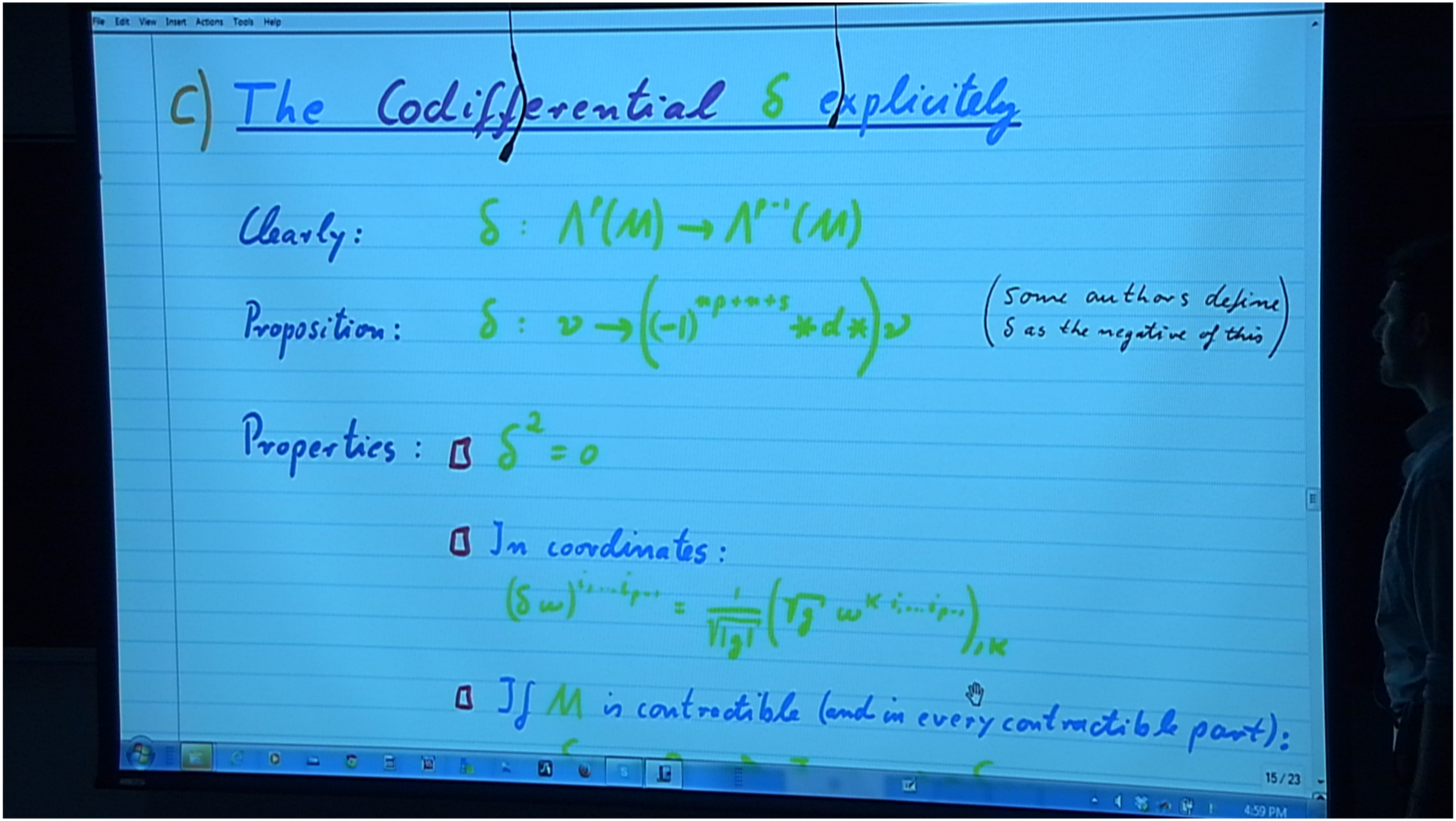
Properties:

- ☐ $\delta^2 = 0$
- ☐ In coordinates:

$$(\delta \omega)^{i_1 \dots i_{p-1}} = \frac{1}{\sqrt{|g|}} \left(\sqrt{g}\, \omega^{k\, i_1 \dots i_{p-1}}\right)_{,k}$$

- ☐ If M is contractible (and in every contractible part):

$$\delta v = 0 \Rightarrow \exists \omega : \; v = \delta \omega$$



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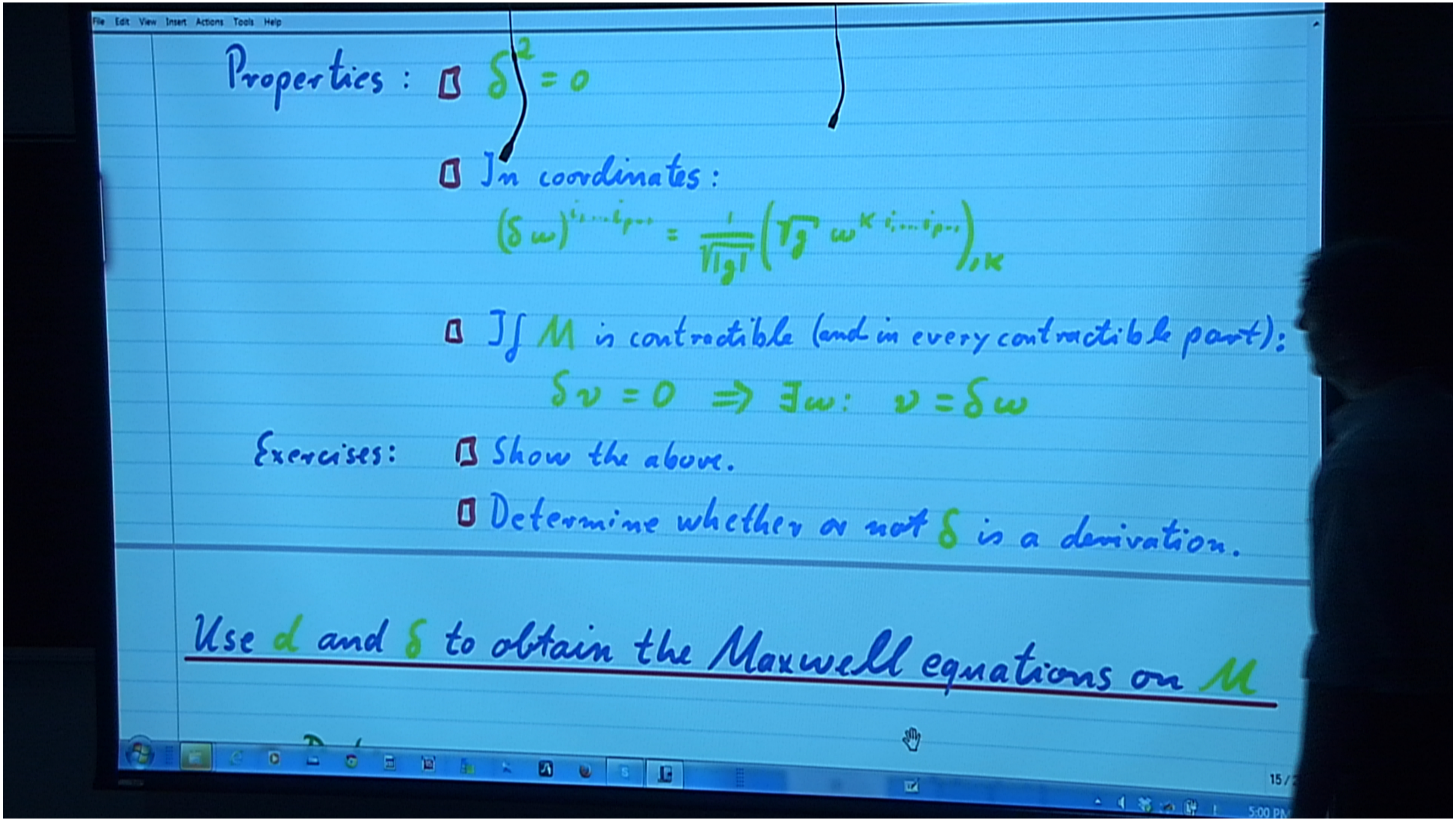

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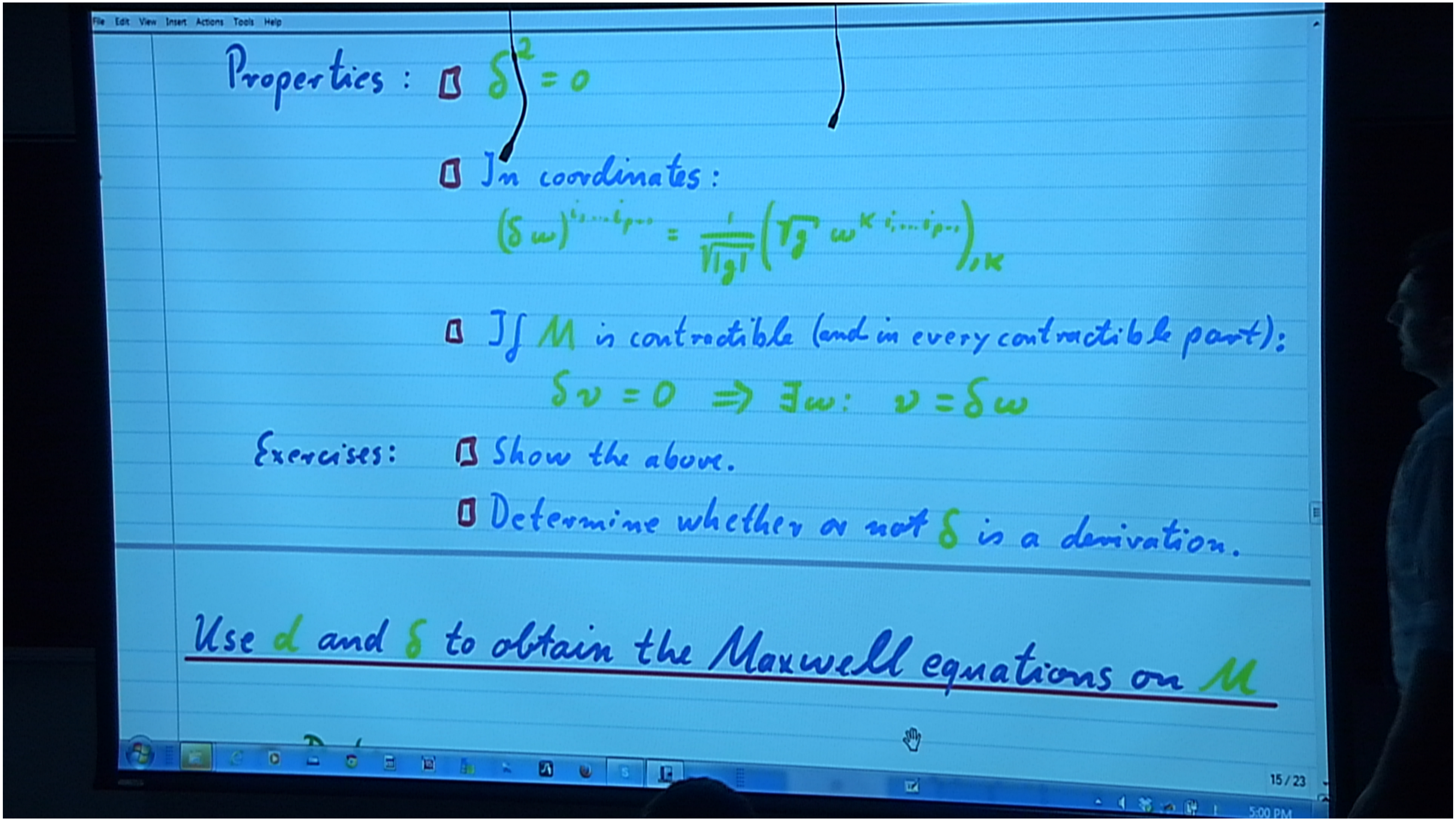
☐ Determine whether or not δ is a derivation.

Use d and δ to obtain the Maxwell equations on M



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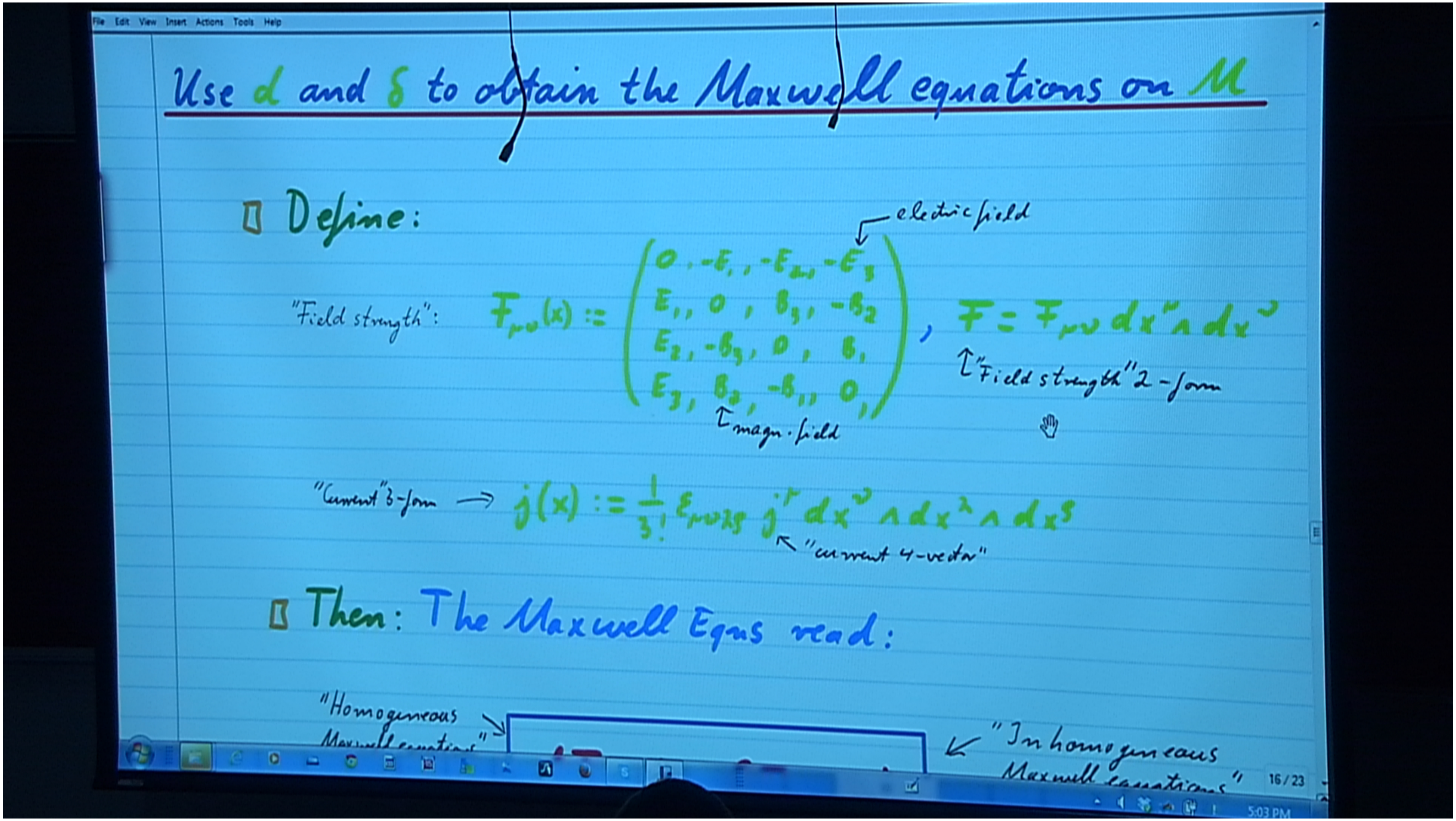


Use d and δ to obtain the Maxwell equations on $\mathcal{M}$

- Define:

"Field strength": $F_{\mu\nu}(x) := \begin{pmatrix} 0, & -E_1, & -E_2, & -E_3 \\ E_1, & 0, & B_3, & -B_2 \\ E_2, & -B_3, & 0, & B_1 \\ E_3, & B_2, & -B_1, & 0 \end{pmatrix}$, $F = F_{\mu\nu}\, dx^\mu \wedge dx^\nu$

electric field

magn. field

"Field strength" 2-form

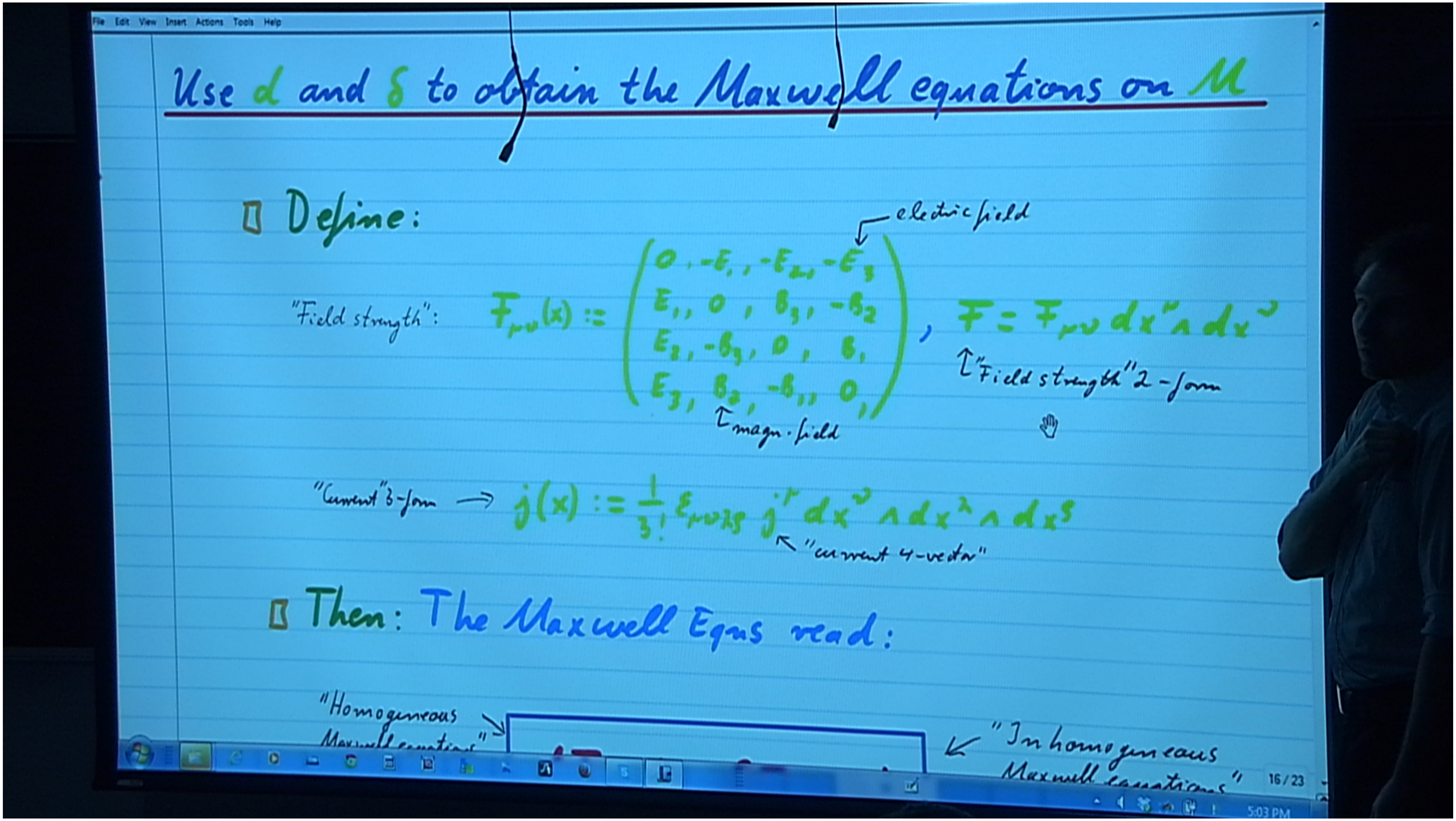
"Current" 3-form → $j(x) := \frac{1}{3!}\, \varepsilon_{\mu\nu\lambda\rho}\, j^\mu\, dx^\nu \wedge dx^\lambda \wedge dx^\rho$

"current 4-vector"

- Then: The Maxwell Eqns read:

"Homogeneous Maxwell equations"

"Inhomogeneous Maxwell equations"



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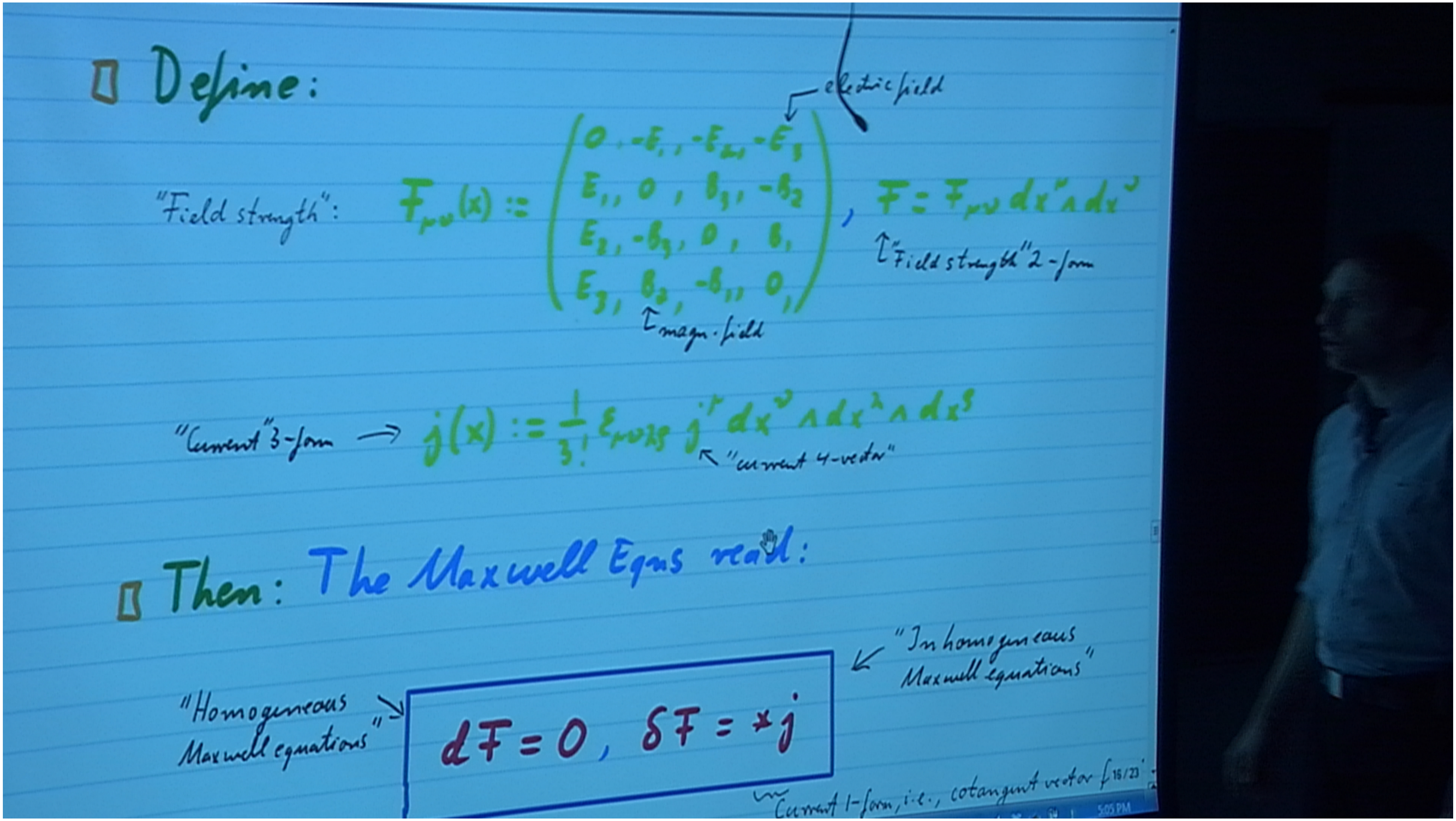

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"Homogeneous Maxwell equations" → $\boxed{dF = 0, \quad \delta F = {}^{*}j}$ ← "Inhomogeneous Maxwell equations"

Current 1-form, i.e., cotangent vector f

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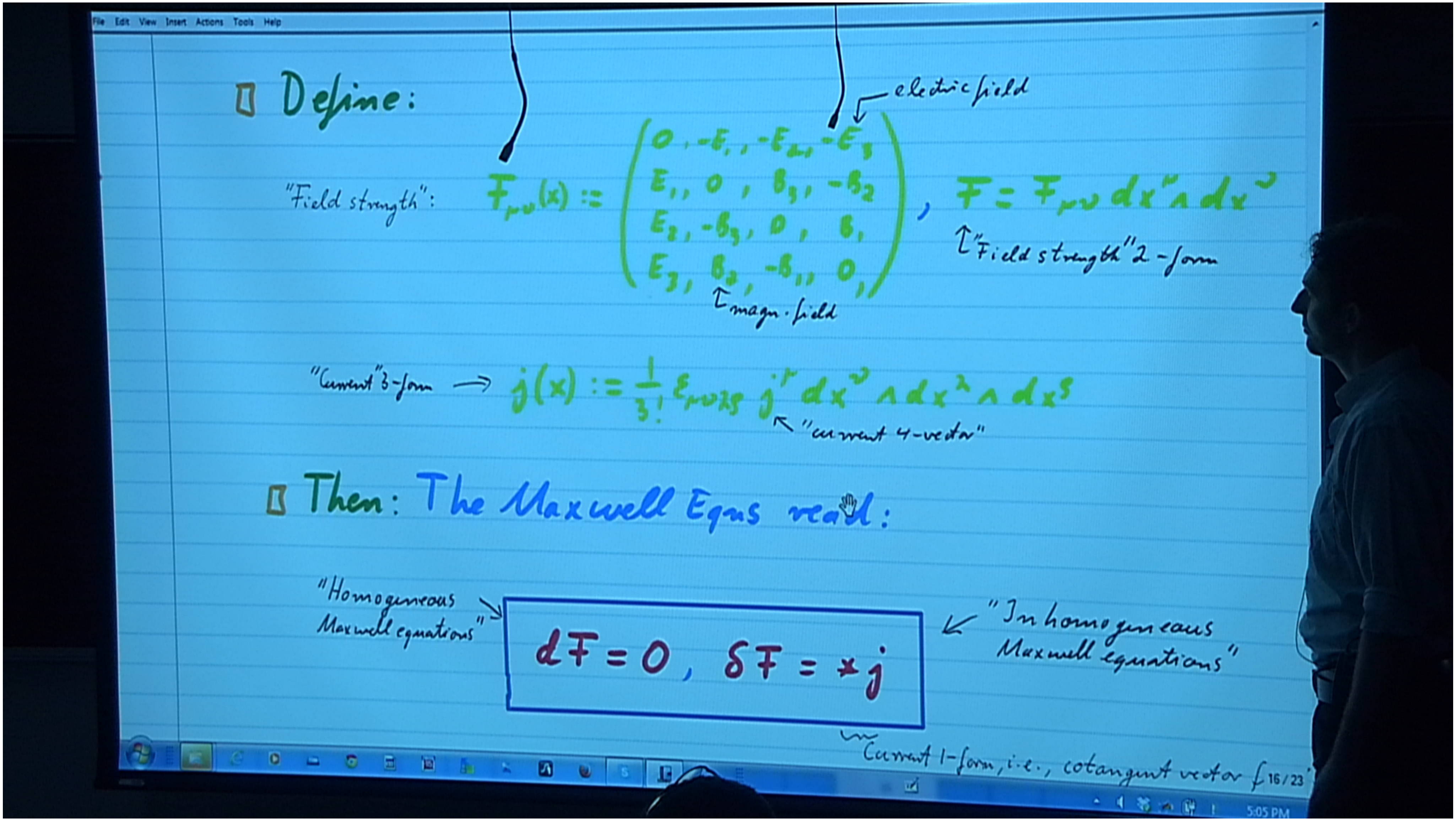

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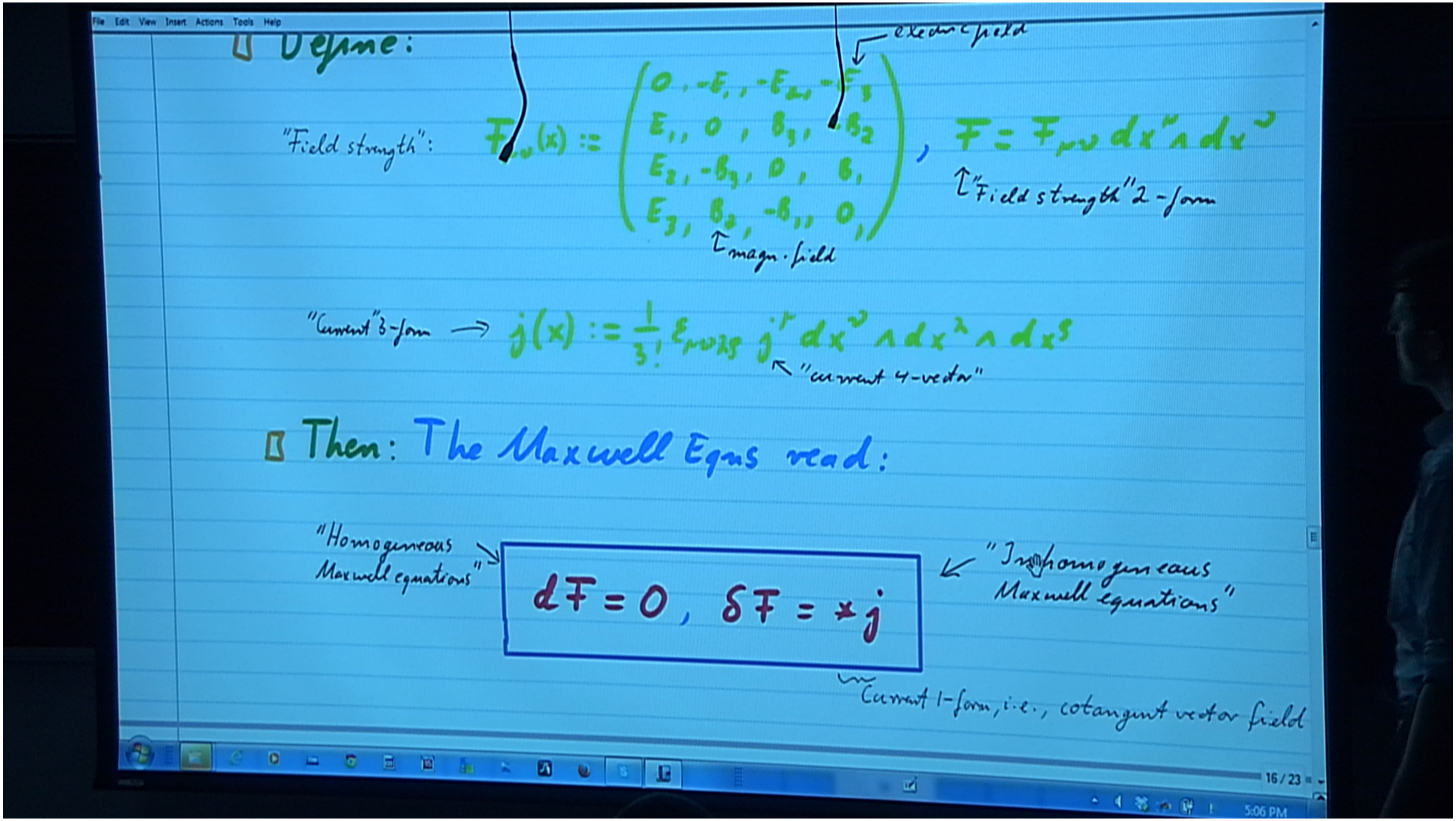

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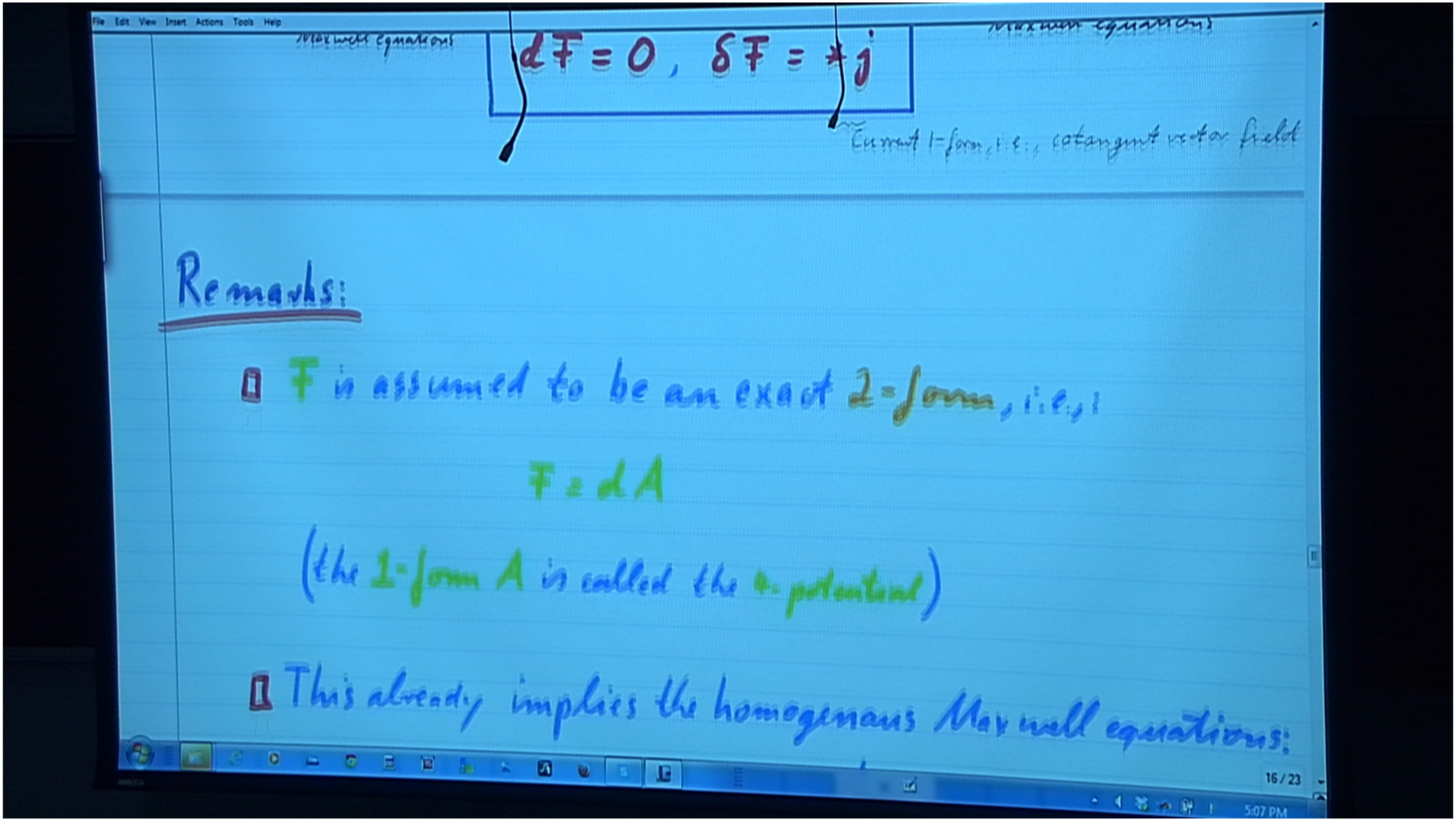

Maxwell equations

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Maxwell equations

Current 1-form, i.e., cotangent vector field

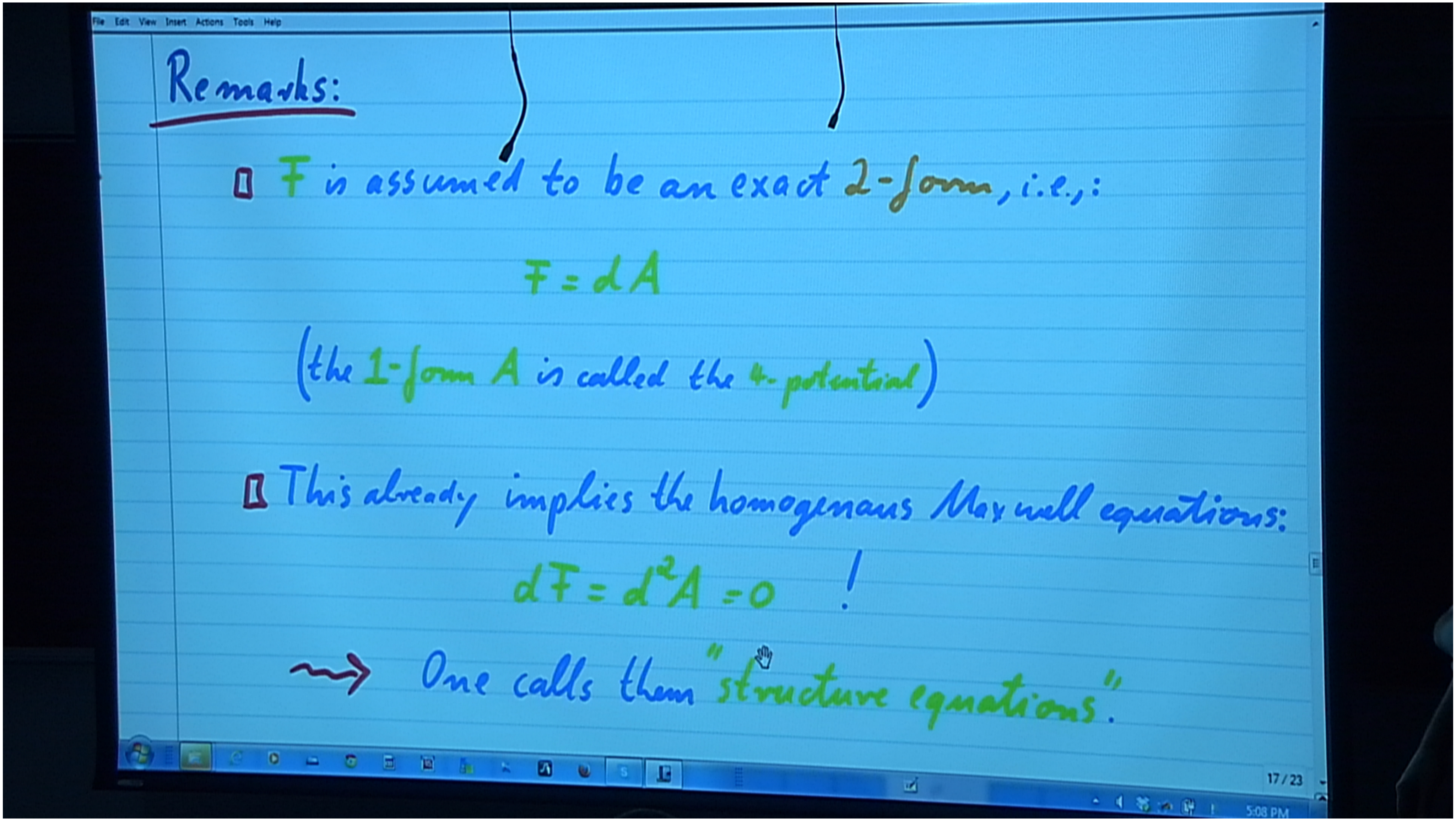
Remarks:

- F is assumed to be an exact 2-form, i.e.,

$$F = dA$$

(the 1-form A is called the 4-potential)

- This already implies the homogenous Maxwell equations:



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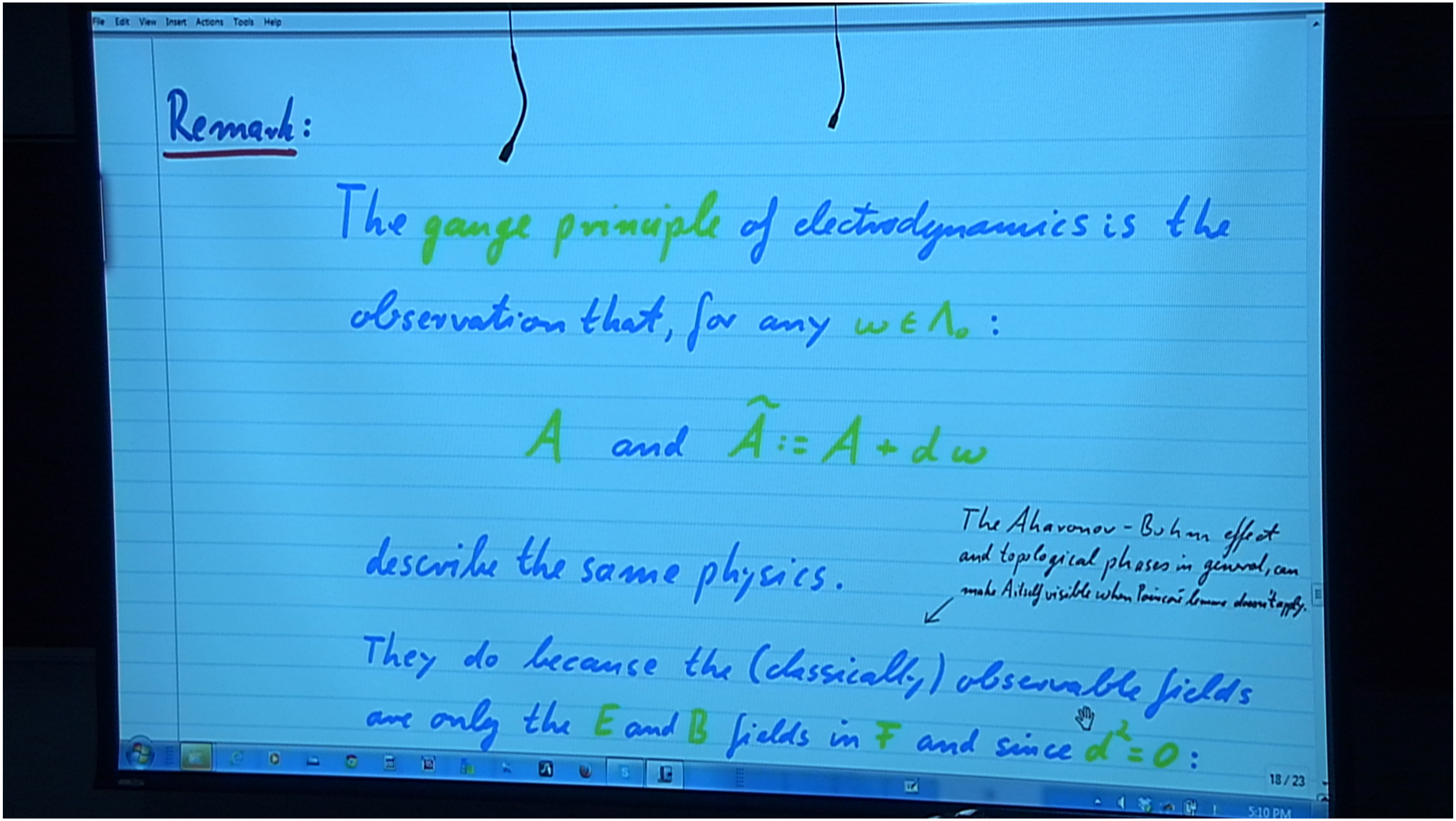
$$dF = d^2 A = 0 \quad !$$

⇝ One calls them "structure equations".



Remark:

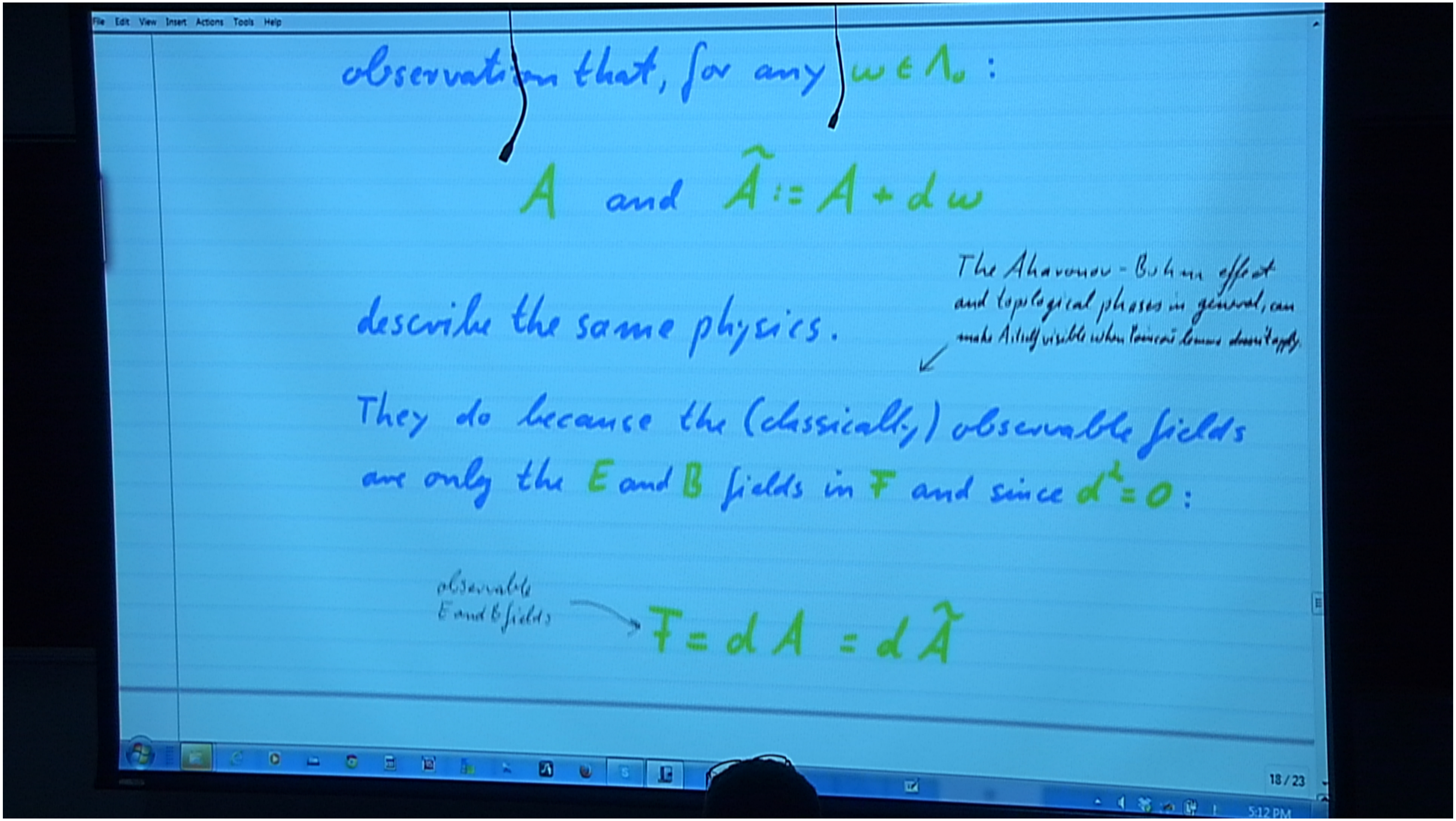
The gauge principle of electrodynamics is the observation that, for any $\omega \in \Lambda_0$:

$$A \text{ and } \tilde{A} := A + d\omega$$

describe the same physics.

The Aharonov - Bohm effect and topological phases in general, can make A itself visible when Poincaré lemma doesn't apply.

They do because the (classically) observable fields are only the E and B fields in F and since $d^2 = 0$:



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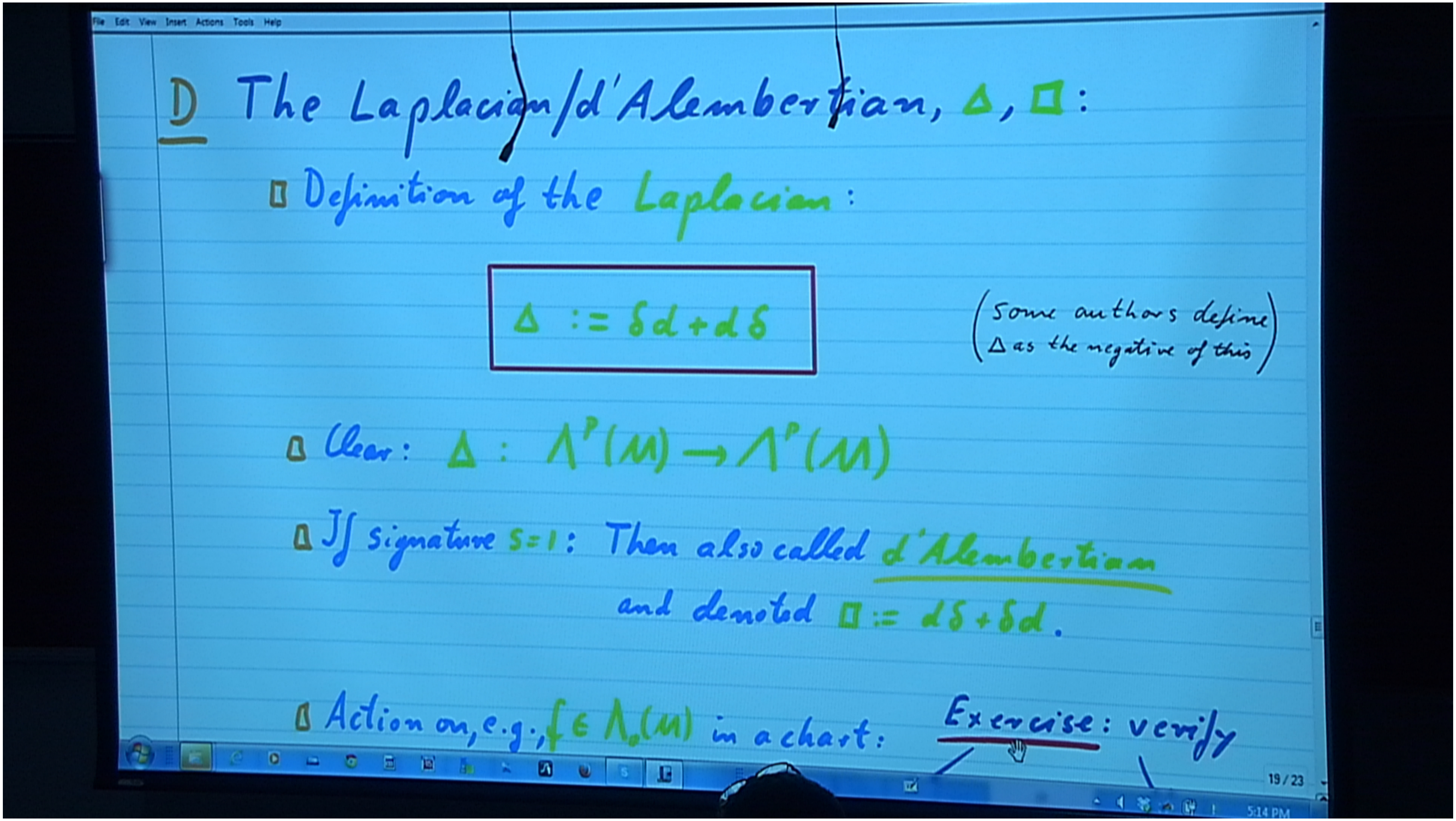
observable E and B fields

$$\xrightarrow{} F = dA = d\tilde{A}$$



D The Laplacian/d'Alembertian, Δ, $\Box$:

- Definition of the Laplacian:

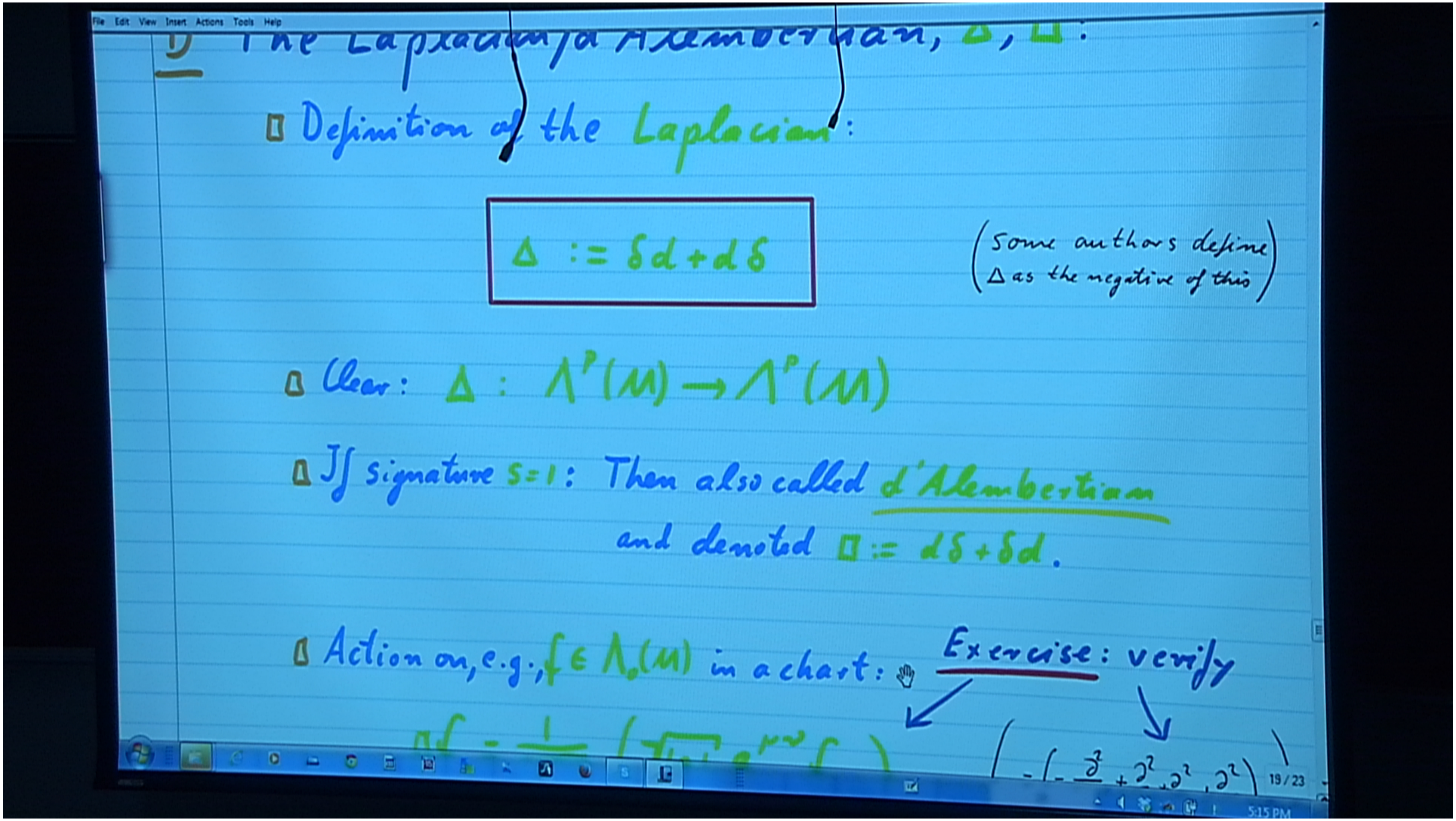
$$\boxed{\Delta := \delta d + d\delta}$$

(Some authors define Δ as the negative of this)

- Clear: $\Delta : \Lambda^p(M) \to \Lambda^p(M)$

- If signature $s=1$: Then also called d'Alembertian and denoted $\Box := d\delta + \delta d$.

- Action on, e.g., $f \in \Lambda_0(M)$ in a chart: Exercise: verify



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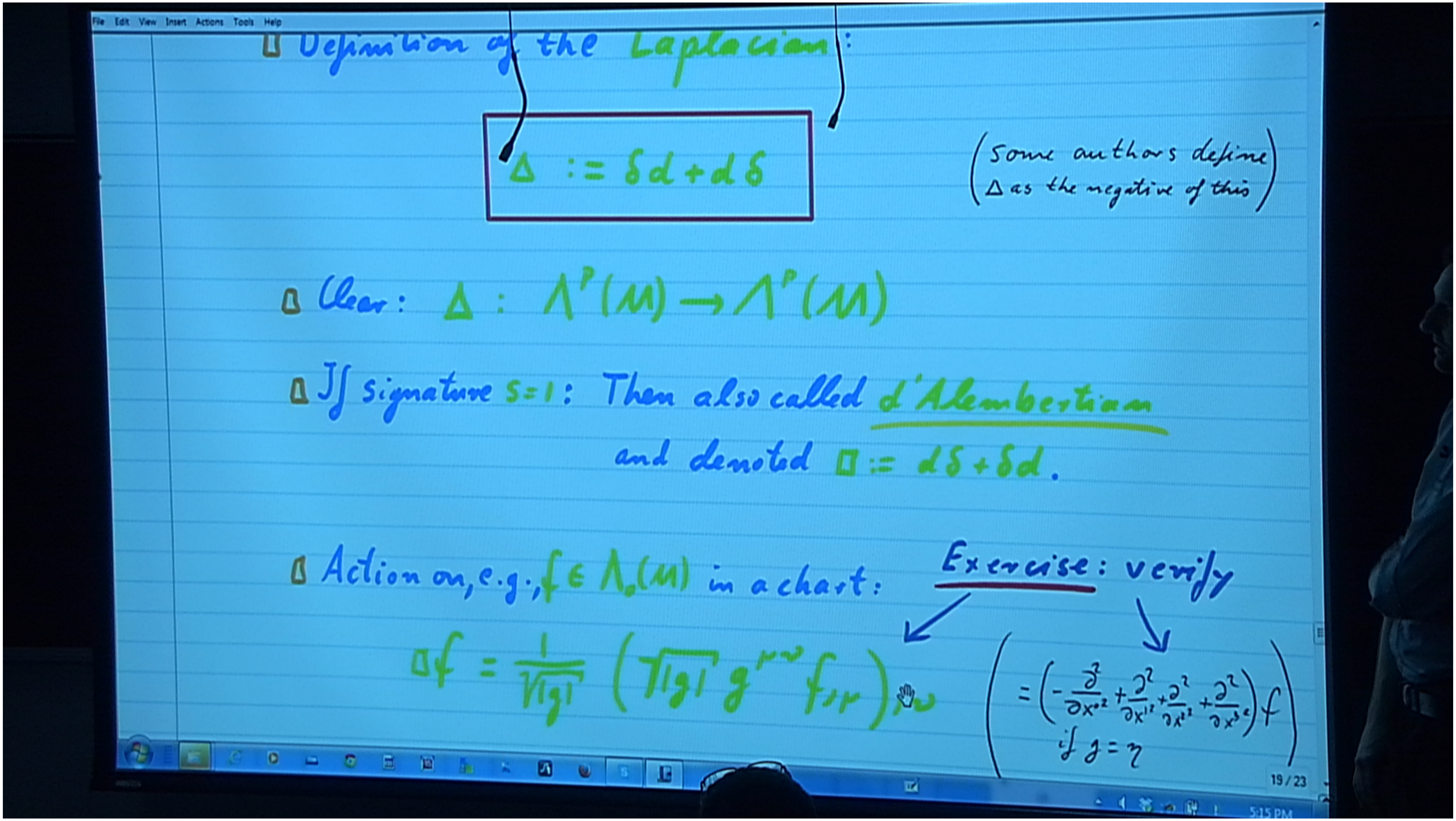

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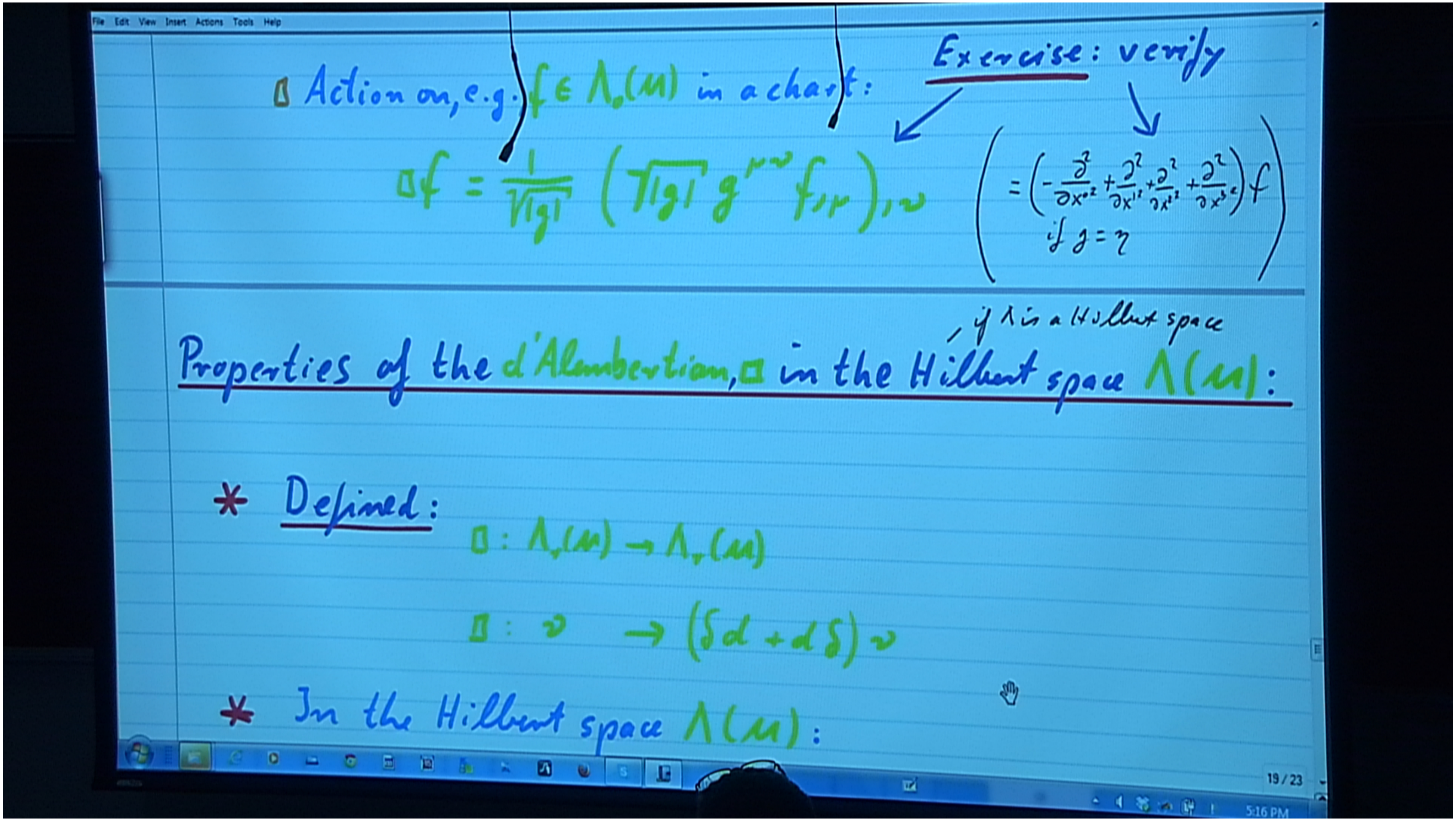

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Properties of the d'Alembertian, □ in the Hilbert space $\Lambda(M)$:

, if Λ is a Hilbert space

* **Defined:**

$$\Box : \Lambda_r(M) \to \Lambda_r(M)$$

$$\Box : \omega \to (\delta d + d\delta)\omega$$

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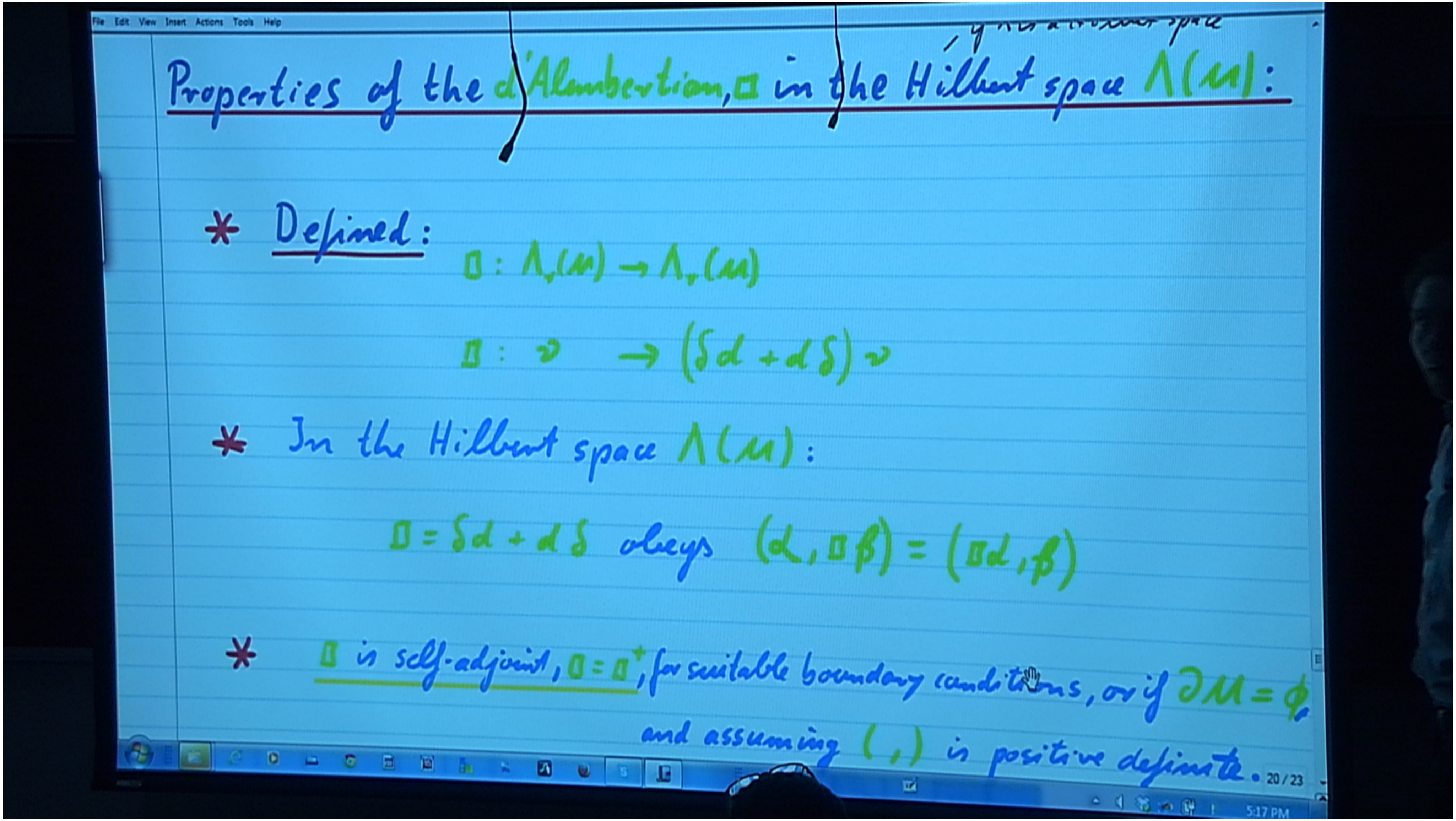

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Properties of the d'Alembertian, $\Box$ in the Hilbert space $\Lambda(M)$:

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* In the Hilbert space $\Lambda(M)$:

$\Box = \delta d + d\delta$ obeys $(\alpha, \Box\beta) = (\Box\alpha, \beta)$

* $\Box$ is self-adjoint, $\Box = \Box^+$, for suitable boundary conditions, or if $\partial M = \phi$, and assuming $(\,,\,)$ is positive definite.

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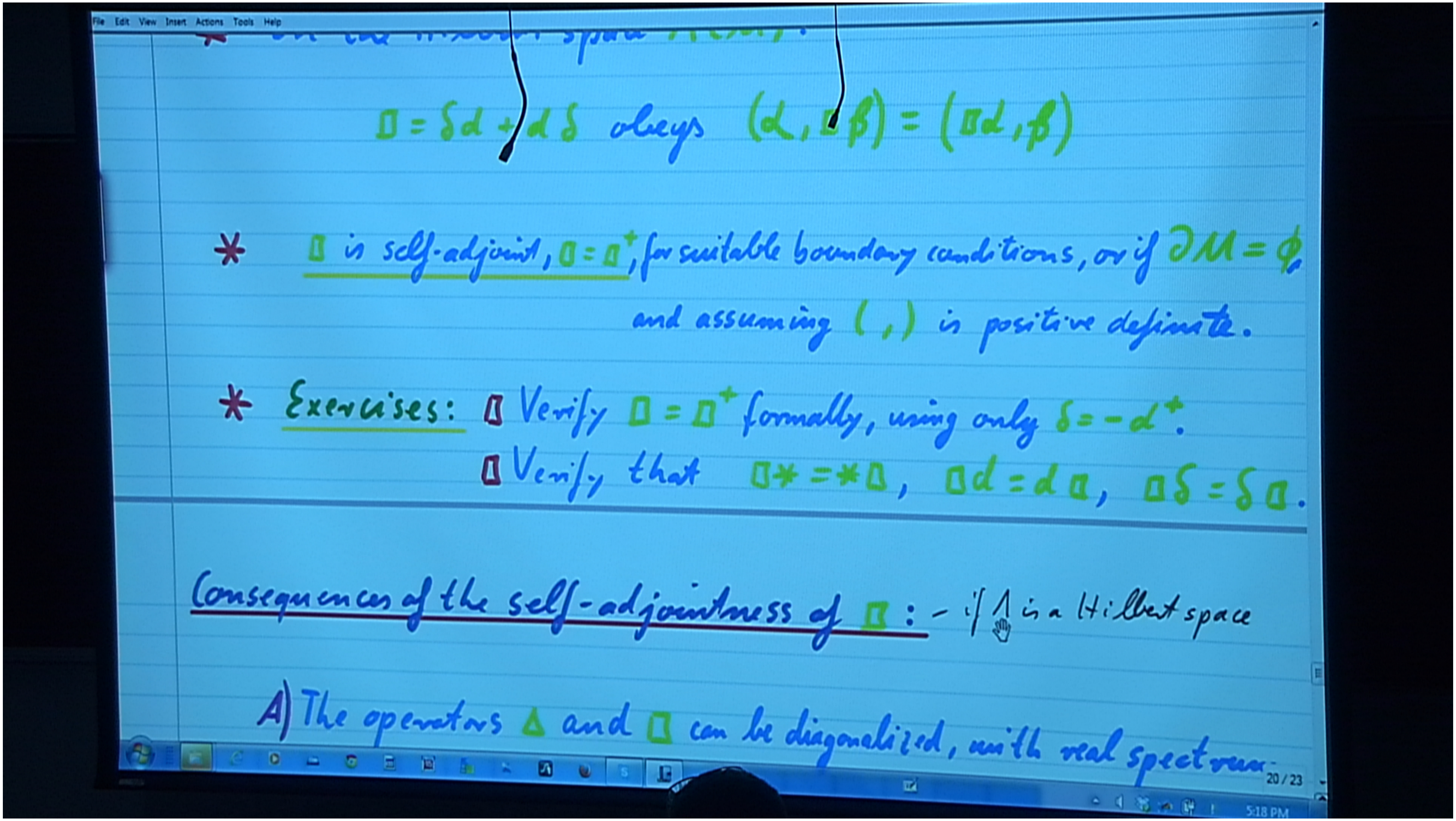

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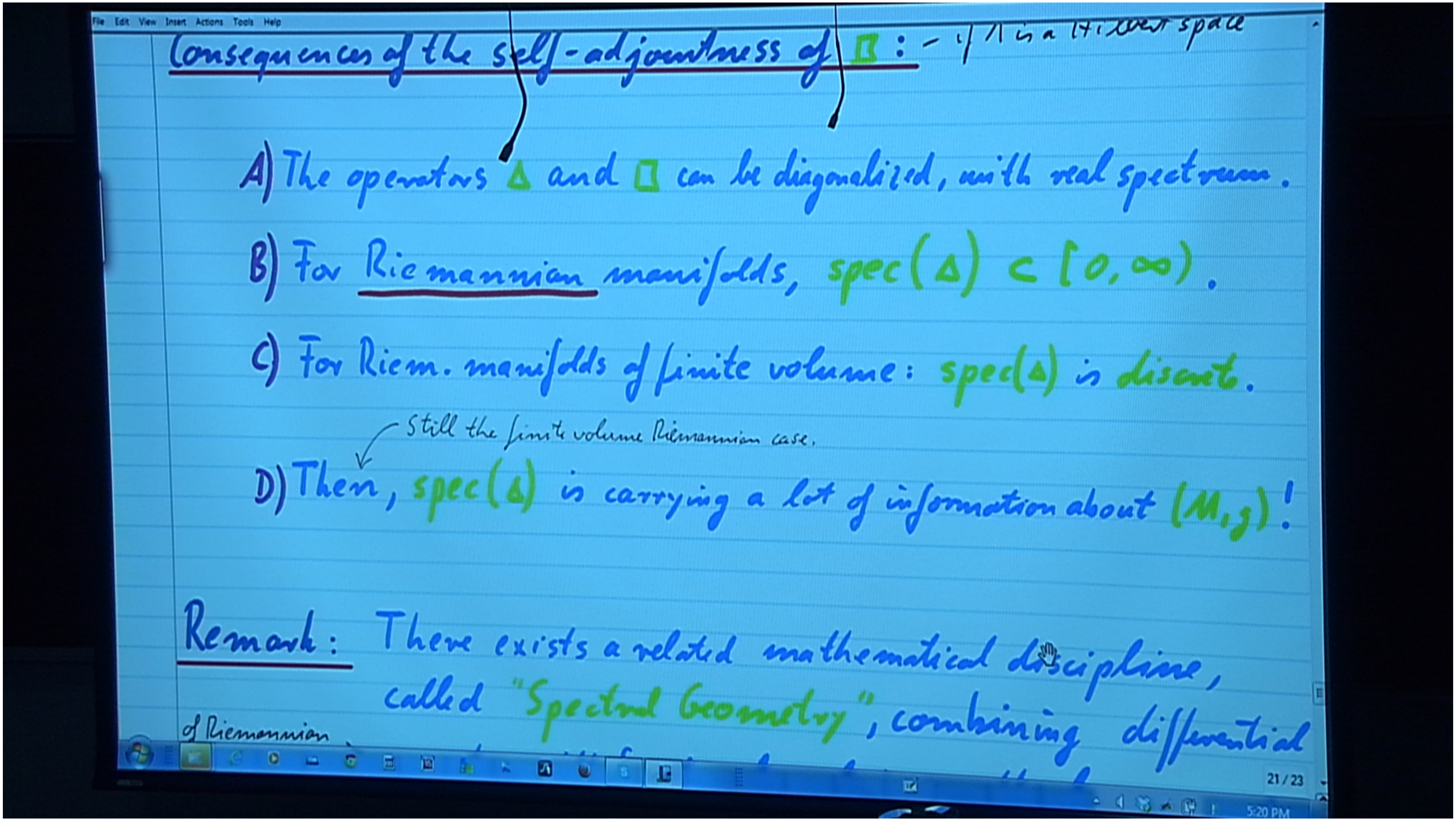
* Exercises: ▯ Verify $\Box = \Box^+$ formally, using only $\delta = -d^+$.
▯ Verify that $\Box * = * \Box$, $\Box d = d \Box$, $\Box \delta = \delta \Box$.

Consequences of the self-adjointness of $\Box$: - if Λ is a Hilbert space

A) The operators Δ and $\Box$ can be diagonalized, with real spectrum.



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A) The operators Δ and $\Box$ can be diagonalized, with real spectrum.

B) For Riemannian manifolds, $\mathrm{spec}(\Delta) \subset [0, \infty)$.

C) For Riem. manifolds of finite volume: $\mathrm{spec}(\Delta)$ is discrete.

Still the finite volume Riemannian case.

D) Then, $\mathrm{spec}(\Delta)$ is carrying a lot of information about (M, g)!

Remark: There exists a related mathematical discipline, called "Spectral Geometry", combining differential

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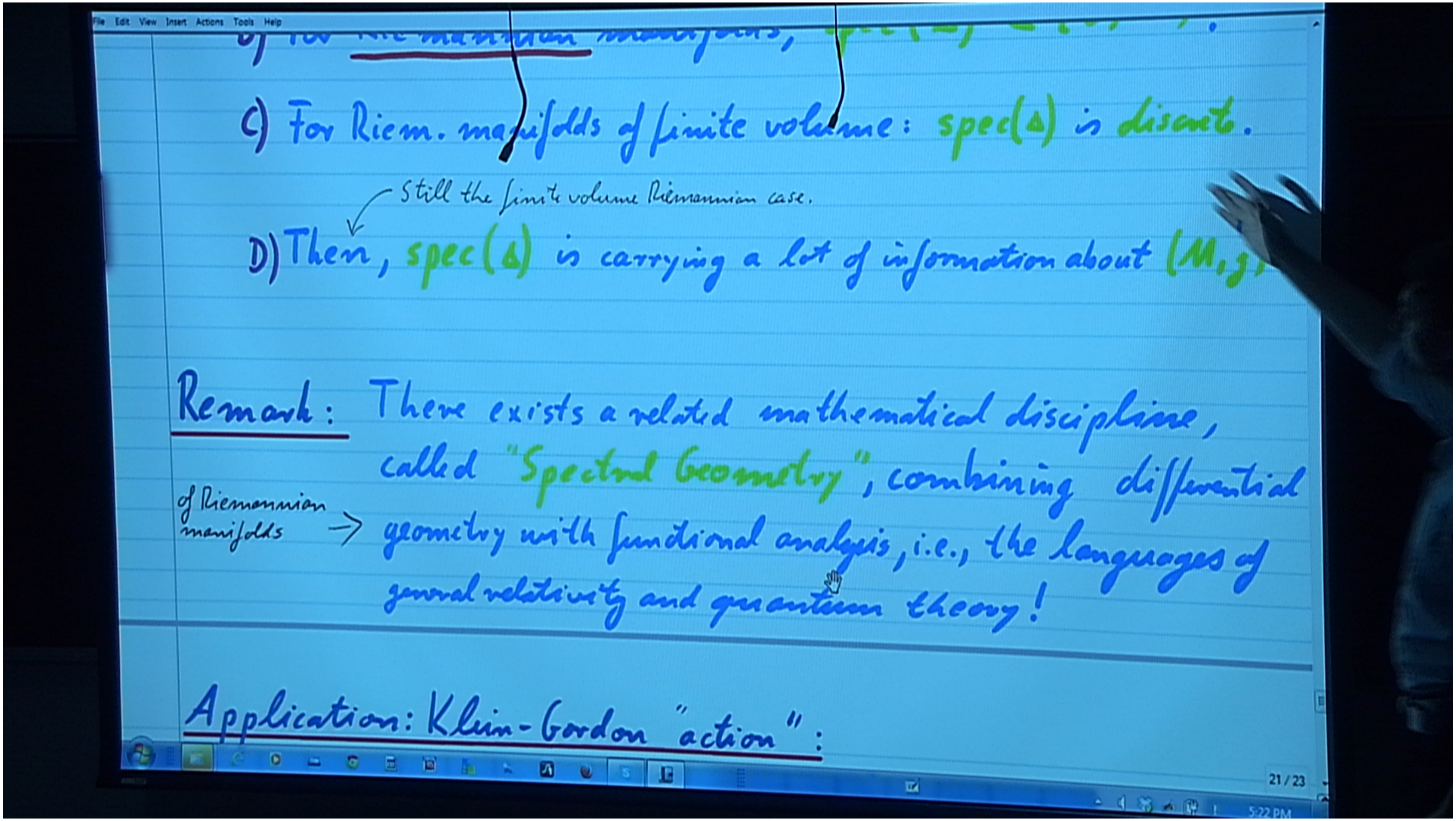

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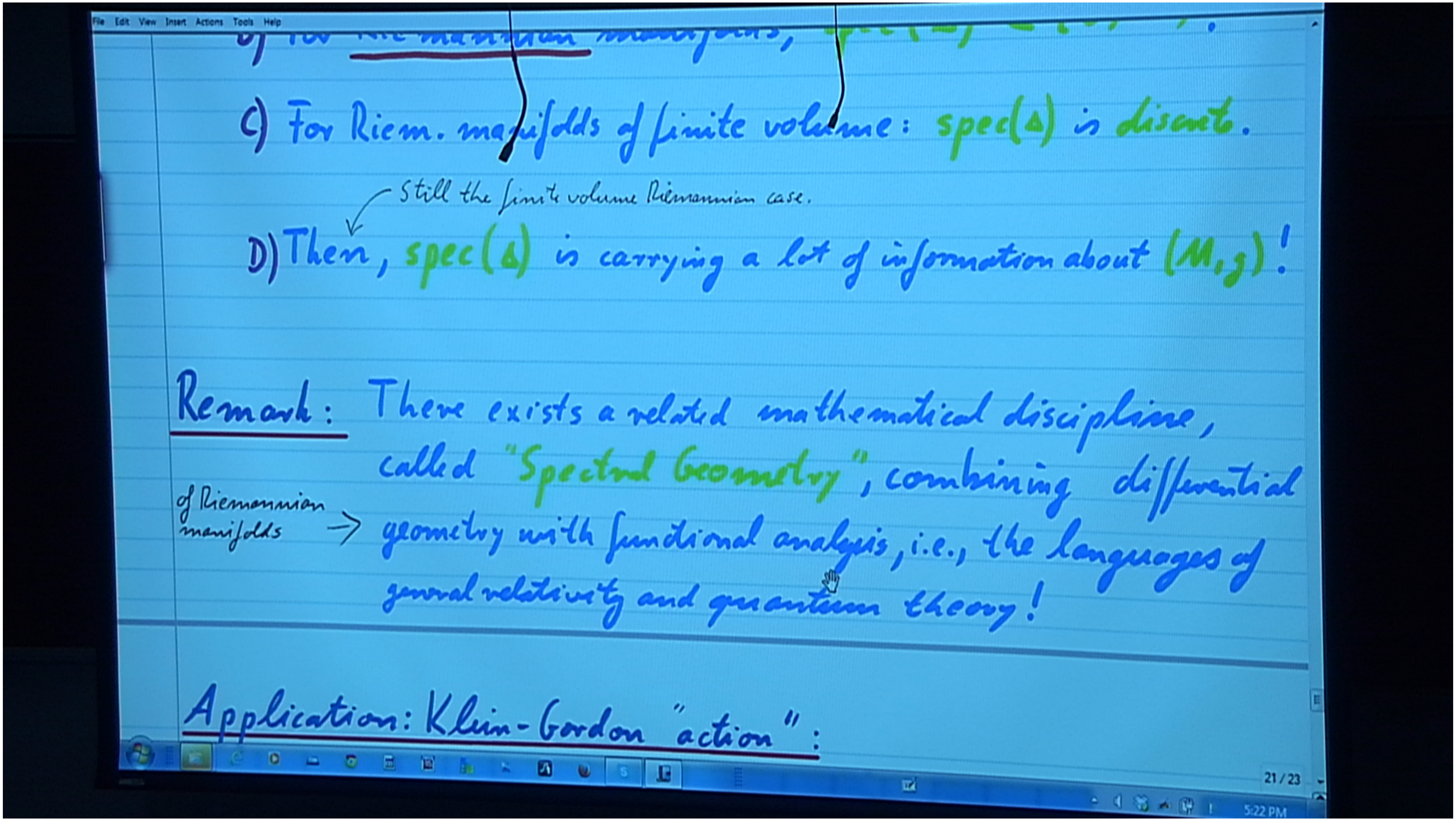
of Riemannian manifolds →

Application: Klein-Gordon "action":



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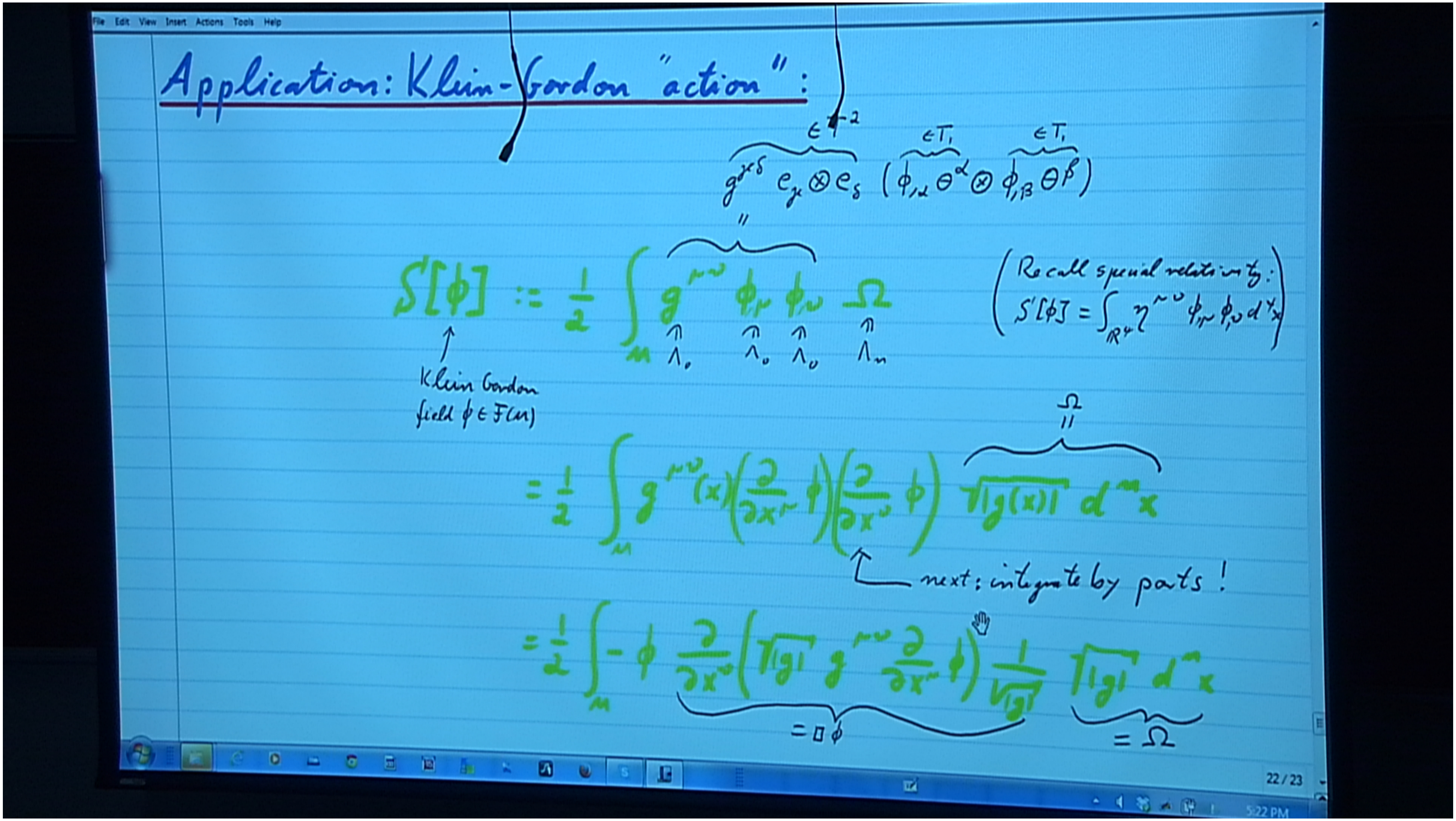


Application: Klein-Gordon "action":

$$\overbrace{g^{\gamma\delta}\, e_\gamma \otimes e_\delta}^{\in T^{2}} \left(\overbrace{\phi_{,\alpha}\,\theta^\alpha}^{\in T_1} \otimes \overbrace{\phi_{,\beta}\,\theta^\beta}^{\in T_1}\right)$$

$$\|$$

$$S[\phi] := \frac{1}{2}\int_M \overbrace{\underset{\overset{\Uparrow}{\Lambda_0}}{g^{\mu\nu}}\ \underset{\overset{\Uparrow}{\Lambda_0}}{\phi_{,\mu}}\ \underset{\overset{\Uparrow}{\Lambda_0}}{\phi_{,\nu}}}\ \underset{\overset{\Uparrow}{\Lambda_n}}{\Omega}$$

↑

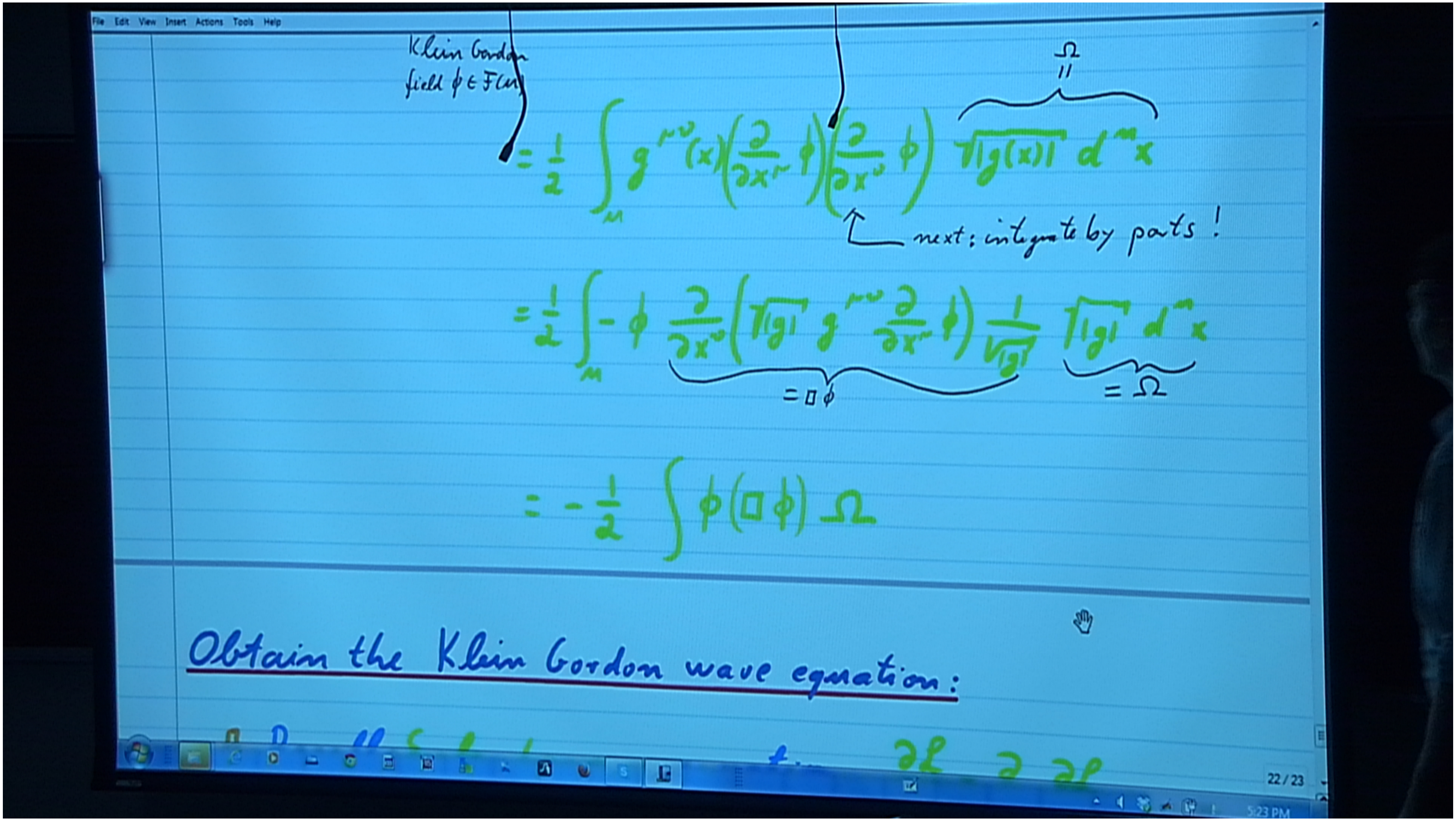
Klein Gordon
field $\phi \in \mathcal{F}(M)$

(Recall special relativity:
$S[\phi] = \int_{\mathbb{R}^4} \eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\, d^4x$)

$$= \frac{1}{2}\int_M g^{\mu\nu}(x)\left(\frac{\partial}{\partial x^\mu}\phi\right)\left(\frac{\partial}{\partial x^\nu}\phi\right) \overset{\overset{\Omega}{\|}}{\overbrace{\sqrt{|g(x)|}\ d^n x}}$$

↑ next: integrate by parts!

$$= \frac{1}{2}\int_M -\phi\ \underbrace{\frac{\partial}{\partial x^\nu}\left(\sqrt{|g|}\ g^{\mu\nu}\frac{\partial}{\partial x^\mu}\phi\right)\frac{1}{\sqrt{|g|}}}_{=\Box\phi}\ \underbrace{\sqrt{|g|}\ d^n x}_{=\Omega}$$



Klein Gordon
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$$= \frac{1}{2}\int_M g^{\mu\nu}(x)\left(\frac{\partial}{\partial x^\mu}\phi\right)\left(\frac{\partial}{\partial x^\nu}\phi\right)\overbrace{\sqrt{|g(x)|}\,d^n x}^{\Omega}$$

next: integrate by parts!

$$= \frac{1}{2}\int_M -\phi\,\underbrace{\frac{\partial}{\partial x^\nu}\left(\sqrt{|g|}\,g^{\mu\nu}\frac{\partial}{\partial x^\mu}\phi\right)\frac{1}{\sqrt{|g|}}}_{=\Box\phi}\,\underbrace{\sqrt{|g|}\,d^n x}_{=\Omega}$$

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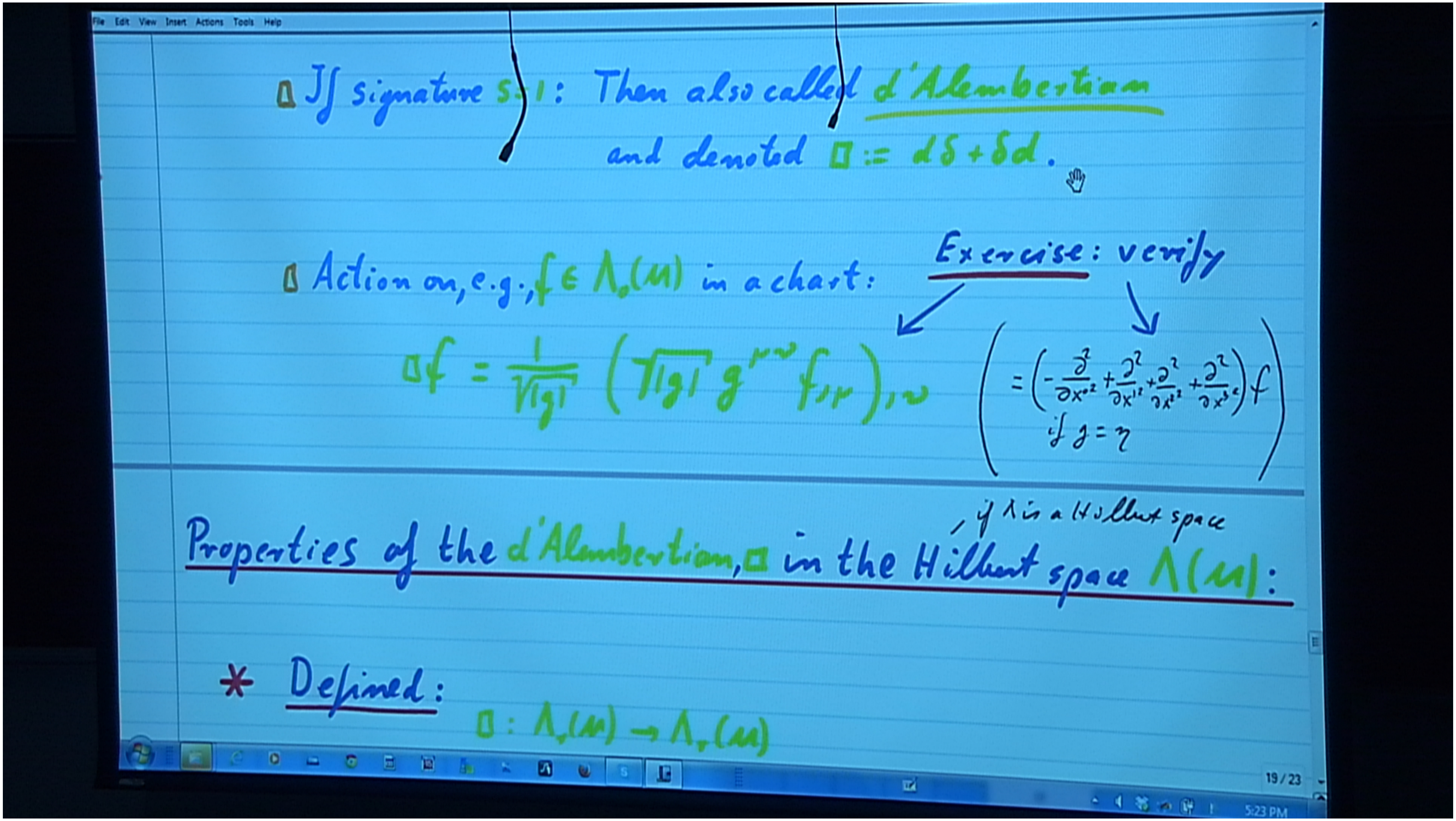
Obtain the Klein Gordon wave equation:



□ Iff signature $s=1$: Then also called d'Alembertian and denoted $\Box := d\delta + \delta d$.

□ Action on, e.g., $f \in \Lambda_0(M)$ in a chart:

Exercise: verify

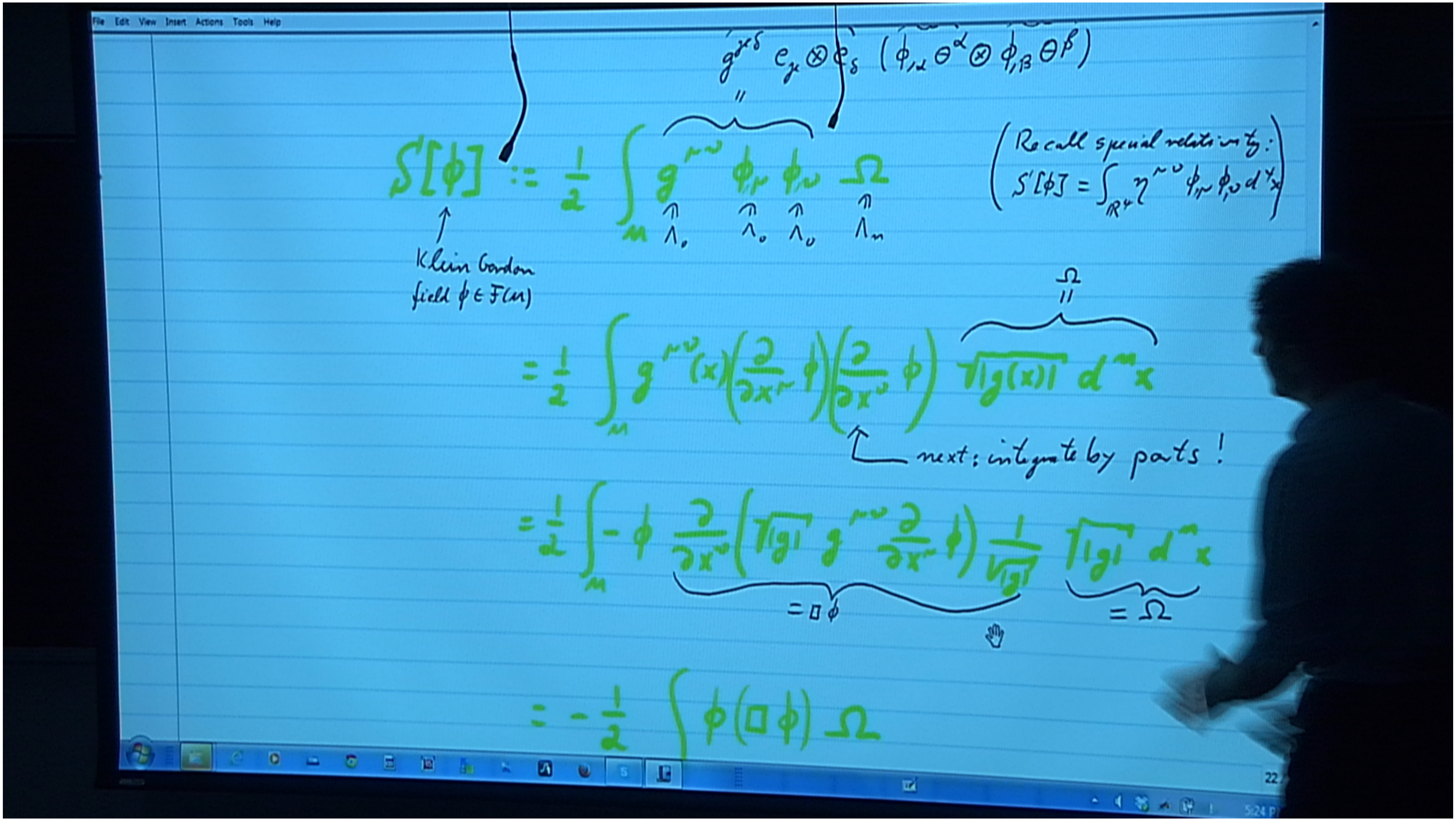
$$\Box f = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|}\, g^{\mu\nu} f_{,\mu} \right)_{,\nu} \qquad \left(= \left(-\frac{\partial^2}{\partial x^{0^2}} + \frac{\partial^2}{\partial x^{1^2}} + \frac{\partial^2}{\partial x^{2^2}} + \frac{\partial^2}{\partial x^{3^2}} \right) f \quad \text{if } g = \eta \right)$$

Properties of the d'Alembertian, □ in the Hilbert space $\Lambda(M)$:

, if Λ is a Hilbert space

* Defined:

$$\Box : \Lambda_r(M) \to \Lambda_r(M)$$



$$g^{\gamma\delta}\, e_\gamma \otimes e_\delta \left(\phi_{,\alpha}\theta^\alpha \otimes \phi_{,\beta}\theta^\beta\right)$$

$$S[\phi] := \frac{1}{2}\int_M \overbrace{\underset{\Lambda_0}{\underset{\Uparrow}{g^{\mu\nu}}}\ \underset{\Lambda_0}{\underset{\Uparrow}{\phi_{,\mu}}}\ \underset{\Lambda_0}{\underset{\Uparrow}{\phi_{,\nu}}}}^{\overset{g^{\gamma\delta} e_\gamma\otimes e_\delta(\phi_{,\alpha}\theta^\alpha\otimes\phi_{,\beta}\theta^\beta)}{\|}}\ \underset{\Lambda_n}{\underset{\Uparrow}{\Omega}}$$

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Klein Gordon field $\phi \in \mathcal{F}(M)$

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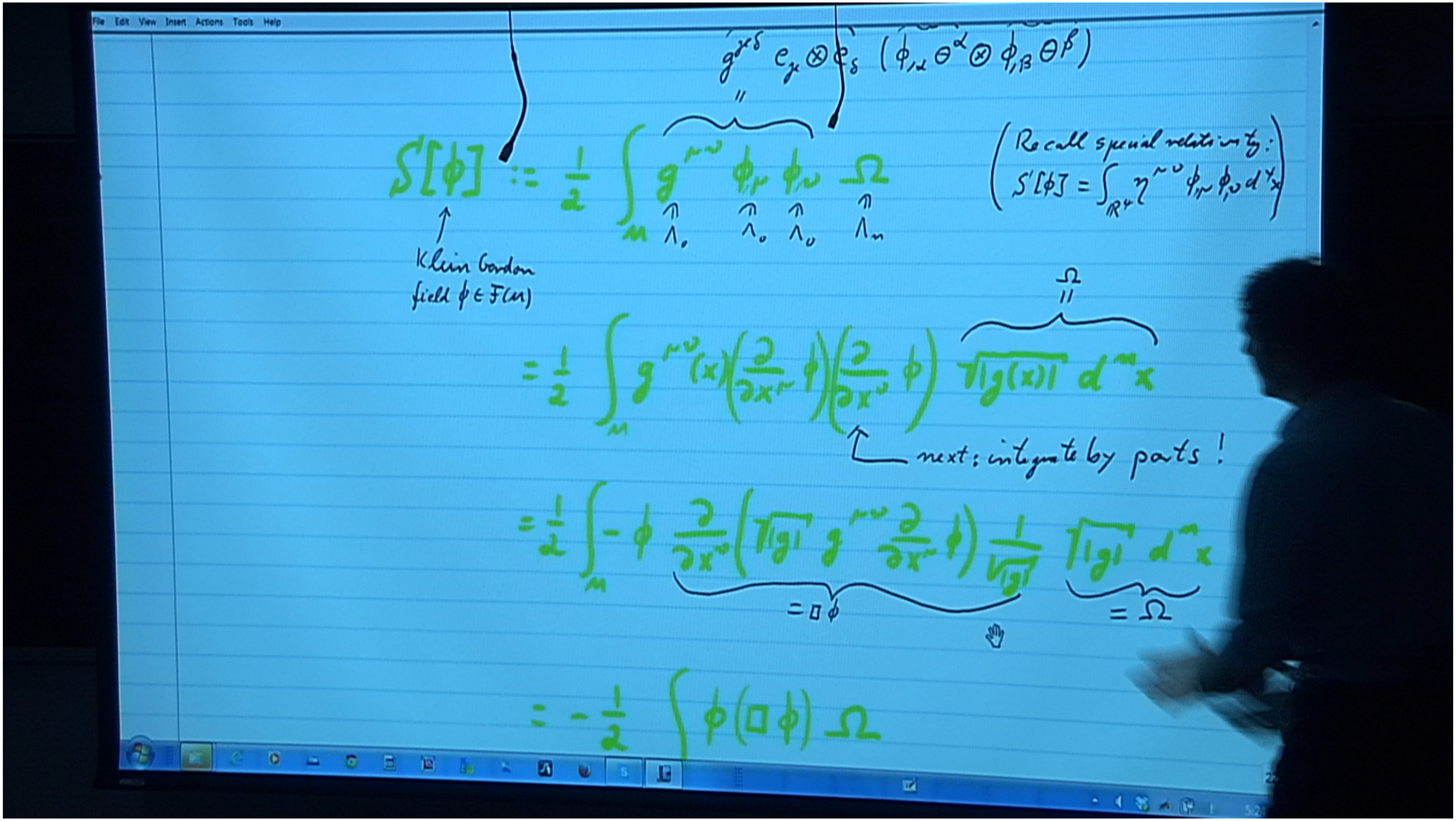

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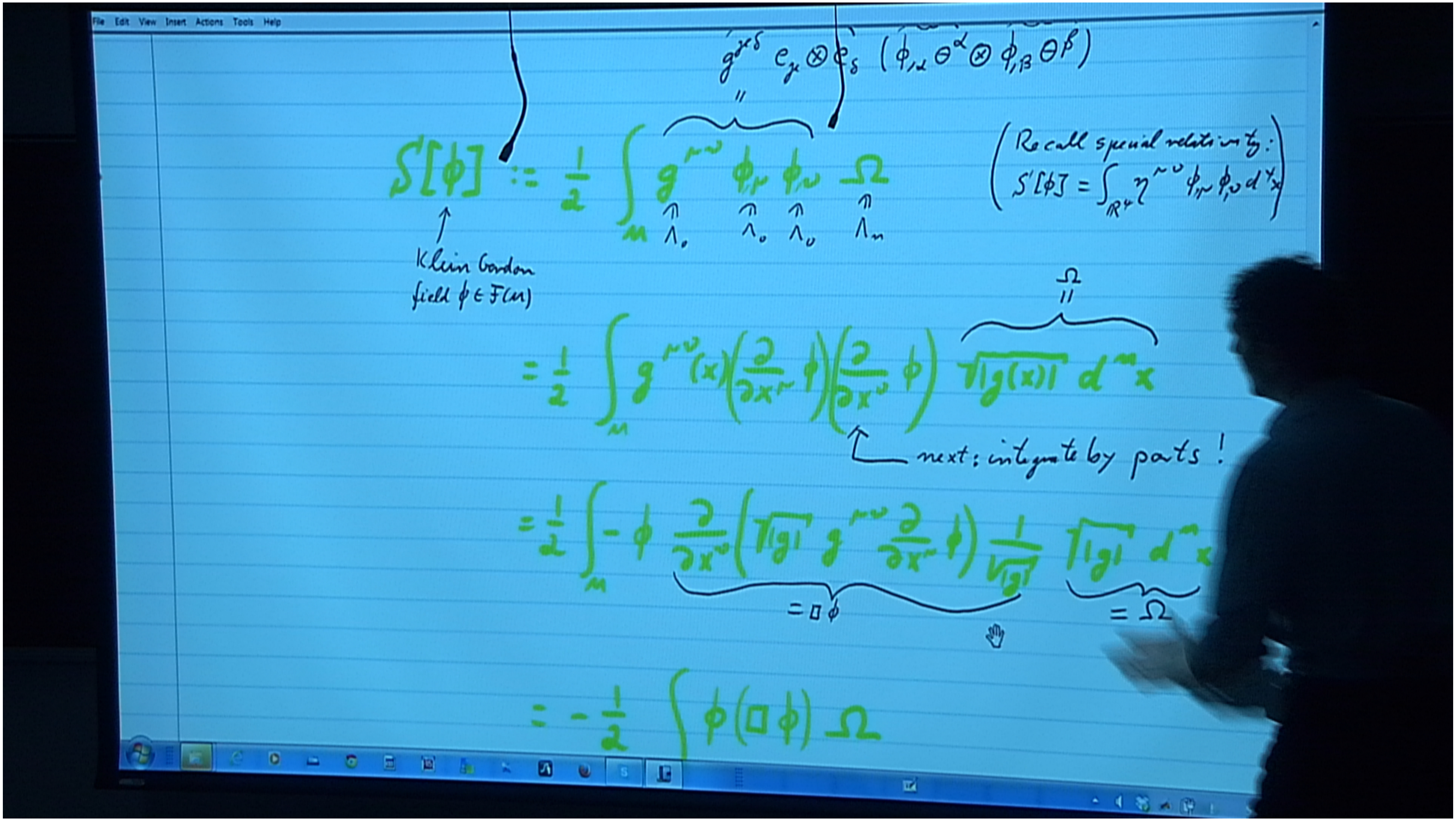

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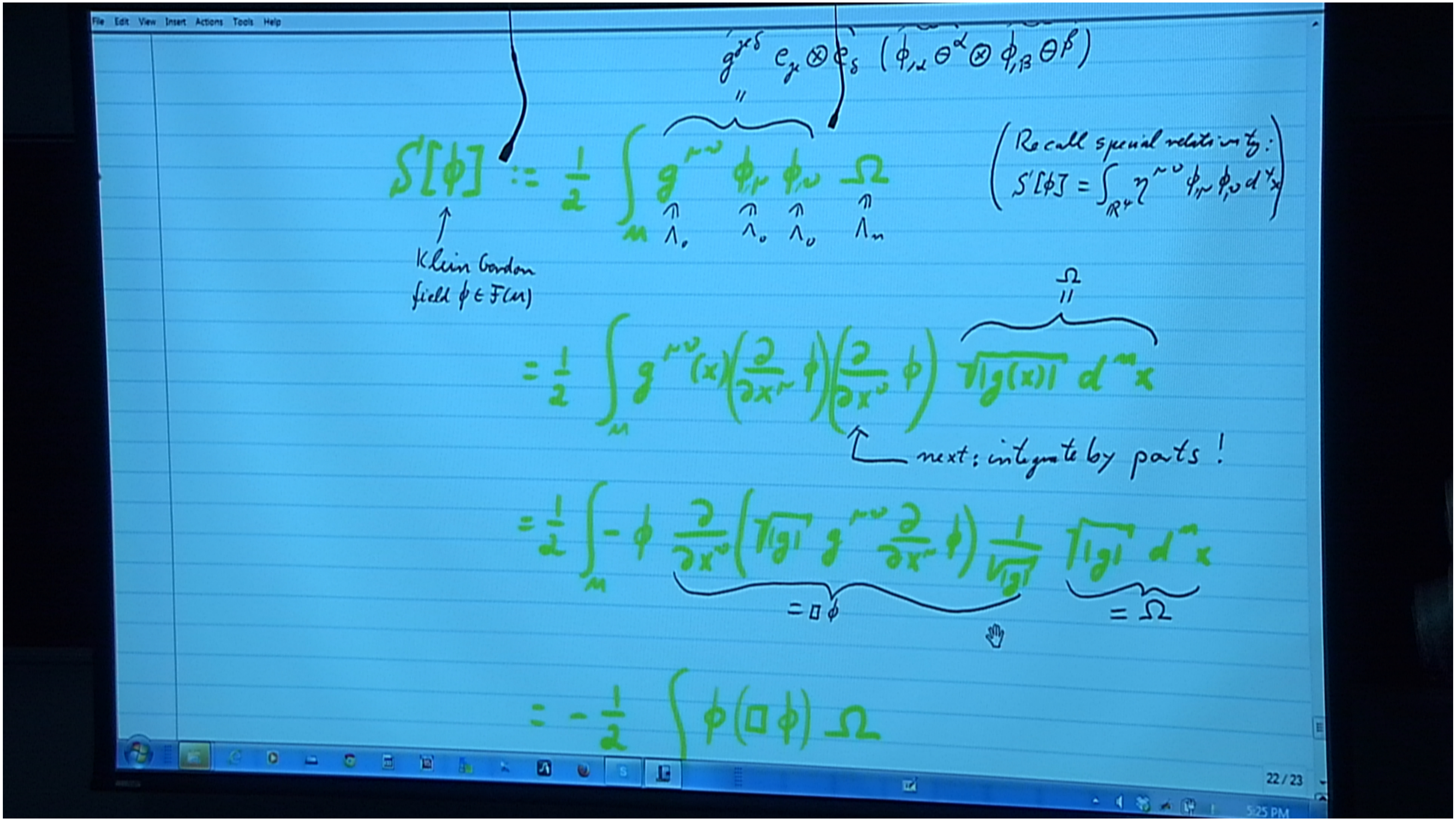

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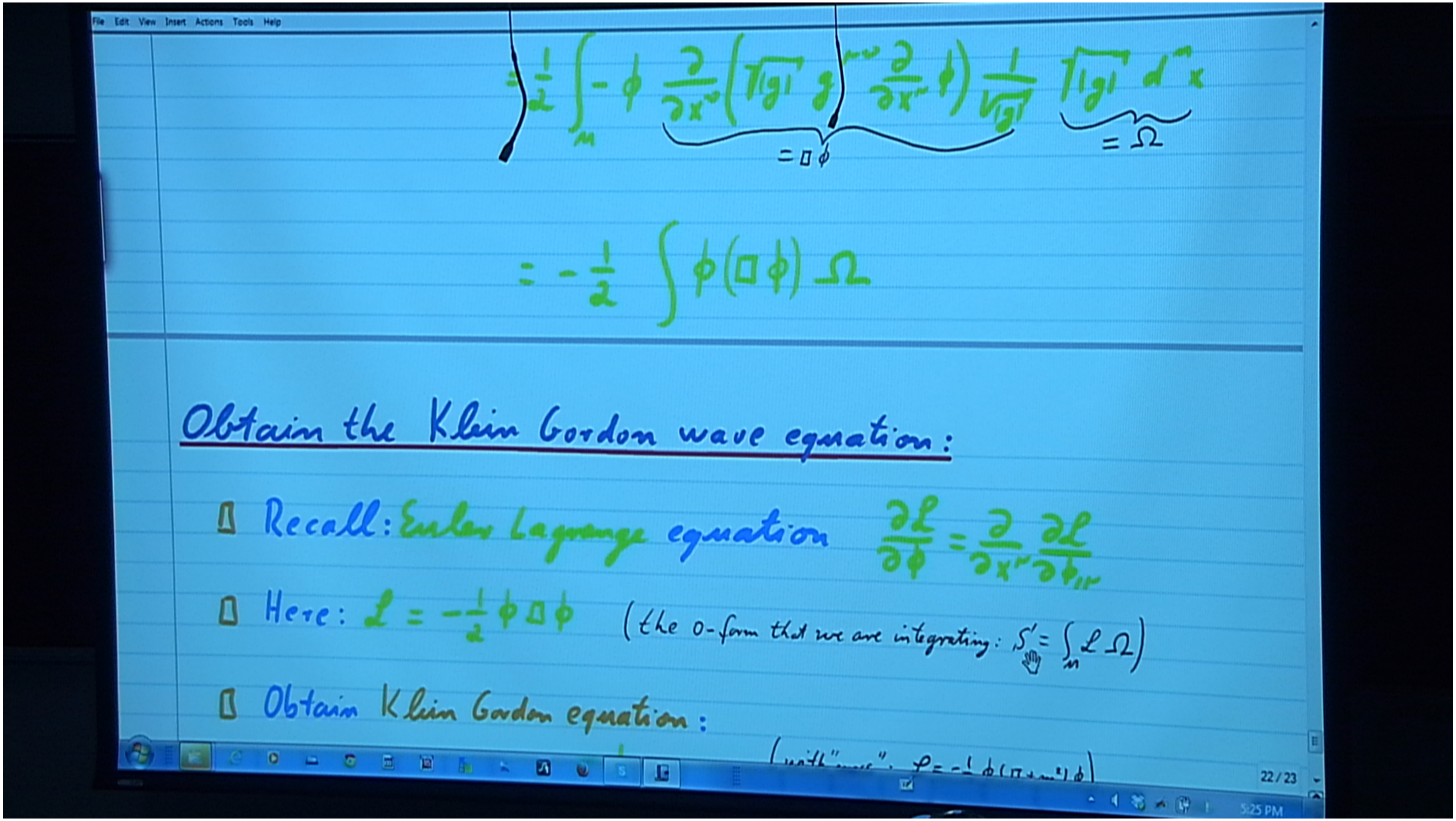

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Obtain the Klein Gordon wave equation:

- Recall: Euler Lagrange equation $\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}$
- Here: $\mathcal{L} = -\frac{1}{2}\phi\Box\phi$ (the 0-form that we are integrating: $S' = \int_M \mathcal{L}\,\Omega$)
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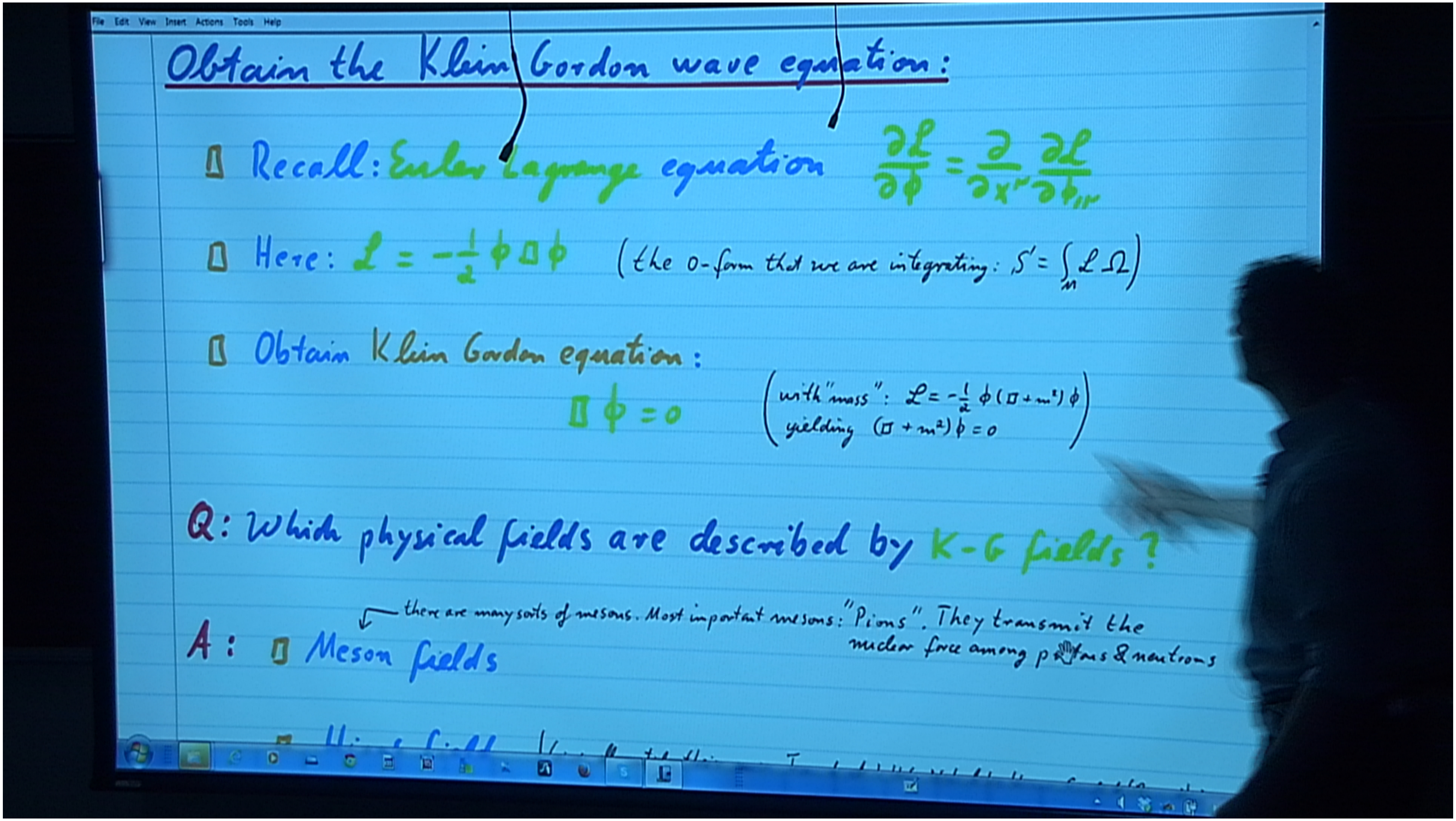
(with "mass": $\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi$)



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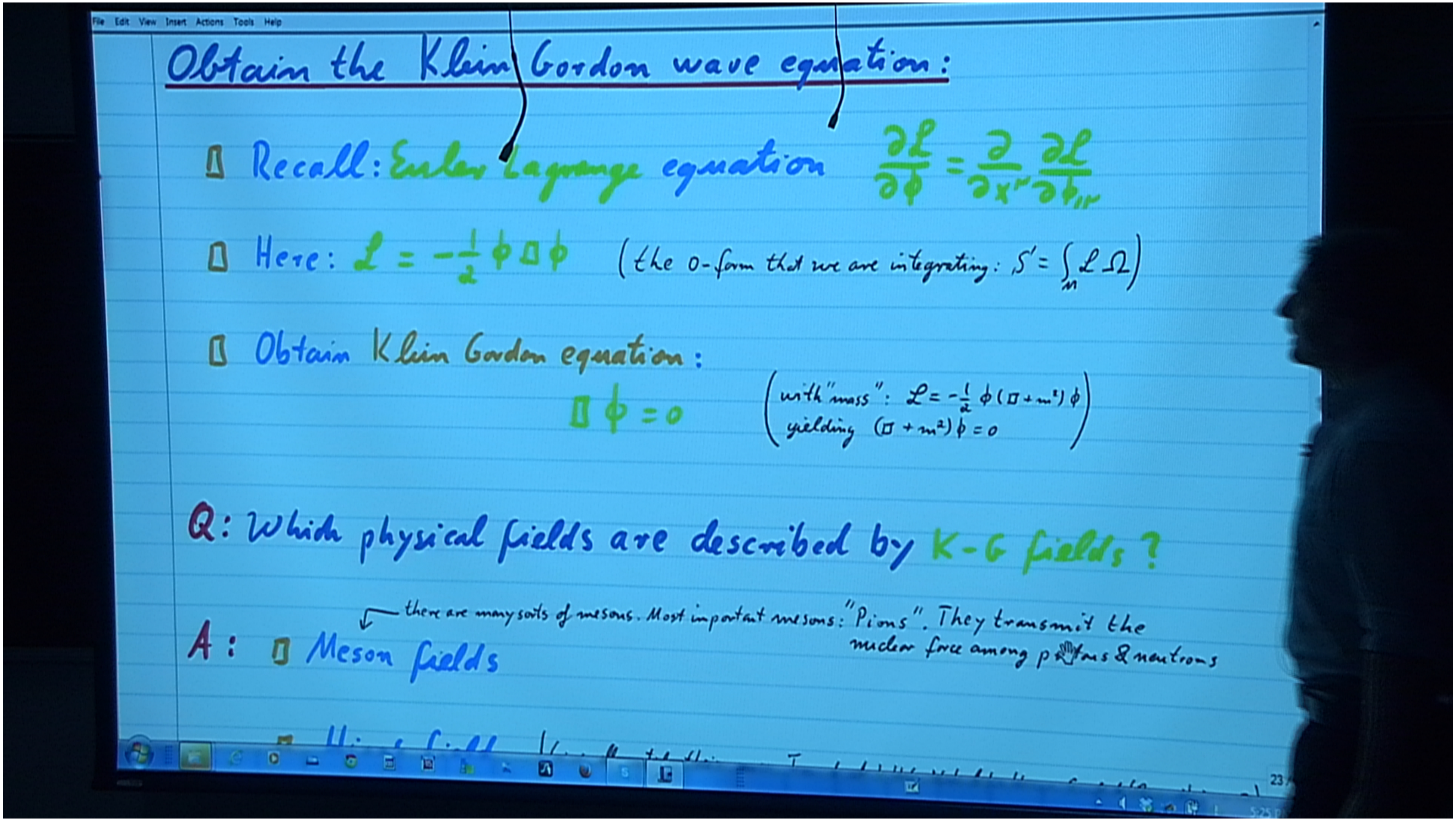
A: ▯ Meson fields ↙ there are many sorts of mesons. Most important mesons: "Pions". They transmit the nuclear force among p[illegible]ons & neutrons



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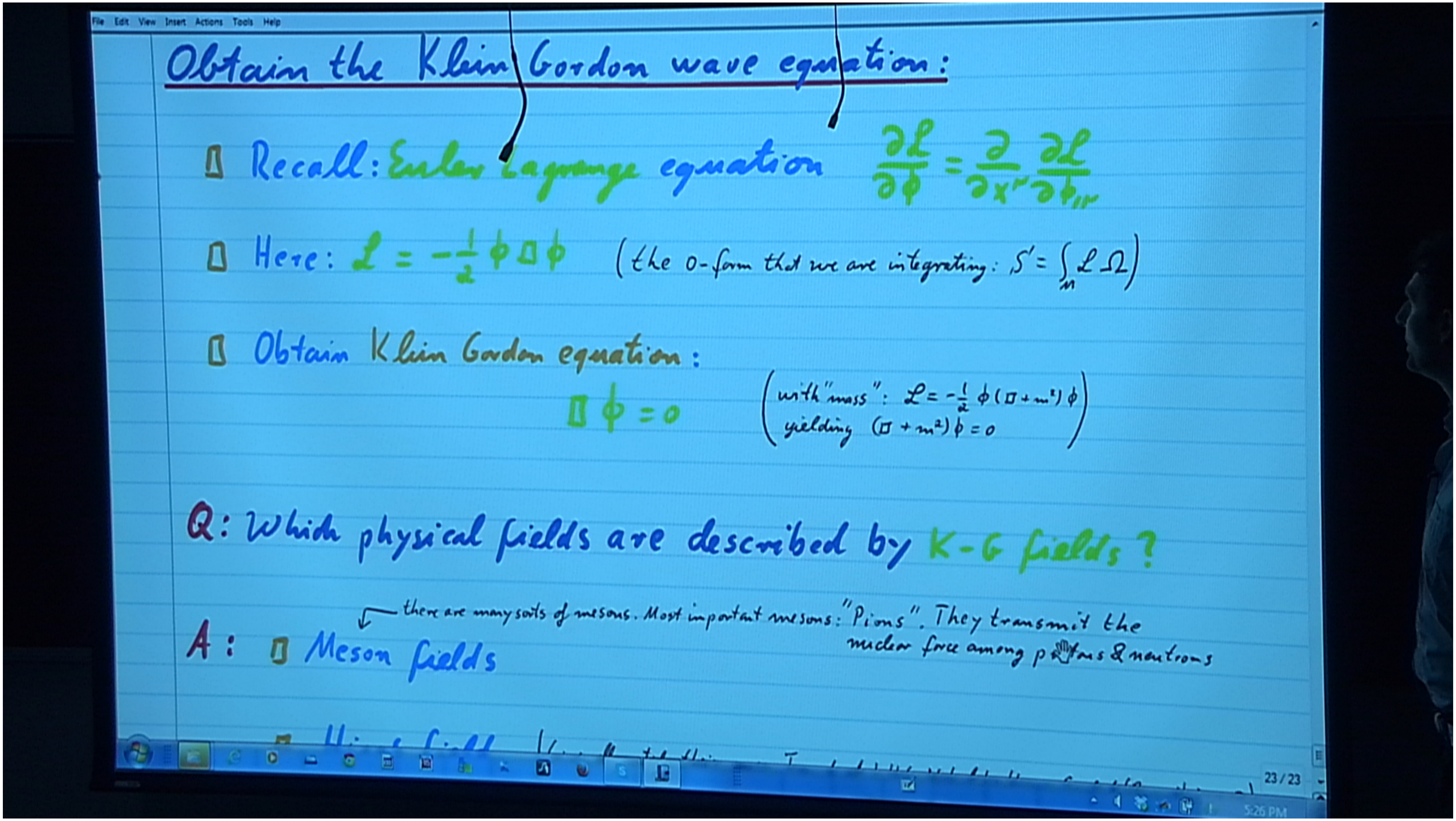

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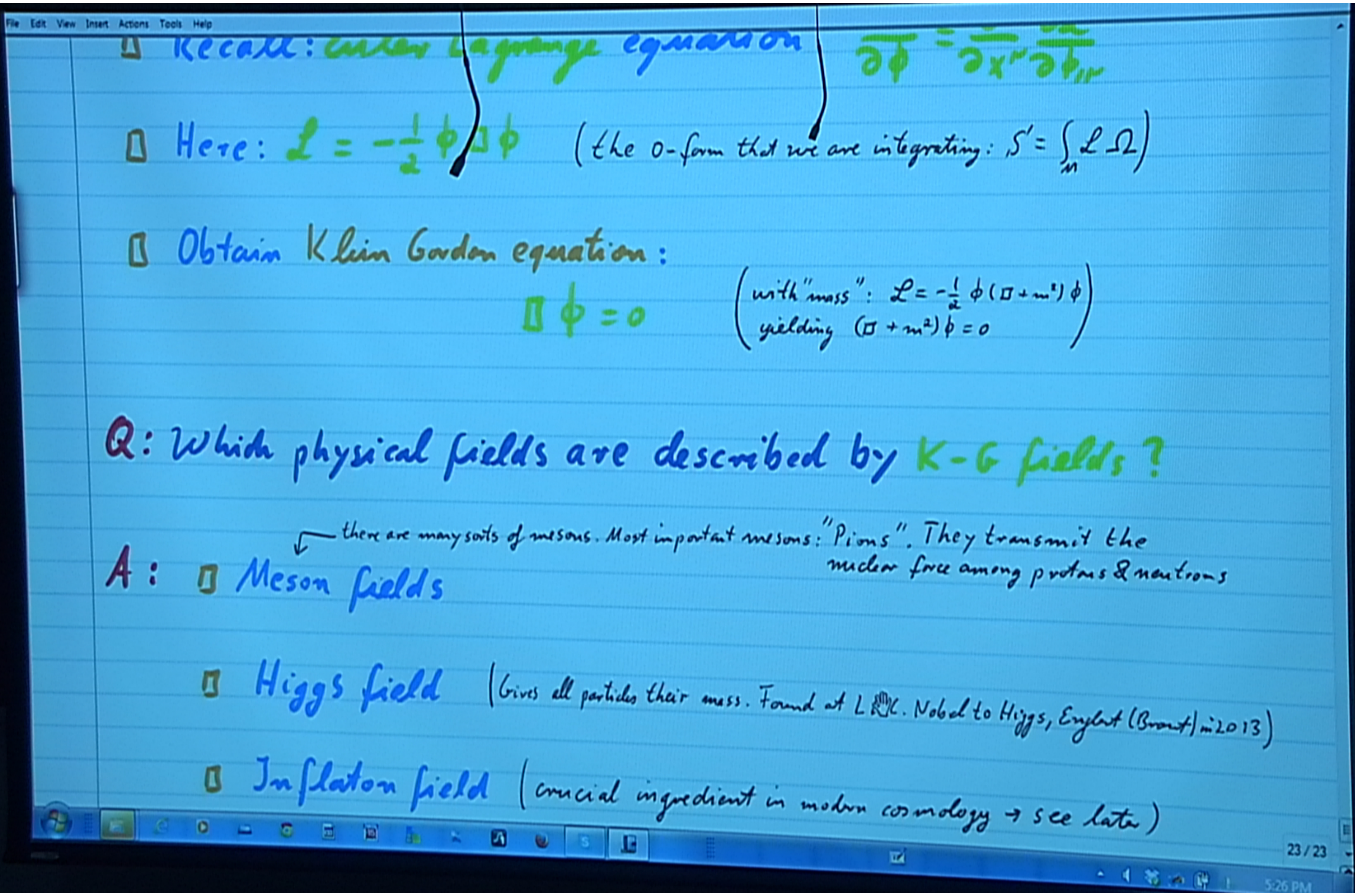

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A: ☐ Meson fields

☐ Higgs field (Gives all particles their mass. Found at LHC. Nobel to Higgs, Englert (Brout) in 2013)

☐ Inflaton field (crucial ingredient in modern cosmology → see later)