

Title: General Relativity for Cosmology - Lecture 3

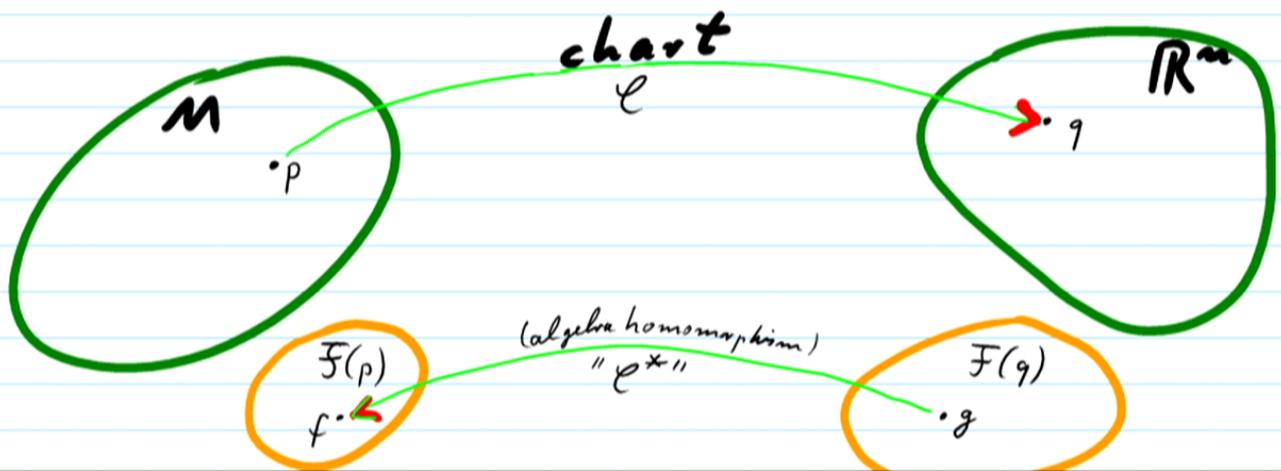
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Abstract:

# GR for Cosmology, Achim Kempf, Fall 13, Lecture 3

Recall: Get concrete handle on the abstract  
 $p \in M$  and  $f \in \mathcal{F}(p)$  and  $\xi \in T_p(M)$   
by using a chart  $\mathcal{C}: M \rightarrow \mathbb{R}^n$ :



Each  $p \in M$  has now concrete image  $q \in \mathbb{R}^n$   
 i.e. it has 'coordinates'.

Each  $f \in \mathcal{F}(p)$  is the image of a concrete function germ  $g \in \mathcal{F}(q)$ .

Each  $\xi \in T_p(M)$  has now a concrete image

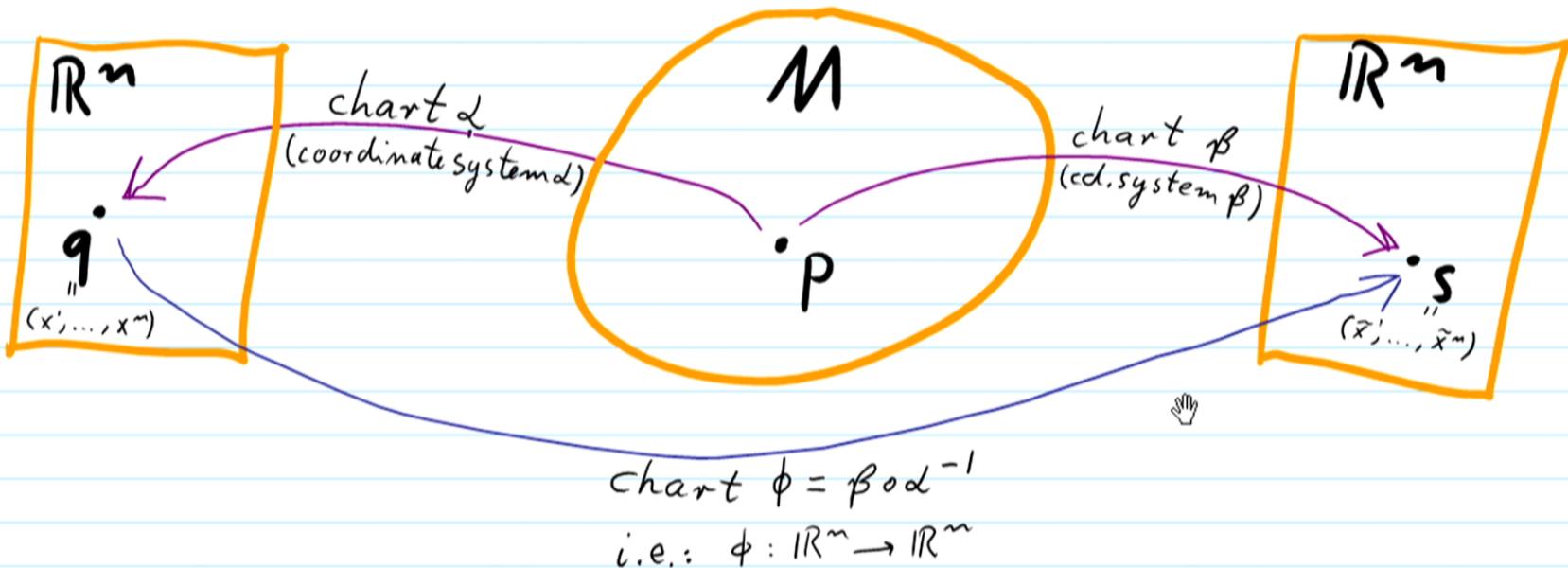
$$\eta \in T_q(\mathbb{R}^n)$$

which we know has the form:

$$\eta = \sum_{i=1}^n \eta_i \frac{\partial}{\partial x^i} \Big|_{x=q}$$

coefficients  $\in \mathbb{R}$

Given a  $p \in M$  and a  $\xi \in T_p(M)$ ,  
 how do their coordinates and coefficients  
 change under a change of charts?



⇒ When changing from cds.  $\alpha$  to cds.  $\beta$ :

1. The coordinates of  $p$  change from

$q = (x^1, \dots, x^m)$  to  $s = (\tilde{x}^1, \dots, \tilde{x}^m)$ :

$$(\tilde{x}^1, \dots, \tilde{x}^m) = \phi(x^1, \dots, x^m)$$

where  $\tilde{x}^i = \phi^i(x^1, \dots, x^m)$ .

2. A function  $f \in \mathcal{F}_p(M)$  has images  $g \in \mathcal{F}_q(\mathbb{R}^m)$

and  $h \in \mathcal{F}_s(\mathbb{R}^m)$ , related by:

$$f(p) = g(q) = h(s) \quad ( \in \mathbb{R} )$$

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$$f(p) = g(q) = h(s) \quad (\in \mathbb{R})$$

and  $h(\tilde{x}^1, \dots, \tilde{x}^m) = g(x^1, \dots, x^m)$  (\*)

where  $\tilde{x}^i = \phi^i(x^1, \dots, x^m)$ .

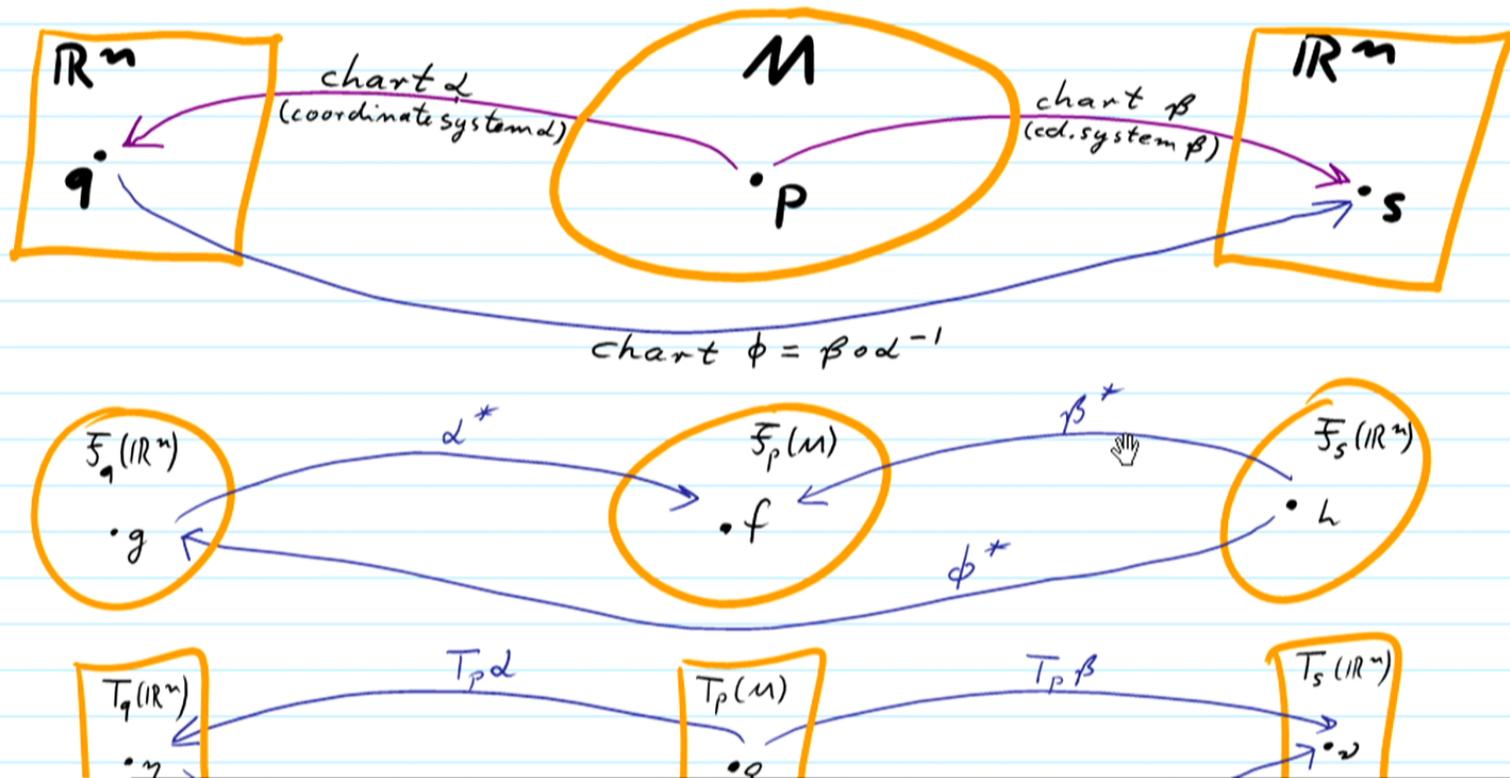
2. A function  $f \in \mathcal{F}_p(\mathcal{M})$  has images  $g \in \mathcal{F}_q(\mathbb{R}^n)$  and  $h \in \mathcal{F}_s(\mathbb{R}^m)$ , related by:

$$f(p) = g(q) = h(s) \quad (s \in \mathbb{R})$$

$$\text{and } h(\tilde{x}^1, \dots, \tilde{x}^m) = g(x^1, \dots, x^m) \quad (*)$$

3. A tangent vector  $\xi \in T_p(\mathcal{M})$  now has two images,  $v \in T(\mathbb{R}^n)$  and  $w \in T(\mathbb{R}^m)$

3. A tangent vector  $\xi \in T_p(M)$  now has two images,  $\eta \in T_q(\mathbb{R}^n)$  and  $v \in T_s(\mathbb{R}^n)$ .



By construction:

$$\xi(f) = \eta(g) = \nu(h) \quad (\in \mathbb{R})$$

⇒ in particular:

$$\sum_{i=1}^m \eta^i \frac{\partial}{\partial x^i} g(x^1, \dots, x^m) \Big|_{x=q} = \sum_{j=1}^m \nu^j \frac{\partial}{\partial \tilde{x}^j} \underbrace{h(\tilde{x}^1, \dots, \tilde{x}^m)}_{g(x^1, \dots, x^m)} \Big|_{\tilde{x}=s} \text{ by } (*)$$

$$= \sum_{j=1}^m \nu^j \frac{\partial x^k}{\partial \tilde{x}^j} \frac{\partial}{\partial x^k} g(x^1, \dots, x^m) \Big|_{x=q}$$

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⇒ in particular :

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$$= \sum_{\substack{j=1 \\ k=1}}^m \nu^j \frac{\partial x^k}{\partial \tilde{x}^j} \Big|_{\tilde{x}=s} \frac{\partial}{\partial x^k} g(x^1, \dots, x^m) \Big|_{x=q}$$

⇒ in particular:

$$\sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i} g(x^1, \dots, x^n) \Big|_{\tilde{x}^k} = \sum_{j=1}^m \nu^j \frac{\partial}{\partial \tilde{x}^j} h(\tilde{x}^1, \dots, \tilde{x}^m) \Big|_{\tilde{x}^k}$$

$g(x, \dots)$

$$= \sum_{\substack{j=1 \\ k=1}}^m \nu^j \frac{\partial x^k}{\partial \tilde{x}^j} \Big|_{\tilde{x}^k} \frac{\partial}{\partial x^k} g(x^1, \dots)$$

Must be true for all  $g$ !

$$\sum_{i=1}^m \eta^i \frac{\partial}{\partial x^i} g(x^1, \dots, x^m) \Big|_{x=q} = \sum_{j=1}^m \nu^j \frac{\partial}{\partial \tilde{x}^j} \underbrace{h(\tilde{x}^1, \dots, \tilde{x}^m)}_{g(x^1, \dots, x^m)} \Big|_{\tilde{x}=s} \text{ by } (*)$$

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Must be true for all  $g$ !

$$\Rightarrow \sum_{i=1}^m \eta^i \frac{\partial}{\partial x^i} = \sum_{\substack{j=1 \\ i=1}}^m \nu^j \frac{\partial x^i}{\partial \tilde{x}^j} \Big|_{\tilde{x}=s} \frac{\partial}{\partial x^i}$$

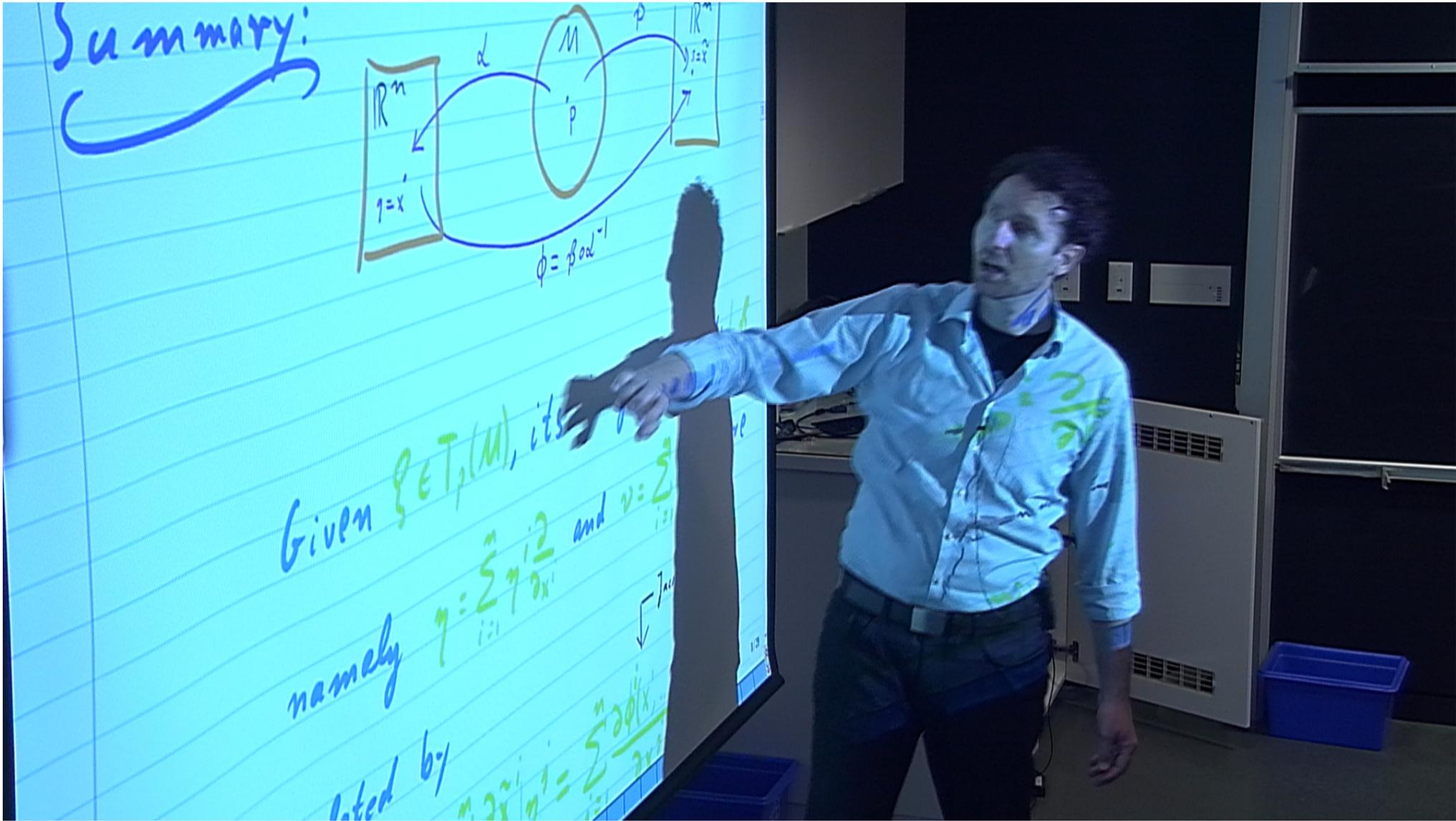
$$\Rightarrow \sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i} = \sum_{j=1}^m v^j \frac{\partial x^i}{\partial \tilde{x}^j} \bigg|_{\tilde{x}=s} \frac{\partial}{\partial x^i}$$

Jacobian matrix  $D\phi^{-1}$   
of  $\phi^{-1}$  at  $s$ .

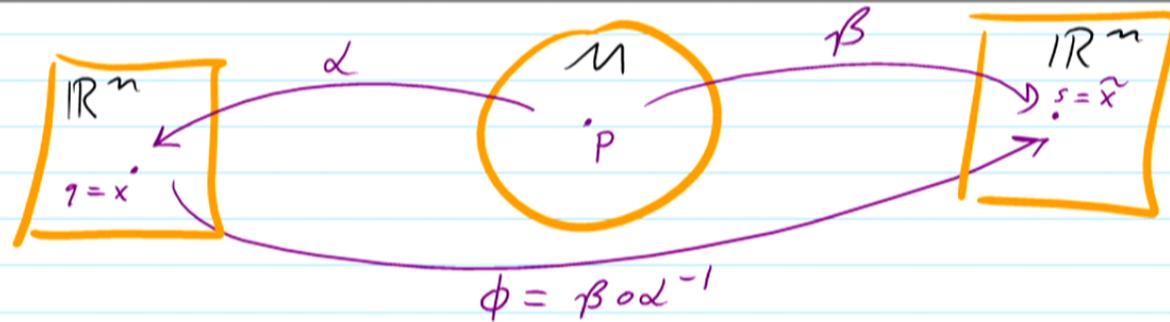
$$\Rightarrow \eta^i = \sum_{j=1}^m \frac{\partial x^i}{\partial \tilde{x}^j} \bigg|_{\tilde{x}=s} v^j$$

Jacobian matrix  $D\phi$   
of  $\phi$  at  $q$ .

$$\Rightarrow \text{conversely: } v^i = \sum_{j=1}^m \frac{\partial \tilde{x}^i}{\partial x^j} \bigg|_{x=q} \eta^j$$



Summary:



Given  $\xi \in T_p(M)$ , its images in charts  $\alpha, \beta$ ,

namely  $\eta = \sum_{i=1}^m \eta^i \frac{\partial}{\partial x^i}$  and  $v = \sum_{i=1}^m v^i \frac{\partial}{\partial \tilde{x}^i}$ , are

related by

$$v^i = \underbrace{\sum_{j=1}^m \frac{\partial \tilde{x}^i}{\partial x^j}}_{\text{Jacobian matrix } D\phi} \bigg|_{x=q} \eta^j = \sum_{j=1}^m \frac{\partial \phi^i(x^1, \dots, x^m)}{\partial x^j} \bigg|_{x=q} \eta^j$$

→ The "physicist's definition of  $T_p(M)$ ":

Def: A tangent vector  $\xi \in T_p(M)$  is a map that assigns to each (germ of a) chart a coefficient vector  $\in \mathbb{R}^n$ , so that if

□  $(\eta^1, \dots, \eta^n)$  is coefficient vector w. resp. to chart  $\alpha$

□  $(v^1, \dots, v^n)$  is coefficient vector w. resp. to chart  $\beta$

then:  $v^i = \sum_{j=1}^n \frac{\partial \tilde{x}^i}{\partial x^j} \eta^j$  with  $\tilde{x} = \phi(x)$

So far, 2 equiv. defs. of  $T_p(M)$ :

In a chart,  $d$ , a tangent vector,  $\xi \in T_p(M)$  is:

• algebraically:  $\sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i} \Big|_{x=d(p)}$

i.e. it is a directional derivative.

Defining property: Leibniz rule.

• physically:  $(\eta^1, \dots, \eta^n)$

i.e. it is just the direction vector.

o physically:  $(\eta', \dots, \eta^m)$

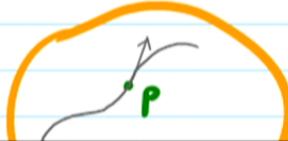
i.e. it is just the direction vector,

Defining property: chart change transformation rule

Finally:

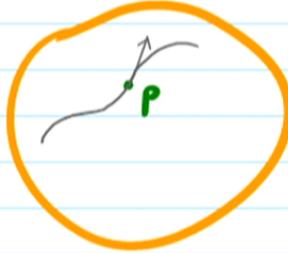
The "geometric definition of  $T_p(\mathcal{M})$ ":

Idea: Tangent vectors as tangents to paths.



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Idea: Tangent vectors as tangents to paths.

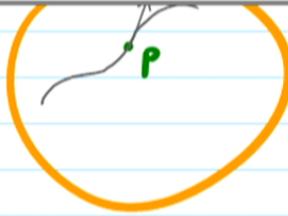


Consider paths in  $M$  that pass through  $p$ :

$$\gamma: \mathbb{R} \rightarrow M$$

$$\gamma(0) = p$$

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$$f \circ \gamma: \mathbb{R} \rightarrow \mathbb{R}$$

Define:

Two diffable paths,  $\gamma_a, \gamma_b$  are called equivalent, if for all  $f \in F_p(M)$ :

$$\left. \frac{d}{dt} (f \circ \gamma_a) \right| = \left. \frac{d}{dt} (f \circ \gamma_b) \right|$$



Define:

Two diffable paths,  $\gamma_a, \gamma_b$  are called equivalent,  
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$$\left. \frac{d}{dt} (f \circ \gamma_a) \right|_{t=0} = \left. \frac{d}{dt} (f \circ \gamma_b) \right|_{t=0} \quad (\otimes)$$

Intuition: Two paths  $\gamma_a, \gamma_b$  are equivalent  
if they have the same 'velocity' at  $p$ :

↑ Note: this includes speed and direction  
because  $(\otimes)$  must hold for all  $f \in F_p(M)$ .

(zoom)  
12

if they have the same 'velocity' at  $p$ :

↑ Note: this includes speed and direction because  $\otimes$  must hold for all  $f \in T_p(M)$ .

Definition:  $T_p(M)^{(geom)}$  is the set of equivalence classes of diffable paths through  $p$ .

Are  $T_p(M)^{(geom)}$  and  $T_p(M)^{(alg)}$  equivalent?   
we'll usually mean  $T_p^{(alg)}(M)$  when we write  $T_p(M)$ .

Yes:  $\square$  Each path  $\gamma$  defines a linear map  $\bar{\gamma}$ :   
really: each equivalence class of diffable paths through  $p$   
 $\bar{\gamma}: F(p) \rightarrow \mathbb{R}$

Are  $T_p(M)^{(\text{geom})}$  and  $T_p(M)^{(\text{alg})}$  equivalent?

Yes:

really: each equivalence class of differentiable paths through  $p$

Each path  $\gamma$  defines a linear map  $\bar{\gamma}$ :

$$\bar{\gamma}: T_p(M) \rightarrow \mathbb{R}^n$$

$$\bar{\gamma}: f \rightarrow \left. \frac{d}{dt} (f \circ \gamma) \right|_{t=0}$$

These  $\bar{\gamma}$  obey the Leibniz rule:

$$\bar{\gamma}(fg) = \left. \frac{d}{dt} (f \cdot g)(\gamma(t)) \right|_{t=0} = \left. \frac{d}{dt} (f(\gamma(t))g(\gamma(t))) \right|_{t=0}$$

105:  
✓

□ Each path  $\gamma$  defines a linear map  $\bar{\gamma}$ :

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$$= \left. \frac{d}{dt} f(\gamma(t)) \right|_{t=0} \overbrace{g(\gamma(0))}^{=p} + f(\gamma(0)) \left. \frac{d}{dt} g(\gamma(t)) \right|_{t=0}$$

$$= \bar{\gamma}(f)g + f\bar{\gamma}(g) \checkmark$$

□  $\Rightarrow \bar{\gamma}$  is an element of  $T_p(M)$  (alg)

# The "Cotangent Space" $T_p(M)^*$ :

Recall:

Given an  $n$ -dimensional vector space  $V$ , the set of linear maps  $\omega: V \rightarrow \mathbb{R}$  forms also an  $n$ -dim. vector space. It is called the "dual space", and denoted  $V^*$ .

Definition:

The dual vector space to  $T_p(M)$

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Definition:

The dual vector space to  $T_p(M)$  is called the Cotangent Space, and denoted  $T_p(M)^*$ .

We notice:

$T_p(M)$  is a vector space of dimension  $n$ .  
 $T_p(M)^*$  is a vector space of dimension  $n$ .

We notice:

For every (germ of a) function at  $p$ ,

$$f \in \mathcal{F}(p)$$

one naturally obtains an element

$$"df \in T_p(M)^*"$$

called the "differential of  $f$ ."

Namely:

$df : T_p(M) \rightarrow \mathbb{R}$  is the linear map:

$$df(v) = v(f)$$

Concretely, in a cds., i.e. in a chart,  $\xi \in T_p(\mathcal{M})$

and  $f \in \mathcal{F}(p)$  correspond to some

$\eta \in T_q(\mathbb{R}^n)$  and  $g \in \mathcal{F}(q)$ . Then:

$$dg: T_q(\mathbb{R}^n) \rightarrow \mathbb{R}$$

$$dg: \eta \rightarrow \eta(g) = \sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i} \Big|_{x=q} g(x^1, \dots, x^n)$$

Recall: Since all  $\eta \in T_q(\mathbb{R}^n)$  take the form  $\eta = \sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i} \Big|_{x=q}$

$\eta \in T_q(\mathbb{R}^n)$  and  $g \in S(q)$ . Recall:

$$dg: T_q(\mathbb{R}^n) \rightarrow \mathbb{R}$$

$$dg: \eta \rightarrow \eta(g) = \sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i} \Big|_{x=q} g(x^1, \dots, x^n)$$

Recall: Since all  $\eta \in T_q(\mathbb{R}^n)$  take the form  $\eta = \sum_{i=1}^n \eta^i \frac{\partial}{\partial x^i} \Big|_{x=q}$

a basis of  $T_q(\mathbb{R}^n)$  is  $\left\{ \frac{\partial}{\partial x^i} \Big|_{x=q} \right\}_{i=1}^n$

Question: What is the dual basis in  $T(\mathbb{R}^n)^*$

Question: What is the dual basis in  $T_q(\mathbb{R}^n)^*$ ?

□ Consider the coordinate functions'  $x^k: \mathbb{R}^n \rightarrow \mathbb{R}$ .

□ Their differentials  $dx^k \in T_q(\mathbb{R}^n)^*$  obey:

$$dx^k: T_q(\mathbb{R}^n) \rightarrow \mathbb{R}$$

$$dx^k: \left. \frac{\partial}{\partial x^i} \right|_{x=q} \rightarrow \left. \frac{\partial}{\partial x^i} x^k \right|_{x=q} = \delta_i^k$$

□  $\Rightarrow$  The dual basis in  $T_q(\mathbb{R}^n)^*$  is given by

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□  $\Rightarrow$  The dual basis in  $T_q(\mathbb{R}^n)^*$  is given by

$$\left\{ dx^k \right\}_{k=1}^n$$

Thus:

Every element  $\omega \in T_q(\mathbb{R}^n)^*$  takes the form:

$$\omega = \sum_{i=1}^n \omega_i dx^i$$

$\uparrow$   
 $\in \mathbb{R}$

and its action is:

$$\omega : T_q(\mathbb{R}^n) \rightarrow \mathbb{R}$$

$$\begin{aligned} \omega : \sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j} &\rightarrow \sum_{i=1}^n \omega_i dx^i \left( \sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j} \right) \\ &= \sum_{i=1}^n \omega_i \sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j} x^i \end{aligned}$$

$\underbrace{\frac{\partial}{\partial x^j} x^i}_{= \delta_{i,j}}$

$$= \sum_{i=1}^n \omega_i \eta^i$$

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$$\Rightarrow \omega \left( \sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j} \right) = \sum_{i=1}^n \omega_i \eta^i \quad (\text{I})$$

In particular: For arbitrary  $g \in T_q(\mathbb{R}^n)$ , its

differential  $dg \in T_q(\mathbb{R}^n)^*$  must be of the form:

$$\Rightarrow \omega \left( \sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j} \right) = \sum_{i=1}^n \omega_i \eta^i \quad (\text{I})$$

$$\sum_{i=1}^n \omega_i \eta^i = \delta_{ij}$$

In particular: For arbitrary  $g \in \mathcal{F}(q)$ , its differential  $dg \in T_q^*(\mathbb{R}^n)$  must be of the form:

$$dg = \sum_{k=1}^n \omega_k dx^k \text{ with suitable } \omega_k \in \mathbb{R}.$$

How to calculate them?

We know:

In particular: For arbitrary  $g \in \mathcal{F}(\gamma)$ , its differential  $dg \in T_\gamma(\mathbb{R}^n)^*$  must be of the form:

$$dg = \sum_{k=1}^n \omega_k dx^k \text{ with suitable } \omega_k \in \mathbb{R}.$$

↑ how to calculate them?

We know:

$$dg(\gamma) = \gamma(g) = \sum_{i=1}^n \gamma^i \underbrace{\frac{\partial}{\partial x^i} g(x)}_{\omega_i} \Big|_{x=\gamma} \quad (\text{II})$$

Compare I, II  $\Rightarrow \omega_i = \frac{\partial}{\partial x^i} g(x) \Big|_{x=\gamma}$

Exercise: (the "pull back" map)

Assume that  $\vartheta \in T_p(M)^*$ , under two charts  $\alpha, \beta$ , as above, corresponds to  $\omega \in T_q(\mathbb{R}^n)^*$  and  $\mu \in T_s(\mathbb{R}^n)^*$  with:

$$\omega = \sum_{i=1}^n \omega_i dx^i \quad \text{and} \quad \mu = \sum_{i=1}^n \mu_i d\tilde{x}^i$$

Show that  $\mu_i = \sum_{j=1}^n \frac{\partial x^j}{\partial \tilde{x}^i} \Big|_{\tilde{x}=s} \omega_j$

Notice that this is the inverse of the Jacobian matrix of  $\beta \circ \alpha^{-1}$  at  $q$

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$\alpha = \sum_{i=1}^m \omega_i \alpha_i$        $\beta = \sum_{i=1}^m \tau_i \alpha_i$

Show that       $\mu_i = \sum_{j=1}^m \frac{\partial x^j}{\partial \tilde{x}^i} \Big|_{\tilde{x}=s} \omega_j$

↑  
Notice that this is the inverse  
of the Jacobian matrix of  $\beta \circ \alpha^{-1}$  at  $q$

Remark: The physicist's definition of  $T_p(M)^*$  uses this.

Some notation and terminology:

□ Elements of  $T_p(M)$  are called **contravariant vectors**

## Some notation and terminology:

- Elements of  $T_p(\mathcal{M})$  are called *contravariant vectors*
- Elements of  $T_p(\mathcal{M})^*$  are called *covariant vectors*
- One often writes symbolically

$$\xi = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i} \Big|_p \quad \text{for } \xi \in T_p(\mathcal{M})$$

$$\omega = \sum_{i=1}^n \omega_i dx^i \quad \text{for } \omega \in T_p(\mathcal{M})^*$$

□ Elements of  $T_p(M)$  are called contravariant vectors

□ Elements of  $T_p(M)^*$  are called covariant vectors

□ One often writes symbolically

$$\xi = \sum_{i=1}^m \xi^i \frac{\partial}{\partial x^i} \Big|_p \quad \text{for } \xi \in T_p(M)$$

$$\omega = \sum_{i=1}^m \omega_i dx^i \quad \text{for } \omega \in T_p(M)^*$$

even without specifying a particular chart.

# Tensors:

Def: A tensor,  $t$ , of rank  $(r, s)$  is an element of

$$T_p(\mathcal{M})_s^r := \underbrace{T_p(\mathcal{M}) \otimes \dots \otimes T_p(\mathcal{M})}_{r \text{ factors}} \otimes \underbrace{T_p(\mathcal{M})^* \otimes \dots \otimes T_p(\mathcal{M})^*}_{s \text{ factors}}$$

In a chart:

$$t = \sum_{\substack{i_1, \dots, i_r \\ j_1, \dots, j_s}} t_{j_1, \dots, j_s}^{i_1, \dots, i_r} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}$$

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 $\mathbb{R}$

Under chart change: (obviously incl. Einstein defined tensors this way)

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Under chart change: (physicists, incl. Einstein, defined tensors this way)

$$\bar{t}_{j_1, \dots, j_s}^{i_1, \dots, i_r} = \sum_{\substack{l_1, \dots, l_r \\ k_1, \dots, k_s}} \frac{\partial \bar{x}^{i_1}}{\partial x^{k_1}} \dots \frac{\partial \bar{x}^{i_r}}{\partial x^{k_r}} \frac{\partial x^{l_1}}{\partial \bar{x}^{j_1}} \dots \frac{\partial x^{l_s}}{\partial \bar{x}^{j_s}} t_{l_1, \dots, l_s}^{k_1, \dots, k_r}$$

In a chart:

$$t = \sum_{\substack{i_1, \dots, i_r \\ j_1, \dots, j_s = 1}}^m t_{j_1, \dots, j_s}^{i_1, \dots, i_r} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}$$

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$$\bar{t}_{j_1, \dots, j_s}^{i_1, \dots, i_r} = \sum_{\substack{i'_1, \dots, i'_r \\ j'_1, \dots, j'_s = 1}}^m \frac{\partial \bar{x}^{i_1}}{\partial x^{k_1}} \dots \frac{\partial \bar{x}^{i_r}}{\partial x^{k_r}} \frac{\partial x^{l_1}}{\partial \bar{x}^{j'_1}} \dots \frac{\partial x^{l_s}}{\partial \bar{x}^{j'_s}} t_{l_1, \dots, l_s}^{k_1, \dots, k_r}$$

Thus:  $T_p(M) = T_p(M)'$  and  $T_p(M)^* = T_p(M)$ , i.e.:

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Thus:  $T_p(M) = T_p(M)'$  and  $T_p(M)^+ = T_p(M)$ , i.e.:

- a tangent vector is a tensor of rank  $(1,0)$
- a cotangent vector is a tensor of rank  $(0,1)$

Consider now all  $(p, T_p(M))$ :

Def: We call  $T(M) := \bigcup_{p \in M} (p, T_p(M))$ ,  
the Tangent bundle. We want to turn it into a  $2n$ -dim. mfd!

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Def:  $T(M)$  is then also called the "total space".

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Def: The map  $\pi: T(M) \rightarrow M$   
 $\pi: (p, T_p(M)) \rightarrow p$  (i.e.:  $\pi^{-1}(p) = T_p(M)$ )

is called the bundle projection.

Def: Recall that all  $T_p(M)$  are  $n$ -dimensional real vector spaces, i.e., are isomorphic to  $\mathbb{R}^n$ . We therefore here call  $\mathbb{R}^n$  the "Standard Fibre".

Remark: One obtains other Fibre bundles

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Remark: One obtains other Fibre bundles by choosing other standard fibres.

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$\square$  Here:  $M$  can be covered with neighborhoods  $U_r$ ,

so that means: there exists a differentiable isomorphism

$\swarrow$  or other standard fibre for other fibre bundles.

$$\pi^{-1}(U_r) \cong U_r \times \mathbb{R}^m$$

$\square$  But fibre bundles are allowed to be globally nontrivial:



For a suitable vector bundle  $B$ , we can have

$$\pi^{-1}(U_1) \cong U_1 \times \mathbb{R}^m$$

$$\pi^{-1}(U_2) \cong U_2 \times \mathbb{R}^m$$

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Def:  $\square$  A tangent vector field:

is a map  $\xi: p \rightarrow \xi_p$

$M$                        $T_p(M)$   
 $\downarrow$                        $\downarrow$

In a chart:  $\xi = \sum_{i=1}^n \xi^i(x) \frac{\partial}{\partial x^i}$

$\square$  A cotangent vector field:

is a map  $\omega: p \rightarrow \omega_p$

$M$                        $T_p(M)^*$   
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In a chart:  $\omega = \sum_{i=1}^n \omega_i(x) dx^i$

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□ A cotangent vector field:  
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□ Similarly, tensor fields:

$t: p \rightarrow t_p$ ,  $t = \sum t_{i_1, \dots, i_s}^{j_1, \dots, j_r}(x) \frac{\partial}{\partial x^{j_1}} \dots \frac{\partial}{\partial x^{j_r}} dx^{i_1} \dots dx^{i_s}$

Def: Each field is called a Section of its fibre bundle.

Definition: For the algebra of  $(C^\infty)$  functions  $M \rightarrow \mathbb{R}$   
we write  $\mathcal{F}(M)$ .

Note: One can show that vector fields  
are the derivations of the algebra  $\mathcal{F}(M)$ :

If  $\xi$  is a contravariant vector field, then

$$\xi: \mathcal{F}(M) \rightarrow \mathcal{F}(M)$$

is linear and obeys the Leibniz rule

$$\xi(fg) = \xi(f)g + f\xi(g)$$

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Important, because of their special transformation properties under chart changes, are these special tensor fields:

## Differential forms:

□ The set  $\Lambda_0 := \mathcal{F}(M)$  is called the set of 0-forms.

□ The set of covariant vector fields