Title: Coalgebras, Models and Logics for Quantum Systems

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Abstract: Coalgebras
are a flexible tool commonly used in computer science to model abstract devices and systems. Coalgebraic models also come with a natural notion of logics for the systems being modelled. In this talk we will introduce coalgebras and aim to illustrate their usefulness for modelling physical systems.
Extending earlier work of Abramsky, we will show how a weakening of the usual morphisms for coalgebras provides the flexibility to model quantum systems in an easy to motivate manner.

We
will then investigate how a natural extension to the usual notion of coalgebraic logic can be used to produce logics for reasoning about quantum systems and protocols. No prior knowledge of coalgebras will be assumed for this talk, and the emphasis throughout will be on examples rather than technical details.

Motivation

Considering Coalgebras for Physics

- Coalgebras provide abstract models of state based systems
- Coalgebraic systems come with a natural notion of logic for reasoning about them
- Applications of coalgebra to modelling physical systems is a relatively new area with scope for innovation

Outline

- Outline of coalgebras and background category theory
- Representation result for the unitary group in a coalgebraic setting
- Introduction to coalgebraic logic
- Application of coalgebraic logic to quantum systems
- Examples all the way

Categories

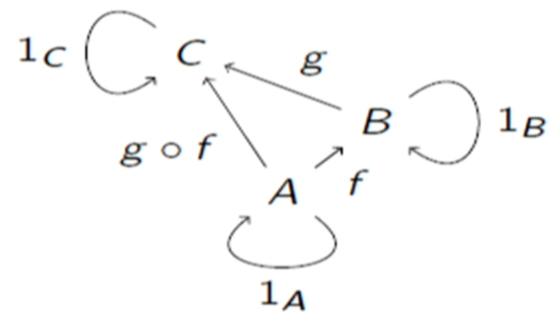
Definition

A category $\mathcal{C}$ consists of:

- A class of **objects** $\text{obj}(\mathcal{C})$
- For each pair of objects A, B, a set of **morphisms** $\mathcal{C}(A, B)$
- For objects A, B, C a composition operation:

$$\mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)$$
$$(g, f) \mapsto g \circ f$$

- For every object A an identity morphism $1_A \in \mathcal{C}(A, A)$



Associativity axiom:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Identity axioms:

$$f \circ 1_A = f, \ 1_B \circ f = f$$

Categories

Examples - Objects and their Transformations

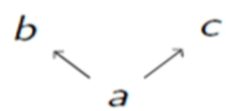
- Sets and (total) functions between them, denoted **Set**
- Sets and partial functions
- Sets and binary relations between them
- Partially ordered sets and monotone functions
- Topological Spaces and continuous functions
- Groups and group homomorphism, monoids and monoid homomorphisms, vector spaces and linear maps...

Categories

Examples - Other Structures

Ordered Structures

Partially ordered set $(\{a, b, c\}, \leq)$ can be represented as:



Functors

Definition

A functor $F : \mathcal{C} \to \mathcal{D}$ consists of:

- A mapping:

$$\begin{aligned} \mathrm{obj}(\mathcal{C}) &\to \mathrm{obj}(\mathcal{D}) \\ A &\mapsto F(A) \end{aligned}$$

- For each pair of $\mathcal{C}$ objects A, B a mapping:

$$\begin{aligned} \mathcal{C}(A,B) &\to \mathcal{D}(F(A),F(B)) \\ f &\mapsto F(f) \end{aligned}$$

With **identity** axiom:

$$F(1_A) = 1_{F(A)}$$

and **composition axiom**:

$$F(g \circ f) = F(g) \circ F(f)$$

Functors

What does Functoriality Mean?

Functors Preserve Equations

We have, for example:

$$g \circ f = i \circ h \Rightarrow F(g) \circ F(f) = F(i) \circ F(h)$$

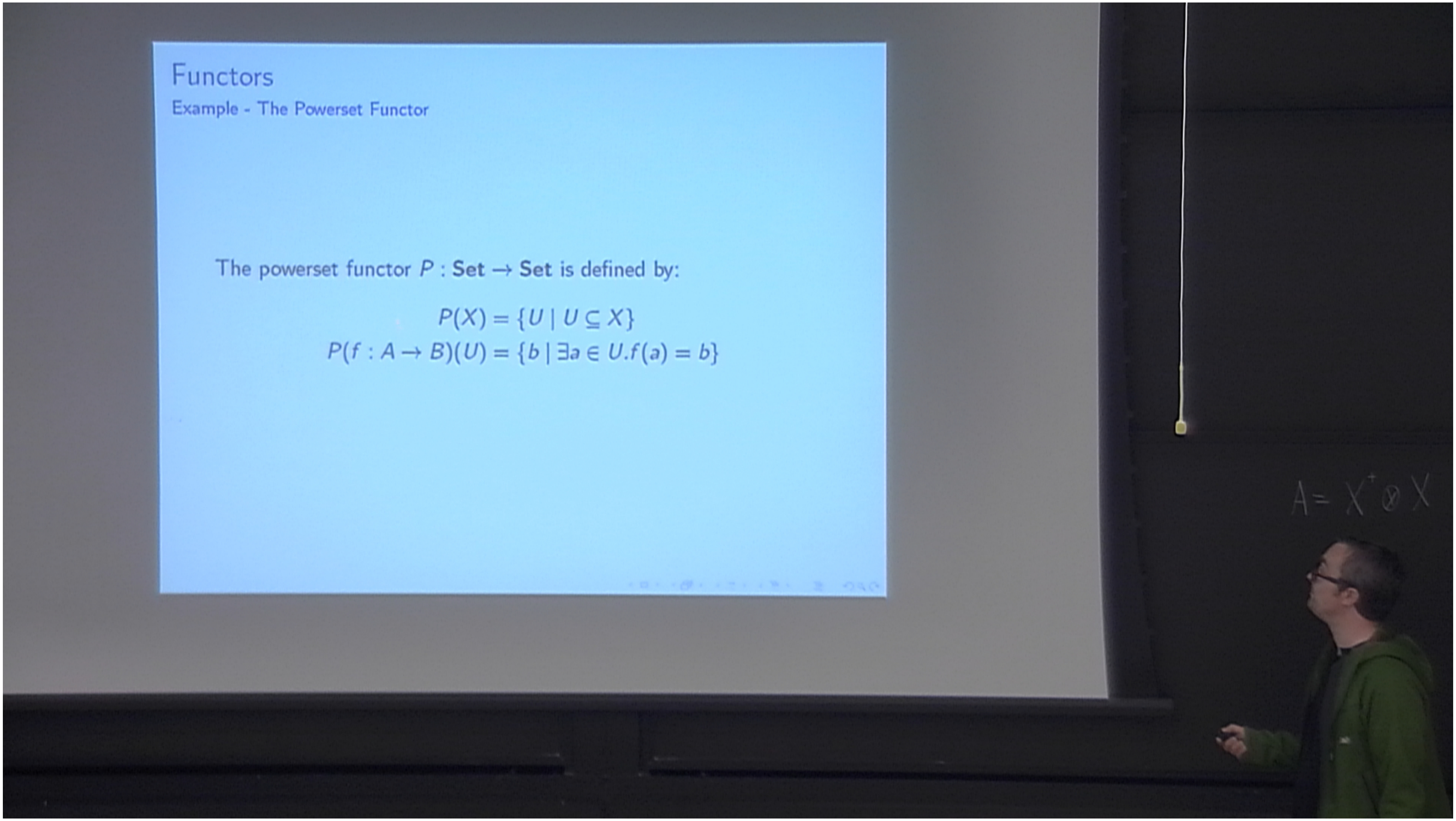
Functors

Simple Examples

- Identity functors $1 : \mathcal{C} \to \mathcal{C}$
- Constant functors, for an object D in category $\mathcal{D}$ define:

$$\begin{aligned} D : \mathcal{C} &\to \mathcal{D} \\ A &\mapsto D \\ f &\mapsto 1_D \end{aligned}$$

- Functors between partial orders as categories are order preserving functors
- Functors between monoids as categories are monoid homomorphisms
- Functors between groups as categories are group homomorphisms


Functors
Example - The Powerset Functor
The powerset functor $P : \mathbf{Set} \to \mathbf{Set}$ is defined by:
$P(X) = \{U \mid U \subseteq X\}$
$P(f : A \to B)(U) = \{b \mid \exists a \in U . f(a) = b\}$

Functors

Example - The Powerset Functor

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Functors

Example - The Contravariant Powerset Functor

Contravariance

A functor F is said to be **contravariant** if it reverses the direction of morphisms.

$$X \xrightarrow{f} Y \xrightarrow{g} Z \;\mapsto\; F(X) \xleftarrow{F(f)} F(Y) \xleftarrow{F(g)} F(Z)$$

Example

The contravariant powerset functor $2 : \mathbf{Set} \to \mathbf{Set}$ is defined by:

$$2(X) = \{U \mid U \subseteq X\}$$
$$2(f) = f^{-1} : 2(Y) \to 2(X)$$

Functors

Example - Finite Distributions

Finite Support

A function $f : X \to [0,1]$ is said to have **finite support** if $f(x) \neq 0$ for only finitely many distinct $x \in X$.

The Finite Distribution Functor

The finite distribution functor D : **Set** $\to$ **Set** is defined by:

$$D(X) = \{d : X \to [0,1] \mid \sum d(x) = 1 \text{ and } d \text{ has finite support}\}$$

$$D(f : X \to Y)(d)(y) = \Sigma_{f(x)=y} d(x)$$

Functors

Example - Cartesian Products

Product Categories

Given categories $\mathcal{C}$ and $\mathcal{D}$, the **product category** $\mathcal{C} \times \mathcal{D}$ is defined as having:

- Objects: Pairs (C, D) with C a $\mathcal{C}$ object and D a $\mathcal{D}$ object
- Morphisms: A morphism of type $(C, D) \to (C', D')$ is a pair (f, g) consisting of a $\mathcal{C}$ morphism $f : C \to C'$ and a $\mathcal{D}$ morphism $g : D \to D'$

Cartesian Products

Cartesian products form a (bi)functor $\times : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$
$$(f \times g)(x, y) = (f(x), g(y))$$

Functors

Example - Cartesian Products

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Functors

Example - Disjoint Unions

Disjoint unions form a (bi)functor $+ : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$:

$$A + B = \{(0, a) \mid a \in A\} \cup \{(1, b) \mid b \in B\}$$
$$(f + g)(0, a) = (0, f(a))$$
$$(f + g)(1, b) = (1, g(b))$$

Functors

Function Spaces with Fixed Domain

If we consider functions with a fixed domain A we get a functor $(-)^A : \mathbf{Set} \to \mathbf{Set}$:

$$X^A = \{f : A \to X\}$$
$$g^A(f : A \to X) = g \circ f : A \to Y$$

Coalgebras

Definition

T-coalgebras

Let $T : \mathcal{C} \to \mathcal{C}$ be an endofunctor. A T-coalgebra is a pair:

$$(X, \gamma : X \to T(X))$$

- X is an object of **abstract states**
- The endofunctor T gives the structure of **observations**
- The morphism γ gives the **dynamics** relating states to observations

Coalgebras

Example - A System with Transitions

$$x \longrightarrow x' \longrightarrow x'' \longrightarrow x''' \longrightarrow \cdots$$

Coalgebras of the identity functor

A coalgebra is a function $\gamma : X \to X$, the dynamics transition each state to a new state.

Coalgebras

Example - A System with Failures

$$x \longrightarrow x' \longrightarrow x'' \longrightarrow x''' \longrightarrow \cdots$$

$$y \longrightarrow y' \longrightarrow y'' \longrightarrow *$$

Coalgebras encoding potential failure

We consider coalgebras of the form:

$$\gamma : X \to \{*\} + X$$

The dynamics transition each state to either a new state, or terminate.

Coalgebras

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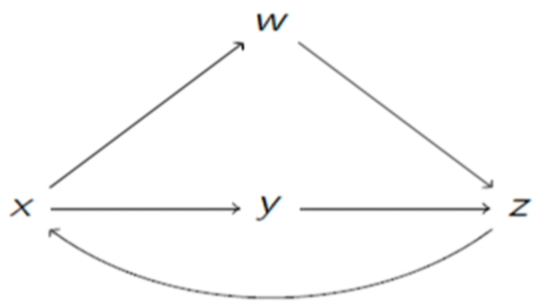
Coalgebras

Example - A System with Non-Deterministic Transitions



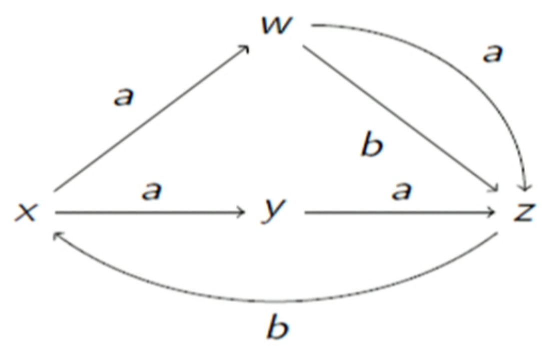
Coalgebras with Non-Determinism

We consider coalgebras of the form:

$$\gamma : X \to P(X)$$

Coalgebras

Example - A System with Non-Determinism and Labelled Transitions



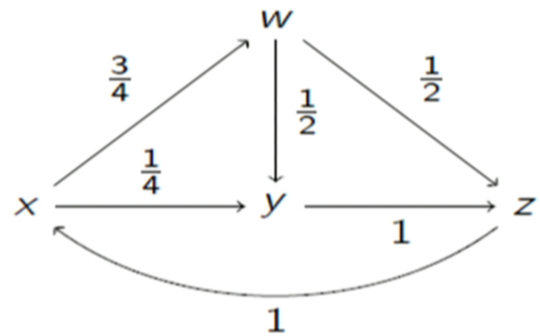
Non-Determinism and Labelled Transitions

We wish to model non-deterministic transitions with labels from a set Σ. We consider coalgebras of the form:

$$\gamma : X \to P(X)^{\Sigma}$$

Coalgebra

Example - A System with Probabilistic Transitions



Finite Probabilistic Behaviour

We consider coalgebras of the form:

$$\gamma : X \to D(X)$$

The Quantum Signature Functor

A Question and Answer System

For a Hilbert space $\mathcal{H}$, let $\mathcal{L}(\mathcal{H})$ be the (orthomodular lattice of) projection operators on $\mathcal{H}$. We then define the quantum signature functor:

$$\begin{aligned} Q : \mathbf{Set} &\to \mathbf{Set} \\ X &\mapsto (\{*\} + (0,1] \times X)^{\mathcal{L}(\mathcal{H})} \end{aligned}$$

The Quantum Coalgebra

We choose a Hilbert space to describe our quantum system. We will then define a coalgebra on the corresponding projective Hilbert space as follows:

$$\gamma_q([|\varphi\rangle])(\hat{P}) := \begin{cases} (\frac{\langle\varphi|\hat{P}|\varphi\rangle}{\langle\varphi|\varphi\rangle}, [\hat{P}|\varphi\rangle]) & \text{if } \langle\varphi \mid \hat{P} \mid \varphi\rangle \neq 0 \\ \star & \text{otherwise} \end{cases}$$

Bisimilarity

Definition

Bisimulation

A relation $R \subseteq X \times Y$ is a **bisimulation** if:

$$\begin{array}{ccccc} X & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & Y \\ \downarrow{\scriptstyle \gamma_X} & & \downarrow{\scriptstyle \gamma_R} & & \downarrow{\scriptstyle \gamma_Y} \\ T(X) & \xleftarrow[T(\pi_1)]{} & T(R) & \xrightarrow[T(\pi_2)]{} & T(Y) \end{array}$$

Two states are said to be **bisimilar** if there is a bisimulation between them.

Informal Intuition

A bisimulation relates states that have the same *observable behaviour*.

Bisimilarity

Examples

- For the identity functor, for all x, y we have $x \sim y$
- For deterministic systems with failure, $x \sim y$ implies:
  1. $x \to * \Leftrightarrow y \to *$
  2. $x \to x' \wedge y \to y' \Rightarrow x' \sim y'$
- For labelled transition systems, $x \sim y$ implies:
  1. $x \xrightarrow{a} x' \Rightarrow \exists y'.y \xrightarrow{a} y' \wedge x' \sim y'$
  2. $y \xrightarrow{a} y' \Rightarrow \exists x'.x \xrightarrow{a} x' \wedge x' \sim y'$
- For our quantum coalgebra each state is in a distinct bisimilarity class, this is referred to as **strong extensionality**

Coalgebra Homomorphisms

A coalgebra homomorphism between coalgebras (X, γ_X) and (Y, γ_Y) is a morphism in the base category such that the following diagram commutes:

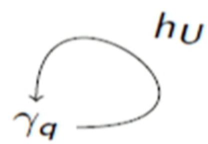
$$\begin{array}{ccc} X & \xrightarrow{\;h\;} & Y \\ {\scriptstyle \gamma_X}\downarrow & & \downarrow{\scriptstyle \gamma_Y} \\ T(X) & \xrightarrow[T(h)]{} & T(Y) \end{array}$$

Preservation of Observable Behaviour

Coalgebra homomorphisms preserve bisimilarity.

A Difficulty

Conflicting Requirements



For our quantum coalgebra γ_q we may wish to model each unitary transformations as coalgebra morphisms $h_U : \gamma_q \to \gamma_q$. We then note:

- If we evolve the state of a physical system, we would expect the observable behaviour to change
- Coalgebra homomorphisms preserve observable behaviour

Natural Transformations

Definition

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** of type $F \Rightarrow G$ is a map between functors, defined as a family $(\alpha_X : F(X) \to G(X))_{X \in \mathrm{obj}(\mathcal{C})}$ of $\mathcal{D}$ morphisms such that diagrams of the following form commute:

$$
\begin{array}{ccc}
F(X) & \xrightarrow{\alpha_X} & G(X) \\
{\scriptstyle F(h)}\downarrow & & \downarrow{\scriptstyle G(h)} \\
F(Y) & \xrightarrow[\alpha_Y]{} & G(Y)
\end{array}
$$

Natural Transformations

Examples

Type	Example
$1 \Rightarrow 1$	$x \mapsto x$
$1 \Rightarrow P$	$x \mapsto \{x\}$
$P \circ P \Rightarrow P$	$U \mapsto \bigcup_{V \in U} V$
$A \times (-) \Rightarrow 1$	$(a, x) \mapsto x$

Intuition

A natural transformation of type $F \Rightarrow G$ can only use the structure given by the functors F, G. In particular it cannot manufacture arbitrary values or have "special cases".

Schrödinger Evolution

We can implement a Schrödinger style evolution of the states of our quantum system in which *states change* and *the observations remain fixed* in the obvious way:

$$h_U : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H})$$
$$|\psi\rangle \mapsto \hat{U}|\psi\rangle$$

By definition this is an operation on a specific state space. We note that this is *not* a coalgebra homomorphism as states are mapped into different bisimilarity classes.

Heisenberg Evolution

We could also adopt a Heisenberg perspective in which the *state remains fixed* and *the observations change* as follows:

$$\hat{P} \mapsto \hat{U}^{\dagger} \hat{P} \hat{U}$$

Precomposition with this map gives a natural automorphism:

$$\alpha^{U} : Q \to Q$$

These automorphisms induce an endofunctor on the category of Q-coalgebras:

$$(X, \gamma) \mapsto (X, \alpha^{U}_{X} \circ \gamma)$$

Weakening Coalgebra Homomorphisms

For signature functor T and G a subgroup of the natural automorphisms of T, we define a category T-**PseudoCoalg**(G) with objects T-coalgebras and morphisms $h : X \to Y$ such that there exist $\alpha, \beta : T \Rightarrow T$ with:

$$
\begin{array}{ccc}
X & \xrightarrow{\;h\;} & Y \\
\gamma_1 \downarrow & & \\
TX & & \downarrow \gamma_2 \\
\alpha_X \downarrow & & \\
TX & \xrightarrow[\;Th\;]{} & TY
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{\;h\;} & Y \\
 & & \downarrow \gamma_2 \\
\gamma_1 \downarrow & & TY \\
 & & \downarrow \beta_Y \\
TX & \xrightarrow[\;Th\;]{} & TY
\end{array}
$$

Representation Results

$$
\begin{array}{ccccc}
\mathcal{P}(\mathcal{H}) & & \xrightarrow{\quad\gamma_q\quad} & & Q(\mathcal{P}(\mathcal{H})) \\
{\scriptstyle h_U}\big\downarrow & & & & \big\downarrow{\scriptstyle Q(h_U)} \\
\mathcal{P}(\mathcal{H}) & \xrightarrow[\gamma_q]{} & Q(\mathcal{P}(\mathcal{H})) & \xrightarrow[\alpha^U]{} & Q(\mathcal{P}(\mathcal{H}))
\end{array}
$$

- That there exists a α^U that the above commutes follows from (physics) covariance between the Schrödinger and Heisenberg pictures
- That there is a distinct h_U for each equivalence class of unitaries follows from Wigner's Theorem
- That these are the only automorphisms of the quantum coalgebra follows from strong extensionality and the h_U being bijections

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Sketch of Propositional Logic

Classical propositional logic consists of a set of proposition variables:

$$\{p, q, ...\}$$

These are combined into formulae using the connectives:

Truth	$\top$
Falsehood	$\bot$
Negation	$\neg$
Conjunction	$\psi \wedge \varphi$
Disjunction	$\psi \vee \varphi$

For example:

$$p \wedge (\neg q \vee r)$$

Modal Logics

Modal logic extends propositional logic with additional **modalities** or **modal operators**.

Examples

Some possible modalities we might consider:

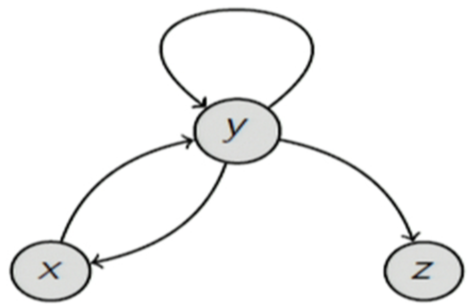
Formula	**Interpretation**
$\Box\varphi$	φ is certain
$\Diamond\varphi$	φ is possible
$\Box_{Alice}\varphi$	Alice knows φ is true
$\Box_{Bob}\varphi$	Bob believes φ is true
$L_p\varphi$	φ is true with probability at least p

An example formula might be:

$$\Diamond p \wedge \Box(q \vee r)$$

Sketch of the Semantics of Modal Logic

We consider a directed graph:



We define the semantics of $\Box$ for a given binary relation R as:

$$x \models \Box\varphi \text{ iff } \forall y.(x \to y) \Rightarrow y \models \varphi$$

e.g.

$$x \not\models \Box\bot \quad y \not\models \Box\bot \quad z \models \Box\bot$$

Modal Logic Coalgebraically

We note a directed graph can be described as a coalgebra:

$$\gamma : X \to P(X)$$

We define the semantics of a formula φ as follows:

$$[\![\varphi]\!] := \{x \mid x \models \varphi\}$$

Define the natural transformation $[\![\Box]\!] : 2 \Rightarrow 2 \circ P$:

$$[\![\Box]\!]_X(U) := \{V \mid V \subseteq U\}$$

Then we can decompose the semantics of the $\Box$ modality as:

$$[\![\Box\varphi]\!] = \gamma^{-1} \circ [\![\Box]\!]_X([\![\varphi]\!])$$

Modal Logic Coalgebraically

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Coalgebraic Modal Logic

Generalizing the previous situation, we consider logics for arbitrary coalgebras:

$$\gamma : X \to T(X)$$

The semantics for a modality $\heartsuit$ is given by a **predicate lifting**:

$$[\![\heartsuit]\!] : 2 \Rightarrow 2 \circ T$$

Given a predicate lifting, our semantics decomposes as:

$$[\![\heartsuit\varphi]\!] = \gamma^{-1} \circ [\![\heartsuit]\!]_X([\![\varphi]\!])$$

A Distribution Based Signature for Quantum Systems

Quantum Signature Functor

For finite dimensional Hilbert space of dimensional n, denote the set of self adjoint operators $\mathcal{A}_n$, and define the endofunctor:

$$Q_n^d := D(\mathbb{R} \times (-))^{\mathcal{A}_n}$$

The Quantum Coalgebra

For n dimensional Hilbert space, we can define a Q_n^d-coalgebra on the set of pure states, giving the probability distributions over measurement outcomes and subsequent states as described by the Hilbert space formalism.

Basic Modalities for the Quantum Distribution Functor

A predicate lifting based on equalities:

$$[\![\mathrm{Eq}_{p,r,\hat{A}}]\!]_X(U) := \{f \mid \Sigma_{u\in U} f(\hat{A})(r,u) = p\}$$

A predicate lifting based on inequalities:

$$[\![\mathrm{Geq}_{p,r,\hat{A}}]\!]_X(U) := \{f \mid \Sigma_{u\in U} f(\hat{A})(r,u) \geq p\}$$

Expressivity

Both these types of modalities are sufficient to distinguish states of any model up to behavioural equivalence. This property is referred to as **expressivity**.

Functors Induced by Natural Transformations

$$\begin{array}{ccccc} X & \xrightarrow{\gamma} & T(X) & \xrightarrow{\alpha_X} & T'(X) \\ \downarrow h & & \downarrow T(h) & & \downarrow T'(h) \\ Y & \xrightarrow[\gamma']{} & T(Y) & \xrightarrow[\alpha_Y]{} & T'(Y) \end{array}$$

Every natural transformation $\alpha : T \Rightarrow T'$ induces a functor between the corresponding categories of coalgebras as follows:

$$\begin{aligned} \alpha^* : \text{T-}\mathbf{Coalg} &\to \text{T'-}\mathbf{Coalg} \\ (X, \gamma) &\mapsto (X, \alpha_X \circ \gamma) \\ h &\mapsto h \end{aligned}$$

Transferring Formulae Between Signatures

For a natural transformation $\alpha : T \Rightarrow T'$ we introduce a corresponding **adaptation** modality with semantics:

$$[\![\alpha\varphi]\!]_{X,\gamma} := [\![\varphi]\!]_{\alpha^*(X,\gamma)}$$

We can think of the modality α as adapting a formula applicable to T'-coalgebras to one applicable to T-coalgebras.

Loose Analogy

Using functors between "simple" models of a logic to build a richer logic is a familiar idea, think of the functors $\exists$ and $\forall$ taking propositions from one context to another in predicate logic.

Unitaries

Example

Recall we defined:

$$Q_n^d := D(\mathbb{R} \times (-))^{\mathcal{A}_n}$$

Every unitary $\hat{U}$ on n dimensional Hilbert space gives a map:

$$\begin{aligned} \mathcal{A}_n &\to \mathcal{A}_n \\ \hat{A} &\mapsto \hat{U}\hat{A}\hat{U}^\dagger \end{aligned}$$

Precomposing with this map gives a natural transformation $Q_n^d \Rightarrow Q_n^d$ giving Heisenberg evolution of the quantum state. So we can write:

$$\hat{U}\varphi$$

for "after unitary $\hat{U}$, φ holds"

Additional Useful Modalities

We also introduce some additional modalities for convenience:

- We encode "projective measurement $\hat{P}$ is certain" as 0-ary modalities:

$$\hat{P}$$

- We encode "After measurement outcome r the post condition is certain to hold" as unary modality:

$$\mathsf{C}_{r,\hat{A}}$$

- Multiple measurement outcomes can be handled with additional (similar) machinery as n-ary modalities:

$$\hat{A}(r_1 \mapsto (-), ..., r_n \mapsto (-))$$

Building Modalities

Given a predicate lifting $[\![\Box_\lambda]\!] : 2 \Rightarrow 2 \circ T$ and a natural transformation $\alpha : T' \Rightarrow T$, the following diagram commutes:

$$
\begin{array}{ccccc}
PX & \xrightarrow{[\![\Box_\lambda]\!]_X} & PT'(X) & \xrightarrow{\alpha_X^{-1}} & PT(X) \\
{\scriptstyle f^{-1}}\uparrow & & \uparrow{\scriptstyle (T(f))^{-1}} & & \uparrow{\scriptstyle (T(f))^{-1}} \\
PY & \xrightarrow[{[\![\Box_\lambda]\!]_Y}]{} & PT'(X) & \xrightarrow[\alpha_Y^{-1}]{} & PT(X)
\end{array}
$$

This gives us a new predicate lifting on T', which we will denote:

$$[\![\Box_\lambda^\alpha]\!] : 2 \Rightarrow 2 \circ T'$$

Local Operations

Example

Recall the natural transformation:

$$\text{Alice} : Q_4^d \Rightarrow Q_2^d$$

This allows us to "lift" modalities from Alice's qubit to the composite system. For example, if we consider the modality $\hat{P}$ for 1 qubit systems with semantics "projective measurement $\hat{P}$ is certain to have a positive outcome", then:

$$\hat{P}^{\text{Alice}}$$

is a 0-ary modality on the composite system describing "performing projective measurement $\hat{P}$ **on Alice's qubit** is certain to have a positive outcome".

A Syntactic Translation

Adequacy

A logic is said to be **adequate** if behaviourally equivalent states cannot be distinguished. Coalgebraic logic with predicate liftings is adequate.

Translation

For a given formula φ, potentially containing adaptation modalities, we can define a new formula ψ with no adaptation modalities such that:

$$[\![\varphi]\!] = [\![\psi]\!]$$

It follows that coalgebraic logic extended with adaptation modalities is adequate.

Quantum Teleportation

Single Shot

Example

A single statement describing quantum teleportation:

$$\hat{P}_{\varphi}^{\text{Alice}} \wedge \hat{P}_{\psi_1}^{\text{Channel}} \Rightarrow \hat{A}_{\text{Bell}}^{\text{Both}}(r_1 \mapsto \text{Bob}\,\hat{\text{U}}_1\,\hat{P}_{\varphi}, \\ r_2 \mapsto \text{Bob}\,\hat{\text{U}}_2\,\hat{P}_{\varphi}, \\ r_3 \mapsto \text{Bob}\,\hat{\text{U}}_3\,\hat{P}_{\varphi}, \\ r_4 \mapsto \text{Bob}\,\hat{\text{U}}_4\,\hat{P}_{\varphi})$$

Quantum Teleportation

Post Selection

Example

We can deal with each measurement outcome in separate statements:

$$\hat{P}^{\text{Alice}}_{\varphi} \wedge \hat{P}^{\text{Channel}}_{\psi_1} \Rightarrow \mathrm{C}^{\text{Both}}_{r_i,\hat{A}_{\text{Bell}}}(\text{Bob}\ \hat{\mathrm{U}}_{\mathrm{i}}\ \hat{P}_{\varphi})$$

Conclusion

- There is a rich theory of modelling state based systems in the computer science literature
- Coalgebraic models of quantum systems offer different possibilities to other formalisms
- There is a lot of scope for innovation in modelling physical situations and theories with coalgebraic techniques