Title: Universal fault-tolerant quantum computation with only transversal gates and error correction

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Abstract: Transversal implementations of encoded unitary gates are highly desirable for fault-tolerant quantum computation. It is known, however, that transversal gates alone cannot be computationally universal. I will show that the limitation on universality can be circumvented using only fault-tolerant error correction, which is already required anyway. This result applies to ``triorthogonal" stabilizer codes, which were recently introduced by Bravyi and Haah for state distillation. I will show that triothogonal codes admit transversal implementation of the controlled-controlled-Z gate, and then demonstrate a transversal Hadamard construction which uses error correction to preserve the codespace. I will also discuss how to adapt the distillation procedure of Bravyi and Haah to Toffoli gates, improving on existing Toffoli distillation schemes.

Universal fault-tolerant quantum computation with only **transversal** gates and error correction

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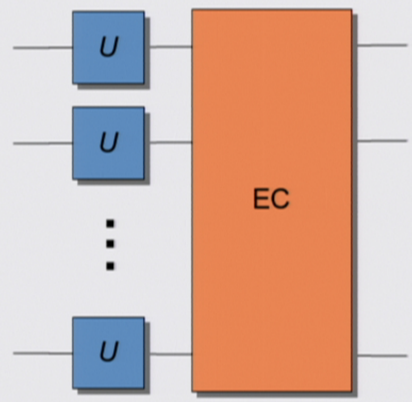

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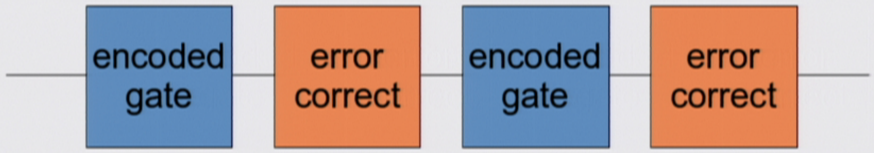
Encoded computing

Fault tolerance

1) encoded gates

2) error correction

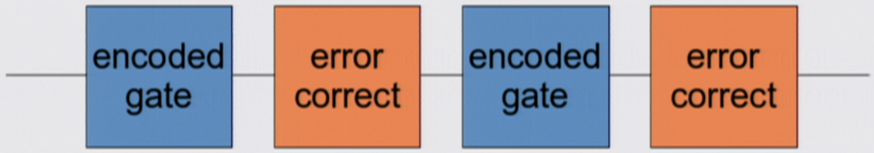


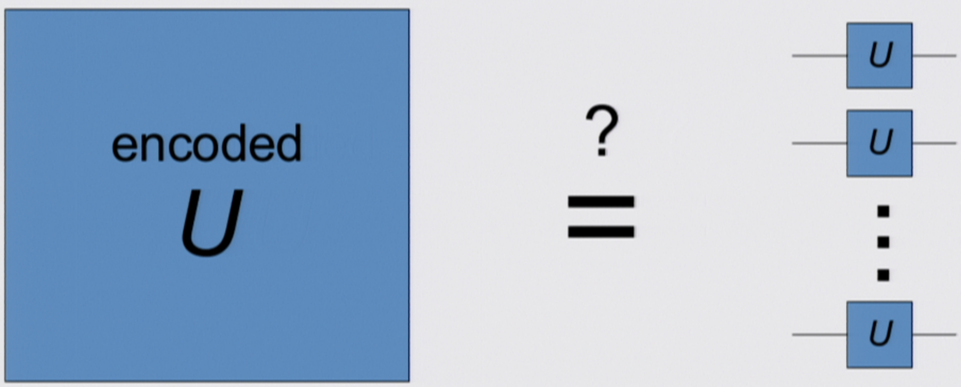
Encoded computing

Fault tolerance

1) encoded gates

2) error correction



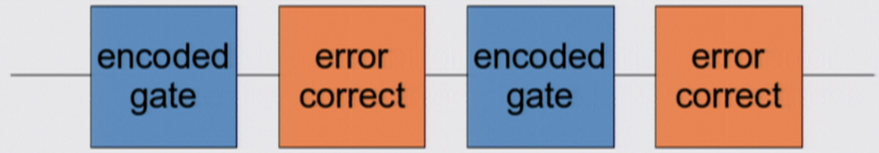


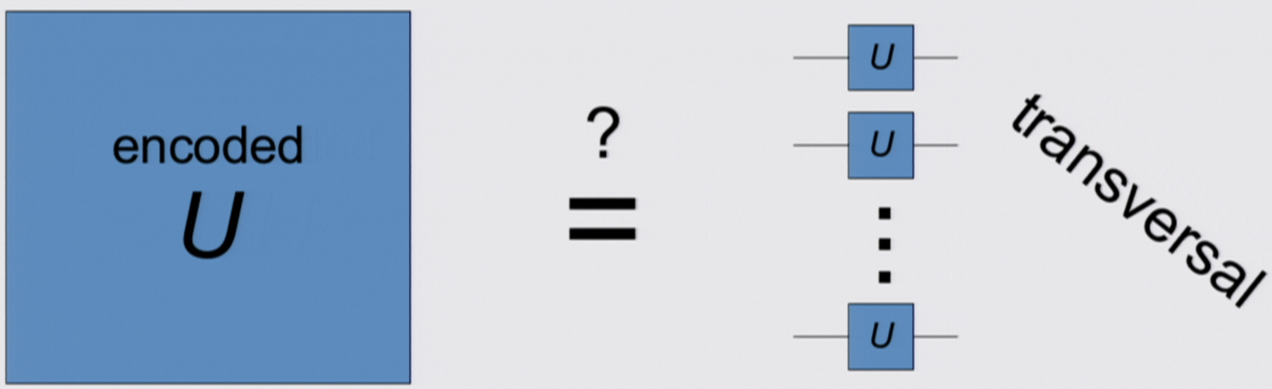
Encoded computing

Fault tolerance

1) encoded gates

2) error correction



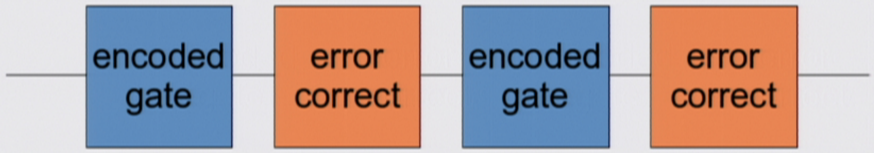


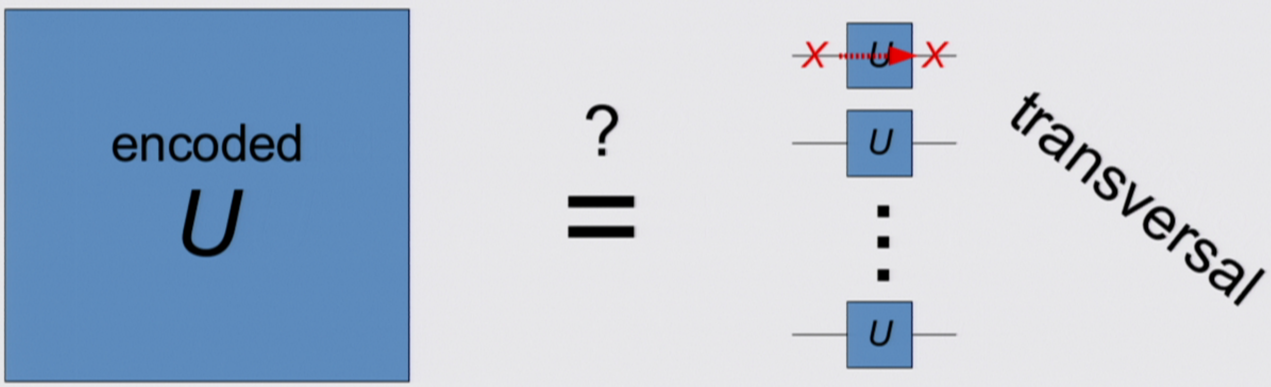
Encoded computing

Fault tolerance

1) encoded gates

2) error correction



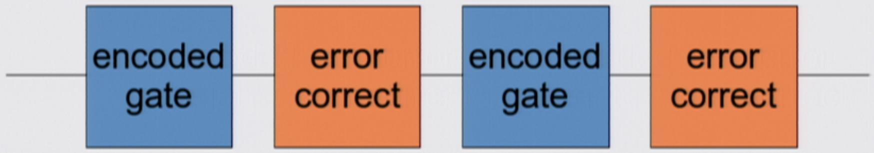


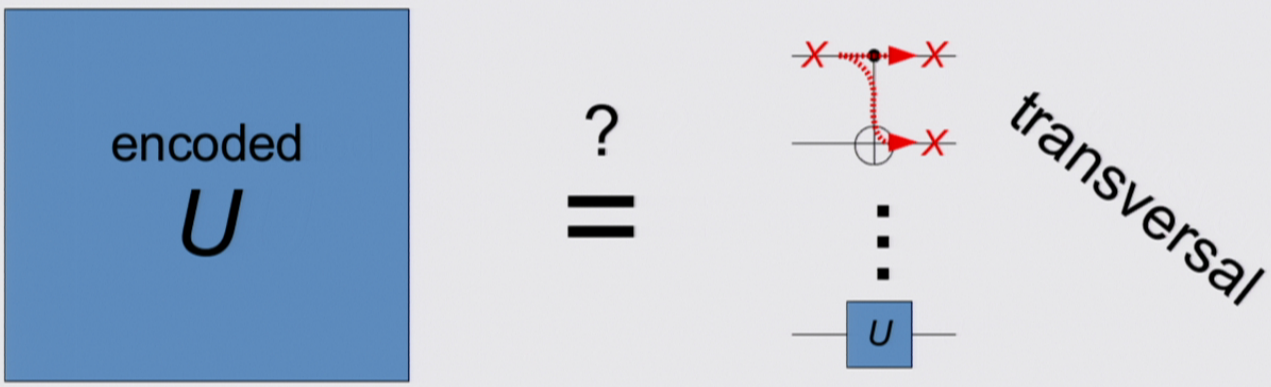
Encoded computing

Fault tolerance

1) encoded gates

2) error correction

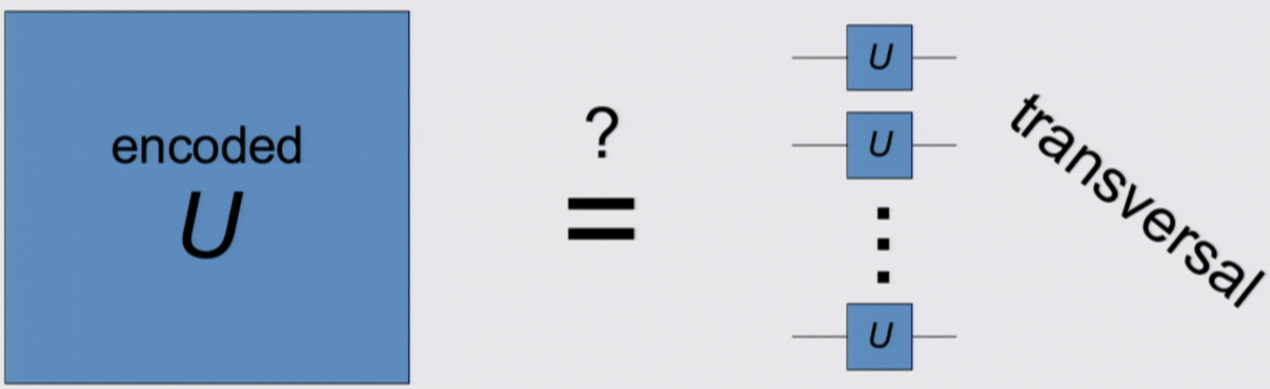




Transversal computing

Theorem [Eastin,Knill 2009]

No quantum code admits transversal implementation of a universal set of encoded gates.



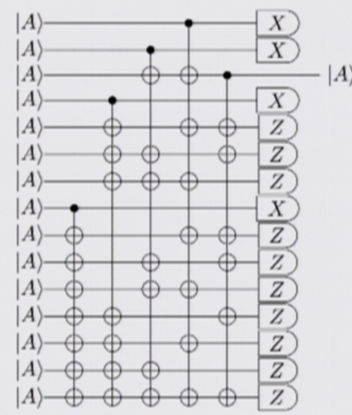
Transversal computing

Theorem [Eastin,Knill 2009]

No quantum code admits transversal implementation of a universal set of encoded gates.

State distillation

~10x cost of transversal gates



Transversal computing

Theorem [Eastin,Knill 2009]

No quantum code admits transversal implementation of a universal set of encoded gates.

Main result

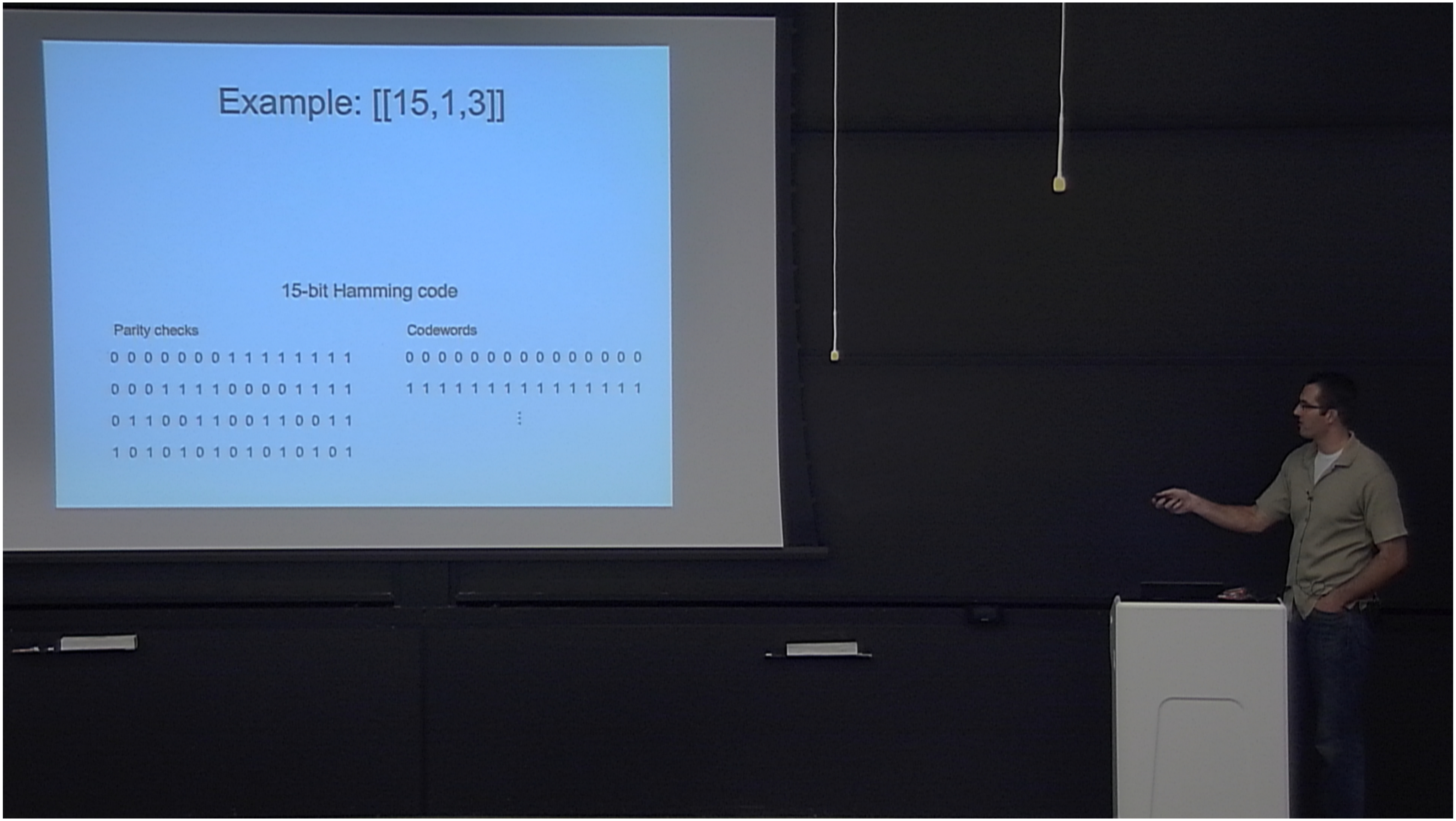
Universality is possible with only transversal gates and *error correction*.

Example: [[15,1,3]]

15-bit Hamming code

Parity checks

0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1


Example: [[15,1,3]]
15-bit Hamming code
Parity checks
0 0 0 0 0 0 0 1 1 1 1 1 1 1 1
0 0 0 1 1 1 1 0 0 0 0 1 1 1 1
0 1 1 0 0 1 1 0 0 1 1 0 0 1 1
1 0 1 0 1 0 1 0 1 0 1 0 1 0 1
Codewords
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
⋮

Example: [[15,1,3]]

15-bit Hamming code

Parity checks

. . . . X X X X X X X X

. . . X X X X . . . . X X X X

. X X . . X X . . X X . . X X

X . X . X . X . X . X . X . X

Codewords

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

⋮

Example: [[15,1,3]]

15-bit Hamming code

~~Parity checks~~ Stabilizers	Codewords
. . . . X X X X X X X X	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
. . . X X X X . . . . X X X X	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
. X X . . X X . . X X . . X X	⋮
X . X . X . X . X . X . X . X	

Example: [[15,1,3]]

15-bit Hamming code

~~Parity checks~~ Stabilizers

```
. . . . . . . X X X X X X X X
. . . X X X X . . . . X X X X
. X X . . X X . . X X . . X X
X . X . X . X . X . X . X . X
-----------------------------
X X X X X X X X X X X X X X X
```

Example: [[15,1,3]]

15-bit Hamming code

~~Parity checks~~ Stabilizers

```
. . . . . . . X X X X X X X X
. . . X X X X . . . . X X X X
. X X . . X X . . X X . . X X
X . X . X . X . X . X . X . X
-----------------------------
X X X X X X X X X X X X X X X
```

Example: [[15,1,3]]

```
. . . . . . . X X X X X X X X        . . . . . . . Z Z Z Z Z Z Z Z
. . . X X X X . . . . X X X X        . . . Z Z Z Z . . . . Z Z Z Z
. X X . . X X . . X X . . X X        . Z Z . . Z Z . . Z Z . . Z Z
X . X . X . X . X . X . X . X        Z . Z . Z . Z . Z . Z . Z . Z
-----------------------------        -----------------------------
X X X X X X X X X X X X X X X        Z Z Z Z Z Z Z Z Z Z Z Z Z Z Z
```

Additional Z rows (shown in red, above the black Z rows):

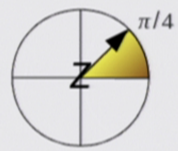
```
Z . . . . . Z . . Z . Z . . .
. Z . . . . Z . Z . . Z . . .
. . Z . . . Z Z . . . Z . . .
. . . Z . . Z . Z Z . . . . .
. . . . Z . Z Z . Z . . . . .
. . . . . Z Z Z Z . . . . . .
```

Example: [[15,1,3]]

```
                                    Z . . . . . Z . . Z . Z . . .
                                    . Z . . . . Z . Z . . Z . . .
                                    . . Z . . . Z Z . . . Z . . .
                                    . . . Z . . Z . Z Z . . . . .
                                    . . . . Z . Z Z . Z . . . . .
                                    . . . . . Z Z Z Z . . . . . .
. . . . . . . X X X X X X X X       . . . . . . . Z Z Z Z Z Z Z Z
. . . X X X X . . . . X X X X       . . . Z Z Z Z . . . . Z Z Z Z
. X X . . X X . . X X . . X X       . Z Z . . Z Z . . Z Z . . Z Z
X . X . X . X . X . X . X . X       Z . Z . Z . Z . Z . Z . Z . Z
-----------------------------       -----------------------------
X X X X X X X X X X X X X X X       Z Z Z Z Z Z Z Z Z Z Z Z Z Z Z
```

Example: [[15,1,3]]

Fact
CNOT and *T* are transversal

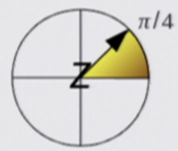
T =



```
Z . . . . . Z . . Z . Z . . .
. Z . . . . Z . Z . . Z . . .
. . Z . . . Z Z . . . Z . . .
. . . Z . . Z . Z Z . . . . .
. . . . Z . Z Z . Z . . . . .
. . . . . Z Z Z Z . . . . . .
```

```
. . . . . . . X X X X X X X X        . . . . . . . Z Z Z Z Z Z Z Z
. . . X X X X . . . . X X X X        . . . Z Z Z Z . . . . Z Z Z Z
. X X . . X X . . X X . . X X        . Z Z . . Z Z . . Z Z . . Z Z
X . X . X . X . X . X . X . X        Z . Z . Z . Z . Z . Z . Z . Z
-----------------------------        -----------------------------
X X X X X X X X X X X X X X X        Z Z Z Z Z Z Z Z Z Z Z Z Z Z Z
```

Example: [[15,1,3]]

Fact
CNOT and *T* are transversal

T =


What about Hadamard?

```
. . . . . . . X X X X X X X X
. . . X X X X . . . . X X X X
. X X . . X X . . X X . . X X
X . X . X . X . X . X . X . X
-----------------------------
X X X X X X X X X X X X X X X
```

```
Z . . . . . Z . . Z . Z . . .
. Z . . . . Z . Z . . Z . . .
. . Z . . . Z Z . . . Z . . .
. . . Z . . Z . Z Z . . . . .
. . . . Z . Z Z . Z . . . . .
. . . . . Z Z Z Z . . . . . .
. . . . . . . Z Z Z Z Z Z Z Z
. . . Z Z Z Z . . . . Z Z Z Z
. Z Z . . Z Z . . Z Z . . Z Z
Z . Z . Z . Z . Z . Z . Z . Z
-----------------------------
Z Z Z Z Z Z Z Z Z Z Z Z Z Z Z
```

Example: [[15,1,3]]

H: $X \mapsto HXH = Z$

$Z \mapsto HZH = X$

What about Hadamard?

```
Z . . . . . Z . . Z . Z . . .
. Z . . . . Z . Z . . Z . . .
. . Z . . . Z Z . . . Z . . .
. . . Z . . Z . Z Z . . . . .
. . . . Z . Z Z . Z . . . . .
. . . . . Z Z Z Z . . . . . .
```

```
. . . . . . . X X X X X X X X      . . . . . . . Z Z Z Z Z Z Z Z
. . . X X X X . . . . X X X X      . . . Z Z Z Z . . . . Z Z Z Z
. X X . . X X . . X X . . X X      . Z Z . . Z Z . . Z Z . . Z Z
X . X . X . X . X . X . X . X      Z . Z . Z . Z . Z . Z . Z . Z
-----------------------------      -----------------------------
X X X X X X X X X X X X X X X      Z Z Z Z Z Z Z Z Z Z Z Z Z Z Z
```

Example: [[15,1,3]]

$$H: X \mapsto HXH = Z$$
$$Z \mapsto HZH = X$$

What about Hadamard?

```
                                    Z . . . . . Z . . Z . Z . . .
                                    . Z . . . . Z . Z . . Z . . .
                                    . . Z . . . Z Z . . . Z . . .
                                    . . . Z . . Z . Z Z . . . . .
                                    . . . . Z . Z Z . Z . . . . .
                                    . . . . . Z Z Z Z . . . . . .
. . . . . . . X X X X X X X X       . . . . . . . Z Z Z Z Z Z Z Z
. . . X X X X . . . . X X X X       . . . Z Z Z Z . . . . Z Z Z Z
. X X . . X X . . X X . . X X       . Z Z . . Z Z . . Z Z . . Z Z
X . X . X . X . X . X . X . X       Z . Z . Z . Z . Z . Z . Z . Z
-----------------------------       -----------------------------
X X X X X X X X X X X X X X X       Z Z Z Z Z Z Z Z Z Z Z Z Z Z Z
```

Example: [[15,1,3]]

```
X . . . . . X . . X . X . . .
. X . . . . X . X . . X . . .
. . X . . . X X . . . X . . .
. . . X . . X . X X . . . . .
. . . . X . X X . X . . . . .
. . . . . X X X X . . . . . .
. . . . . . . X X X X X X X X     . . . . . . . Z Z Z Z Z Z Z Z
. . . X X X X . . . . X X X X     . . . Z Z Z Z . . . . Z Z Z Z
. X X . . X X . . X X . . X X     . Z Z . . Z Z . . Z Z . . Z Z
X . X . X . X . X . X . X . X     Z . Z . Z . Z . Z . Z . Z . Z
-----------------------------     -----------------------------
X X X X X X X X X X X X X X X     Z Z Z Z Z Z Z Z Z Z Z Z Z Z Z
```

Example: [[15,1,3]]

```
X . . . . . X . . X . X . . .      ± Z . . . . . Z . . Z . Z . . .
. X . . . . X . X . . X . . .
. . X . . . X X . . . X . . .
. . . X . . X . X X . . . . .
. . . . X . X X . X . . . . .
. . . . . X X X X . . . . . .
. . . . . . . X X X X X X X X      . . . . . . . Z Z Z Z Z Z Z Z
. . . X X X X . . . . X X X X      . . . Z Z Z Z . . . . Z Z Z Z
. X X . . X X . . X X . . X X      . Z Z . . Z Z . . Z Z . . Z Z
X . X . X . X . X . X . X . X      Z . Z . Z . Z . Z . Z . Z . Z
-----------------------------      -----------------------------
X X X X X X X X X X X X X X X      Z Z Z Z Z Z Z Z Z Z Z Z Z Z Z
```

Example: [[15,1,3]]

X . . . . . X . . X . X . . .	± Z . . . . . Z . . Z . Z . . .
. X . . . . X . X . . X . . .	
. . X . . . X X . . . X . . .	
. . . X . . X . X X	
. . . . X . X X . X	
. . . . . X X X X	
. . . . X X X X X X X X	. . . . Z Z Z Z Z Z Z Z
. . . X X X X . . . . X X X X	. . . Z Z Z Z . . . . Z Z Z Z
. X X . . X X . . X X . . X X	. Z Z . . Z Z . . Z Z . . Z Z
X . X . X . X . X . X . X . X	Z . Z . Z . Z . Z . Z . Z . Z
X X X X X X X X X X X X X X X	Z Z Z Z Z Z Z Z Z Z Z Z Z Z Z

Example: [[15,1,3]]

```
                                    ± Z . . . . . Z . . Z . Z . . .
                                    ± . Z . . . . Z . Z . . Z . . .
                                    ± . . Z . . . Z Z . . . Z . . .
                                    ± . . . Z . . Z . Z Z . . . . .
                                    ± . . . . Z . Z Z . Z . . . . .
                                    ± . . . . . Z Z Z Z . . . . . .
. . . . . . . X X X X X X X X         . . . . . . . Z Z Z Z Z Z Z Z
. . . X X X X . . . . X X X X         . . . Z Z Z Z . . . . Z Z Z Z
. X X . . X X . . X X . . X X         . Z Z . . Z Z . . Z Z . . Z Z
X . X . X . X . X . X . X . X         Z . Z . Z . Z . Z . Z . Z . Z
-----------------------------         -----------------------------
X X X X X X X X X X X X X X X         Z Z Z Z Z Z Z Z Z Z Z Z Z Z Z
```

Example: [[15,1,3]]

Result
CNOT, *T*, *H* are transversal

```
Z . . . . . Z . . Z . Z . . .
. Z . . . . Z . Z . . Z . . .
. . Z . . . Z Z . . . Z . . .
. . . Z . . Z . Z Z . . . . .
. . . . Z . Z Z . Z . . . . .
. . . . . Z Z Z Z . . . . . .
```

```
. . . . . . . X X X X X X X X      . . . . . . . Z Z Z Z Z Z Z Z
. . . X X X X . . . . X X X X      . . . Z Z Z Z . . . . Z Z Z Z
. X X . . X X . . X X . . X X      . Z Z . . Z Z . . Z Z . . Z Z
X . X . X . X . X . X . X . X      Z . Z . Z . Z . Z . Z . Z . Z
-----------------------------      -----------------------------
X X X X X X X X X X X X X X X      Z Z Z Z Z Z Z Z Z Z Z Z Z Z Z
```

Example: [[15,1,3]]

Result
CNOT, *T*, *H* are transversal

Universality!

```
. . . . . . . X X X X X X X X
. . . X X X X . . . . X X X X
. X X . . X X . . X X . . X X
X . X . X . X . X . X . X . X
-----------------------------
X X X X X X X X X X X X X X X
```

```
Z . . . . . Z . . Z . Z . . .
. Z . . . . Z . Z . . Z . . .
. . Z . . . Z Z . . . Z . . .
. . . Z . . Z . Z Z . . . . .
. . . . Z . Z Z . Z . . . . .
. . . . . Z Z Z Z . . . . . .
. . . . . . . Z Z Z Z Z Z Z Z
. . . Z Z Z Z . . . . Z Z Z Z
. Z Z . . Z Z . . Z Z . . Z Z
Z . Z . Z . Z . Z . Z . Z . Z
-----------------------------
Z Z Z Z Z Z Z Z Z Z Z Z Z Z Z
```

Triorthogonal codes

Triorthogonal matrix

$$|f_i \cdot f_j| = 0 \text{ mod } 2$$

$$|f_i \cdot f_j \cdot f_k| = 0 \text{ mod } 2$$

$$G = \begin{pmatrix} \longleftarrow & f_1 & \longrightarrow \\ \longleftarrow & f_2 & \longrightarrow \\ & \vdots & \\ \longleftarrow & f_m & \longrightarrow \end{pmatrix}$$

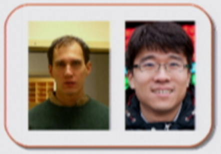
Triorthogonal codes

Triorthogonal matrix

$$|f_i \cdot f_j| = 0 \text{ mod } 2$$

$$|f_i \cdot f_j \cdot f_k| = 0 \text{ mod } 2$$

$$G = \begin{pmatrix} 0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix}$$



Triorthogonal codes

Brayvi, Haah 2012

Stabilizer code

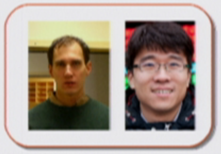
	G	$G^{\perp}$
even-weight row	X stabilizer	Z stabilizer
odd-weight row	logical X, Z	

Triorthogonal matrix

$$|f_i \cdot f_j| = 0 \text{ mod } 2$$

$$|f_i \cdot f_j \cdot f_k| = 0 \text{ mod } 2$$

$$G = \begin{pmatrix} 0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix}$$



Triorthogonal codes

Brayvi, Haah 2012

Stabilizer code

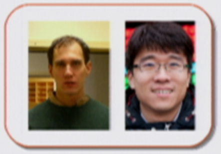
	G	$G^{\perp}$
even-weight row	X stabilizer	Z stabilizer
odd-weight row	logical X, Z	

Triorthogonal matrix

$$|f_i \cdot f_j| = 0 \text{ mod } 2$$

$$|f_i \cdot f_j \cdot f_k| = 0 \text{ mod } 2$$

$$G = \begin{pmatrix} 0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix}$$



Triorthogonal codes

Brayvi, Haah 2012

Stabilizer code

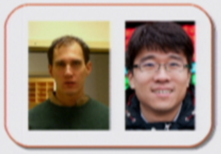
	G	$G^{\perp}$
even-weight row	X stabilizer	Z stabilizer
odd-weight row	logical X, Z	

Triorthogonal matrix

$$|f_i \cdot f_j| = 0 \text{ mod } 2$$

$$|f_i \cdot f_j \cdot f_k| = 0 \text{ mod } 2$$

$$G = \begin{pmatrix} 0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix}$$



Triorthogonal codes

Brayvi, Haah 2012

Stabilizer code

	G	$G^{\perp}$
even-weight row	*X* stabilizer	*Z* stabilizer
odd-weight row	logical *X*, *Z*	

Explicit constructions

- [[49,1,5]]
- [[3*k*+8,*k*,2]] (*k* even)

Triorthogonal matrix

$|f_i \cdot f_j| = 0$ mod 2

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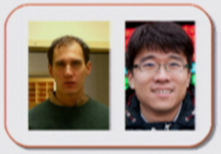

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Brayvi, Haah 2012

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even-weight row	*X* stabilizer	*Z* stabilizer
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Explicit constructions

- [[49,1,5]]
- [[3*k*+8,*k*,2]] (*k* even)

Theorem
Any triorthogonal code admits transversal *T*, up to (diagonal) Clifford corrections.

Triorthogonal matrix

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$$|f_i \cdot f_j \cdot f_k| = 0 \text{ mod } 2$$

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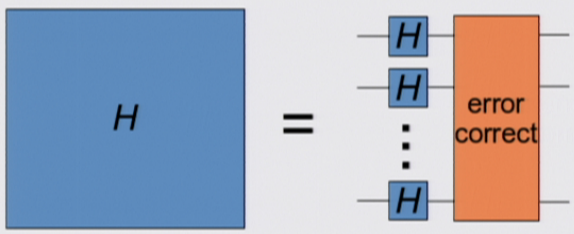
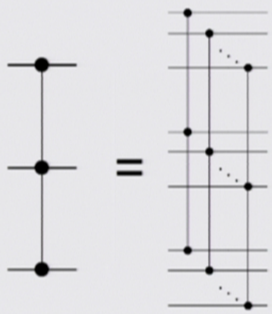
Triorthogonal codes

Claim

For any triorthogonal code:





Triorthogonal codes

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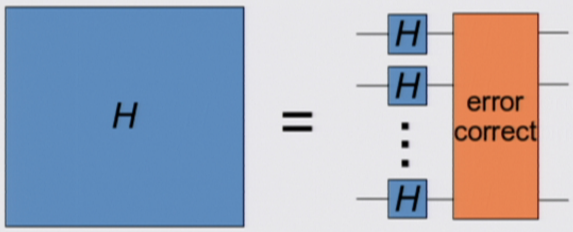
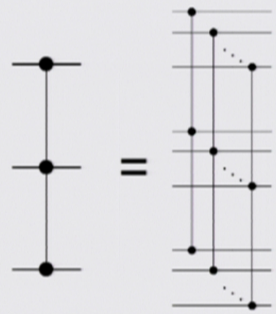
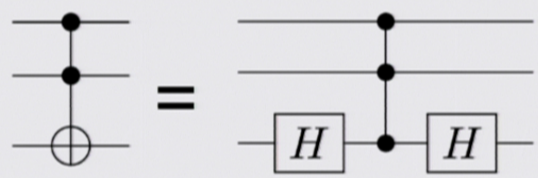
For any triorthogonal code:



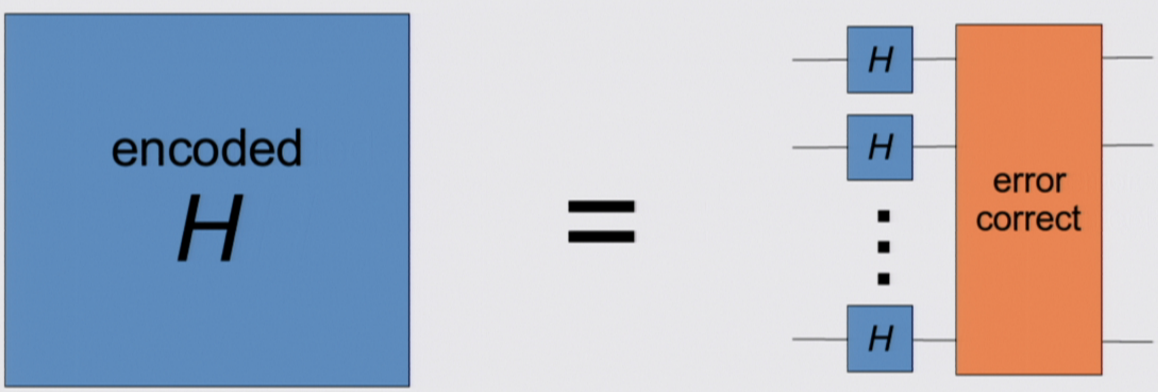


Theorem [Shi 2003]

Toffoli and Hadamard are universal for quantum computation



Transversal Hadamard



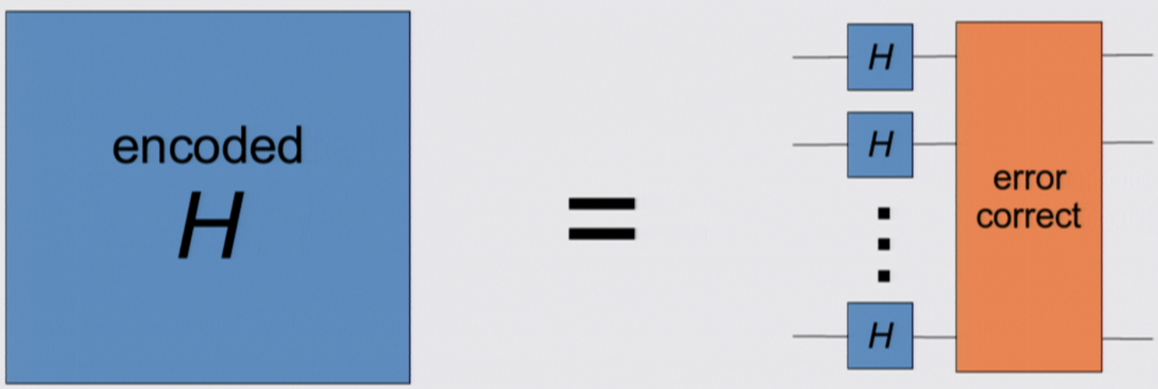
Transversal Hadamard

X stabilizers

$$\begin{pmatrix} \\ \text{even-weight rows of } G \\ \hline \text{odd-weight rows of } G \end{pmatrix}$$

Z stabilizers

$$\begin{pmatrix} G^{\perp} / \text{even-weight rows of } G \\ \cdots\cdots\cdots\cdots \\ \text{even-weight rows of } G \\ \hline \text{odd-weight rows of } G \end{pmatrix}$$



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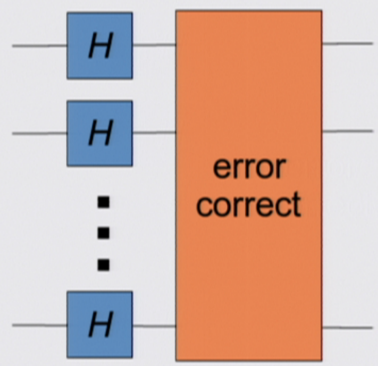
For each generator of $G^{\perp}$ / even-weight rows of G:

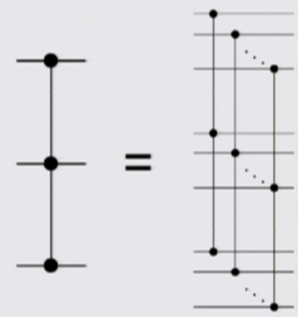
1) Measure the generator
2) If the outcome is -1, apply an X correction



Transversal CCZ

Theorem

$$(-1)^{abc}|abc\rangle \;:=$$

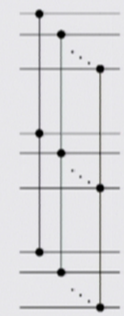


Transversal CCZ

Codewords

$$\begin{pmatrix} \mathcal{G}_0 \\ \text{span of} \\ \text{even-weight rows of } G \\ \hline f_\star\text{: odd-weight row of } G \end{pmatrix}$$

Theorem

$(-1)^{abc}|abc\rangle$:=  = 

Transversal CCZ

Codewords

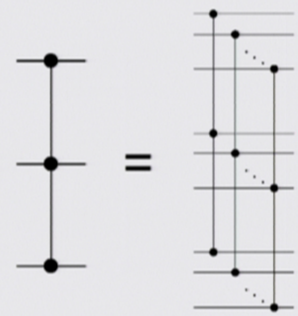
$\mathcal{G}_0$

span of
even-weight rows of G

$f_\star$: odd-weight row of G

$\mathcal{G}_1 := \{f_\star + g : g \in \mathcal{G}_0\}$

Theorem

$(-1)^{abc}|abc\rangle$:= 

Transversal CCZ

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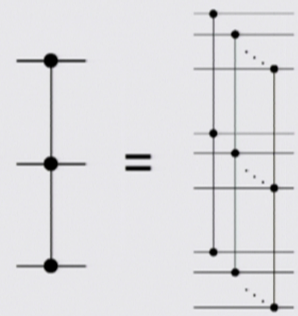
$$|\overline{a}\rangle = \frac{1}{\sqrt{|\mathcal{G}_a|}} \sum_{g \in \mathcal{G}_a} |g\rangle$$

Theorem

$(-1)^{abc}|abc\rangle$:= (circuit) = (circuit)



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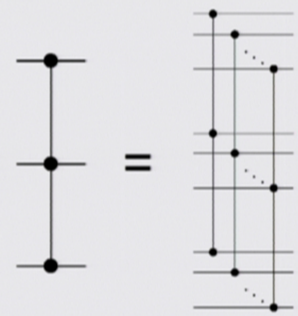
Theorem

$(-1)^{abc}|abc\rangle$:=



Proof

$$\mathrm{CCZ}^{\otimes n}\,|\overline{a,b,c}\rangle = \sum_{g\in\mathcal{G}_a, h\in\mathcal{G}_b, i\in\mathcal{G}_c} \mathrm{CCZ}^{\otimes n}\,|g,h,i\rangle$$

Transversal CCZ

Codewords

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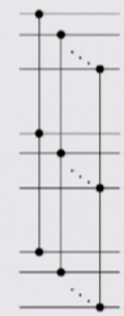
$(-1)^{abc}|abc\rangle$:=  = 

Proof

$$\mathrm{CCZ}^{\otimes n} \, |\overline{a,b,c}\rangle = \sum_{g \in \mathcal{G}_a, h \in \mathcal{G}_b, i \in \mathcal{G}_c} (-1)^{|g \cdot h \cdot i|} \, |g,h,i\rangle$$

Transversal CCZ

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$$g \cdot h \cdot i = (af_\star + g') \cdot (bf_\star + h') \cdot (cf_\star + i')$$

$$g', h', i' \in \mathcal{G}_0$$

Transversal CCZ

Codewords

$$\left(\frac{\mathcal{G}_0 \text{ span of even-weight rows of } G}{f_\star\text{: odd-weight row of } G} \right)$$

$$\mathcal{G}_1 := \{ f_\star + g : g \in \mathcal{G}_0 \}$$

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Triorthogonality

$$|f_i \cdot f_j| = 0 \text{ mod } 2$$

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Theorem

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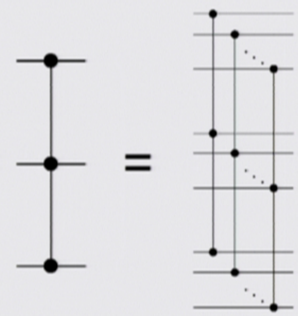

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$g \cdot h \cdot i = (af_\star + g') \cdot (bf_\star + h') \cdot (cf_\star + i')$

$= abc(f_\star \cdot f_\star \cdot f_\star)$ + even-weight terms

Transversal CCZ

Codewords

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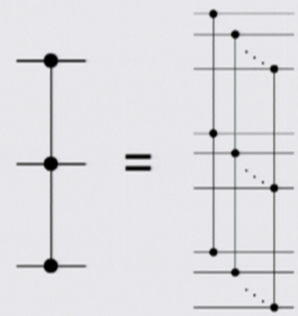

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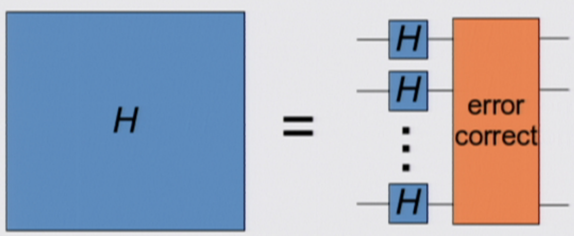
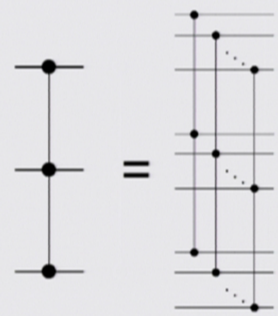
$$= abc(f_\star \cdot f_\star \cdot f_\star) + \text{even-weight terms}$$

Outlook for triorthogonal codes

- [[15,1,3]] threshold error rate ~0.01%
- Performance is worse under locality constraints
- Thresholds unknown for other codes





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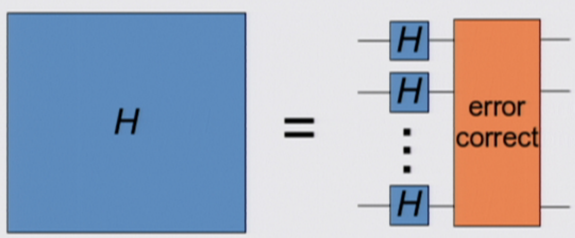
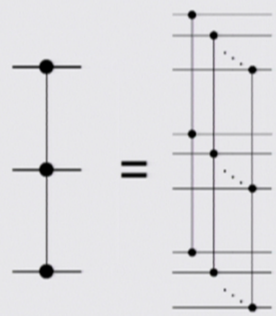
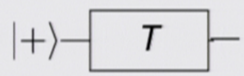




State distillation

Goal

Prepare the state



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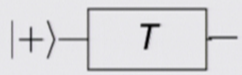
Prepare the state 

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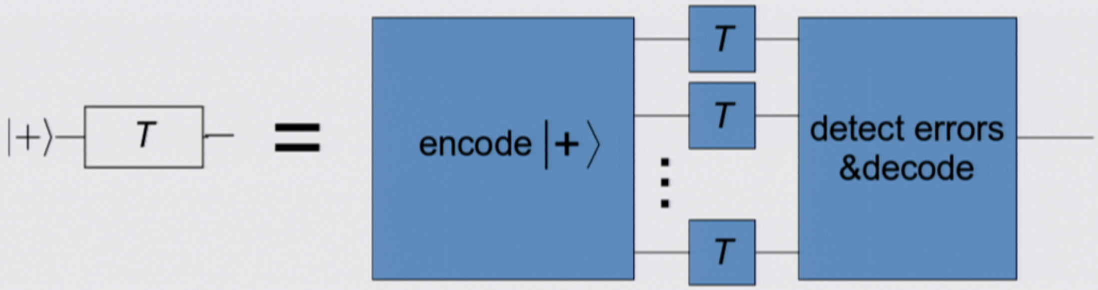

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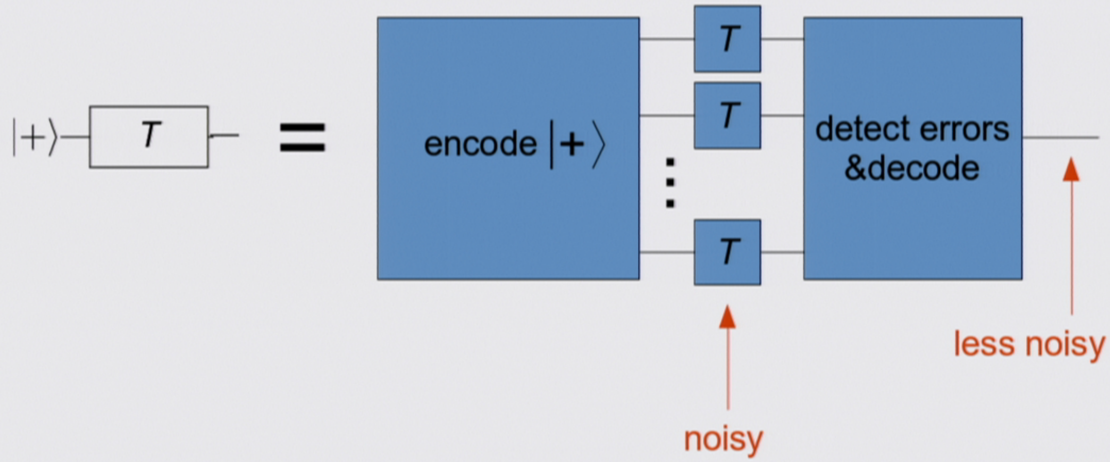
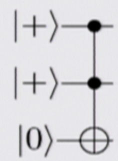


Toffoli distillation

Goal

Prepare the state



Toffoli distillation

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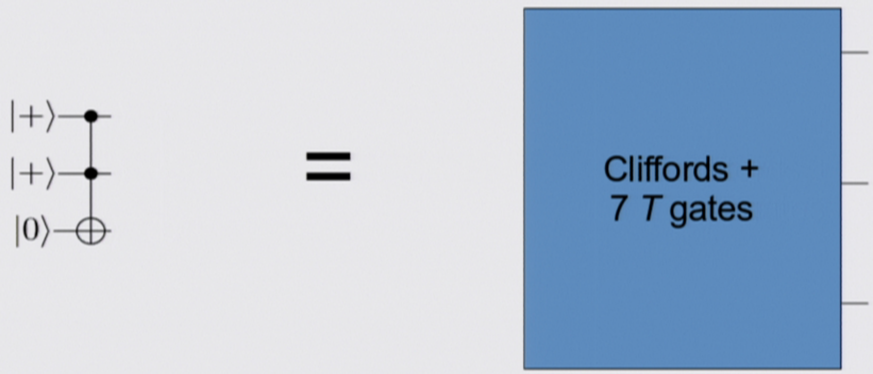

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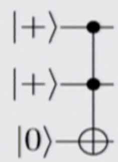

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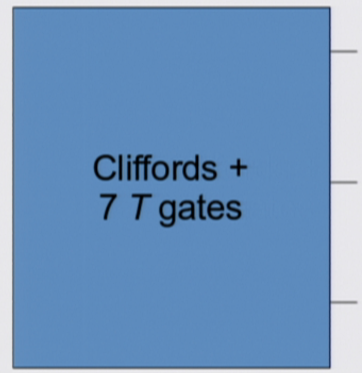
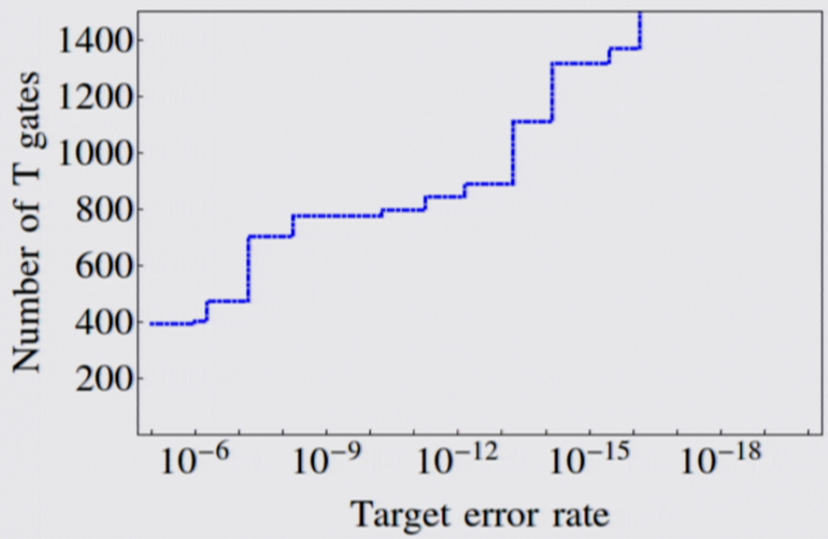


Average cost to produce one Toffoli



Toffoli distillation

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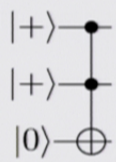

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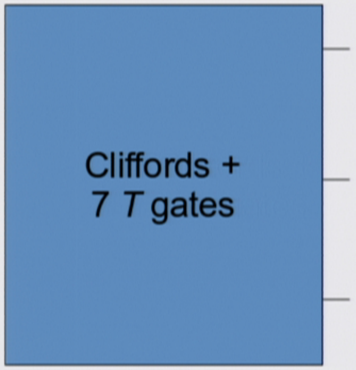

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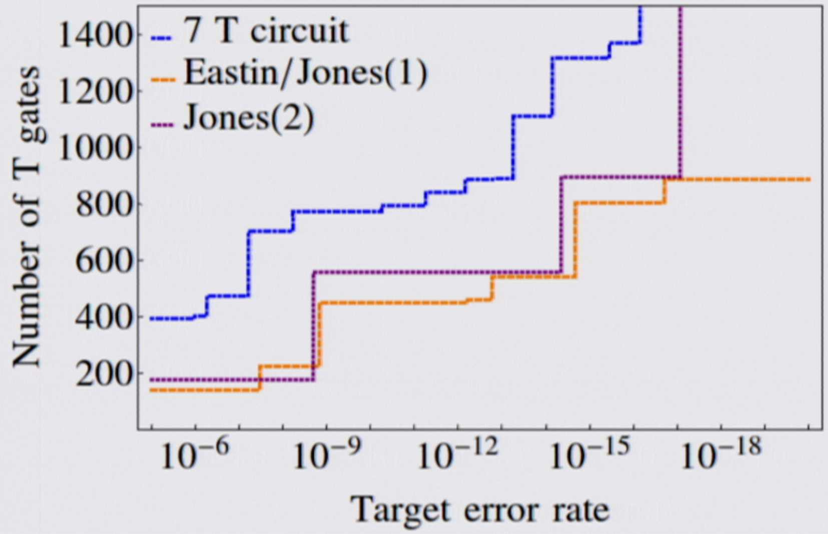
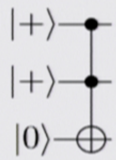


Toffoli distillation

Goal
Prepare the state



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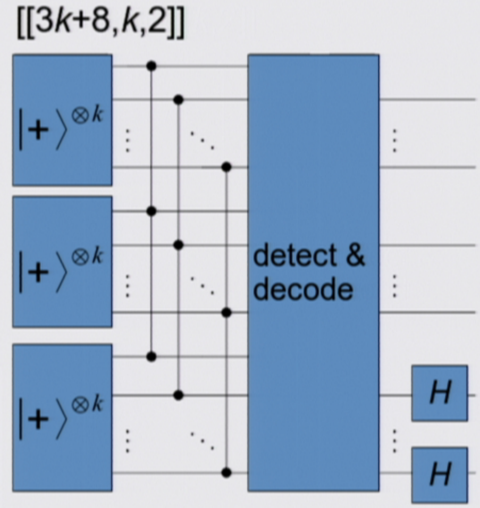

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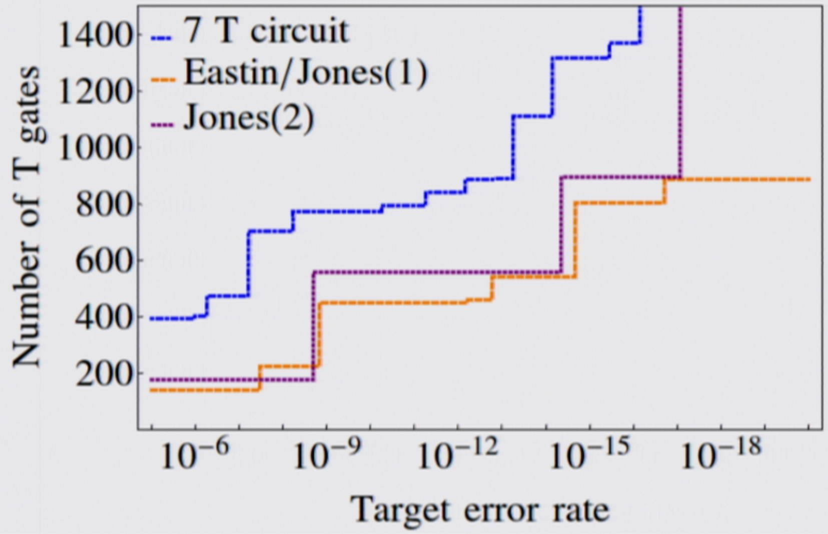

Toffoli distillation

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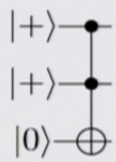

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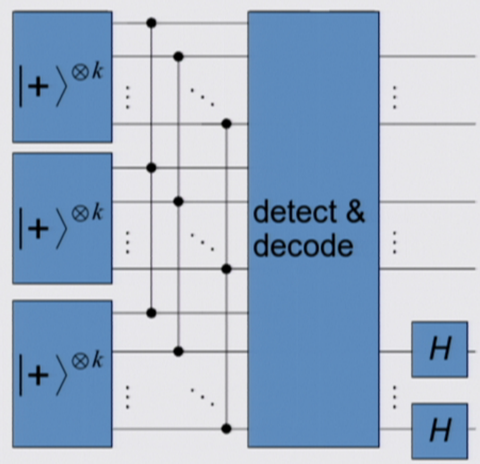
[[3*k*+8,*k*,2]]



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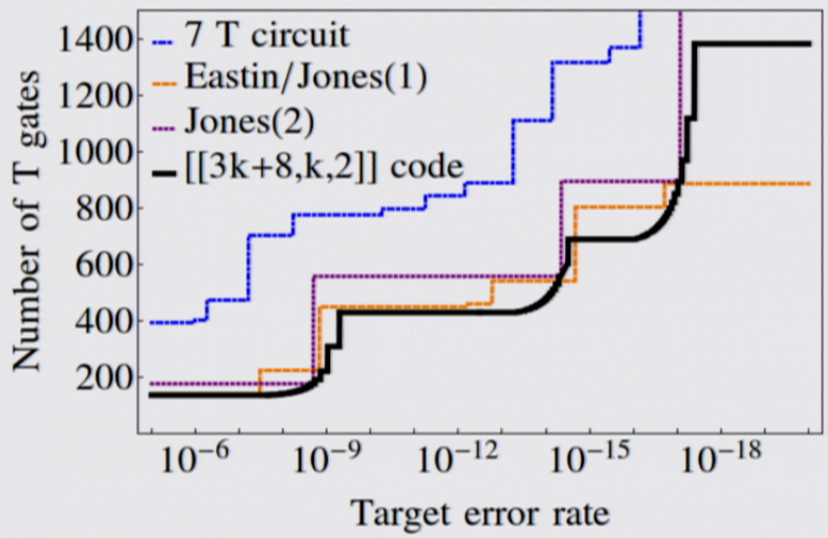
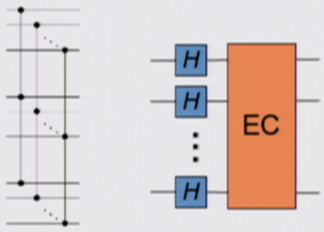
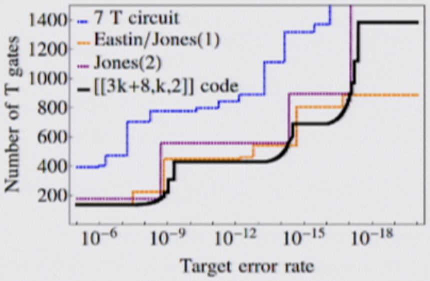


Summary

- Triorthogonal codes admit transversal CCZ & Hadamard (with EC)
- Improved Toffoli distillation





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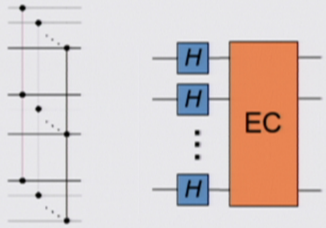
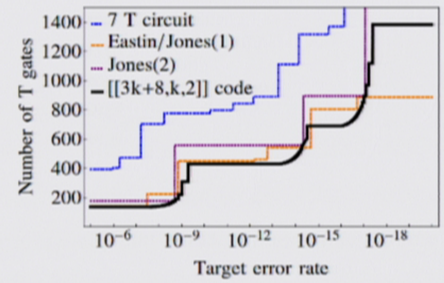
- Improved Toffoli distillation





Open questions

- Thresholds for existing triorthogonal codes?
- More (and better) triorthogonal codes?
- Other ways to eliminate distillation?
  - [Bombin, Martin-Delgado 2007]
  - [B, Chhajlany, Horodecki, M-D 2013]

  - [Jochym-O'Connor, Laflamme 2013]