Title: Quantum Observables as Real-valued Functions and Quantum Probability

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Abstract: Quantum observables are commonly described by self-adjoint operators on a Hilbert space H. I will show that one can equivalently describe observables by real-valued functions on the set P(H) of projections, which we call q-observable functions. If one regards a quantum observable as a random variable, the corresponding q-observable function can be understood as a quantum quantile function, generalising the classical notion. I will briefly sketch how q-observable functions relate to the topos approach to quantum theory and the process called daseinisation. The topos approach provides a generalised state space for quantum systems that serves as a joint sample space for all quantum observables. This is joint work with Barry Dewitt.

Quantum Observables as Real-valued Functions and Quantum Probability

Quantum Foundations Seminar
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All knowledge degenerates into probability.

David Hume, in *A treatise of Human Nature* (1739)

Introduction

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- While this is mathematically natural, it is much less clear what addition means physically.
- E.g., what is the interpretation of $\hat{p} + \hat{q}$?
- Let $\hat{H} = \hat{E}_{kin} + \hat{E}_{pot}$. Even if we know sp $\hat{E}_{kin}$ and sp $\hat{E}_{pot}$, together with all the corresponding eigenspaces, this does not give us sp $\hat{H}$ and the eigenspaces of $\hat{H}$.

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The relation between linear aspects and spectral aspects of self-adjoint operators is notoriously difficult.

Order

In this talk, I will emphasise *order* over *linearity*, providing another perspective. This allows us to write all self-adjoint operators as functions (*not* expectation value functions, *not* Wigner functions).

I will show how this representation relates to probability theory, and that there is a kind of *joint sample space* for all quantum observables, contrary to ordinary wisdom.

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The talk is based on:

- AD, B. Dewitt, "Self-adjoint Operators as Functions I: Lattices, Galois Connections, and the Spectral Order", arXiv:1208.4724
- AD, B. Dewitt, "Self-adjoint Operators as Functions II: Quantum Probability", arXiv:1210.5747

In the papers, we treat von Neumann algebras and unbounded operators. Here just $\mathcal{B}(\mathcal{H})$ and bounded operators.

Posets

As a reminder:

Definition

A **partially ordered set** (or **poset**) is a set X with a binary relation $\leq$, the **partial order**, which is

(a) reflexive: $x \leq x$ for all $x \in X$,

(b) antisymmetric: if $x \leq y$ and $y \leq x$, then $x = y$,

(c) transitive: if $x \leq y$ and $y \leq z$, then $x \leq z$.

A poset X is **bounded** if there are a **bottom element** 0 and a **top element** 1 in X such that $0 \leq x$ and $x \leq 1$ for all $x \in X$.

Examples: the subsets $P(Y)$ of a set Y with inclusion $\subseteq$ as partial order; $\mathbb{R}$ with the usual order (which is a **total order**), ...

Meets and joins

Definition

Let X be a poset, and let $x, y \in X$. The **meet** $x \wedge y$ is the greatest lower bound of x and y in X (if it exists), that is, $x \wedge y \in X$,

$$x \wedge y \leq x, \quad x \wedge y \leq y,$$

and if $z \leq x, y$, then $z \leq x \wedge y$. A poset in which all binary meets exist is called a **meet-semilattice**. If any family $(x_i)_{i \in I}$ has a meet $\bigwedge_{i \in I} x_i$ in X, then X is called a **complete meet-semilattice**.

Lattices (2)

Definition

If a poset X is both a meet-semilattice and a join-semilattice, then X is called a **lattice**. If all meets and joins exist, then X is **complete**.

Lattices (2)

Examples:

- The power set $P(Y)$ of a set Y is a bounded complete lattice, with intersections as meets and unions as joins. $P(Y)$ is distributive.
- The real numbers with the usual order form a distributive lattice $\mathbb{R}$, where meets are infima and joins are suprema. $\mathbb{R}$ is neither bounded nor complete: e.g. $\bigvee_{r\in\mathbb{R}} r$ does not exist in $\mathbb{R}$.

Boundedly complete lattices

Some lattices are not bounded, but 'almost' complete:

Definition

A lattice is **boundedly** (or **conditionally**) **complete** if every family of elements that has a lower bound has a greatest lower bound (meet), and every family that has an upper bound has a least upper bound (join).

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Example: $\mathbb{R}$.

The boundedly complete lattice $\mathbb{R}$ can be made complete by adding a bottom element $-\infty$ and a top element ∞, that is,

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}.$$

We call $\overline{\mathbb{R}}$ the **extended reals**.

Orthocomplements

Definition

Let L be a bounded lattice. An **orthocomplementation function on** L is a map $' : L \to L$, $x \mapsto x'$ such that

- $x' \vee x = 1$, $x' \wedge x = 0$ (complement law),
- $x'' = x$ (involution law),
- If $x \leq y$, then $y' \leq x'$ (order-reversal).

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An **orthomodular lattice** L is an ortholattice such that for any $x, y \in L$ with $x \leq y$, it holds that $x \vee (x' \wedge y) = y$. This is the **orthomodularity law**.

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Example: The projection operators on a Hilbert space $\mathcal{H}$ form a nondistributive, complete, orthomodular lattice $\mathcal{P}(\mathcal{H})$.

Galois connections

Definition

If $(P, \leq)$ and $(Q, \leq)$ are two posets and $f : P \to Q$, $g : Q \to P$ are order-preserving (monotone) maps such that

$$\forall p \in P \; \forall q \in Q : f(p) \leq q \quad \text{iff} \quad p \leq g(q),$$

then (P, Q, f, g) form a **Galois connection**. f is called the **left adjoint** and g the **right adjoint** (in the *categorical* sense). f determines g uniquely and vice versa.

The linear order

Let $\hat{A}, \hat{B} \in \mathcal{B}(\mathcal{H})_{\mathsf{sa}}$ be self-adjoint operators. Usually, one uses the **linear order** on self-adjoint operators:

$$\hat{A} \leq \hat{B} :\Longleftrightarrow \hat{B} - \hat{A} \text{ positive.}$$

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Useful order in many respects, but Kadison ('51) showed that two self-adjoint operators $\hat{A}, \hat{B} \in \mathcal{B}(\mathcal{H})_{sa}$ have a meet $\hat{A} \wedge \hat{B}$ if and only if $\hat{A} \leq \hat{B}$ or $\hat{B} \leq \hat{A}$, so $(\mathcal{B}(\mathcal{H})_{\mathrm{sa}}, \leq)$ is very far from being a lattice (it is an **anti-lattice**).

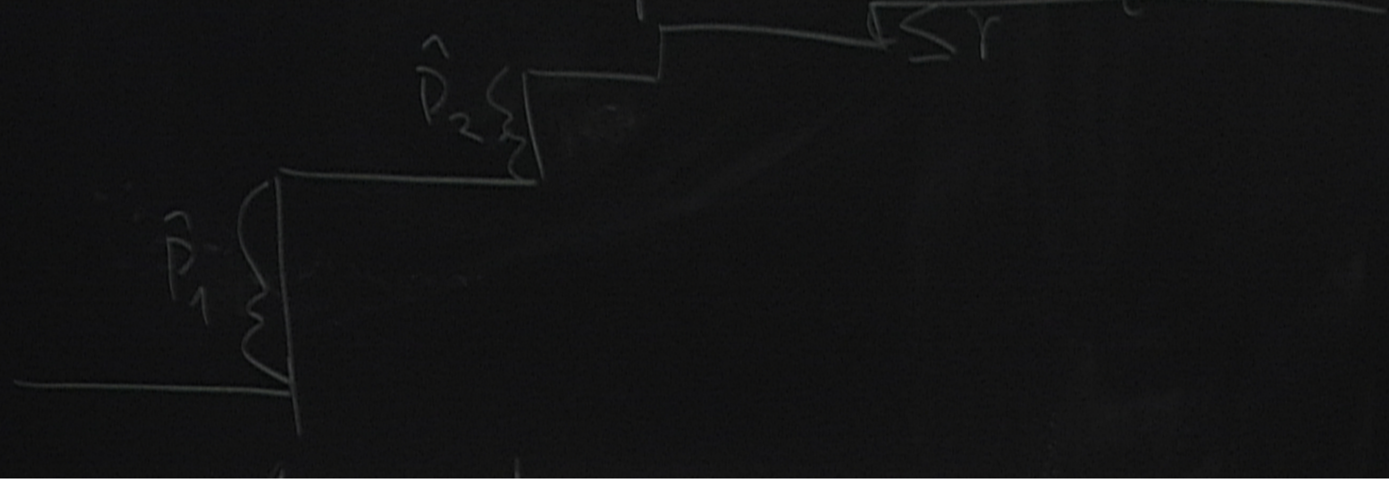
The spectral order

Olson ('71) introduced the **spectral order** on the self-adjoint operators on a Hilbert space: if $\hat{A}, \hat{B} \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ and $\hat{E}^{\hat{A}} = (\hat{E}^{\hat{A}}_r)_{r\in\mathbb{R}}$, $\hat{E}^{\hat{B}} = (\hat{E}^{\hat{B}}_r)_{r\in\mathbb{R}}$ are their **spectral families**, then

$$\hat{A} \leq_s \hat{B} :\Longleftrightarrow (\forall r \in \mathbb{R} : \hat{E}^{\hat{A}}_r \geq \hat{E}^{\hat{B}}_r).$$

$$\hat{A} = \sum_{i=1}^{n} a_i \hat{P}_i$$

$$\hat{E}^{\hat{A}}_{r} = \sum_{i \in r} \hat{P}_i$$



q-observable functions

We remedy this by using the extended reals $\overline{\mathbb{R}}$ and extend $\hat{E}^{\hat{A}}$ canonically by setting $\hat{E}^{\hat{A}}_{-\infty} := \hat{0}$ and $\hat{E}^{\hat{A}}_{\infty} := \hat{1}$. Clearly, the extended spectral family

$$\hat{E}^{\hat{A}} : \overline{\mathbb{R}} \longrightarrow \mathcal{P}(\mathcal{H}).$$

is uniquely determined by the non-extended one. (But now we have a map preserving all meets between complete meet-semilattices.) We define:

Definition

The q-**observable function of** $\hat{A} \in \mathcal{B}(\mathcal{H})_{\text{sa}}$ is the left adjoint

$$o^{\hat{A}} : \mathcal{P}(\mathcal{H}) \longrightarrow \overline{\mathbb{R}}$$

of the extended spectral family $\hat{E}^{\hat{A}} : \overline{\mathbb{R}} \to \mathcal{P}(\mathcal{H})$.

Some properties

The adjoint functor theorem gives the concrete form of the left adjoint:

$$\forall \hat{P} \in \mathcal{P}(\mathcal{H}) : o^{\hat{A}}(\hat{P}) = \inf\{r \in \overline{\mathbb{R}} \mid \hat{E}_r^{\hat{A}} \geq \hat{P}\}.$$

This means that $o^{\hat{A}}(\hat{P})$ is the smallest value r such that the subspace spanned by all spectral spaces of $\hat{A}$ for spectral values $\leq r$ contains the subspace that $\hat{P}$ projects onto.

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Note that $o^{\hat{A}}(\hat{0}) = -\infty$, but $o^{\hat{A}}(\hat{P}) \geq \min(\operatorname{sp}\hat{A})$ if $\hat{P} > \hat{0}$.

Abstract characterisation

Definition

Let $o : \mathcal{P}(\mathcal{H}) \to \overline{\mathbb{R}}$ be a function that

- preserves joins, i.e., $o(\bigvee_{i \in I} \hat{P}_i) = \sup_{i \in I} o(\hat{P}_i)$ for all families $(\hat{P}_i)_{i \in I} \subseteq \mathcal{P}(\mathcal{H})$,
- $o(\mathcal{P}_0(\mathcal{H})) = K$ is compact.

Such an o is called an **abstract q-observable function**.

Note that there is no reference to a linear operator in this definition.

One can show that each such function determines a unique extended right-continuous spectral family $\hat{E}^o : \overline{\mathbb{R}} \to \mathcal{P}(\mathcal{H})$ and hence a self-adjoint operator $\hat{A}^o \in \mathcal{B}(\mathcal{H})_{\text{sa}}$, so abstract q-observable functions are q-observable functions and vice versa.

Self-adjoint operators as functions

Let $QO(\mathcal{P}(\mathcal{H}), \overline{\mathbb{R}})$ denote the set of all q-observable functions, and let $SF(\overline{\mathbb{R}}, \mathcal{P}(\mathcal{H}))$ denote the set of all bounded, right-continuous, extended spectral families with values in $\mathcal{P}(\mathcal{H})$. We have so far:

Proposition

There are bijections $\mathcal{B}(\mathcal{H})_{\mathrm{sa}} \simeq SF(\overline{\mathbb{R}}, \mathcal{P}(\mathcal{H})) \simeq QO(\mathcal{P}(\mathcal{H}), \overline{\mathbb{R}})$.

Daseinisation

In the **topos approach to quantum theory**, one considers approximations of self-adjoint operators w.r.t. the spectral order. Let $\hat{A} \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}}$, and let V be a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$, then

$$\delta^o_V(\hat{A}) := \bigwedge \{\hat{B} \in V_{\mathrm{sa}} \mid \hat{B} \geq_s \hat{A}\}.$$

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This is a self-adjoint operator in V approximating $\hat{A}$ 'from above' in the spectral order. $\delta^o_V(\hat{A})$ is called the **(outer) daseinisation of $\hat{A}$ to V**. One can show:

Proposition

$o^{\delta^o_V(\hat{A})} = o^{\hat{A}}|_{\mathcal{P}(V)} : \mathcal{P}(V) \to \overline{\mathbb{R}}$, *where $\mathcal{P}(V)$ denotes the lattice of projections in V.*

Rescalings

There is a limited form of functional calculus for q-observable functions:

Proposition

If $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ *is a join-preserving function such that* $f(\mathbb{R}) \subseteq \mathbb{R}$*, then, for all* $\hat{A} \in \mathcal{B}(\mathcal{H})_{sa}$*, it holds that*

$$o^{f(\hat{A})} = f(o^{\hat{A}}).$$

Probabilistic interpretation

In probability and statistics, a random variable or stochastic variable is a variable whose value is subject to variations due to chance (i.e. randomness, in a mathematical sense).

From Wikipedia, 'Random variable'

Random variables

We consider classical probability for a moment.

Let Ω be a **sample space**, $B(\Omega)$ its **Borel (measurable) subsets**. Let $A : \Omega \to \mathbb{R}$ be a classical **random variable**, i.e., a Borel function, and let $\mu : B(\Omega) \to [0, 1]$ be a **probability measure**.

To calculate the probability that the outcome of a 'measurement' of A lies in a Borel set $\Delta \subset \mathbb{R}$ in the 'state' μ, we form

$$\mu(A^{-1}(\Delta)).$$

Note that we use the **inverse image function** $A^{-1} : B(\mathbb{R}) \to B(\Omega)$ of the random variable. A^{-1} maps Borel subsets of outcomes to Borel subsets of the sample space.

Quantile functions

A classical CDF C^A can be extended to $\overline{\mathbb{R}}$ canonically and then becomes a meet-preserving map. Hence, it has a left adjoint

$$\begin{aligned} q^A : [0,1] &\longrightarrow \overline{\mathbb{R}} \\ r &\longmapsto \inf\{s \in \overline{\mathbb{R}} \mid C^A(s) \geq r\}. \end{aligned}$$

The function q^A is well-known in classical probability and is called the **quantile function of the random variable** A.

$B(\Omega)$-CDFs and $B(\Omega)$-quantile functions

What if there is no probability measure? Given a random variable $A : \Omega \to \overline{\mathbb{R}}$, we can still define

$$\begin{aligned} \tilde{C}^A : \overline{\mathbb{R}} &\longrightarrow B(\Omega) \\ r &\longmapsto A^{-1}([-\infty, r]), \end{aligned}$$

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L-CDFs and L-quantile functions

We can now generalise: let L be a complete meet-semilattice, and let $A^{-1} : B(\overline{\mathbb{R}}) \to L$ be a meet-preserving map such that $A^{-1}(\emptyset) = \bot_L$. We consider such a map A^{-1} as the **inverse image of a generalised random variable**.

L-CDFs and L-quantile functions

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Note that we do *not* need to define a function $A : (\text{Points of } L) \to \overline{\mathbb{R}}$, although we assume that such a generalised random variable exists 'in spirit'. Then

$$\begin{aligned} \tilde{C}^A : \overline{\mathbb{R}} &\longrightarrow L \\ r &\longmapsto A^{-1}([-\infty, r]), \end{aligned}$$

is called the L-**CDF of** A,

Spectral measures

We now show that all these aspects of classical probability theory have analogues in the quantum case. Much of this is well-known, but we also show some new aspects.

Let $\hat{A} \in \mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ be a self-adjoint operator. In quantum probability, $\hat{A}$ is interpreted as a **quantum random variable** and defines a **projection-valued measure**, the **spectral measure of** $\hat{A}$: as the spectral theorem shows, $\hat{A}$ gives (and is given by) a map

$$e^{\hat{A}} : B(\mathrm{sp}\,\hat{A}) \longrightarrow \mathcal{P}(\mathcal{H}),$$

where $B(\mathrm{sp}\,\hat{A})$ are the Borel subsets of the spectrum of $\hat{A}$.

Gelfand transforms as random variables

A self-adjoint operator $\hat{A}$ is not a real-valued function, so it is not the direct analogue of a random variable $A : \Omega \to \overline{\mathbb{R}}$.

First, we need an analogue of the sample space Ω. This is no problem as long as we consider only one operator $\hat{A}$: consider the commutative algebra $V_{\hat{A}}$, the smallest von Neumann algebra that contains $\hat{A}$.

$V_{\hat{A}}$ has a **Gelfand spectrum** $\Sigma_{V_{\hat{A}}}$, which is nothing but the space of pure states on $V_{\hat{A}}$.

The set of clopen (i.e., closed and open) subsets of $\Sigma_{V_{\hat{A}}}$, denoted $\mathcal{C}l(\Sigma_{V_{\hat{A}}})$, is a complete Boolean algebra. Moreover, there is an isomorphism of complete Boolean algebras

$$\alpha_{V_{\hat{A}}} : \mathcal{P}(V_{\hat{A}}) \longrightarrow \mathcal{C}l(\Sigma_{V_{\hat{A}}}).$$

Hence, we can take $\Sigma_{V_{\hat{A}}}$ as our sample space.

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Gelfand transforms as random variables

The function

$$\begin{aligned} \overline{A} : \Sigma_{V_{\hat{A}}} &\longrightarrow \operatorname{sp} \hat{A} \subset \mathbb{R} \\ \lambda &\longmapsto \overline{A}(\lambda) = \lambda(\hat{A}) \end{aligned}$$

is called the **Gelfand transform of** $\hat{A}$ (w.r.t. $V_{\hat{A}}$). It is the analogue of a classical random variable.

Quantum quantile functions

A quantum random variable $\hat{A}$ determines a spectral measure $e^{\hat{A}}$ with values in the projection lattice $\mathcal{P}(\mathcal{H})$, which in particular is a complete meet-semilattice. We define

$$\begin{aligned} \hat{E}^{\hat{A}} : \overline{\mathbb{R}} &\longrightarrow \mathcal{P}(\mathcal{H}) \\ r &\longmapsto e^{\hat{A}}([-\infty, r]), \end{aligned}$$

so the spectral family $\hat{E}^{\hat{A}} = (\hat{E}^{\hat{A}}_r)_{r\in\overline{\mathbb{R}}}$ is the $\mathcal{P}(\mathcal{H})$-CDF of $\hat{A}$. It has a left adjoint,

$$\begin{aligned} o^{\hat{A}} : \mathcal{P}(\mathcal{H}) &\longrightarrow \overline{\mathbb{R}} \\ \hat{P} &\longmapsto \inf\{r \in \overline{\mathbb{R}} \mid \hat{E}^{\hat{A}}_r \geq \hat{P}\}, \end{aligned}$$

which is the q-observable function of $\hat{A}$. We have shown:

The q-observable function $o^{\hat{A}}$ is the **quantum quantile function** *of the quantum random variable $\hat{A}$.*

Comparison classical – quantum probability

Sample space	Ω	$\mathcal{H}$
Random variable	$A : \Omega \to \text{im}\, A \subset \overline{\mathbb{R}}$	$\hat{A} \in \mathcal{B}(\mathcal{H})_{\text{sa}}$
Inv. im. of random var.	$A^{-1} : B(\overline{\mathbb{R}}) \to B(\Omega)$	$e^{\hat{A}} : B(\overline{\mathbb{R}}) \to \mathcal{P}(\mathcal{H})$
L-CDF	$\tilde{C}^A : \mathbb{R} \to B(\Omega)$	$\hat{E}^{\hat{A}} : \mathbb{R} \to \mathcal{P}(\mathcal{H})$
L-quantile function	$\tilde{q}^A : B(\Omega) \to \overline{\mathbb{R}}$	$o^{\hat{A}} : \mathcal{P}(\mathcal{H}) \to \overline{\mathbb{R}}$
State (probab. meas.)	$\mu : B(\Omega) \to [0,1]$	$\mu_\rho : \mathcal{P}(\mathcal{H}) \to [0,1]$
CDF	$C^A = \mu \circ \tilde{C}^A : \mathbb{R} \to [0,1]$	$C^{\hat{A}} = \mu_\rho \circ \hat{E}^{\hat{A}} : \mathbb{R} \to [0,1]$
Quantile function	$q^A : [0,1] \to \overline{\mathbb{R}}$	$q^{\hat{A}} : [0,1] \to \overline{\mathbb{R}}$

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Quantile function	$q^A : [0,1] \to \overline{\mathbb{R}}$	$q^{\hat{A}} : [0,1] \to \overline{\mathbb{R}}$

A quantum sample space

Is there a suitable sample space for the quantum side, in analogy to the Gelfand spectrum Σ_V of an abelian von Neumann algebra V? Such a sample space Σ should

- generalise the Gelfand spectrum Σ_V to the nonabelian von Neumann algebra $\mathcal{B}(\mathcal{H})$,

A quantum sample space

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- generalise the Gelfand spectrum Σ_V to the nonabelian von Neumann algebra $\mathcal{B}(\mathcal{H})$,
- come equipped with a family of measurable subsets, analogous to the clopen subsets $\mathcal{C}l(\Sigma_V)$ of Σ_V,
- serve as a common domain for the random variables, and hence as a common codomain for the associated spectral measures,
- serve as a domain for the states of $\mathcal{B}(\mathcal{H})$, seen as probability measures.

The topos approach to quantum theory provides such a generalised sample space, in the form of the **spectral presheaf** $\underline{\Sigma}$ of a von Neumann algebra $\mathcal{N}$. We will only consider the case $\mathcal{N} = \mathcal{B}(\mathcal{H})$ here.

The spectral presheaf

But how can there be such a sample space? As is well known, there is *no* joint sample space for noncommuting quantum observables.

Technically this means (in our formulation) that the noncommutative von Neumann algebra $\mathcal{B}(\mathcal{H})$ has no Gelfand spectrum $\Sigma_{\mathcal{B}(\mathcal{H})}$.

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The idea is to generalise from sets to objects in a topos. In particular, the **spectral presheaf** $\underline{\Sigma}$ of $\mathcal{B}(\mathcal{H})$ is defined as follows:

- for each commutative von Neumann subalgebra $V \subset \mathcal{B}(\mathcal{H})$, let $\underline{\Sigma}_V := \Sigma_V$, the Gelfand spectrum of V,
- for all inclusions $i_{V'V} : V' \hookrightarrow V$, let $\underline{\Sigma}(i_{V'V}) : \underline{\Sigma}_V \to \underline{\Sigma}_{V'}$ be the function sending $\lambda \in \underline{\Sigma}_V$ to its restriction $\lambda|_{V'} \in \underline{\Sigma}_{V'}$.

The spectral presheaf

But how can there be such a sample space? As is well known, there is *no* joint sample space for noncommuting quantum observables.

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Clopen subobjects

The analogue of the measurable subsets $B(\Omega)$ of a classical sample space Ω are the **clopen subobjects** of the quantum sample space $\underline{\Sigma}$:

A subpresheaf $\underline{S}$ of $\underline{\Sigma}$ is called **clopen** if for all commutative $V \subset \mathcal{N}$, the set $\underline{S}_V \subseteq \underline{\Sigma}_V$ is clopen.

Proposition

The clopen subojects of the quantum sample space $\underline{\Sigma}$ form a complete bi-Heyting algebra $\mathrm{Sub_{cl}}(\underline{\Sigma})$.

A bi-Heyting algebra is a comparatively mild generalisation of a Boolean algebra (different from an orthomodular lattice such as $\mathcal{P}(\mathcal{H})$ – this has consequences for quantum logic).

Inverse images of random variables

We need the inverse image of a quantum random variable. In the topos setting, this should be a map

$$\breve{A}^{-1} : B(\overline{\mathbb{R}}) \longrightarrow \mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma}) \tag{1}$$

from Borel subsets of outcomes to measurable subsets of the quantum sample space.

Probability measures on $\underline{\Sigma}$

Let $\mathcal{V}(\mathcal{H})$ be the set of commutative von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$, partially ordered by inclusion, and let $[0,1]_{\mathcal{V}(\mathcal{H})}$ be the set of antitone (order-reversing) functions from $\mathcal{V}(\mathcal{H})$ to the unit interval.

Definition

A **probability measure on the quantum sample space** $\underline{\Sigma}$ is a map

$$\mu : \mathrm{Sub_{cl}}(\underline{\Sigma}) \longrightarrow [0,1]_{\mathcal{V}(\mathcal{H})}$$

such that

(1) $\mu(\underline{\Sigma}) = 1_{\mathcal{V}(\mathcal{H})}$, the constant function with value 1 on all $V \in \mathcal{V}(\mathcal{H})$,

(2) for all $\underline{S}, \underline{T} \in \mathrm{Sub_{cl}}(\underline{\Sigma})$, it holds that $\mu(\underline{S}) + \mu(\underline{T}) = \mu(\underline{S} \vee \underline{T}) + \mu(\underline{S} \wedge \underline{T})$.

Quantum states as probability measures

Let $\mathcal{H}$ be a Hilbert space, $\rho : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a quantum state, pure or mixed. One can show:

Theorem

If $\dim \mathcal{H} \geq 3$, *there is a bijection*

$$p : \mathcal{S}(\mathcal{B}(\mathcal{H})) \longrightarrow \mathcal{M}(\underline{\Sigma})$$

between $\mathcal{S}(\mathcal{B}(\mathcal{H}))$, *the convex space of states of* $\mathcal{B}(\mathcal{H})$, *and* $\mathcal{M}(\underline{\Sigma})$, *the convex set of probability measures on the quantum sample space* $\underline{\Sigma}$.

This means that in the topos formulation, we can think of quantum states as probability measures on the quantum sample space $\underline{\Sigma}$. The clopen subobjects take the role of the measurable subsets.

Comparison classical – quantum probability in topos form

Sample space	Ω	$\underline{\Sigma}$
Inv. im. of random var.	$A^{-1}: B(\overline{\mathbb{R}}) \to B(\Omega)$	$\breve{A}^{-1}: B(\overline{\mathbb{R}}) \to \mathrm{Sub_{cl}}(\underline{\Sigma})$
L-CDF	$\tilde{C}^A: \mathbb{R} \to B(\Omega)$	$E^{\breve{A}}: \mathbb{R} \to \mathrm{Sub_{cl}}(\underline{\Sigma})$
L-quantile function	$\tilde{q}^A: B(\Omega) \to \overline{\mathbb{R}}$	$o^{\breve{A}}: \mathrm{Sub_{cl}}(\underline{\Sigma}) \to \overline{\mathbb{R}}$
State (probab. meas.)	$\mu: B(\Omega) \to [0,1]$	$\mu_\rho: \mathrm{Sub_{cl}}(\underline{\Sigma}) \to [0,1]_{\mathcal{V}(\mathcal{H})}$
CDF	$C^A = \mu \circ \tilde{C}^A: \overline{\mathbb{R}} \to [0,1]$	$c^{\breve{A}} = \min_V(\mu_\rho \circ E^{\breve{A}}): \overline{\mathbb{R}} \to [0,1]$
Quantile function	$q^A: [0,1] \to \overline{\mathbb{R}}$	$q^{\breve{A}}: [0,1] \to \overline{\mathbb{R}}$

Thanks for listening!