Title: The local Callan-Symanzik equation: structure and applications

Date: Sep 06, 2013 01:00 PM

URL: http://pirsa.org/13090067

Abstract: The local Callan-Symanzik equation describes the response of a quantum field theory to local scale transformations in the presence of background sources. The consistency conditions associated with this anomalous equation can be used to derive powerful constraints on RG flows. We will discuss various aspects of the equation and present new results regarding the structure of the anomaly. We then use the equation to write correlation functions of the trace of the energy-momentum tensor off-criticality.

Characterizing RG flows: The Callan-Symanzik equation

The basic idea:

- Global scale transformation symmetry can be broken explicitly by quantum effects.
- The non-invariance can be compensated by modifications of the parameters.

Implications:

- Scale transformation can be translated into a flow in parameter space.

Characterizing RG flows: The Callan-Symanzik equation

The basic idea:

- Global scale transformation symmetry can be broken explicitly by quantum effects.
- The non-invariance can be compensated by modifications of the parameters.

Implications:

- Scale transformation can be translated into a flow in parameter space.

Characterizing RG flows: Finiteness of T (Brown & Collins 1980)

The basic idea:

- **Local** scale transformations are encoded in $T = T^{\mu}_{\mu}$.
- Put the theory in a curved background.
- Compute T in terms of bare composite operators and gravitational terms.
- **Consistency condition**: T is finite.

Implications:

- Non-trivial constraints on the structure of counterterms.

Characterizing RG flows: Finiteness of T (Brown & Collins 1980)

The basic idea:

- **Local** scale transformations are encoded in $T = T^{\mu}_{\mu}$.
- Put the theory in a curved background.
- Compute T in terms of bare composite operators and gravitational terms.
- **Consistency condition**: T is finite.

Implications:

- Non-trivial constraints on the structure of counterterms.

Characterizing RG flows: Finiteness of T with sources (Jack & Osborn 1990)

The basic idea:

- Local scale transformations are encoded in $T = T^{\mu}_{\mu}$.
- Put the theory in a curved background, **and promote the couplings to be** x **dependent sources** $\lambda(x)$.
- Compute T in terms of bare composite operators, gravitational terms and derivatives of sources (e.g. $(\nabla\lambda)^4$).
- **Consistency condition**: T is finite.

Implications:

- Non-trivial constraints on the anomaly coefficients.

$$\mu\frac{d}{d\mu}\tilde{a}(\lambda) = \frac{1}{8}\chi^{g}_{IJ}\beta^{I}\beta^{J}$$

If χ^{g}_{IJ} is positive definite – **irreversibility of RG flow**!

Characterizing RG flows: Finiteness of T with sources (Jack & Osborn 1990)

The basic idea:

- Local scale transformations are encoded in $T = T^{\mu}_{\mu}$.
- Put the theory in a curved background, **and promote the couplings to be x dependent sources** $\lambda(x)$.
- Compute T in terms of bare composite operators, gravitational terms and derivatives of sources (e.g. $(\nabla\lambda)^4$).
- **Consistency condition**: T is finite.

Implications:

- Non-trivial constraints on the anomaly coefficients.

$$\mu\frac{d}{d\mu}\tilde{a}(\lambda) = \frac{1}{8}\chi^{g}_{IJ}\beta^{I}\beta^{J}$$

If χ^{g}_{IJ} is positive definite – **irreversibility of RG flow**!

Characterizing RG flows: Finiteness of T with sources (Jack & Osborn 1990)

The basic idea:

- Local scale transformations are encoded in $T = T^{\mu}_{\mu}$.
- Put the theory in a curved background,
  and promote the couplings to be x dependent sources $\lambda(x)$.
- Compute T in terms of bare composite operators,
  gravitational terms and derivatives of sources (e.g. $(\nabla\lambda)^4$).
- **Consistency condition**: T is finite.

Implications:

- Non-trivial constraints on the anomaly coefficients.

$$\mu\frac{d}{d\mu}\tilde{a}(\lambda) = \frac{1}{8}\chi^{g}_{IJ}\beta^{I}\beta^{J}$$

If χ^{g}_{IJ} is positive definite – **irreversibility of RG flow**!

Characterizing RG flows: The Local Callan-Symanzik equation (Osborn 91)

The basic idea:

- Consider local scale transformation in the presence of background sources.
- **A Callan-Symanzik symmetry**: Local scale transformations can be compensated by local transformations of the background sources.
- **Consistency condition**: The symmetry is abelian.

Implications:

- Reproduced the results of Jack & Osborn.
- The constraints are independent of the details of the model and the regularization scheme.

The Local Callan-Symanzik symmetry: ingredients

- Local scale transformations implemented using a background metric.

$$\begin{aligned} \eta^{\mu\nu} &\rightarrow g^{\mu\nu}(x) \\ \delta_\sigma g^{\mu\nu}(x) &= 2\sigma(x) g^{\mu\nu}(x) \end{aligned}$$

- The local transformation of the parameters implemented by promoting them to background fields

$$\begin{aligned} \lambda^I &\rightarrow \lambda^I(x) \\ \delta_\sigma \lambda^I(x) &= -\sigma(x)\beta^I(\lambda(x)) \end{aligned}$$

- The symmetry generator:

$$\left(\delta_\sigma g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}(x)} + \delta_\sigma \lambda^I \frac{\delta}{\delta \lambda^I(x)} + \dots\right) = \sigma\left(2g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}(x)} - \beta^I \frac{\delta}{\delta \lambda^I(x)} + \dots\right)$$

The Local Callan-Symanzik symmetry: ingredients

- Local scale transformations implemented using a background metric.

$$\begin{aligned} \eta^{\mu\nu} &\rightarrow g^{\mu\nu}(x) \\ \delta_\sigma g^{\mu\nu}(x) &= 2\sigma(x) g^{\mu\nu}(x) \end{aligned}$$

- The local transformation of the parameters implemented by promoting them to background fields

$$\begin{aligned} \lambda^I &\rightarrow \lambda^I(x) \\ \delta_\sigma \lambda^I(x) &= -\sigma(x)\beta^I(\lambda(x)) \end{aligned}$$

- The symmetry generator:

$$\left(\delta_\sigma g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}(x)} + \delta_\sigma \lambda^I \frac{\delta}{\delta \lambda^I(x)} + \dots\right) = \sigma \left(2 g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}(x)} - \beta^I \frac{\delta}{\delta \lambda^I(x)} + \dots\right)$$

Background fields: a useful concept

- **Sources for renormalized operators**:
  One can define $\mathcal{W}[g, \lambda, \ldots] = -i \log \mathcal{Z}[g, \lambda, \ldots]$
  a renormalized generating functional for correlation functions of composite operators

$$\frac{1}{\sqrt{-g}} \frac{\delta \mathcal{W}}{\delta \lambda^I(x)} = [\mathcal{O}_I(x)] \qquad \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{W}}{\delta g^{\mu\nu}(x)} = [T_{\mu\nu}(x)]$$

- **Spurions**:
  If an interaction breaks a symmetry explicitly,
  one can formally restore the symmetry by assigning the background sources with transformation properties.

Background fields: a useful concept

- **Sources for renormalized operators**:
  One can define $\mathcal{W}[g, \lambda, \ldots] = -i \log \mathcal{Z}[g, \lambda, \ldots]$
  a renormalized generating functional for correlation functions of composite operators

$$\frac{1}{\sqrt{-g}} \frac{\delta \mathcal{W}}{\delta \lambda^I(x)} = [\mathcal{O}_I(x)] \qquad \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{W}}{\delta g^{\mu\nu}(x)} = [T_{\mu\nu}(x)]$$

- **Spurions**:
  If an interaction breaks a symmetry explicitly,
  one can formally restore the symmetry by assigning the background sources with transformation properties.

The background dilaton: a redundant but useful notation

Introduce a background metric $g^{\mu\nu}(x)$,
and use the redundant notation

$$g^{\mu\nu}(x) = e^{2\tau(x)}\bar{g}^{\mu\nu}(x)$$

$\tau(x)$ is a source for T:

$$\frac{\delta}{\delta\tau(x)}\mathcal{W}[g]\Big| \quad = \quad 2g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}(x)}\mathcal{W}[g]\Big| \quad = \quad [T^{\mu}_{\mu}(x)]$$

An effective action for the dilaton
$\Rightarrow$ bookkeeping device for n-point functions of $T \equiv T^{\mu}_{\mu}$.

The background dilaton: a redundant but useful notation

Introduce a background metric $g^{\mu\nu}(x)$,
and use the redundant notation

$$g^{\mu\nu}(x) = e^{2\tau(x)}\bar{g}^{\mu\nu}(x)$$

$\tau(x)$ is a source for T:

$$\frac{\delta}{\delta\tau(x)}\mathcal{W}[g]\Big| \quad = \quad 2g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}(x)}\mathcal{W}[g]\Big| \quad = \quad [T^{\mu}_{\mu}(x)]$$

An effective action for the dilaton
$\Rightarrow$ bookkeeping device for n-point functions of $T \equiv T^{\mu}_{\mu}$.

The local Callan-Symanzik equation

Define a symmetry generator

$$\Delta_\sigma^{CS} = \int d^4x\sigma(x)\left(\frac{\delta}{\delta\tau(x)} - \beta^I\frac{\delta}{\delta\lambda^I(x)} + \dots\right)$$

$\mathcal{W}$ is invariant up to a local anomaly

$$\Delta_\sigma^{CS}\mathcal{W}[g,\lambda,\dots] = \int dx\sigma\mathcal{A}[g,\lambda,\dots]$$

- The anomaly $\mathcal{A}$ is the most general scalar which can be written using the sources and their derivatives.
- Can be written in an operator form

$$T(x) = \beta^I\left[\mathcal{O}_I(x)\right] + \dots$$

Consistency conditions

- Constraints on the coefficients in the symmetry generator

$$\left[\Delta^{CS}_{\sigma_2}, \Delta^{CS}_{\sigma_1}\right] = 0$$

- Constraints on the anomaly coefficients

$$\left[\Delta^{CS}_{\sigma_2}, \Delta^{CS}_{\sigma_1}\right] \mathcal{W} \quad = \quad \Delta^{CS}_{\sigma_2}\left(\int dx_1 \sigma_1 \mathcal{A}\right) - \Delta^{CS}_{\sigma_1}\left(\int dx_2 \sigma_2 \mathcal{A}\right) = 0$$

Consistency conditions - implications

$$\left[\Delta^{CS}_{\sigma_2}, \Delta^{CS}_{\sigma_1}\right] \mathcal{W} = \Delta^{CS}_{\sigma_2}\left(\int dx_1 \sigma_1 \mathcal{A}\right) - \Delta^{CS}_{\sigma_1}\left(\int dx_2 \sigma_2 \mathcal{A}\right) = 0$$

- Leads to ~ 10 differential equations.
- Example:

$$\mu \frac{d}{d\mu}\tilde{a}(\lambda) = \frac{1}{8}\chi^g_{IJ}\beta^I\beta^J$$

  If χ^g_{IJ} is positive definite – **irreversibility of RG flow**!

- What about the other constraints on the anomaly ?
- **New result:**
  a new formulation of the anomaly.
  $\Rightarrow$ most of the other equations do not constrain the RG flow.
  Only one additional non-trivial consistency condition
  (related to F^2 anomaly).

Consistency conditions - implications

$$\left[\Delta^{CS}_{\sigma_2}, \Delta^{CS}_{\sigma_1}\right]\mathcal{W} = \Delta^{CS}_{\sigma_2}\left(\int dx_1\sigma_1\mathcal{A}\right) - \Delta^{CS}_{\sigma_1}\left(\int dx_2\sigma_2\mathcal{A}\right) = 0$$

- Leads to ~ 10 differential equations.
- Example:

$$\mu\frac{d}{d\mu}\tilde{a}(\lambda) = \frac{1}{8}\chi^g_{IJ}\beta^I\beta^J$$

If χ^g_{IJ} is positive definite – **irreversibility of RG flow**!

- What about the other constraints on the anomaly ?
- **New result:**
a new formulation of the anomaly.
$\Rightarrow$ most of the other equations do not constrain the RG flow.
Only one additional non-trivial consistency condition
(related to F^2 anomaly).

Characterizing RG flows: Dispersion relations for correlators of T (Komargodski & Schwimmer 2011)

The basic idea:

- Study the response of the system to **local** scale transformations.
  $=$ compute **correlators** of T ("dilaton scattering amplitudes")
- Use analyticity and unitarity to write dispersion relations
  with positivity constraints.

Implications:

- Comparing between two limits of the flow, across non-perturbative regimes
  (a theorem, KS 2011).
- Constraining the asymptotic limits of perturbative RG flows
  (All perturbative RG flows end in conformal fixed points, LPR, 2011)

Dilaton scattering amplitudes – issues

- KS used the dilaton effective action at the CFT fixed points.
  The effective action is determined uniquely by the Weyl anomaly (Wess-Zumino action)
- But, when comparing two CFTs, one of them might not be "improved" ($T \sim T + \Box \mathcal{O}$)
  How does it affect the "scattering amplitude"?
- LPR used the effective action off-criticality.
  They claimed that the imaginary part of the dilaton scattering amplitude is controlled by the β function.
  Is it obvious?

We will use the local CS equation to answer these questions.

Dilaton scattering amplitudes – issues

- KS used the dilaton effective action at the CFT fixed points. The effective action is determined uniquely by the Weyl anomaly (Wess-Zumino action)
- But, when comparing two CFTs, one of them might not be "improved" ($T \sim T + \Box \mathcal{O}$) **How does it affect the "scattering amplitude"?**
- LPR used the effective action off-criticality. They claimed that the imaginary part of the dilaton scattering amplitude is controlled by the β function. **Is it obvious?**

We will use the local CS equation to answer these questions.

Dilaton effective action and the local CS equation

- The local CS equation can be used to compute correlators of T off-criticality:

$$\langle T(x)\rangle = \beta^I \langle \mathcal{O}_I(x)\rangle + \mathcal{A}(x)$$

$$\langle T(x_n)\dots T(x_1)\rangle \sim \beta^{I_m}\dots\beta^{I_1}\langle \mathcal{O}_{I_m}(x_m)\dots\mathcal{O}_{I_1}(x_1)\rangle + \textit{anomaly related contact terms}$$

- The dilaton effective action is a convenient bookkeeping device for correlators of T.
- **New result:**
  a systematic approach for this computation.
- Requires the reformulation of the anomaly.

Main points of the talk

- The local CS equation can be used to characterize RG flows
  1. by studying the consistency conditions.
  2. as a tool for computing correlators of T.

  There is a lot of "know-how" involved in the process....

- We have improved the formulation of the anomaly, in a way which isolates only the interesting constraints.

- The new formulation allowed us to give analytic expressions for the dilaton effective action off-criticality.

- We proved that the "scattering amplitudes" involved in the a theorem and LPR are indeed insensitive to lower dimension operators.

Outline

- Introduction
- The local Callan-Symanzik equation
- The local Callan-Symanzik anomaly
- n point functions of T (dilaton effective action)

The set-up

- Consider a 4D fixed point.
- Put the theory in a curved background metric $g^{\mu\nu}(x)$.
- Assign a dimensionless source $\lambda^I(x)$ to each of the marginal operators.
- Define a renormalized generating functional $\mathcal{W}$.
- Consider a background

$$\begin{aligned} \bar{g}^{\mu\nu} &= \eta^{\mu\nu} \\ \nabla_\mu \lambda^I &= 0 \\ \left|\lambda^I - \lambda^{I*}\right| &\ll 1 \end{aligned}$$

The β functions vanish at $\lambda^I = \lambda^{I*}$

The set-up

- The local CS equation:

$$\int d^4x\sigma\left(\frac{\delta}{\delta\tau(x)}-\beta^I\frac{\delta}{\delta\lambda^I(x)}\right)\mathcal{W}[g,\lambda]=\int dx\sigma\mathcal{A}$$

- In general there are other operators $\mathcal{O}_\alpha(x)$ of dimension d_α.
  We could add sources m^α of dimension $4-d_\alpha$ to all these operators and write

$$\int d^4x\sigma\left(\frac{\delta}{\delta\tau(x)}-\beta^I\frac{\delta}{\delta\lambda^I(x)}+m^\beta\left(d^\alpha_\beta+\gamma^\alpha_\beta\right)\frac{\delta}{\delta m^\alpha(x)}\right)\mathcal{W}[g,\lambda^I,m^\alpha]$$
$$=\int dx\sigma\mathcal{A}$$

  In a background where $m^\alpha=0$ this will have no effect on our computations.

- Except...

The set-up

- The local CS equation:

$$\int d^4x\sigma \left(\frac{\delta}{\delta\tau(x)} - \beta^I \frac{\delta}{\delta\lambda^I(x)} \right) \mathcal{W}[g,\lambda] = \int dx\sigma\mathcal{A}$$

- In general there are other operators $\mathcal{O}_\alpha(x)$ of dimension d_α.
  We could add sources m^α of dimension $4 - d_\alpha$ to all these operators and write

$$\int d^4x\sigma \left(\frac{\delta}{\delta\tau(x)} - \beta^I \frac{\delta}{\delta\lambda^I(x)} + m^\beta \left(d^\alpha_\beta + \gamma^\alpha_\beta\right) \frac{\delta}{\delta m^\alpha(x)} \right) \mathcal{W}[g,\lambda^I,m^\alpha]$$
$$= \int dx\sigma\mathcal{A}$$

  In a background where $m^\alpha = 0$ this will have no effect on our computations.

- Except...

The set-up

- When writing the basis of renormalized dimension 4 scalar operators, we must take into account

$$[\mathcal{O}_I(x)]\,,\quad \nabla_\mu\,[J^\mu_A]\,,\quad \nabla^2\,[\mathcal{O}_a]$$

- We therefore have to add more background fields $A^A_\mu(x)$ and $m^a(x)$:

$$\frac{1}{\sqrt{-g}}\frac{\delta\mathcal{W}}{\delta m^a(x)} = [\mathcal{O}_a(x)] \qquad\qquad \frac{1}{\sqrt{-g}}\frac{\delta\mathcal{W}}{\delta A^A_\mu(x)} = \left[J^A_\mu(x)\right]$$

- All derivatives are promoted to be covariant derivatives:

$$\nabla_\mu \;=\; \partial_\mu + A_\mu$$

- evaluate all derivatives in the background

$$\begin{aligned} \bar{g}^{\mu\nu} &= \eta^{\mu\nu} \\ m = A = \nabla_\mu\lambda^I &= 0 \\ \left|\lambda^I - \lambda^{I*}\right| &\ll 1 \end{aligned}$$

- Now we are ready to write the most general symmetry allowed by dimensional analysis:

The local CS symmetry

$$\Delta_\sigma^{CS}(x) \quad = \quad \Delta_\sigma^{W}(x) - \Delta_\sigma^{\beta}(x)$$

where

$$\Delta_\sigma^W = \int d^4x \left[\sigma \frac{\delta}{\delta\tau(x)}\right]$$

$$\Delta_\sigma^\beta = \int d^4x \Bigg[\sigma \left(\beta^I \frac{\delta}{\delta\lambda^I(x)} + \rho_I^A \nabla_\mu \lambda^I \frac{\delta}{\delta A_\mu^A(x)}\right) - \nabla_\mu \sigma \left(S^A \frac{\delta}{\delta A_\mu^A(x)}\right)$$
$$- \sigma \left(m^b \left(2\delta_b^a + \gamma_b^a\right) + \frac{1}{3}\eta^a R + d_I^a \nabla^2 \lambda^I + \frac{1}{2}\epsilon_{IJ}^a \nabla_\mu \lambda^I \nabla^\mu \lambda^J\right) \frac{\delta}{\delta m^a(x)}$$
$$+ \nabla_\mu \sigma \left(\theta_I^a \nabla^\mu \lambda^I \frac{\delta}{\delta m^a(x)}\right) - \nabla^2 \sigma \left(t^a \frac{\delta}{\delta m^a(x)}\right)\Bigg]$$

The local CS equation

$$\boxed{\Delta_\sigma^{CS} \mathcal{W} = \int dx \sigma \mathcal{A}}$$

The local CS equation - operator form

$$\int d^4x \Big[\sigma \left(\frac{\delta}{\delta \tau(x)} - \beta^I \frac{\delta}{\delta \lambda^I(x)} - \rho_I^A \nabla_\mu \lambda^I \frac{\delta}{\delta A_\mu^A(x)} \right) + \nabla_\mu \sigma \left(S^A \frac{\delta}{\delta A_\mu^A(x)} \right)$$
$$+ \sigma \left(m^b \left(2\delta_b^a + \gamma_b^a \right) + \frac{1}{3} \eta^a R + d_I^a \nabla^2 \lambda^I + \frac{1}{2} \epsilon_{IJ}^a \nabla_\mu \lambda^I \nabla^\mu \lambda^J \right) \frac{\delta}{\delta m^a(x)}$$
$$- \nabla_\mu \sigma \left(\theta_I^a \nabla^\mu \lambda^I \frac{\delta}{\delta m^a(x)} \right) + \nabla^2 \sigma \left(t^a \frac{\delta}{\delta m^a(x)} \right) \Big] \mathcal{W} = \int dx \sigma \mathcal{A}$$

The operator form of the equation:

($\sigma(x) = \delta(x)$, flat background)

$$T = \beta^I [\mathcal{O}_I] - S^A \nabla_\mu [J_A^\mu] - t^a \nabla^2 [\mathcal{O}_a]$$

The running of the renormalized operators:

$$(\mathcal{D} - 4)[\mathcal{O}_I] = \partial_J \beta^I [\mathcal{O}_J] - \rho_I^A \nabla_\mu [J_A^\mu] - d_I^a \nabla^2 [\mathcal{O}_a]$$
$$(\mathcal{D} - 2)[\mathcal{O}_a] = \gamma_a^b [\mathcal{O}_b]$$

(Comment: the currents are renormalized because the symmetry is explicitly broken by the sources)

The local CS equation - operator form

$$\int d^4x \Big[\sigma \left(\frac{\delta}{\delta \tau(x)} - \beta^I \frac{\delta}{\delta \lambda^I(x)} - \rho_I^A \nabla_\mu \lambda^I \frac{\delta}{\delta A_\mu^A(x)} \right) + \nabla_\mu \sigma \left(S^A \frac{\delta}{\delta A_\mu^A(x)} \right)$$
$$+ \sigma \left(m^b \left(2\delta_b^a + \gamma_b^a \right) + \frac{1}{3}\eta^a R + d_I^a \nabla^2 \lambda^I + \frac{1}{2}\epsilon_{IJ}^a \nabla_\mu \lambda^I \nabla^\mu \lambda^J \right) \frac{\delta}{\delta m^a(x)}$$
$$- \nabla_\mu \sigma \left(\theta_I^a \nabla^\mu \lambda^I \frac{\delta}{\delta m^a(x)} \right) + \nabla^2 \sigma \left(t^a \frac{\delta}{\delta m^a(x)} \right) \Big] \mathcal{W} = \int dx \sigma \mathcal{A}$$

The operator form of the equation:

($\sigma(x) = \delta(x)$, flat background)

$$T = \beta^I [\mathcal{O}_I] - S^A \nabla_\mu [J_A^\mu] - t^a \nabla^2 [\mathcal{O}_a]$$

The running of the renormalized operators:

$$(\mathcal{D} - 4)[\mathcal{O}_I] = \partial_J \beta^I [\mathcal{O}_J] - \rho_I^A \nabla_\mu [J_A^\mu] - d_I^a \nabla^2 [\mathcal{O}_a]$$
$$(\mathcal{D} - 2)[\mathcal{O}_a] = \gamma_a^b [\mathcal{O}_b]$$

(Comment: the currents are renormalized because the symmetry is explicitly broken by the sources)

The local CS symmetry - details

$$\int d^4x \Big[\sigma \left(\frac{\delta}{\delta\tau(x)} - \beta^I \frac{\delta}{\delta\lambda^I(x)} - \rho_I^A \nabla_\mu \lambda^I \frac{\delta}{\delta A_\mu^A(x)} \right) + \nabla_\mu \sigma \left(S^A \frac{\delta}{\delta A_\mu^A(x)} \right)$$

$$+ \sigma \left(m^b \left(2\delta_b^a + \gamma_b^a \right) + \frac{1}{3}\eta^a R + d_I^a \nabla^2 \lambda^I + \frac{1}{2}\epsilon_{IJ}^a \nabla_\mu \lambda^I \nabla^\mu \lambda^J \right) \frac{\delta}{\delta m^a(x)}$$

$$- \nabla_\mu \sigma \left(\theta_I^a \nabla^\mu \lambda^I \frac{\delta}{\delta m^a(x)} \right) + \nabla^2 \sigma \left(t^a \frac{\delta}{\delta m^a(x)} \right) \Big]$$

1. Renormalization and improvement schemes.
2. Ambiguities.
3. Consistency conditions.
4. Transformation properties of functions.

Improvement scheme

A theory is defined up to improvement of the energy-momentum tensor

$$\begin{aligned} T_{\mu\nu} &\sim T_{\mu\nu} + \frac{1}{3}\left(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu\right)\mathcal{O}_a \\ T &\sim T + \Box\mathcal{O}_a \end{aligned}$$

In a curved background, this is determined by

$$\mathcal{W} \supset \int \sqrt{-g}d^4xR\mathcal{O}_a$$

This effect is taken into account by

$$T = \beta^I\left[\mathcal{O}_I\right] \quad - S^A\nabla_\mu\left[J^\mu_A\right] \quad -t^a\Box\left[\mathcal{O}_a\right]$$

t^a describes the choice of "improvement" scheme in the theory.
("unimproved fixed point" = non-zero t^a)

Ambiguities in the presence of global symmetries

Add the Ward identity to the local CS equation

$$\Delta_\alpha^{Global}\mathcal{W} = \int d^4x \left[\alpha^A (T_A\lambda)^I \frac{\delta}{\delta\lambda^I(x)} - \nabla_\mu \alpha^A \frac{\delta}{\delta A_\mu^A(x)}\right]\mathcal{W} = 0$$

$$\Rightarrow \left(\Delta_\sigma^{CS} - \Delta_\alpha^{Global}\right)\mathcal{W} = \int dx \sigma \mathcal{A}$$

choosing $\alpha^A = \sigma w^A(\lambda)$ we can rewrite the symmetry generator as

$$\Delta_\sigma^\beta + \Delta_{\sigma\omega}^{Global} = \int d^4x \left[\sigma\left(\beta^I + (\omega^A T_A\lambda)^I\right)\frac{\delta}{\delta\lambda^I(x)}\right.$$
$$\left. + \sigma\left(\rho_I^A - \partial_I\omega^A\right)\nabla_\mu\lambda^I \frac{\delta}{\delta A_\mu^A(x)} - \nabla_\mu\sigma\left(\left(S^A + \omega^A\right)\frac{\delta}{\delta A_\mu^A(x)}\right) + \dots\right.$$

S^A can be set to zero.

Ambiguities in the presence of global symmetries

The β function is ambiguous.

$$\beta^I \to \beta^I + (\omega^A T^A \lambda)^I \qquad \rho_I^A \to \rho_I^A - \partial_I \omega^A \qquad S^A \to S^A + \omega^A$$

Invariant functions

$$\begin{aligned} B^I &= \beta^I - \left(S^A T^A \lambda\right)^I \\ P_I^A &= \rho_I^A + \partial_I S^A \end{aligned}$$

choosing the gauge $\omega^A = -S^A$

$$T = \beta^I [\mathcal{O}_I] + S^A \nabla_\mu [J_A^\mu] \qquad \to \qquad T = B^I [\mathcal{O}_I]$$

$$\text{CFT} \Leftrightarrow B^I = 0$$

There are CFTs with non-zero β functions (FGS, 2012).

The Weyl anomaly

This classification cannot be used in our formalism, because $a(\lambda)E_4$ is not a total derivative.

Generalizing the classification in the presence of background sources:

- "type B": Manifestly consistent
  (variation contains no derivative of σ, so the commutator always vanishes).
- "type A": Not consistent in the presence of background sources,
  unless imposing non-trivial relations between the different anomalies.

The consistency conditions for the local CS anomaly

One of these consistency condition has physical significance.

$$\frac{1}{\sqrt{-g}}\sigma\mathcal{A} \quad = \quad \ldots\sigma\left(aE_4 + \frac{1}{2}\chi^g_{IJ}G^{\mu\nu}\nabla_\mu\lambda^I\nabla_\nu\lambda^J\right) + \nabla^\mu\sigma\left(G_{\mu\nu}w_I\nabla^\nu\lambda^I\right)\ldots$$

The vanishing of the $\sigma_{[1}\nabla_\mu\sigma_{2]}G^{\mu\nu}\nabla_\nu\lambda^I$:

$$\mathcal{L}[w_I] \quad = \quad -8\partial_I a + \chi^g_{IJ}B^J$$

multiplying by B^I:

$$\mu\frac{d}{d\mu}\tilde{a} = B^I\partial_I\tilde{a} = \frac{1}{8}\chi^g_{IJ}B^I B^J$$

where $\tilde{a} = a + \frac{1}{8}B^J w_J$.
If χ^g_{IJ} is positive definite then we have a function which changes monotonously along the RG flow.

What about all the other equations??

Do they have interesting implications?

The consistency conditions for the local CS anomaly

One of these consistency condition has physical significance.

$$\frac{1}{\sqrt{-g}}\sigma\mathcal{A} \quad = \quad \ldots\sigma\left(aE_4 + \frac{1}{2}\chi^g_{IJ}G^{\mu\nu}\nabla_\mu\lambda^I\nabla_\nu\lambda^J\right) + \nabla^\mu\sigma\left(G_{\mu\nu}w_I\nabla^\nu\lambda^I\right)\ldots$$

The vanishing of the $\sigma_{[1}\nabla_\mu\sigma_{2]}G^{\mu\nu}\nabla_\nu\lambda^I$:

$$\mathcal{L}[w_I] \quad = \quad -8\partial_I a + \chi^g_{IJ}B^J$$

multiplying by B^I:

$$\mu\frac{d}{d\mu}\tilde{a} = B^I\partial_I\tilde{a} = \frac{1}{8}\chi^g_{IJ}B^IB^J$$

where $\tilde{a} = a + \frac{1}{8}B^J w_J$.
If χ^g_{IJ} is positive definite then we have a function which changes monotonously along the RG flow.

What about all the other equations??

Do they have interesting implications?

Solving the consistency conditions

Step 1: Remove scheme dependent anomalies

- The coefficients $d, U_I, V_{IJ}, S_{(IJ)}, T_{IJK}, k_a, j_{aI}$ can be set to zero.
- Most differential equations are replaced by algebraic constraints! e.g.

$$q_a - \frac{1}{2}\left(\cancel{B^I\partial_I k_a - \gamma_a^b k_b} + r_{aI}B^I\right) = 0$$

Step 2: Impose algebraic constraints

- The coefficients $\beta_c, Y_I, \chi_I^e, \chi_{IJ}^f, \chi_{IJ}^a, \chi_{IJK}^b, q_a, r_{aI}$ can be eliminated.
- We find the "generalized Weyl anomaly":

$$\mathcal{A} = \mathcal{A}_{R^2} + \mathcal{A}_{W^2} + \mathcal{A}_{E_4} + \mathcal{A}_{F^2} + \mathcal{A}_{\nabla^2 R}$$

- Only ~ 2 consistency conditions left.

Solving the consistency conditions

Step 1: Remove scheme dependent anomalies

- The coefficients $d, U_I, V_{IJ}, S_{(IJ)}, T_{IJK}, k_a, j_{aI}$ can be set to zero.
- Most differential equations are replaced by algebraic constraints! e.g.

$$q_a - \frac{1}{2}\left(\cancel{B^I\partial_I k_a - \gamma_a^b k_b} + r_{aI} B^I\right) = 0$$

Step 2: Impose algebraic constraints

- The coefficients $\beta_c, Y_I, \chi_I^e, \chi_{IJ}^f, \chi_{IJ}^a, \chi_{IJK}^b, q_a, r_{aI}$ can be eliminated.
- We find the "generalized Weyl anomaly":

$$\mathcal{A} \quad = \quad \mathcal{A}_{R^2} + \mathcal{A}_{W^2} + \mathcal{A}_{E_4} + \mathcal{A}_{F^2} + \mathcal{A}_{\nabla^2 R}$$

- Only ~ 2 consistency conditions left.

The generalized W^2 anomaly

$$\frac{1}{\sqrt{-g}}\mathcal{A}_{W^2} \quad = \quad -cW^2$$

- The only difference with respect to the Weyl anomaly - c is a function of λ.
- Manifestly consistent (W^2 is invariant, c transforms without derivatives) - type B anomaly.

The generalized R^2 anomaly

$$\frac{1}{\sqrt{-g}}\sigma\mathcal{A}_{R^2} = \sigma\left(\frac{1}{2}b_{ab}M^aM^b + \frac{1}{2}b_{aIJ}M^a\Pi^{IJ} + \frac{1}{4}b_{IJKL}\Pi^{IJ}\Pi^{KL}\right)$$

- The "meaning" of the consistency conditions:
  The most general bilinear scalar constructed from Π and M.
- Manifestly consistent - type B anomaly.
- Unimproved fixed points have an R^2 anomaly

$$\frac{1}{\sqrt{-g}}\mathcal{A}_{R^2}\Big|_{\nabla\lambda=B=m=0} = \frac{1}{72}b_{ab}t^at^bR^2$$

(relevant for a theorem and Buican's conjecture)

The generalized R^2 anomaly

$$\frac{1}{\sqrt{-g}}\sigma\mathcal{A}_{R^2} \quad = \quad \sigma\left(\frac{1}{2}b_{ab}M^aM^b + \frac{1}{2}b_{aIJ}M^a\Pi^{IJ} + \frac{1}{4}b_{IJKL}\Pi^{IJ}\Pi^{KL}\right)$$

- The "meaning" of the consistency conditions:
  The most general bilinear scalar constructed from Π and M.
- Manifestly consistent - type B anomaly.
- Unimproved fixed points have an R^2 anomaly

$$\frac{1}{\sqrt{-g}}\mathcal{A}_{R^2}\Big|_{\nabla\lambda=B=m=0} \quad = \quad \frac{1}{72}b_{ab}t^at^bR^2$$

(relevant for a theorem and Buican's conjecture)

The generalized E_4 anomaly

$$
\begin{aligned}
\frac{1}{\sqrt{-g}}\sigma\mathcal{A}_{E_4} \quad = \quad & \sigma\left(aE_4 + \chi^g_{IJ}\left(\frac{1}{2}J_{\mu\nu}\nabla^\mu\lambda^I\nabla^\nu\lambda^J - \frac{1}{4}U^I_K\Lambda^K\Lambda^J\right)\right) \\
& +\nabla^\mu\sigma\left(w_I G_{\mu\nu}\nabla^\nu\lambda^I\right) + \frac{1}{2}\partial_{[J}w_{I]}\Lambda^I\left(\Delta^{CS}_\sigma\Lambda^J\right) \\
& +\frac{1}{2}\sigma\overline{\chi}^g_{IJK}\Omega^{IJK}
\end{aligned}
$$

- a, χ^g_{IJ} and w_I are related by a differential equation:

$$\mathcal{L}[w_I] \quad = \quad -8\partial_I a + \chi^g_{IJ}B^J$$

This is a genuine constraint on the QFT. Irreversibility!

$$
\begin{aligned}
\Lambda^I &= \left(U^{-1}\right)^I_J\left(\nabla^2\lambda^J + \tfrac{1}{6}B^J R\right) & \qquad J_{\mu\nu} &= G_{\mu\nu} + \tfrac{R}{6}g_{\mu\nu} \\
\Omega^{IJK} &= \left(\Pi^{IJ} + \tfrac{1}{2}B^{(I}\Lambda^{J)}\right)\Lambda^K & \qquad \overline{\chi}^g_{IJK} &= -\partial_{(J}\chi^g_{KI)} + \tfrac{1}{2}\partial_K\chi^g_{IJ}
\end{aligned}
$$

The generalized F^2 anomaly

$$\begin{aligned}\frac{1}{\sqrt{-g}}\sigma\mathcal{A}_{F^2} &= \sigma\left(\frac{1}{4}\kappa_{AB}F^A_{\mu\nu}F^{B\mu\nu}+\frac{1}{2}\zeta_{AIJ}F^A_{\mu\nu}\nabla^\mu\lambda^I\nabla^\nu\lambda^J\right)\\ &+\nabla^\mu\sigma\left(\eta_{AI}F^A_{\mu\nu}\nabla^\nu\lambda^I\right)+\frac{1}{2}\eta_{A[I}P^A_{J]}\Lambda^I\left(\Delta^{CS}_\sigma\Lambda^J\right)\\ &+\sigma\left(\frac{1}{2}P^A_I\zeta_{AJK}+\eta_{AI}\partial_{[J}P^A_{K]}\right)\Omega^{IJK}\end{aligned}$$

- κ_{AB}, ζ_{AIJ} and η_{AI} are related by the equations:

$$\begin{aligned}\mathcal{L}[\eta_{AI}] &= \kappa_{AB}P^B_I+\zeta_{AIJ}B^J-\chi^g_{IJ}(T_A\lambda)^J\\ 0 &= \eta_{AI}B^I+w_I(T_A\lambda)^I\end{aligned}$$

- Resemblance to the E_4 anomaly,
- The Lie derivative of the second equation is the consequence of the other two consistency conditions.
- Is there interesting information in this equation?

The generalized F^2 anomaly

$$
\begin{aligned}
\frac{1}{\sqrt{-g}}\sigma\mathcal{A}_{F^2} &= \sigma\left(\frac{1}{4}\kappa_{AB}F^A_{\mu\nu}F^{B\mu\nu}+\frac{1}{2}\zeta_{AIJ}F^A_{\mu\nu}\nabla^\mu\lambda^I\nabla^\nu\lambda^J\right)\\
&\quad +\nabla^\mu\sigma\left(\eta_{AI}F^A_{\mu\nu}\nabla^\nu\lambda^I\right)+\frac{1}{2}\eta_{A[I}P^A_{J]}\Lambda^I\left(\Delta^{CS}_\sigma\Lambda^J\right)\\
&\quad +\sigma\left(\frac{1}{2}P^A_I\zeta_{AJK}+\eta_{AI}\partial_{[J}P^A_{K]}\right)\Omega^{IJK}
\end{aligned}
$$

- κ_{AB}, ζ_{AIJ} and η_{AI} are related by the equations:

$$
\begin{aligned}
\mathcal{L}[\eta_{AI}] &= \kappa_{AB}P^B_I+\zeta_{AIJ}B^J-\chi^g_{IJ}(T_A\lambda)^J\\
0 &= \eta_{AI}B^I+w_I(T_A\lambda)^I
\end{aligned}
$$

- Resemblance to the E_4 anomaly,
- The Lie derivative of the second equation is the consequence of the other two consistency conditions.
- Is there interesting information in this equation?

The local CS anomaly – summary

- Most of the consistency conditions are eliminated when using covariant functions to write the anomaly.
- One of the remaining equations is related to the irreversibility of the flow, the interpretation of the other is still unclear.
- The CS anomaly in 3 dimensions (Nakayama 2013) can be simplified by using the analogues of Π and M.
- The new form of the anomaly is a good starting point for computing the dilaton effective action.

The generalized F^2 anomaly

$$
\begin{aligned}
\frac{1}{\sqrt{-g}}\sigma\mathcal{A}_{F^2} &= \sigma\left(\frac{1}{4}\kappa_{AB}F^A_{\mu\nu}F^{B\mu\nu}+\frac{1}{2}\zeta_{AIJ}F^A_{\mu\nu}\nabla^\mu\lambda^I\nabla^\nu\lambda^J\right)\\
&\quad+\nabla^\mu\sigma\left(\eta_{AI}F^A_{\mu\nu}\nabla^\nu\lambda^I\right)+\frac{1}{2}\eta_{A[I}P^A_{J]}\Lambda^I\left(\Delta^{CS}_\sigma\Lambda^J\right)\\
&\quad+\sigma\left(\frac{1}{2}P^A_I\zeta_{AJK}+\eta_{AI}\partial_{[J}P^A_{K]}\right)\Omega^{IJK}
\end{aligned}
$$

- κ_{AB}, ζ_{AIJ} and η_{AI} are related by the equations:

$$
\begin{aligned}
\mathcal{L}[\eta_{AI}] &= \kappa_{AB}P^B_I+\zeta_{AIJ}B^J-\chi^g_{IJ}(T_A\lambda)^J\\
0 &= \eta_{AI}B^I+w_I(T_A\lambda)^I
\end{aligned}
$$

- Resemblance to the E_4 anomaly,
- The Lie derivative of the second equation is the consequence of the other two consistency conditions.
- Is there interesting information in this equation?

1 and 2-point functions of T (dilaton effective action)

$$\Delta_\tau^W \mathcal{W} = \Delta_\tau^\beta \mathcal{W} + \int d^4x \tau \mathcal{A}$$

$$\begin{aligned}
\Delta_\tau^W \Delta_\tau^W \mathcal{W} &= \Delta_\tau^W \Delta_\tau^\beta \mathcal{W} + \Delta_\tau^W \int d^4x \tau \mathcal{A} \\
&= \Delta_\tau^\beta \Delta_\tau^W \mathcal{W} + \left[\Delta_\tau^W, \Delta_\tau^\beta\right] \mathcal{W} + \Delta_\tau^W \int d^4x \tau \mathcal{A} \\
&= \underbrace{\Delta_\tau^\beta \Delta_\tau^\beta \mathcal{W} + \left[\Delta_\tau^W, \Delta_\tau^\beta\right] \mathcal{W}}_{\mathcal{D}_2 \mathcal{W}} + \underbrace{\Delta_\tau^\beta \int d^4x \tau \mathcal{A} + \Delta_\tau^W \int d^4x \tau \mathcal{A}}_{\mathcal{C}_2}
\end{aligned}$$

$\mathcal{D}_2\mathcal{W}$: contribution from the composite operators of the theory.
$\mathcal{C}_2$: anomaly related ultra-local terms.

$$\begin{aligned}
\mathcal{D}_2 \mathcal{W}\Big| &= \int d^4x \sqrt{-g}\, \tau(x) \int d^4y \sqrt{-g}\, \tau(y) B^I(x) B^J(y) \, \langle \mathcal{O}_I(x) \mathcal{O}_J(y) \rangle \\
&\quad + \int dx \sqrt{-g}\, \tau^2(x) B^I \partial_I B^J \langle \mathcal{O}_J(x) \rangle \, . \\
\mathcal{C}_2 &= \int d^4x \sqrt{-g} \nabla^2 \tau \nabla^2 \tau (2d + B^I U_I)
\end{aligned}$$

1 and 2-point functions of T (dilaton effective action)

$$\Delta_\tau^W \mathcal{W} = \Delta_\tau^\beta \mathcal{W} + \int d^4x \tau \mathcal{A}$$

$$\begin{aligned}
\Delta_\tau^W \Delta_\tau^W \mathcal{W} &= \Delta_\tau^W \Delta_\tau^\beta \mathcal{W} + \Delta_\tau^W \int d^4x \tau \mathcal{A} \\
&= \Delta_\tau^\beta \Delta_\tau^W \mathcal{W} + \left[\Delta_\tau^W, \Delta_\tau^\beta\right] \mathcal{W} + \Delta_\tau^W \int d^4x \tau \mathcal{A} \\
&= \underbrace{\Delta_\tau^\beta \Delta_\tau^\beta \mathcal{W} + \left[\Delta_\tau^W, \Delta_\tau^\beta\right] \mathcal{W}}_{\mathcal{D}_2 \mathcal{W}} + \underbrace{\Delta_\tau^\beta \int d^4x \tau \mathcal{A} + \Delta_\tau^W \int d^4x \tau \mathcal{A}}_{\mathcal{C}_2}
\end{aligned}$$

$\mathcal{D}_2 \mathcal{W}$: contribution from the composite operators of the theory.
$\mathcal{C}_2$: anomaly related ultra-local terms.

$$\begin{aligned}
\mathcal{D}_2 \mathcal{W}\big| &= \int d^4x \sqrt{-g}\, \tau(x) \int d^4y \sqrt{-g}\, \tau(y) B^I(x) B^J(y) \left\langle \mathcal{O}_I(x) \mathcal{O}_J(y) \right\rangle \\
&\quad + \int dx \sqrt{-g}\, \tau^2(x) B^I \partial_I B^J \left\langle \mathcal{O}_J(x) \right\rangle . \\
\mathcal{C}_2 &= \int d^4x \sqrt{-g} \nabla^2 \tau \nabla^2 \tau (2d + B^I U_I)
\end{aligned}$$

n-point functions of T (dilaton effective action)

$$\underbrace{\Delta_\tau^W \ldots \Delta_\tau^W}_{n} \mathcal{W} = \mathcal{D}_n \mathcal{W} + \mathcal{C}_n$$

where we used the recursive definitions

$$\begin{aligned} \mathcal{D}_n &= \mathcal{D}_{n-1} \Delta_\tau^\beta + \left[\Delta_\tau^W, \mathcal{D}_{n-1} \right] \\ \mathcal{C}_n &= \Delta_\tau^W \mathcal{C}_{n-1} + \mathcal{D}_{n-1} \int dx \tau \mathcal{A} \end{aligned}$$

$\langle T(x_1) \ldots T(x_n) \rangle =$

Anomaly related contact terms

Generalized W^2 anomaly

Vanishes in a flat background.

Generalized F^2 anomaly

Vanishes in the $S^A = 0$ gauge, and flat background, due to the consistency condition

$$\Delta_\tau^{CS} A_\mu^A \Big| = -\tau P_I^A \nabla_\mu \lambda^I \Big| = 0$$

$$\Delta_\tau^{CS} \Delta_\tau^{CS} A_\mu^A \Big| = \tau \nabla_\mu \tau B^I P_I^A \Big| = 0$$

$$\Delta_\sigma^\beta = \int d^4x \Big[\sigma \Big(B^I \frac{\delta}{\delta \lambda^I(x)} + P_I^A \nabla_\mu \lambda^I \frac{\delta}{\delta A_\mu^A(x)} + \dots \Big)$$

Anomaly related contact terms

Generalized R^2 anomaly

$$\Gamma[\tau] \supset \sum_{k=1}^{\infty} \frac{1}{k!} \tilde{b}_k \int dx \tau^k \left(\nabla^2 \tau - (\nabla \tau)^2\right)^2$$

Generalized $\nabla^2 R$ anomaly

$$\Gamma[\tau] \supset -\tilde{d} \int dx \left(\nabla^2 \tau - (\nabla \tau)^2\right)^2$$

Both types of interactions vanish when using the on-shell condition

$$\nabla^2 \phi \propto \nabla^2 \tau - (\nabla \tau)^2 = 0$$

$(e^{-\tau} = 1 + \phi)$

Anomaly related contact terms

Generalized E_4 anomaly

$$\Gamma[\tau] \supset \sum_{k=0}^{\infty} \frac{1}{k!} \left(B^I \partial_I\right)^k \tilde{a} \int dx \tau^k \left(-4\nabla^2\tau\nabla_\mu\tau\nabla^\mu\tau + 2\left(\nabla_\mu\tau\nabla^\mu\tau\right)^2\right)$$
$$-\frac{3}{8}\sum_{k=0}^{\infty} \frac{1}{k!} \left(B^I \partial_I\right)^{k+1} \left(B^I w_I\right) \int dx \tau^k \left(\nabla_\mu\tau\nabla^\mu\tau\right)^2$$

$$= \tilde{a} \int dx \left(-4\nabla^2\tau\nabla_\mu\tau\nabla^\mu\tau + 2\left(\nabla_\mu\tau\nabla^\mu\tau\right)^2\right) + O(B^2)$$

The corrections to the fixed point WZ action begin at order $(B^I)^2$.
(need to use the consistency condition $B^I \partial_I \tilde{a} = \frac{1}{8}\chi^g_{IJ} B^I B^J$).

The couplings of the dilaton to composite operators

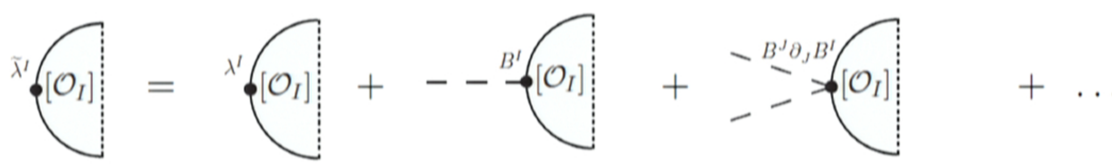
$$\Gamma[\tau] \supset \sum_{n=0}^{\infty} \frac{1}{n!}\mathcal{D}_n \mathcal{W}$$
$$= \exp\left\{\int d^4x \sum_{k=1}^{\infty} \tau^{k_j}\left(\frac{v^I_{k_j}}{k_j!}\frac{\delta}{\delta\lambda^I(x)} + \dots\right)\right\}\mathcal{W}$$

v^I_k: the coefficients of a coupling of k dilatons to $[\mathcal{O}_I]$

$$v^I_K = (B^J\partial_J)^{k-1}B^I$$

Agrees with the standard procedure of absorbing the dilaton into the renormalization scale

$$\lambda^I\mathcal{O}_I \rightarrow \widetilde{\lambda}^I(e^\tau\mu)\mathcal{O}_I = \lambda^I\mathcal{O}_I + \tau B^I\mathcal{O}_I + \frac{\tau^2}{2}B^J\partial_J B^I\mathcal{O}_I \dots$$



The couplings of the dilaton to composite operators

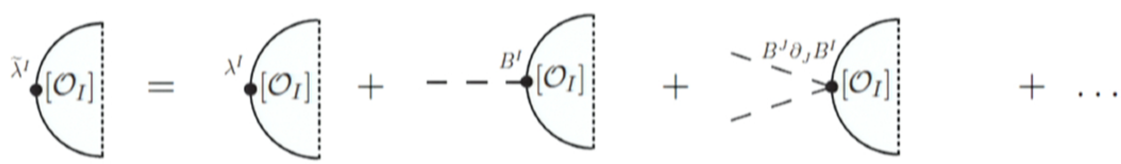
$$\Gamma[\tau] \supset \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{D}_n \mathcal{W}$$
$$= \exp\left\{ \int d^4x \sum_{k=1}^{\infty} \tau^{k_j} \left(\frac{v^I_{k_j}}{k_j!} \frac{\delta}{\delta \lambda^I(x)} + \ldots \right) \right\} \mathcal{W}$$

v^I_k: the coefficients of a coupling of k dilatons to $[\mathcal{O}_I]$

$$v^I_K = (B^J \partial_J)^{k-1} B^I$$

Agrees with the standard procedure of absorbing the dilaton into the renormalization scale

$$\lambda^I \mathcal{O}_I \rightarrow \widetilde{\lambda}^I(e^\tau \mu)\mathcal{O}_I = \lambda^I \mathcal{O}_I + \tau B^I \mathcal{O}_I + \frac{\tau^2}{2} B^J \partial_J B^I \mathcal{O}_I \ldots$$



The couplings of the dilaton to composite operators

Coupling to dim 3 vectors

These couplings are eliminated when working in the "gauge" $S^A = 0$, using B^I instead of β^I.

Coupling to dim 2 operators

Derivative couplings. More complicated...

$$\Gamma[\phi] \quad \supset \quad \exp\left\{\int d^4x \left(-\phi\, B^I \frac{\delta}{\delta\lambda^I(x)} + \frac{\phi^2}{2}\left(B^J\left(\delta^I_J + \partial_J B^I\right)\frac{\delta}{\delta\lambda^I(x)} + \frac{1}{2}B^J\theta^a_J\nabla^2\frac{\delta}{\delta m^a(x)}\right)\right.\right.$$
$$+ (1-\phi)\nabla^2\phi\; t^a\frac{\delta}{\delta m^a(x)}$$
$$\left.\left. - \phi\nabla^2\phi\left(2\eta^a + B^I\left(\frac{1}{2}\theta^a_I - \partial_I t^a\right)\right)\frac{\delta}{\delta m^a(x)} + \dots\right)\right\}\mathcal{W}\Big|$$

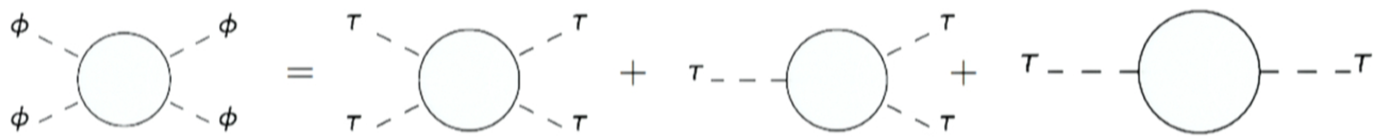
$(e^{-\tau} = 1 + \phi)$

Notice the importance of the on-shell condition $\nabla^2\phi = 0$!
Eliminates t^a which could be of order 1.

Example - Dilaton scattering and irreversibility of RG flow

Consider the 4 point function of ϕ with on-shell kinematics

$$\phi(x) = e^{-\tau(x)} - 1 \qquad , \qquad A(s) = \frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}\mathcal{W}$$



- Close enough to the fixed point B^I is a good expansion parameter.
- The leading contribution

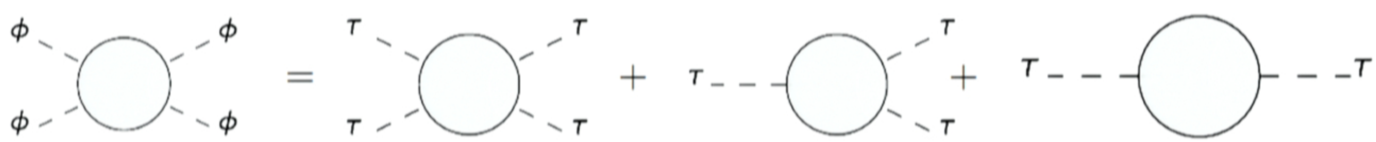
$$A(s) \quad \propto \quad s^2\left(\left(\tilde{a} + O(B^2)\right) + \left(\frac{1}{2}B^I B^J \mathcal{G}_{IJ} + O(B^3)\right)\ln s/\mu^2\right)$$

where $\mathcal{G}_{IJ}$ is a matrix in parameter space related to the 2 point functions $\langle \mathcal{O}_I \mathcal{O}_J \rangle$ and $\theta^a_I \theta^b_J \langle \mathcal{O}_a \mathcal{O}_b \rangle$.

Example - Dilaton scattering and irreversibility of RG flow

Consider the 4 point function of ϕ with on-shell kinematics

$$\phi(x) = e^{-\tau(x)} - 1 \qquad , \qquad A(s) = \frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}\mathcal{W}$$



- Close enough to the fixed point B^I is a good expansion parameter.
- The leading contribution

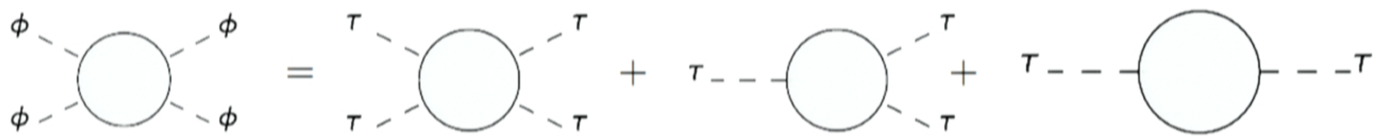
$$A(s) \quad \propto \quad s^2 \left((\tilde{a} + O(B^2)) + \left(\frac{1}{2} B^I B^J \mathcal{G}_{IJ} + O(B^3) \right) \ln s/\mu^2 \right)$$

where $\mathcal{G}_{IJ}$ is a matrix in parameter space related to the 2 point functions $\langle \mathcal{O}_I \mathcal{O}_J \rangle$ and $\theta^a_I \theta^b_J \langle \mathcal{O}_a \mathcal{O}_b \rangle$.

Example - Dilaton scattering and irreversibility of RG flow

Consider the 4 point function of ϕ with on-shell kinematics

$$\phi(x) = e^{-\tau(x)} - 1 \qquad , \qquad A(s) = \frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}\mathcal{W}$$



- Close enough to the fixed point B^I is a good expansion parameter.
- The leading contribution

$$A(s) \quad \propto \quad s^2\left((\tilde{a} + O(B^2)) + \left(\frac{1}{2}B^I B^J \mathcal{G}_{IJ} + O(B^3)\right)\ln s/\mu^2\right)$$

where $\mathcal{G}_{IJ}$ is a matrix in parameter space related to the 2 point functions $\langle\mathcal{O}_I\mathcal{O}_J\rangle$ and $\theta^a_I\theta^b_J\langle\mathcal{O}_a\mathcal{O}_b\rangle$.

Example: Dilaton scattering and irreversibility of RG flow

$$A(s) \quad \propto \quad s^2 \left((\widetilde{a} + O(B^2)) + \left(\frac{1}{2} B^I B^J \mathcal{G}_{IJ} + O(B^3) \right) \ln s/\mu^2 \right)$$

- This expression is non-trivial:
  1. The corrections to $\widetilde{a}$ begin at order B^2.
  2. All the non-local contributions begin at order B^3.
- In a unitary theory $\mathcal{G}_{IJ}$ is positive definite.
- The amplitude is independent of μ:

$$0 \quad = \quad \mu \frac{d}{d\mu} A(s) = \mu \frac{d}{d\mu} \widetilde{a} - B^I B^J \mathcal{G}_{IJ} + O(B^3)$$

Conclusion:
In a unitary theory, the change in $\widetilde{a}(\lambda)$ is monotonous
$\Rightarrow$ Irreversibility of RG flow.

Main points of the talk

- The local CS equation can be used to characterize RG flows
  1. by studying the consistency conditions.
  2. as a tool for computing correlators of T.

  There is a lot of "know-how" involved in the process....

- We have improved the notations of the formalism, and the formulation of the anomaly, in a way which isolates only the interesting constraints.

- The new formulation allowed us to give analytic expressions for the dilaton effective action off-criticality, in terms of the coefficients in the equation.

- We proved that the "scattering amplitudes" involved in the a theorem and LPR are indeed insensitive to lower dimension operators.

Open questions

- SUSY
- Consistency conditions in the presence of chiral anomalies.
- What can we learn from the constraints on the F^2 anomaly?
- Can we say something unrelate to irreversibility?
  e.g., can we constrain accidental symmetries?

Thank you