Title: From dlogs to dilogs

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Abstract: In this talk, I will describe a new form of hidden simplicity in the planar scattering amplitudes of N=4 super-Yang-Mills theory, notably that the loop integrands can be expressed in dlog form. I will explain how this form arises geometrically from computing the scattering amplitudes using a holomorphic Wilson loop in twistor space. I will also describe a systematic method for evaluating such integrals and use it to obtain a new formula for the 1-loop MHV amplitude. These techniques can be extended to higher loop amplitudes and may lead to a simple and efficient method for computing scattering amplitudes more generally.

Overview

- I will consider on-shell, color-ordered scattering amplitudes in the planar limit of N=4 super-Yang-Mills theory
- The amplitudes have a lot of hidden symmetry and many remarkable properties.
- In this talk, I will describe a new form of hidden simplicity:

  Loop integrands can be written in dlog form!

Example: 1-loop MHV amplitude

$$K_{ij} = -\frac{1}{4\pi^2}\int \mathrm{d}\ln s_0 \,\mathrm{d}\ln t_0 \,\mathrm{d}\ln s \,\mathrm{d}\ln t$$

where

$$s_0 = \bar{s}_0$$

$$t_0 = \bar{t}_0$$

$$s = -\frac{\bar{t}\,(a_{i-1j} - v) + a_{i-1j-1} - v}{\bar{t}\,(a_{i\,j} - v) + a_{ij-1} - v} \qquad v = s_0 - t_0$$

- Systematic method for doing the integrals naturally incorporates Feynman iε and mass reg

- For 1-loop MHV, the generic contribution is

$$K_{ij} = \mathrm{Li}_2\left(\frac{a_{ij}}{v_*}\right) + \mathrm{Li}_2\left(\frac{a_{i-1j-1}}{v_*}\right) - \mathrm{Li}_2\left(\frac{a_{i-1j}}{v_*}\right) - \mathrm{Li}_2\left(\frac{a_{ij-1}}{v_*}\right) + c.c.$$

where $Li_2(x) = -\int_0^x \ln(1-x')d\ln x'$

- Readily extends to higher loops

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Why N=4 sYM?

- Maximal supersymmetry
- Conformal
- Dual to type IIB string theory on AdS_5xS^5
- Believed to be exactly solvable (in planar limit)
- Toy model for QCD

Spinor-Helicity

- 4d null momentum:

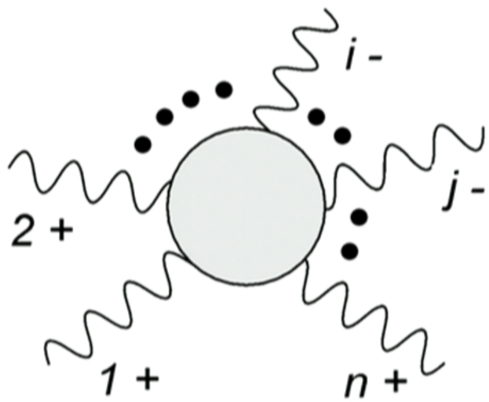
$$p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}$$

where $\alpha = 0, 1$ and $\dot{\alpha} = \dot{0}, \dot{1}$

- Gluon amplitudes can be written in terms of these spinors, leading to very simple expressions.

MHV Amplitudes

At tree-level:



$$\mathcal{A}_n = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}$$

(Parke,Taylor)

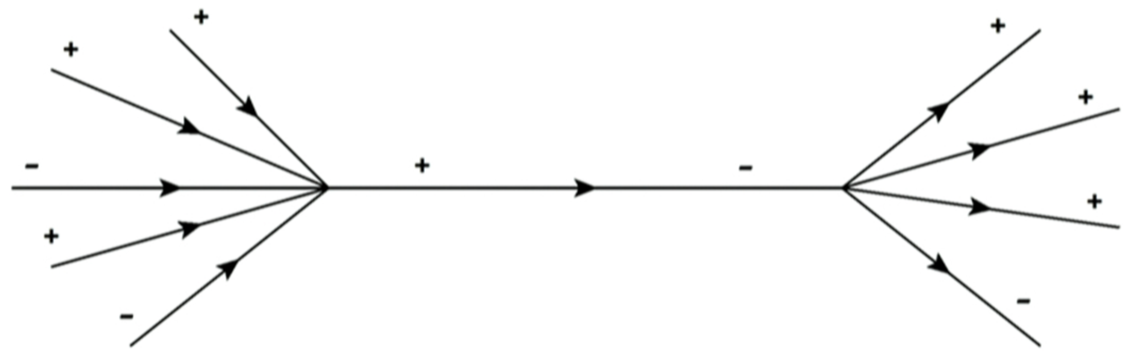
where $\langle ij \rangle = \epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta$

CSW Formalism

- Use tree-level MHV amplitudes as Feynman vertices for constructing tree-level non-MHV amplitudes.

  (Cachazo,Svrcek,Witten)

- Example: NMHV amplitude



Superamplitudes

- Supermomentum: $q^{a\alpha} = \lambda^{\alpha}\eta^{a}$ $\quad a = 1, 2, 3, 4$

- Tree-level MHV superamplitude:

$$A_n^{MHV} = \frac{\delta^4(p)\,\delta^8(q)}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle}$$

- N^kMHV superamplitude:

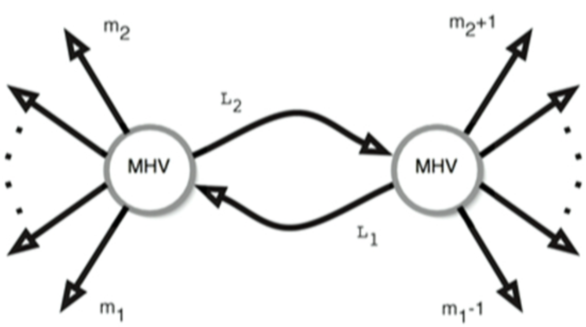
$$A_n^{N^kMHV} = \frac{\delta^4(p)\,\delta^8(q)}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle} M_n^k$$

where M_n^k has fermionic degree 4k

(super) CSW Formalism

- Use tree-level MHV superamplitudes as Feynman vertices to construct tree-level non-MHV amplitudes

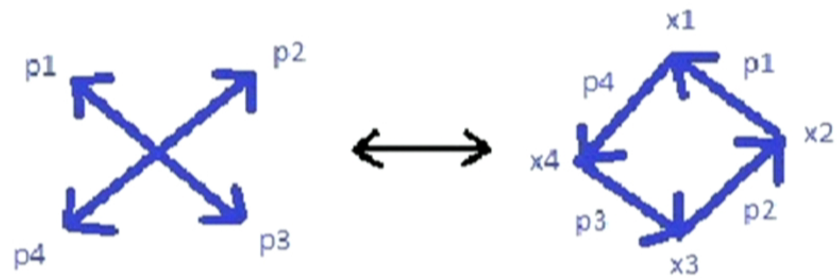
- Can also use these vertices to construct loop amplitudes (Brandhuber,Spence,Travaglini)



Dual Conformal Symmetry

- Dual variables:

$$x_i - x_{i+1} = p_i$$



- Tree-level amplitudes and loop integrands transform covariantly when

$$x_i \to x_i^{-1}$$

(Drummond,Henn,Korchemsky,Smirnov,Sokatchev;
Brandhuber, Heslop,Travaglini)

Amplitude/Wilson Loop Duality

- Dual conformal symmetry can be extended to dual superconformal symmetry by defining fermionic dual variables:

$$\theta_i - \theta_{i+1} = q_i$$

- Dual superconformal symmetry corresponds to ordinary superconformal symmetry of a null-polygonal Wilson-loop

- This Wilson loop is dual to the planar S-matrix!

(Alday,Maldacena;Drummond,Henn,Korchemsky,Sokatchev; Brandhuber,Heslop,Travaglini;Mason,Skinner;Caron-Huot)

Momentum Twistors

- Make dual conformal symmetry manifest:

$$\begin{pmatrix} Z^A \\ \chi^a \end{pmatrix}, \quad Z^A = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot\alpha} \end{pmatrix}$$

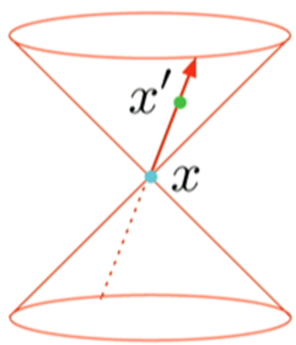
- Incidence relations: (Penrose)

$$\mu^{\dot\alpha} = -i x^{\dot\alpha\alpha}\lambda_\alpha, \quad \chi^a = -i\theta^{a\alpha}\lambda_\alpha$$

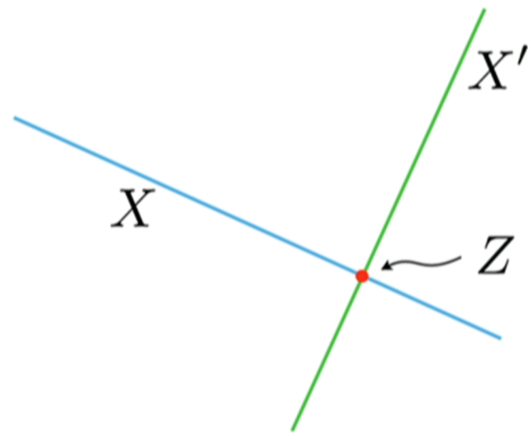
- Momentum conservation automatic (Hodges)

Spacetime vs Twistor Space

Space-time



Twistor Space

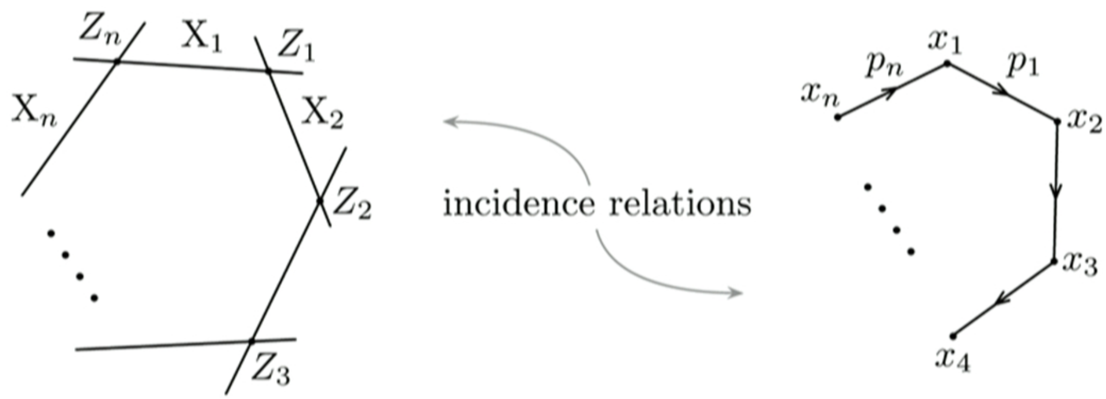


Point in spacetime ⟷ CP^1 in twistor space

Point in twistor space ⟷ null ray in spacetime

Twistor Wilson Loop

- Null polygon in spacetime corresponds to polygon in twistor space:



- Expectation value of the twistor Wilson loop computes planar S-matrix! (Mason,Skinner)

N=4 sYM in Twistor Space

- Superfield:

$$\mathcal{A} = g^+ + \chi^a \tilde{\psi}_a + \frac{1}{2}\chi^a\chi^b\phi_{ab} + \epsilon_{abcd}\chi^a\chi^b\chi^c\left(\frac{1}{3!}\psi^d + \frac{1}{4!}\chi^d g^-\right)$$

- Twistor action: (Boels,Mason,Skinner)

$$S[\mathcal{A}] = \frac{i}{2\pi}\int D^{3|4}Z\,\mathrm{Tr}\left(\mathcal{A}\wedge\bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) \longleftarrow \text{self-dual sector}$$

$$+ g^2\int d^{4|8}x \log\det\left(\left(\bar{\partial}+\mathcal{A}\right)\big|_X\right) \longleftarrow \text{MHV expansion}$$

- Axial Gauge: $\bar{Z}_* \cdot \mathcal{A} = 0$
- Feynman rules correspond to CSW formalism!

$$\ln \det(\bar{\partial} + A)$$

$$\mathrm{tr} \ln\left[\bar{\partial}\left(1 + \bar{\partial}^{-1} A\right)\right]$$

$$\mathrm{tr} \ln \bar{\partial} + \mathrm{tr} \ln\left(1 + \bar{\partial}^{-1} A\right)$$

Feynman Rules

- Propagator:

$$\Delta(Z,Z')= \frac{1}{2\pi i}\int_{\mathbb{C}^2}\frac{\mathrm{d}u}{u}\frac{\mathrm{d}v}{v}\bar{\delta}^{4|4}(Z+uZ_*+vZ')$$

where $\delta^{4|4}(Z) = \Pi_{A=1}^{4}\bar{\delta}\left(Z^A\right)\Pi_{a=1}^{4}\chi^a$

- MHV Vertices:

$$\int_{\mathbb{M}\times(\mathbb{CP}^1)^n}\frac{\mathrm{d}^{4|4}Z_A\,\mathrm{d}^{4|4}Z_B}{\mathrm{Vol\ GL}(2)}\prod_{i=1}^{n}\frac{D\sigma_i}{(\sigma_i\,\sigma_{i+1})}$$

where $\sigma = (\sigma^0,\sigma^1)$ are homogeneous coordinates for the line (Z_A, Z_B) and

$$D\sigma = (\sigma d\sigma)\,,\quad (\sigma_i\,\sigma_j) = \sigma_i^0\sigma_j^1 - \sigma_i^1\sigma_j^0$$

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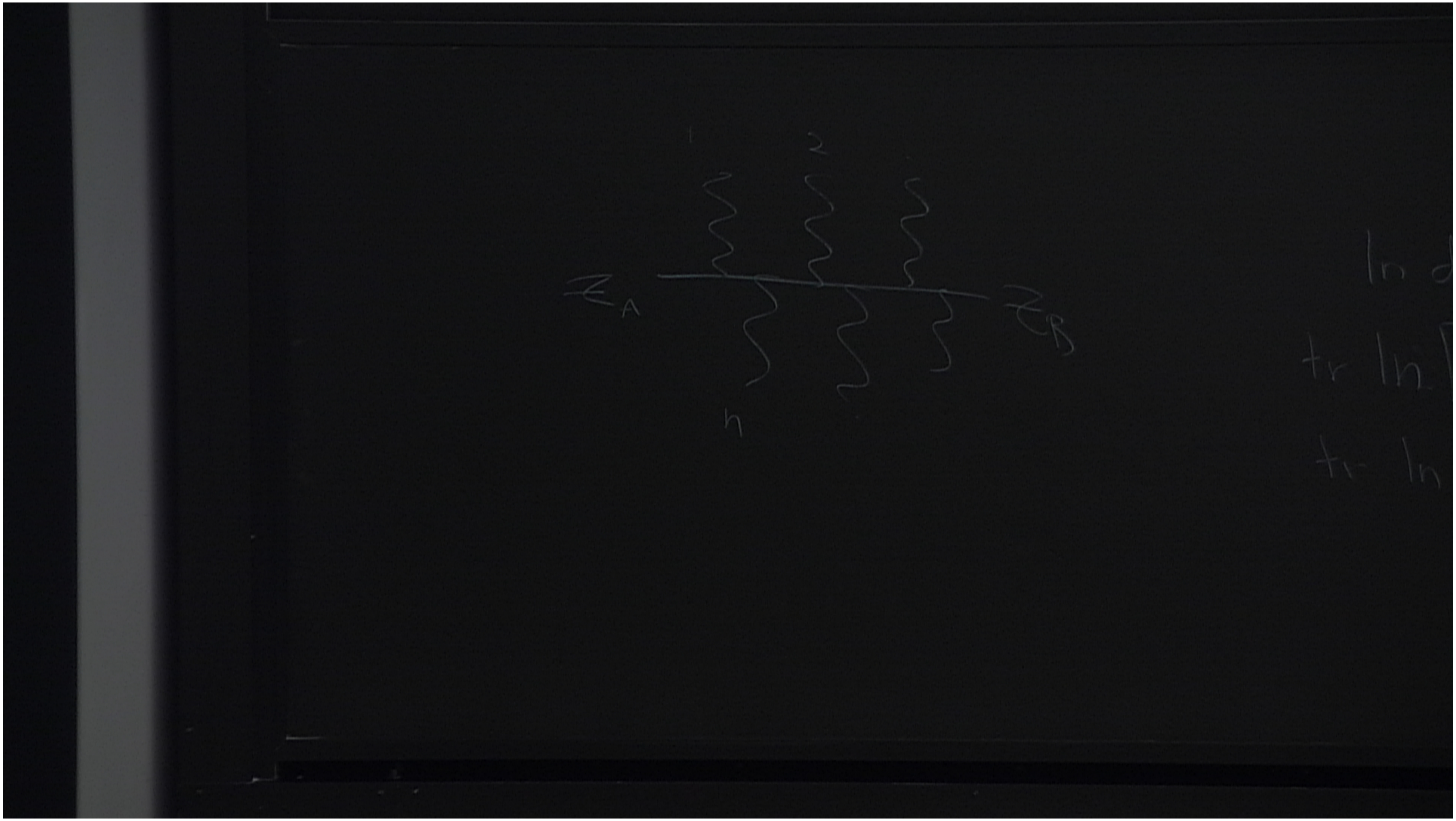

1
2
n

Back to the Twistor WL

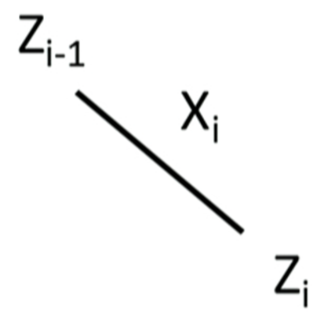
- Wilson loop expectation value:

$$\langle W[C] \rangle \propto \int \mathcal{D}\mathcal{A} e^{-S[\mathcal{A}]} W[C]$$

where W[C] follows from parallel transport:

- For each side (Z_{i-1}, Z_i) of the twistor polygon, find $H_i(Z)$ such that

$$\left(\bar{\partial} + \mathcal{A}\right) H_i|_{X_i} = 0, \;\; H_i\left(Z_{i-1}\right) = 1$$



- Then

$$W[C] := \mathrm{Tr} \prod_{i=1}^{n} H_i(Z_i)$$

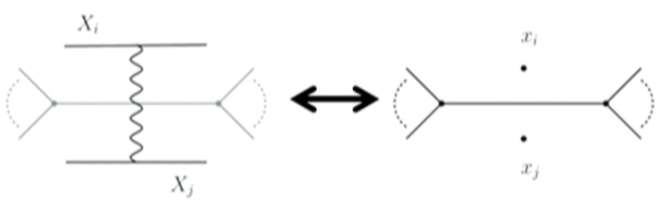
Planar Duality

- Amplitude Diagrams vs Wilson loop diagrams:

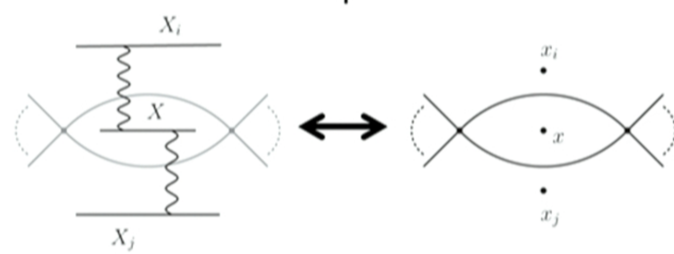
Amplitude	Wilson Loop
# of legs	# of sides
# of loops	# of MHV vertices
MHV degree	# of propagators – 2 x (# of MHV vertices)

- Examples:

Tree-level NMHV:

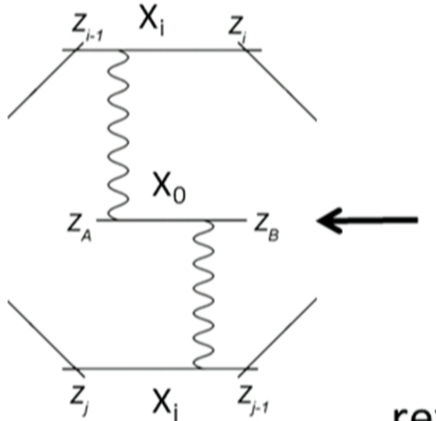


1-loop MHV:



Example: 1-loop MHV

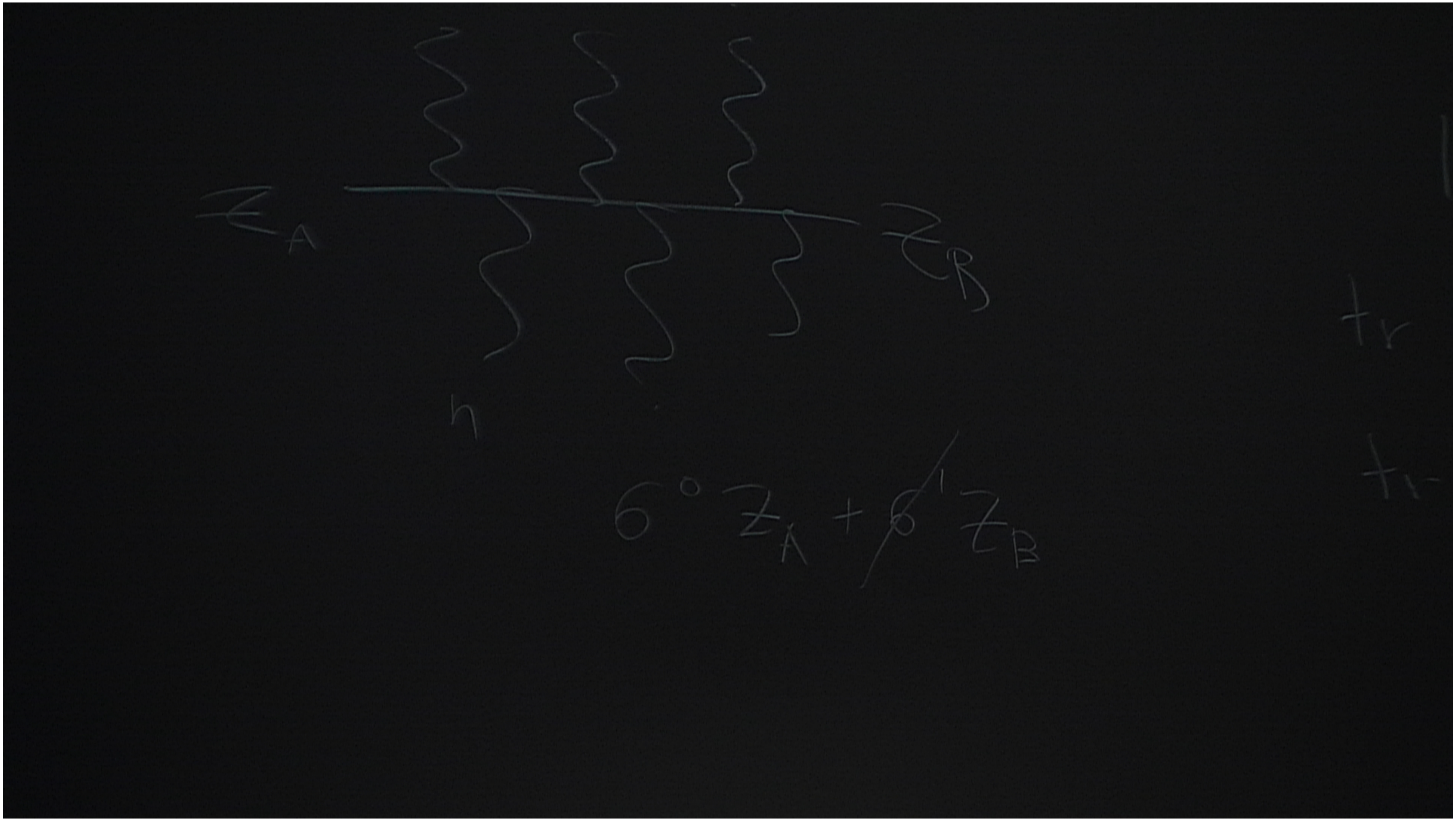
- Twistor Wilson loop diagram: (Kermit)

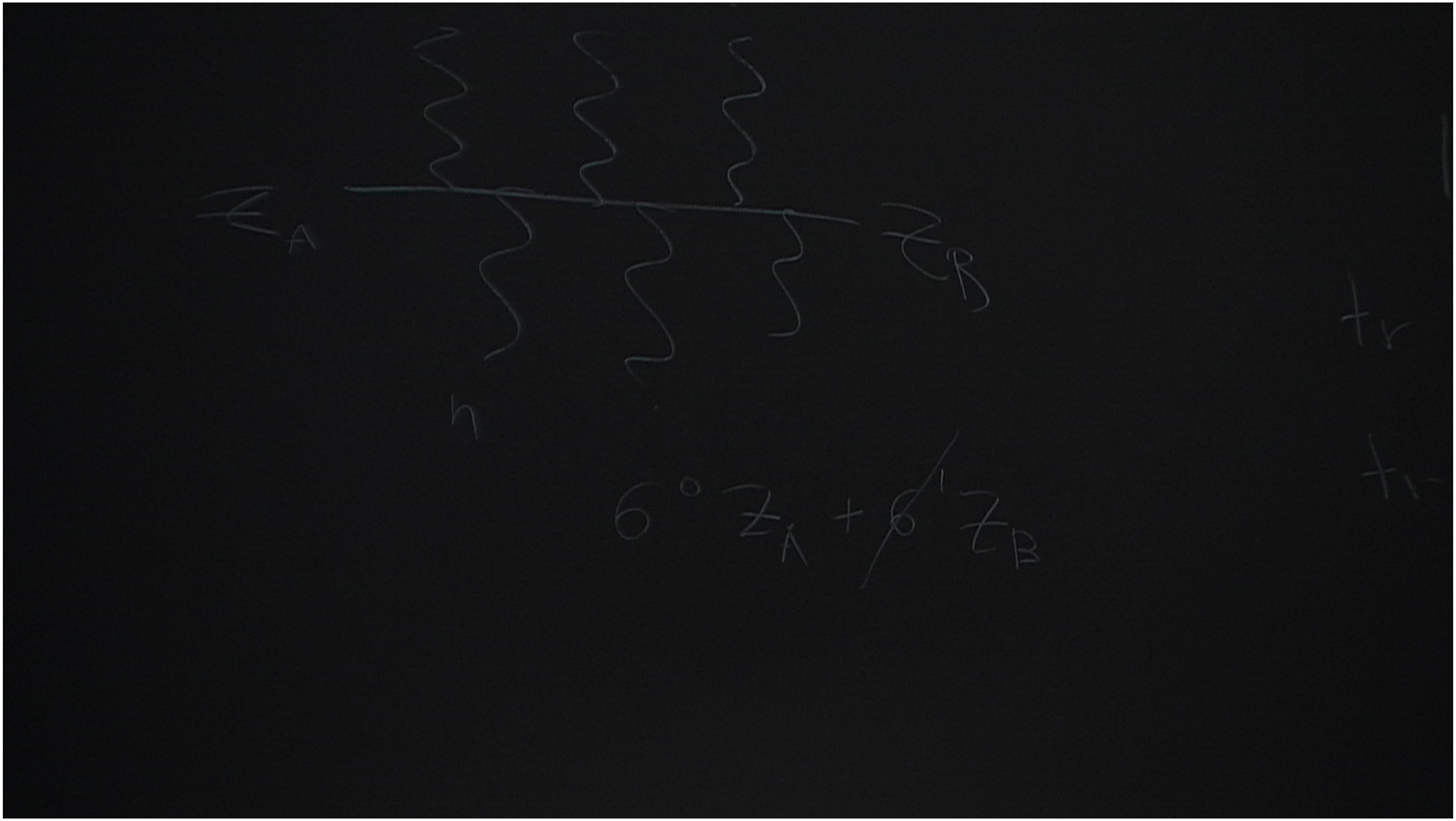


- Feynman rules give

reference twistor

$$-\frac{1}{4\pi^2}\int \frac{ds_1}{s_1}\frac{dt_1}{t_1}\frac{\mathrm{d}s_2}{s_2}\frac{\mathrm{d}t_2}{t_2}\int \mathrm{d}^{4|4}Z_A\,\mathrm{d}^{4|4}Z_B\,\bar{\delta}^{4|4}\left(Z_A - Z_* - s_1 Z_{i-1} - t_1 Z_i\right)\bar{\delta}^{4|4}\left(Z_B - Z_* - s_2 Z_{j-1} - t_2 Z_j\right)$$


Z_A
Z_B
h


Z_A
Z_B
h
$\sigma^0 Z_A + \sigma^1 Z_B$

- Integrating Z_A and Z_B against delta functions leaves us with

$$-\frac{1}{4\pi^2}\int \frac{ds_1}{s_1}\frac{dt_1}{t_1}\frac{ds_2}{s_2}\frac{dt_2}{t_2}$$

and the constraints

$$Z_A = Z_* + s_1 Z_{i-1} + t_1 Z_i\,, \qquad Z_B = Z_* + s_2 Z_{j-1} + t_2 Z_j$$

- MHV vertex corresponds to point in real Minkowski space:

$$Z_A\cdot\bar{Z}_A = Z_B\cdot\bar{Z}_B = Z_A\cdot\bar{Z}_B = 0$$

- Let

$$(s_1,t_1) = -\frac{i}{s_0\,(1+s)}\,(1,s)\,,\quad (s_2,t_2) = -\frac{i}{t_0\,(1+t)}\,(1,t)$$

Reality constraints determine the contour:

$$Z_A \cdot \bar{Z}_A = 0 \to s_0 = \bar{s}_0$$

$$Z_B \cdot \bar{Z}_B = 0 \to t_0 = \bar{t}_0$$

$$Z_A \cdot \bar{Z}_B = 0 \to$$

$$s = -\frac{\bar{t}\,(a_{i-1j} - v) + a_{i-1j-1} - v}{\bar{t}\,(a_{i\,j} - v) + a_{ij-1} - v}, \quad v = s_0 - t_0$$

where

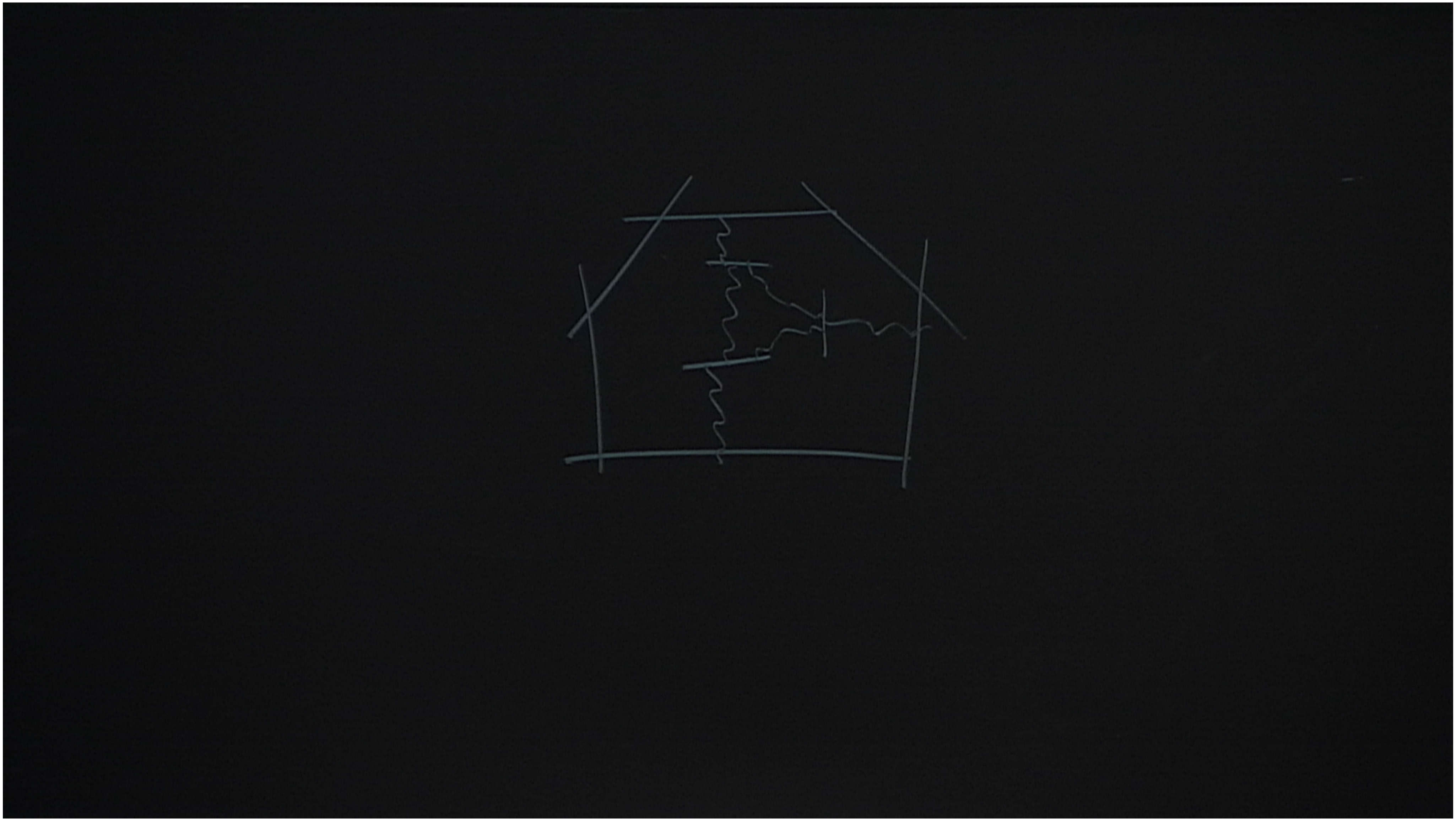
$$a_{ij} = iZ_i \cdot \bar{Z}_j$$

and we used

$$\bar{Z}_* \cdot Z_* = \bar{Z}_i \cdot Z_i = \bar{Z}_{i-1} \cdot Z_i = 0 \qquad \bar{Z}_* \cdot Z_i = 1$$

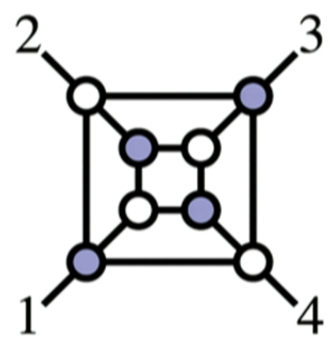
Summary

- Dlog form of loop integrands comes directly from the twistor Wilson loop
- The integrand of an L-loop N^k MHV amplitude can be expressed in terms of 4(L+k) dlogs and k $\delta^{4|4}$'s
- Integration variables correspond to insertion points of propagators on MHV vertices or edges of the twistor Wilson loop
- Integration contour determined by reality constraints



On-Shell Diagrams

- Another dlog form for planar loop integrands of N=4 sYM follows from using on-shell diagrams:



- Integration variables correspond to BCFW shifts

(Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka)

Integration

- For 1-loop MHV, we have

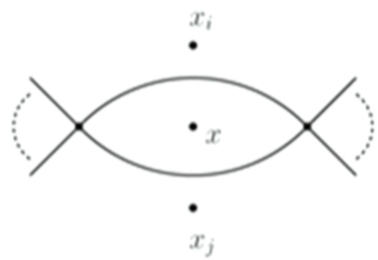
$$K_{ij} = -\frac{1}{4\pi^2} \int \mathrm{d}\ln s_0 \; \mathrm{d}\ln t_0 \; \mathrm{d}\ln s \; \mathrm{d}\ln t$$

- Poles in s_0 and t_0 are real, so require regularization

- This can be achieved using Feynman iε prescription

iε Prescription

- In region momentum space, Kermit corresponds to



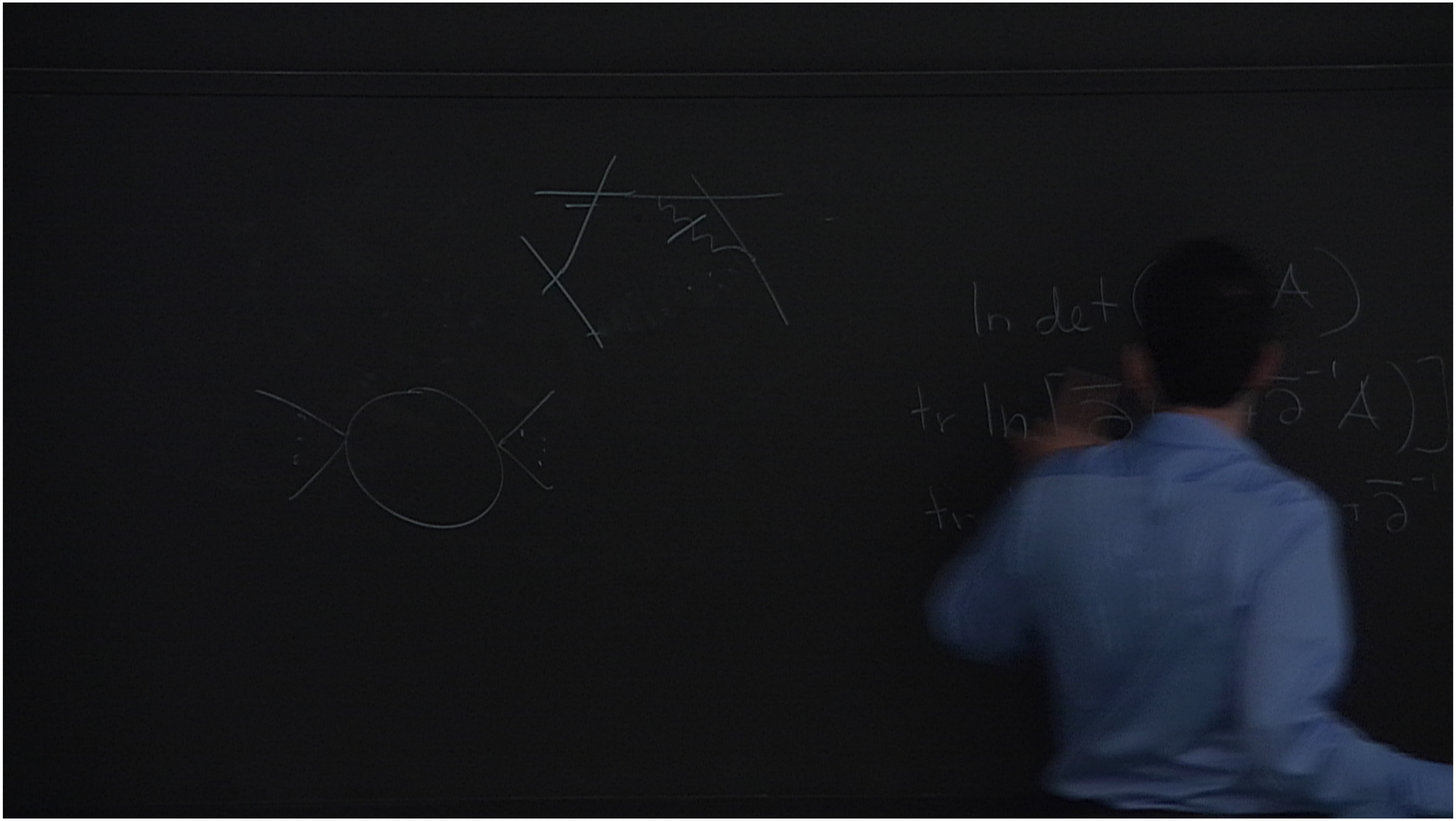
- Propagator momenta given by

$$(x-x_i)^2 = \frac{\langle Z_A Z_B Z_{i-1} Z_i \rangle}{\langle Z_A Z_B \rangle \langle Z_{i-1} Z_i \rangle} = \frac{s_0}{f_i}, \quad (x-x_j)^2 = \frac{\langle Z_A Z_B Z_{j-1} Z_j \rangle}{\langle Z_A Z_B \rangle \langle Z_{j-1} Z_j \rangle} = \frac{t_0}{f_j}$$

- Hence,

$$(x-x_i)^2 \to (x-x_i)^2 + i\epsilon \longrightarrow s_0 \to s_0 + i\epsilon f_i$$

$$(x-x_j)^2 \to (x-x_j)^2 + i\epsilon \longrightarrow t_0 \to t_0 + i\epsilon f_j$$


ln det
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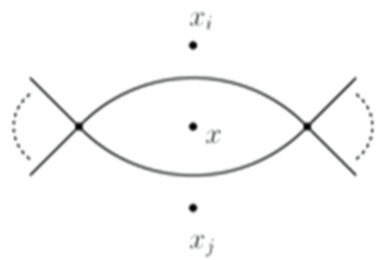

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- Hence,

$$(x-x_i)^2 \to (x-x_i)^2 + i\epsilon \quad \longrightarrow \quad s_0 \to s_0 + i\epsilon f_i$$

$$(x-x_j)^2 \to (x-x_j)^2 + i\epsilon \quad \longrightarrow \quad t_0 \to t_0 + i\epsilon f_j$$

- Remarkably, this reduces to

$$\frac{1}{2\pi i}\int_{v_*}^{\infty} d\ln v \int d\ln t d\ln s(v,\bar{t}), \quad v_* > 0$$

$$\frac{1}{2\pi i}\int_{-\infty}^{v_*} d\ln v \int d\ln t d\ln s(v,\bar{t}), \quad v_* < 0$$

$$v_* = \frac{X_i \cdot X_j}{\bar{Z}_* \cdot X_i \cdot X_j \cdot Z_*} = \frac{a_{ij}a_{i-1j-1} - a_{ij-1}a_{i-1j}}{a_{ij} + a_{i-1j-1} - a_{i-1j} - a_{ij-1}}$$

- Choose $v_* > 0$ for definiteness

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- Choose $v_* > 0$ for definiteness

Generic Diagrams

- Evaluate (s,t) integrals using Stokes theorem:

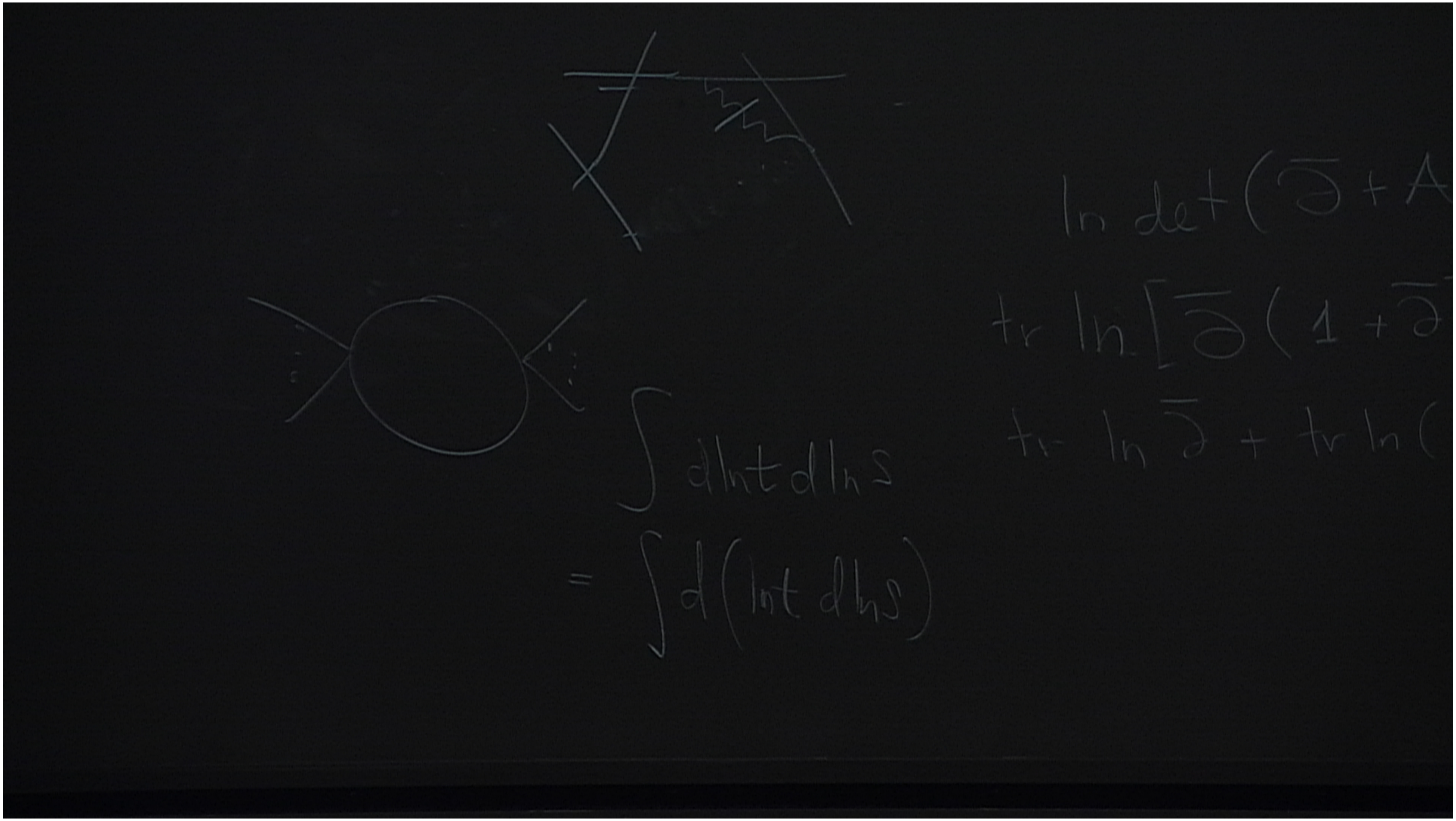
$$\int d\ln t d\ln s\,(v,\bar{t}) = 2\pi i \ln \left| \frac{(a_{i-1j} - v)\,(a_{ij-1} - v)}{(a_{i-1j-1} - v)\,(a_{ij} - v)} \right|^2$$

- Doing v integral finally gives

$$K_{ij} = \mathrm{Li}_2\left(\frac{a_{ij}}{v_*}\right) + \mathrm{Li}_2\left(\frac{a_{i-1j-1}}{v_*}\right) - \mathrm{Li}_2\left(\frac{a_{i-1j}}{v_*}\right) - \mathrm{Li}_2\left(\frac{a_{ij-1}}{v_*}\right) + c.c.$$

- Dual conformal symmetry manifest (up to choice of Z_*)

- Nontrivially agrees with previous results (Brandhuber,Spence, Travaglini)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tr ln [∂̄ (1 + ∂̄
tr ln ∂̄ + tr ln (
∫ d ln t d ln s
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- Dual conformal symmetry manifest (up to choice of Z_*)
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Mass Regularization

- This gives $$\frac{1}{2\pi i}\int_0^\infty \mathrm{d}v \int_\Gamma \frac{\mathrm{d}t}{t}\frac{\mathrm{d}s}{s}\frac{1}{v+(i\epsilon-m^2)\,(f_i-f_{i+1})}$$

where $$f_i - f_{i+1} = -\frac{1}{2}\frac{\left(v + x_{i\,i+2}^2\,|1+t^{-1}|^{-2}\right)^2}{v\,x_{i\,i+2}^2\,|1+t^{-1}|^{-2}}$$

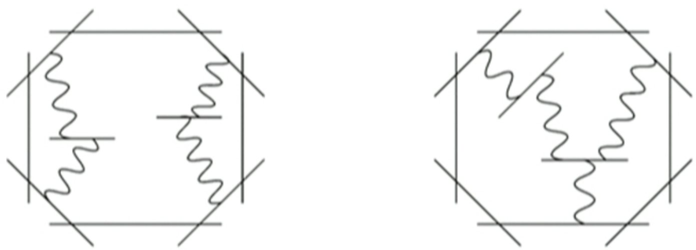
- In the limit m → 0, we find

$$-\frac{1}{4}\ln^2\left(\frac{m^2}{x_{ii+2}^2}\right) + finite$$

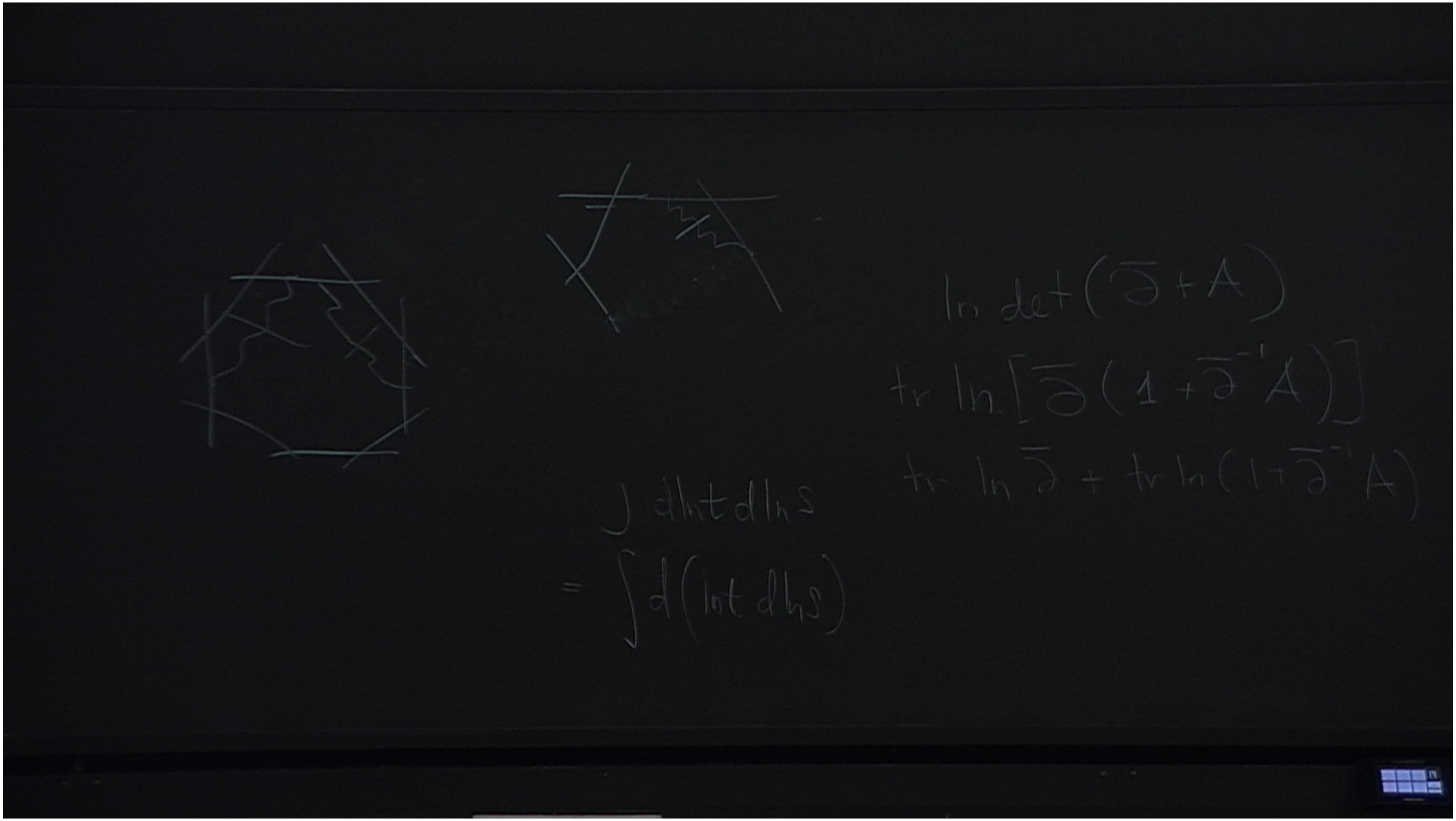
which agrees with previous results (Alday,Henn,Plefka,Schuster)

Higher Loops

- Example: 2-loop MHV



- In general, L-loop MHV integrand will have 4L dlogs
  - 2L real variables ⟶ use Feynman iε prescription
  - 2L complex variables ⟶ use Stokes theorem


ln det(∂̄ + A)
tr ln[∂̄(1 + ∂̄⁻¹A)]
tr ln ∂̄ + tr ln(1 + ∂̄⁻¹A)
∫ d lnt d ln s
= ∫ d(lnt d ln s)

Conclusions

- Loop integrands of planar amplitudes in N=4 sYM can be expressed in dlog form
- The twistor Wilson loop naturally gives a dlog form and provides a simple geometric interpretation
- The 1-loop MHV amplitude can be systematically computed from the dlog form
- The result for generic diagrams is simple and dual conformal invariant up to the choice of reference twistor
- This method readily extends to higher loops

Future Directions

- Higher loops and higher MHV degree
- Relation to on-shell diagram approach
- Other theories:
  - ABJM
  - N<4 sYM
- Nonplanar sector