Title: 13/14 PSI - Quantum Theory - Lecture 4

Date: Sep 12, 2013 10:30 AM

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Abstract:

Composite Systems & Entanglement

$\mathcal{H}_A$ for system A

& $\mathcal{H}_B$ " " B

Then $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$|\psi\rangle \in \mathcal{H}_A$

$|\phi\rangle \in \mathcal{H}_B$

$|\chi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$

O.N. basis for $\mathcal{H}_A$

$\{|a_\ell\rangle\}$

O.N. basis for $\mathcal{H}_B$

$\{|b_k\rangle\}$

Then we can construct an O.N product basis for $\mathcal{H}$ as follows

$$\{ |a_\ell\rangle \otimes |b_k\rangle \}$$

$$\ell \in \{1, \dots, \dim(\mathcal{H}_A)\},$$
$$k \in \{1, \dots, \dim(\mathcal{H}_B)\}$$

- A state in $\mathcal{H}$ of the form

$$|\omega\rangle = |\psi\rangle \otimes |\phi\rangle$$

is a product state

- An arbitrary state $|\chi\rangle \in \mathcal{H}$ admits

$$|\chi\rangle = \sum_{\ell,k} c_{\ell,k} |a_\ell\rangle \otimes |b_k\rangle$$

In general $|\chi\rangle$ can be entangled.

We say $|\chi\rangle$ is entangled if it does not admit an expression of the form $|\chi\rangle = |\psi\rangle \otimes |\phi\rangle$ for some $|\psi\rangle \in \mathcal{H}_A$ & $|\phi\rangle \in \mathcal{H}_B$.

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Representations

Given some O.N. basis for $\mathcal{H}_A$

$$|\psi\rangle = \sum_{\ell=1}^{\dim(\mathcal{H}_A)} \psi_\ell \, |a_\ell\rangle \longmapsto \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix}$$

$$\psi_\ell \equiv \langle a_\ell | \psi \rangle$$

Si

Similarly

$$|\phi\rangle = \sum_{k=1}^{\dim(\mathcal{H}_{vs})} \phi_k \, |b_k\rangle$$

$$\mapsto \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \end{pmatrix}$$

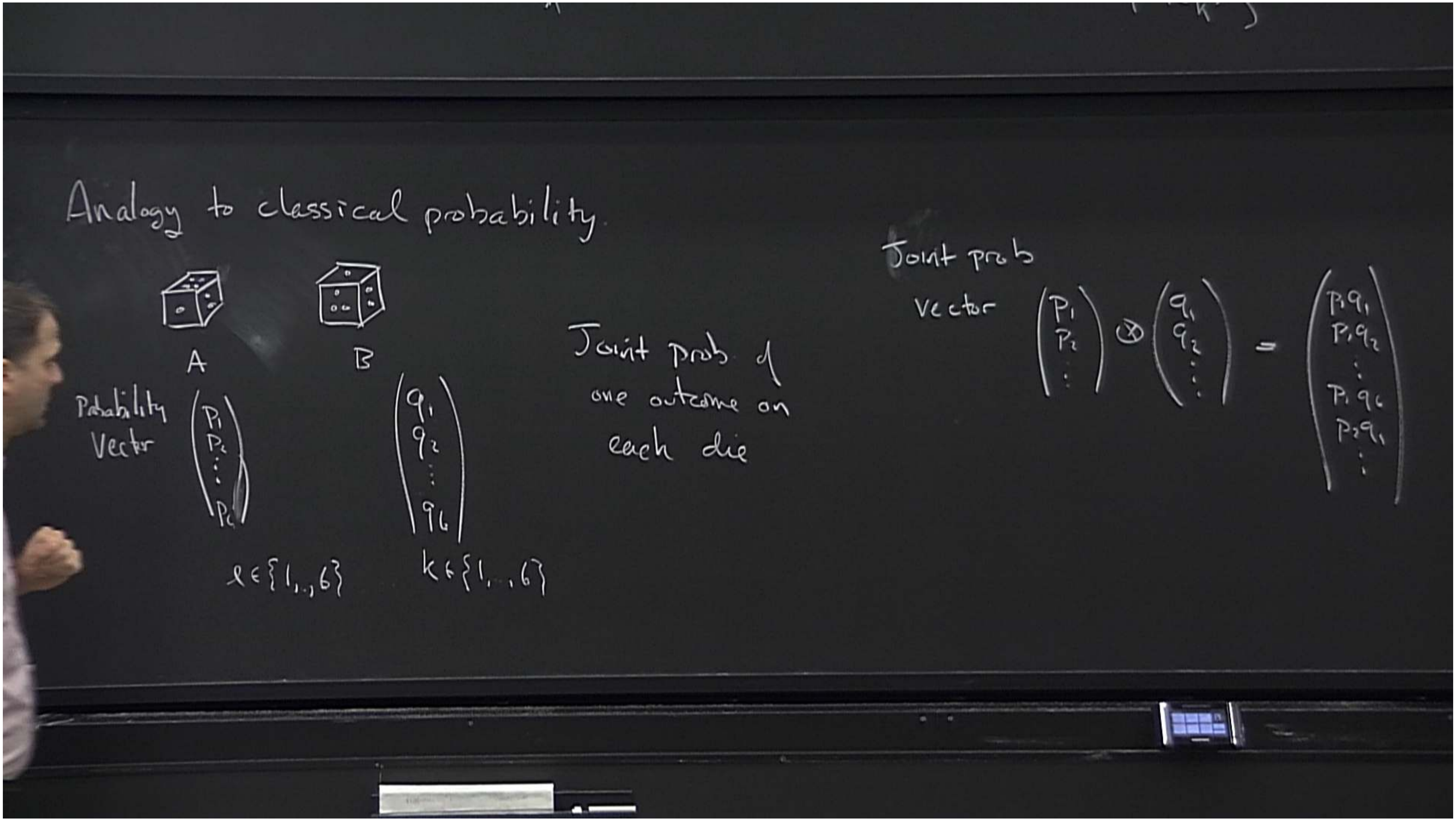
$$\phi_k = \langle b_k | \phi \rangle$$

Tensor Product

$$|\psi\rangle \otimes |\phi\rangle \mapsto \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix} \otimes \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$$

Tensor Product

$$|\psi\rangle \otimes |\phi\rangle \longmapsto \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix} \otimes \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \psi_1 \cdot \phi_1 \\ \psi_1 \cdot \phi_2 \\ \vdots \\ \psi_2 \phi_1 \\ \psi_2 \phi_2 \\ \vdots \end{pmatrix}$$


Analogy to classical probability
A
B
Probability Vector
$p_1, p_2, \ldots, p_6$
$q_1, q_2, \ldots, q_6$
$l \in \{1,..,6\}$
$k \in \{1,\ldots,6\}$
Joint prob of one outcome on each die
Joint prob vector
$\begin{pmatrix} p_1 \\ p_2 \\ \vdots \end{pmatrix} \otimes \begin{pmatrix} q_1 \\ q_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} p_1 q_1 \\ p_1 q_2 \\ \vdots \\ p_1 q_6 \\ p_2 q_1 \\ \vdots \end{pmatrix}$

Analogy to classical probability

A B

Probability Vector $\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_6 \end{pmatrix}$ $\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_6 \end{pmatrix}$

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$|a_\ell\rangle\}$

for $\mathcal{H}_B$

$\{|b_k\rangle\}$

$\ell \in \{1, \dots, \dim(\mathcal{H}_A)\}$,

$k \in \{1, \dots, \dim(\mathcal{H}_B)\}$

$|\chi\rangle \in \mathcal{H}$ admits

$|\chi\rangle = \sum_{\ell,k} c_{\ell,k} |a_\ell\rangle \otimes |b_k\rangle$

$$\otimes \begin{pmatrix} q_1 \\ q_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} p_1 q_1 \\ p_1 q_2 \\ \vdots \\ p_1 q_6 \\ p_2 q_1 \\ \vdots \end{pmatrix}$$

In general given

$$|\chi\rangle = \sum_{\ell,k} c_{\ell,k} |a_\ell\rangle \otimes |b_k\rangle$$

$$\longmapsto \begin{pmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{21} \\ c_{22} \\ \vdots \end{pmatrix}$$

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ON basis for $\mathcal{H}_B$

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Then we can construct an ON product basis for $\mathcal{H}$ as follows

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In general given

$$|x\rangle = \sum_{l,k} c_{l,k} \, |a_l\rangle \otimes |b_k\rangle$$

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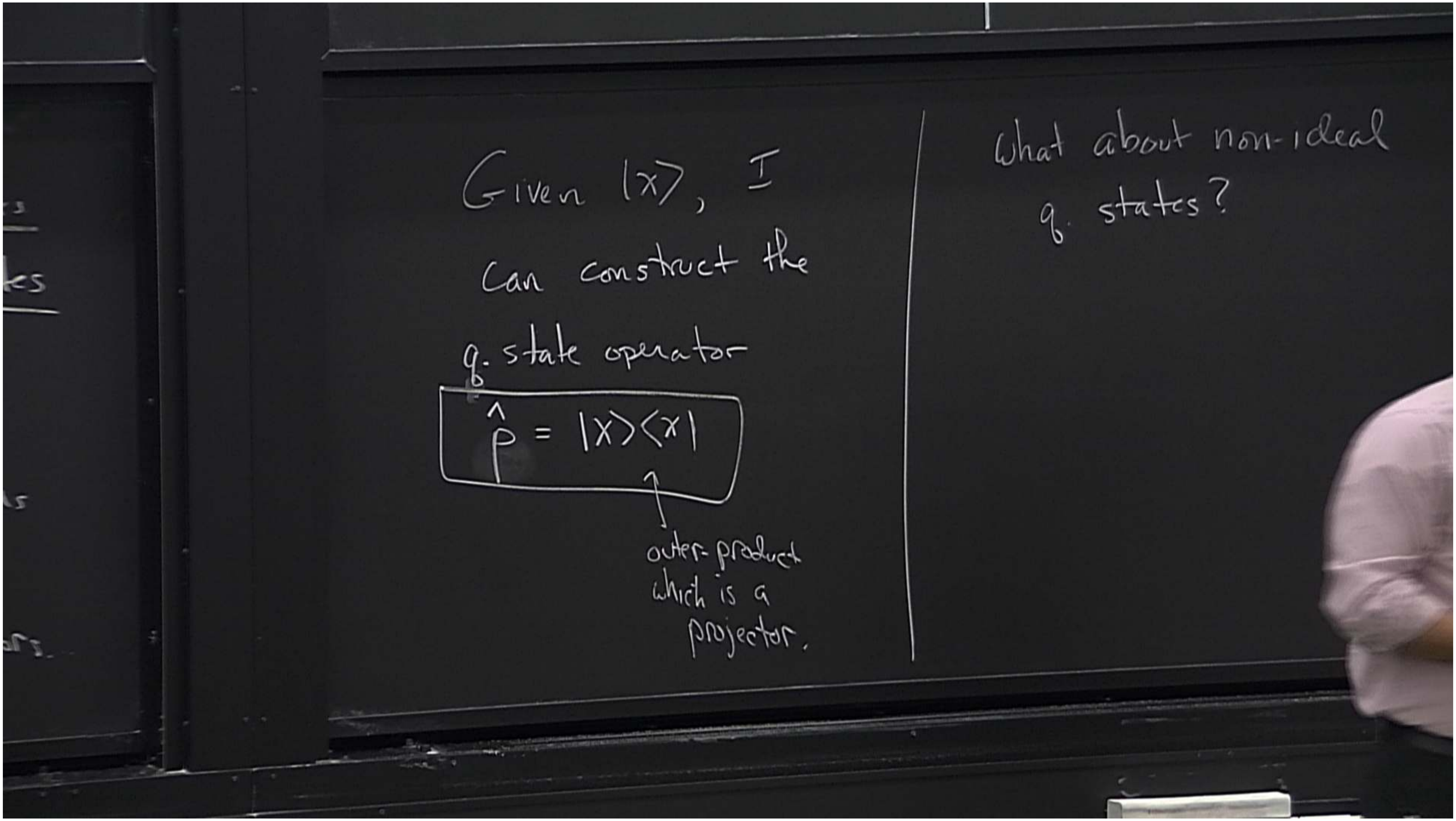
Density operators & mixed states

Axiom 1:

Ideal preparations (q. states) are rank-1 projectors.

Given $|x\rangle$, I can construct the q. state operator

$$\boxed{\hat{\rho} = |x\rangle\langle x|}$$


Given $|x\rangle$, I
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$\hat{\rho} = |x\rangle\langle x|$
outer-product
which is a
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What about non-ideal
q. states?

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What about non-ideal

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2 paradigms
that require more
general kinds of
q. state operators.

ppose I have
dence that
pares $|\chi_i\rangle$
th probability p_i

sume a measurement
$\hat{A} = \sum_\ell a_\ell |a_\ell\rangle\langle a_\ell|$

Outcome probability for case i.

$$\Pr(a_\ell|i) = \mathrm{Tr}\left(|\chi_i\rangle\langle\chi_i|\, |a_\ell\rangle\langle a_\ell| \right)$$

So clearly $\Pr(a_\ell)$ not conditioned on i,

is $\Pr(a_\ell) = \sum_i p_i \Pr(a_\ell|i)$

$$= \sum_i p_i \mathrm{Tr}\left(|\chi_i\rangle\langle\chi_i|\, |a_\ell\rangle\langle a_\ell| \right)$$

$$= \mathrm{Tr}\left[\left(\sum_i p_i |\chi_i\rangle\langle\chi_i| \right) |a_\ell\rangle\langle a_\ell| \right]$$

Suppose I have a device that prepares $|\chi_i\rangle$ with probability p_i.

Assume a measurement of $\hat{A} = \sum_\ell a_\ell |a_\ell\rangle\langle a_\ell|$

Outcome probability for case i.

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$$\Pr(a_\ell) = \sum_i p_i \Pr(a_\ell|i)$$

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$$= \mathrm{Tr}\left[\left(\sum_i p_i |\chi_i\rangle\langle\chi_i|\right)|a_\ell\rangle\langle a_\ell|\right]$$

Identify $\hat{\rho} = \sum_i p_i \, |\chi_i\rangle\langle\chi_i|$

↳ a density operator.

- A general q. state / preparation is a density operator.
- A <u>density operator</u> is a non-negative linear operator satisfying $\mathrm{Tr}(\hat{\rho}) = 1$.

$p_i \; |\chi_i\rangle\langle\chi_i|$

ensity operator.

/preparation

erator.

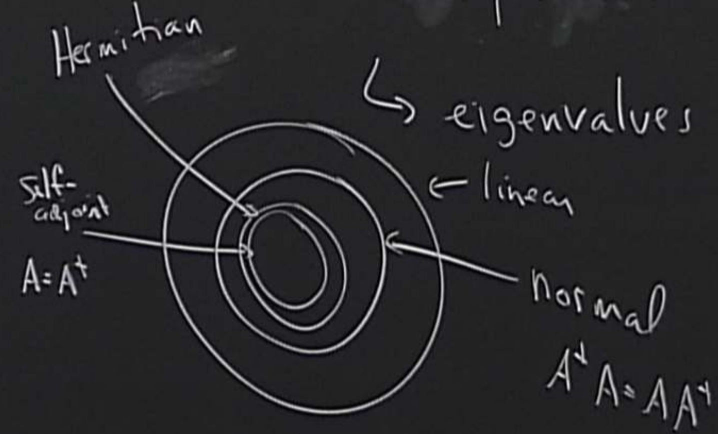
is a non-negative

sfying $\mathrm{Tr}(\hat{\rho}) = 1$.

· A non-negative linear operator $\hat{\rho}$ satisfies

$$\langle \omega | \hat{\rho} | \omega \rangle \geq 0 \quad \forall \, |\omega\rangle \in \mathcal{H}$$

↳ eigenvalues are all non-negative.



Hermitian $\langle \hat{A}\phi, \psi \rangle = \langle \phi, \hat{A}\psi \rangle$

$\forall \; |\psi\rangle, |\phi\rangle \in D(\hat{A})$

$\forall \; |\omega\rangle \in \mathcal{H}$

Self-adjoint $A^{\dagger} = A \iff$ (i) Hermitian

(ii) $D(\hat{A}) = D(\hat{A}^{\dagger})$

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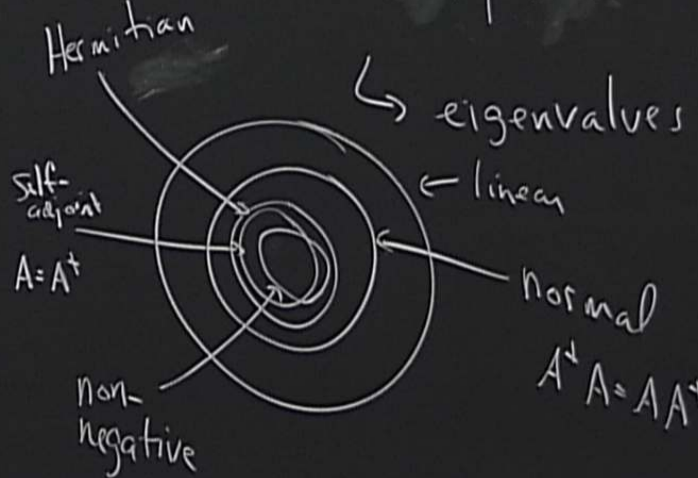

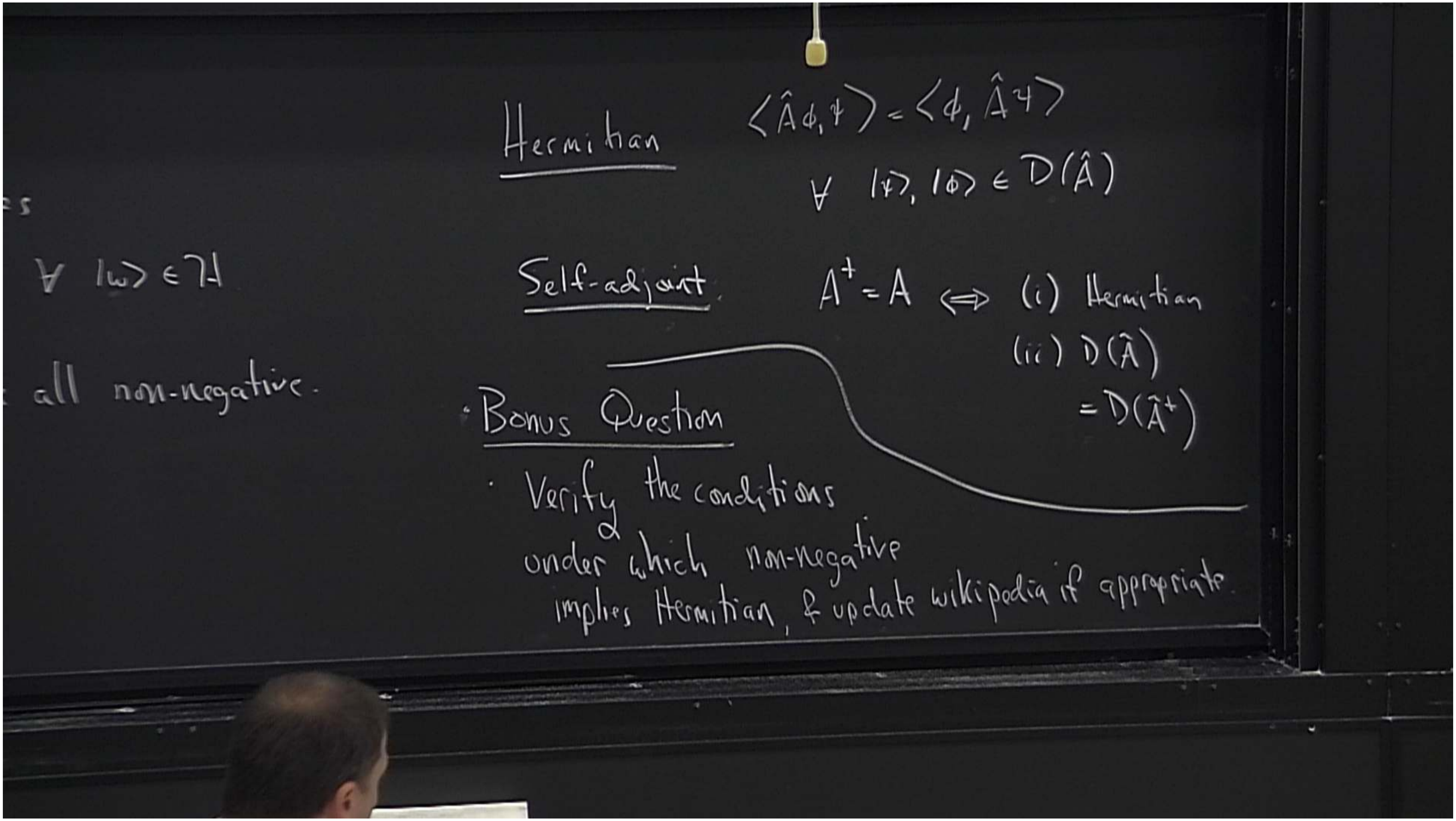
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$\forall \; |\omega\rangle \in \mathcal{H}$
all non-negative.
Self-adjoint $A^{\dagger} = A \iff$ (i) Hermitian
(ii) $D(\hat{A}) = D(\hat{A}^{\dagger})$
Bonus Question
Verify & the conditions
under which non-negative
implies Hermitian, & update wikipedia if appropriate.

ii) Suppose I have systems

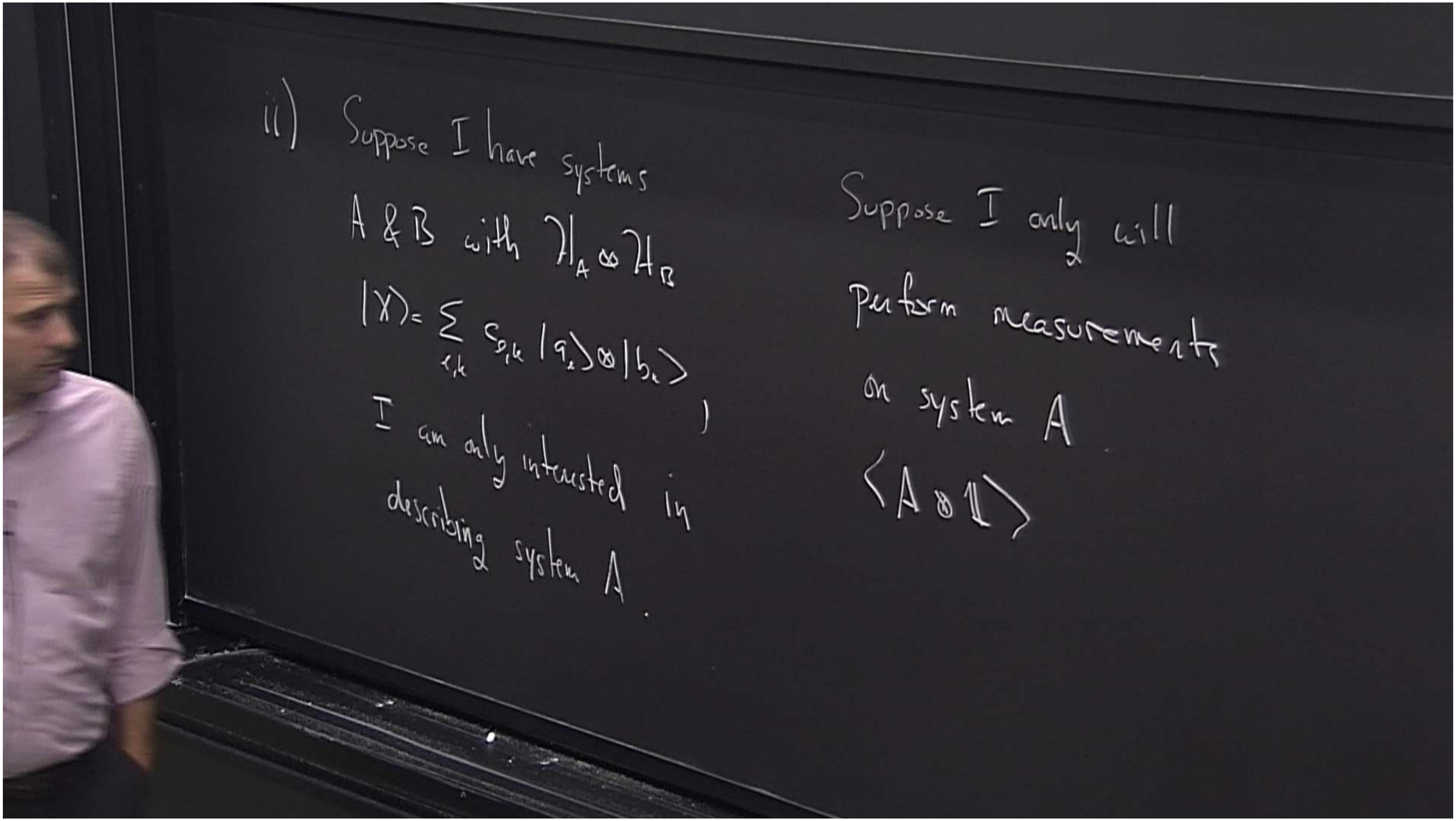
A & B with $\mathcal{H}_A \otimes \mathcal{H}_B$

$$|\chi\rangle = \sum_{\ell,k} c_{\ell,k} \, |a_\ell\rangle \otimes |b_k\rangle$$

ii) Suppose I have systems A & B with $\mathcal{H}_A \otimes \mathcal{H}_B$

$$|\chi\rangle = \sum_{l,k} c_{l,k} |a_l\rangle \otimes |b_k\rangle,$$

I am only interested in describing system A.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I am only interested in
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Suppose I only will
perform measurements
on system A
$\langle A \otimes \mathbb{1} \rangle$

Suppose I have systems A & B with $\mathcal{H}_A \otimes \mathcal{H}_B$

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I am only interested in describing system A.

Suppose I only will perform measurements on system A.

$$\langle A \otimes \mathbb{1} \rangle = \mathrm{Tr}\left[|\chi\rangle\langle\chi| \cdot (A \otimes \mathbb{1}) \right]$$

Suppose I have systems

A & B with $\mathcal{H}_A \otimes \mathcal{H}_B$

$$|\chi\rangle = \sum_{\ell,k} c_{\ell,k} \, |a_\ell\rangle \otimes |b_k\rangle$$

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Suppose I only will perform measurements on system A.

$$\langle A \otimes \mathbb{1} \rangle = \mathrm{Tr}\left[|\chi\rangle\langle\chi| \cdot (A \otimes \mathbb{1}) \right]$$

OR

$$\Pr(a_\ell) = \mathrm{Tr}\left[|\chi\rangle\langle\chi| \left(|a_\ell\rangle\langle a_\ell| \otimes \mathbb{1} \right) \right]$$

ii) Suppose I have systems A & B with $\mathcal{H}_A \otimes \mathcal{H}_B$

$$|\chi\rangle = \sum_{\ell,k} c_{\ell,k} |a_\ell\rangle \otimes |b_k\rangle ,$$

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will

urements

A

$= \mathrm{Tr}\left[|x\rangle\langle x| \cdot (A \otimes \mathbb{1}) \right]$

$= \mathrm{Tr}\left[|x\rangle\langle x| \left(|a_i\rangle\langle a_i| \otimes \mathbb{1} \right) \right]$

Require assigning an operator $\hat{\rho}_A$ to system A

such that, $\forall\, \hat{A} \in \mathcal{L}(\mathcal{H}_A)$ = linear operators acting on $\mathcal{H}_A$.

$\langle x_i|$

erator.

ation

egative

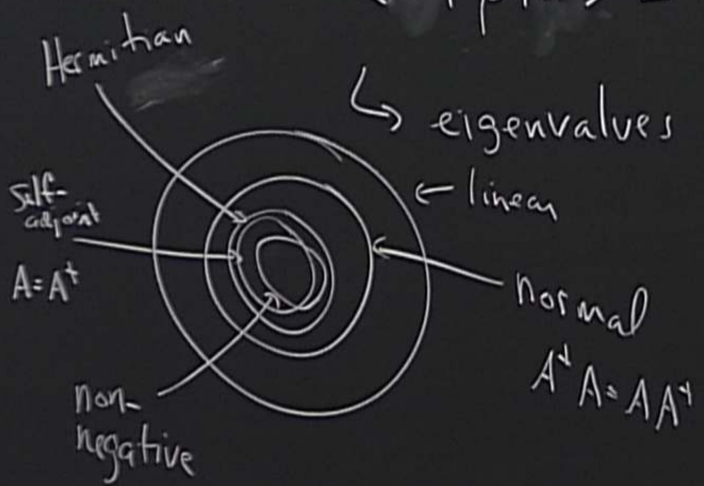
$(\hat{\rho})=1$.

· A non-negative linear operator $\hat{\rho}$ satisfies

$\hat{\rho} \in \mathcal{L}(\mathcal{H})$.

$$\langle \omega | \hat{\rho} | \omega \rangle \geq 0 \quad \forall \; |\omega\rangle \in \mathcal{H}$$

↳ eigenvalues are all non-negative.



Hermitian

Self-adjoint

· Bonus Questio

· Verify the c under which implies Herm

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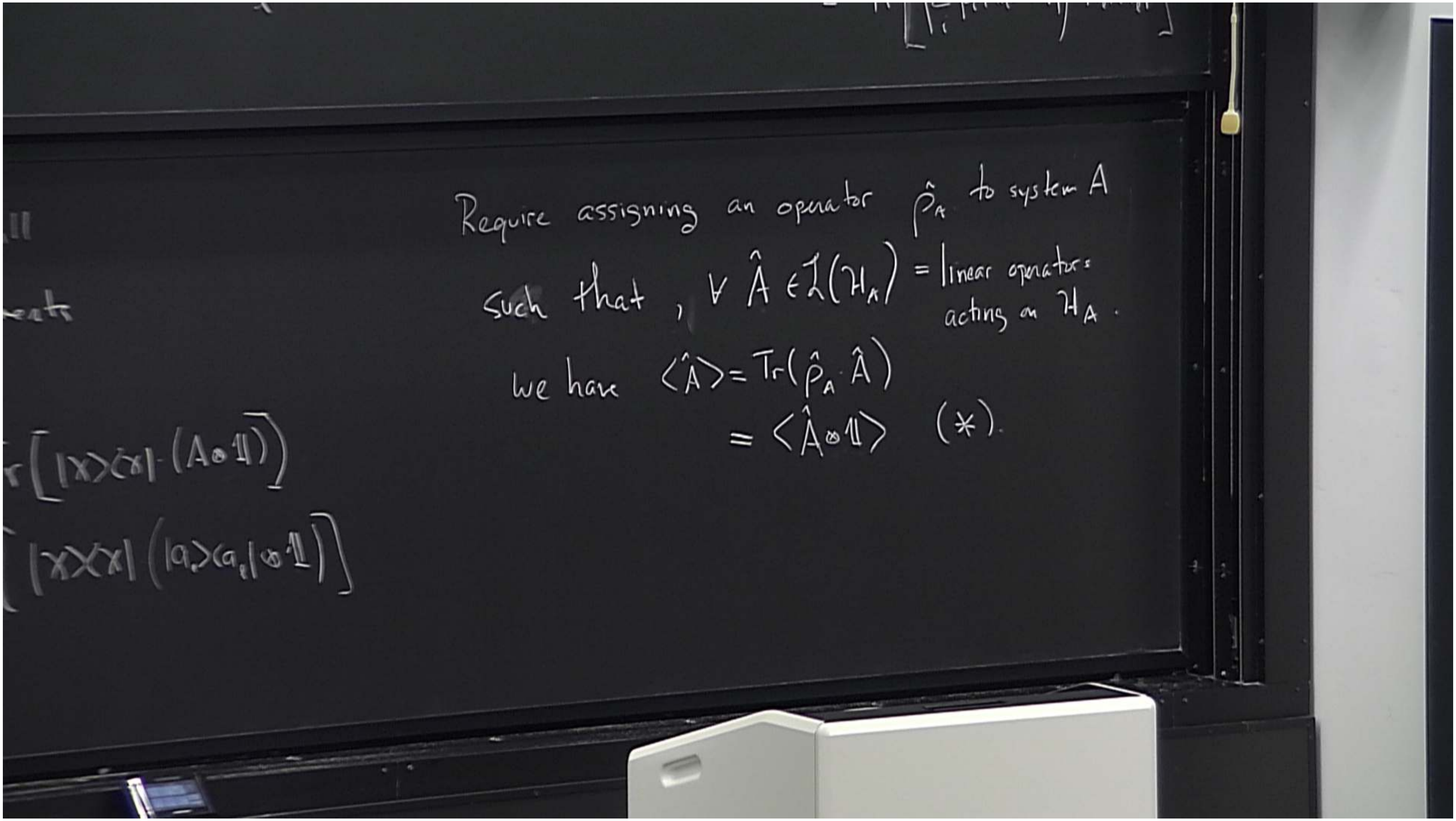
we have $\langle \hat{A} \rangle = \mathrm{Tr}(\hat{\rho}_A \cdot \hat{A})$

$= \langle \hat{A} \otimes \mathbb{1} \rangle$

$\rangle\langle x| \cdot (A \otimes \mathbb{1}) \big) \big]$

$x| \big(|a_e\rangle\langle a_e| \otimes \mathbb{1} \big) \big]$



Require assigning an operator $\hat{\rho}_A$ to system A
such that, $\forall \hat{A} \in \mathcal{L}(\mathcal{H}_A)$ = linear operators acting on $\mathcal{H}_A$.
we have $\langle \hat{A} \rangle = \mathrm{Tr}(\hat{\rho}_A \cdot \hat{A})$
$= \langle \hat{A} \otimes \mathbb{1} \rangle$ (*).

Recall, given $\hat{\rho}^A \in \mathcal{L}(\mathcal{H}_A)$

$$\hat{\rho}^A \longmapsto \begin{bmatrix} \rho^A_{11} & \rho^A_{12} & \rho^A_{13} \\ \rho^A_{21} & \rho^A_{22} & \\ & & \ddots \end{bmatrix}$$

where $\rho^A_{ij} = \langle a_i | \hat{\rho}^A | a_j \rangle$

$$\langle \hat{A} \otimes \mathbb{1}_B \rangle = \sum_{\substack{l', k' \\ l, k}} \langle a_{l'} | \otimes \langle b_{k'} | \; \hat{A} \otimes \hat{\mathbb{1}}_B | a_l, b_k \rangle \langle a_l, b_k | \, |\chi\rangle\langle\chi| \, a_{l'} \rangle \otimes | b_{k'} \rangle.$$

where $$\hat{\mathbb{1}}_{AB} = \sum_{l, k} | a_l, b_k \rangle \langle a_l, b_k |$$

$$\langle \hat{A} \otimes \mathbb{1}_B \rangle = \sum_{\substack{l'k' \\ l,k}} \langle a_{l'} | \otimes \langle b_{k'} | \; \hat{A} \otimes \hat{\mathbb{1}}_B \, | a_l, b_k \rangle \langle a_l, b_k | \, | \chi \rangle \langle \chi | \, a_{l'} \rangle \otimes | b_{k'} \rangle .$$

where $$\hat{\mathbb{1}}_{AB} = \sum_{l,k} | a_l, b_k \rangle \langle a_l, b_k |$$

where $\mathbb{1}_{AB}$ $\sum_{l,k}$

$$= \sum_{\substack{l,k \\ l',k'}} A_{l'l}\, \delta_{k,k'}\, \rho^{AB}_{lk,l'k'} \qquad , \qquad \hat{\rho}^{AB} = |x\rangle\langle x|$$

$$= \sum_{l,l'} A_{l',l} \underbrace{\left(\sum_{k} \rho^{AB}_{lk,l'k} \right)}_{\equiv\, \rho^{A}_{l,l'}} \qquad \stackrel{?}{=} \mathrm{Tr}(\hat{A}\, \rho^{A})$$

$$= \sum_{l,l'} \langle a_l | \hat{A} | a_{l'} \rangle \langle a_{l'} | \rho^{A} | a_l \rangle$$

$$= \sum_{l,l'} A_{l',l}\, \rho^{A}_{l\, l'}$$

We find we can identify

$$\hat{\rho}^A := \mathrm{Tr}_B[\hat{\rho}]$$

where Tr_B is called <u>partial</u> <u>trace</u>.

$\hat{\rho} \in \mathcal{L}(\mathcal{H}_{AB})$

$$\mathrm{Tr}_B(\hat{\rho}) := \sum_k \langle b_k | \hat{\rho} | b_k \rangle$$

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i) $\mathrm{Tr}(\cdot) : \mathcal{L}(\mathcal{H}) \longrightarrow \mathbb{C}$

ii) $\mathrm{Tr}_B(\ ) : \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \longrightarrow \mathcal{L}(\mathcal{H}_A)$

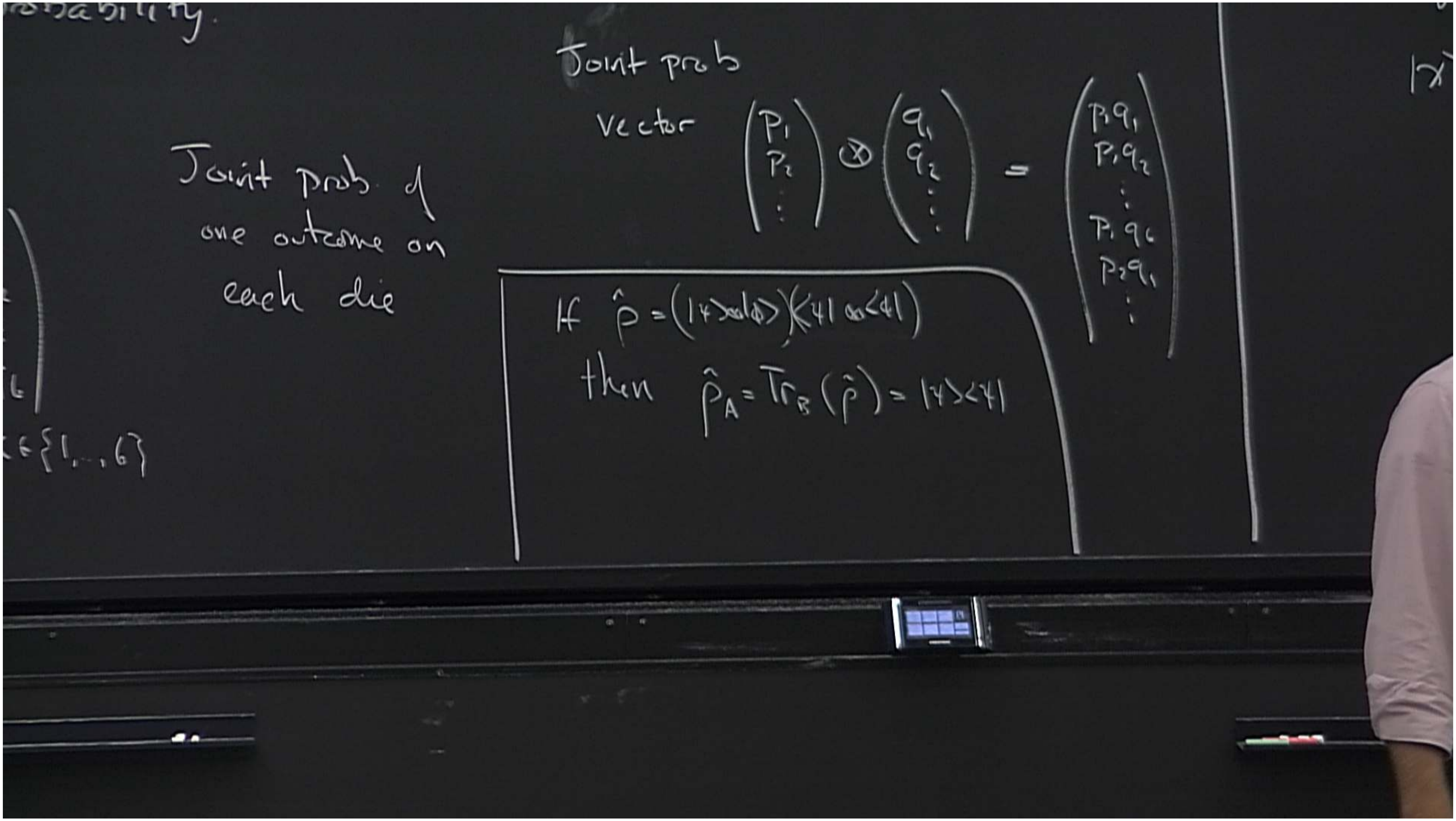
$$\hat{\rho}^A := Tr_B[\hat{\rho}]$$

where Tr_B is called partial trace.

i) $Tr(\cdot) : \mathcal{L}(\mathcal{H}) \to \mathbb{C}$

ii) $Tr_B(\ ) : \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_A)$

The set of objects $\{\rho^A\}$ obtained via partial trace is the set of density operators.


Joint prob vector
$\begin{pmatrix} p_1 \\ p_2 \\ \vdots \end{pmatrix} \otimes \begin{pmatrix} q_1 \\ q_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} p_1 q_1 \\ p_1 q_2 \\ \vdots \\ p_1 q_6 \\ p_2 q_1 \\ \vdots \end{pmatrix}$
Joint prob of one outcome on each die
If $\hat{\rho} = (|\psi\rangle \otimes |\phi\rangle)(\langle\psi| \otimes \langle\phi|)$
then $\hat{\rho}_A = \mathrm{Tr}_B(\hat{\rho}) = |\psi\rangle\langle\psi|$

$$\otimes \begin{pmatrix} q_1 \\ q_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} p_1 q_1 \\ p_1 q_2 \\ \vdots \\ p_1 q_6 \\ p_2 q_1 \\ \vdots \end{pmatrix}$$

$\langle\psi| \otimes \langle\psi|)$

$(\hat{\rho}) = |\psi\rangle\langle\psi|$

In general given

$$|\chi\rangle = \sum_{l,k} c_{l,k} \, |a_l\rangle \otimes |b_k\rangle \longmapsto \begin{pmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{21} \\ c_{22} \\ \vdots \end{pmatrix}$$

Pure states are defined by 3 equivalent conditions

(i) $\hat{\rho} = |\psi\rangle\langle\psi|$ for some $|\psi\rangle \in \mathcal{H}$

(ii) $\hat{\rho}^2 = \hat{\rho}$, (iii) $\mathrm{Tr}(\hat{\rho}^2) = 1$.