Title: 13/14 PSI - Relativity - Lecture 15

Date: Sep 27, 2013 09:00 AM

URL: http://pirsa.org/13090029

Abstract:

Extending Schwarzchild into the Black Hole Interior

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

Extending Schwarzchild into the Black Hole Interior

$$ds^2 = -\left(1-\frac{r_s}{r}\right)dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2$$

$r = r_s$ metric is singular : shall show this is just a coordinate singularity

For $r < r_s$

$r = r_s$ metric is singular : shall show this is just a coordinate singularity

For $r < r_s$, coordinate r is time like, $-t$ is spacelike

Extending Schwarzschild into the Black Hole Interior

$$ds^2 = -\left(1-\frac{r_s}{r}\right)dt^2 + \left(1-\frac{r_s}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2$$

$r = r_s$ metric is singular : shall show this is just a coordinate singularity

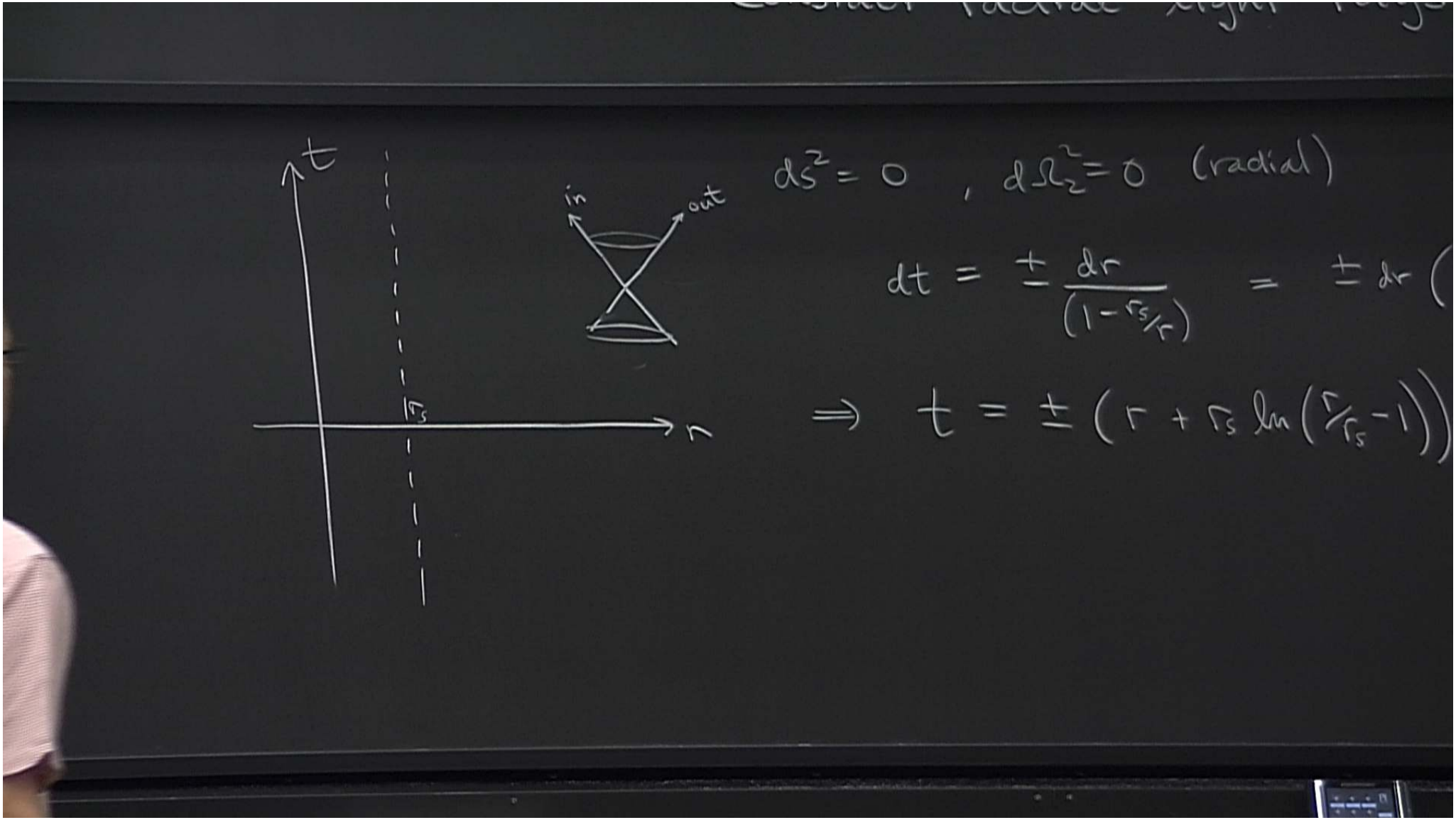
For $r < r_s$, coordinate r is time like, $-t$ is spacelike

Consider radial light rays (null geodesics)

$ds^2 = 0 \quad , \quad d\Omega_2^2 = 0 \quad \text{(radial)}$

$$dt = \pm \frac{dr}{(1 - r_s/r)} = \pm dr \left(1 + \frac{r_s}{r - r_s}\right)$$

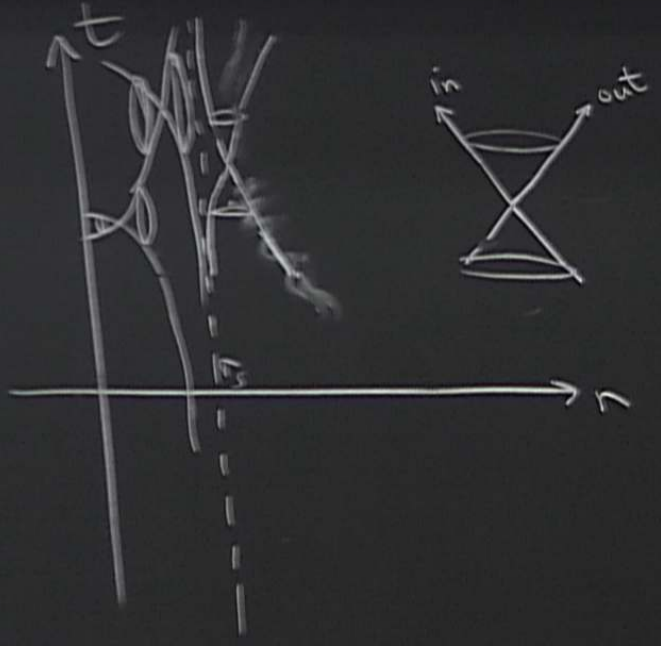
$$\Rightarrow \quad t = \pm \left(r + r_s \ln\left(\frac{r}{r_s} - 1\right)\right) + \text{const} \qquad r > r_s$$


t
r
r_s
in
out
$ds^2 = 0$, $d\Omega_2^2 = 0$ (radial)
$dt = \pm \frac{dr}{(1 - r_s/r)} = \pm dr$
$\Rightarrow t = \pm \left(r + r_s \ln\left(\frac{r}{r_s} - 1 \right) \right)$

$ds^2 = 0$ (radial)

$$= \pm \frac{dr}{(1 - r_s/r)} = \pm dr \left(1 + \frac{r_s}{r - r_s}\right)$$

$$t = \pm \left(r + r_s \ln\left(\frac{r}{r_s} - 1\right)\right) + \text{const} \quad \begin{Bmatrix} \text{out} \\ \text{in} \end{Bmatrix} \qquad r > r_s$$



$ds^2 = 0 \quad , \quad d\Omega_2^2 = 0 \quad \text{(radial)}$

$$dt = \pm \frac{dr}{(1 - r_s/r)} = \pm dr \left(1 + \right.$$

$$\Rightarrow \quad t = \pm \left(r + r_s \ln\left(r/r_s - 1\right)\right) +$$

$$= \pm \Big(\underbrace{r + r_s \ln\left(1 - r/r_s\right)}_{r \to 0 \;\sim\; -\frac{r^2}{2r_s}} \Big) +$$

Change coordinates to Ingoing Eddington-Finkelstein coordinates

Change coordinates to Ingoing Eddington-Finkelstein coordinates

– coordinates in which the ingoing null rays are $u = \text{const.}$

Change coordinates to Ingoing Eddington-Finkelstein coordinates

- coordinates in which the ingoing null rays are $u = \text{const.}$

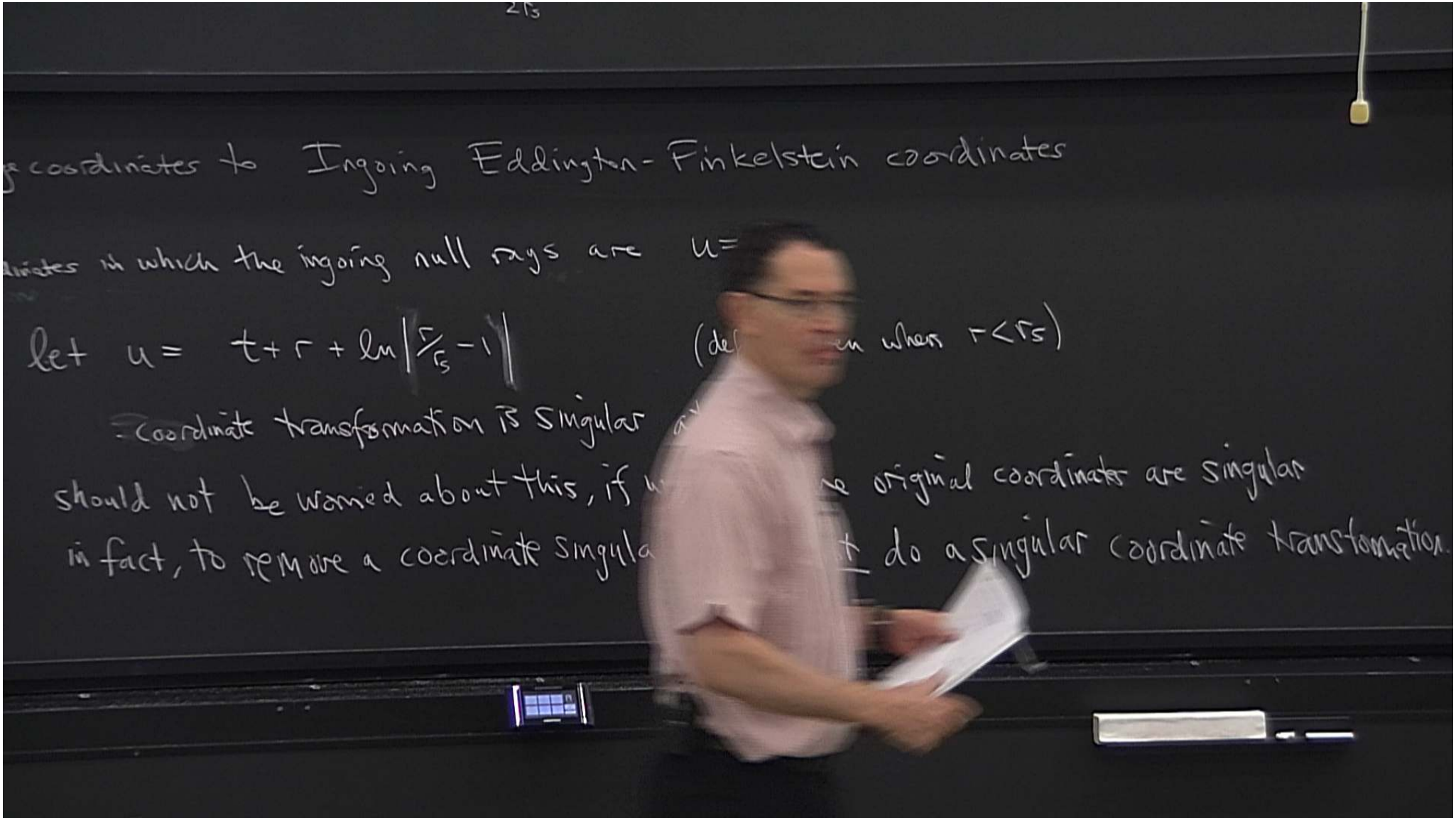
let $u = t + r + \ln\left(\frac{r}{r_s} - 1\right)$

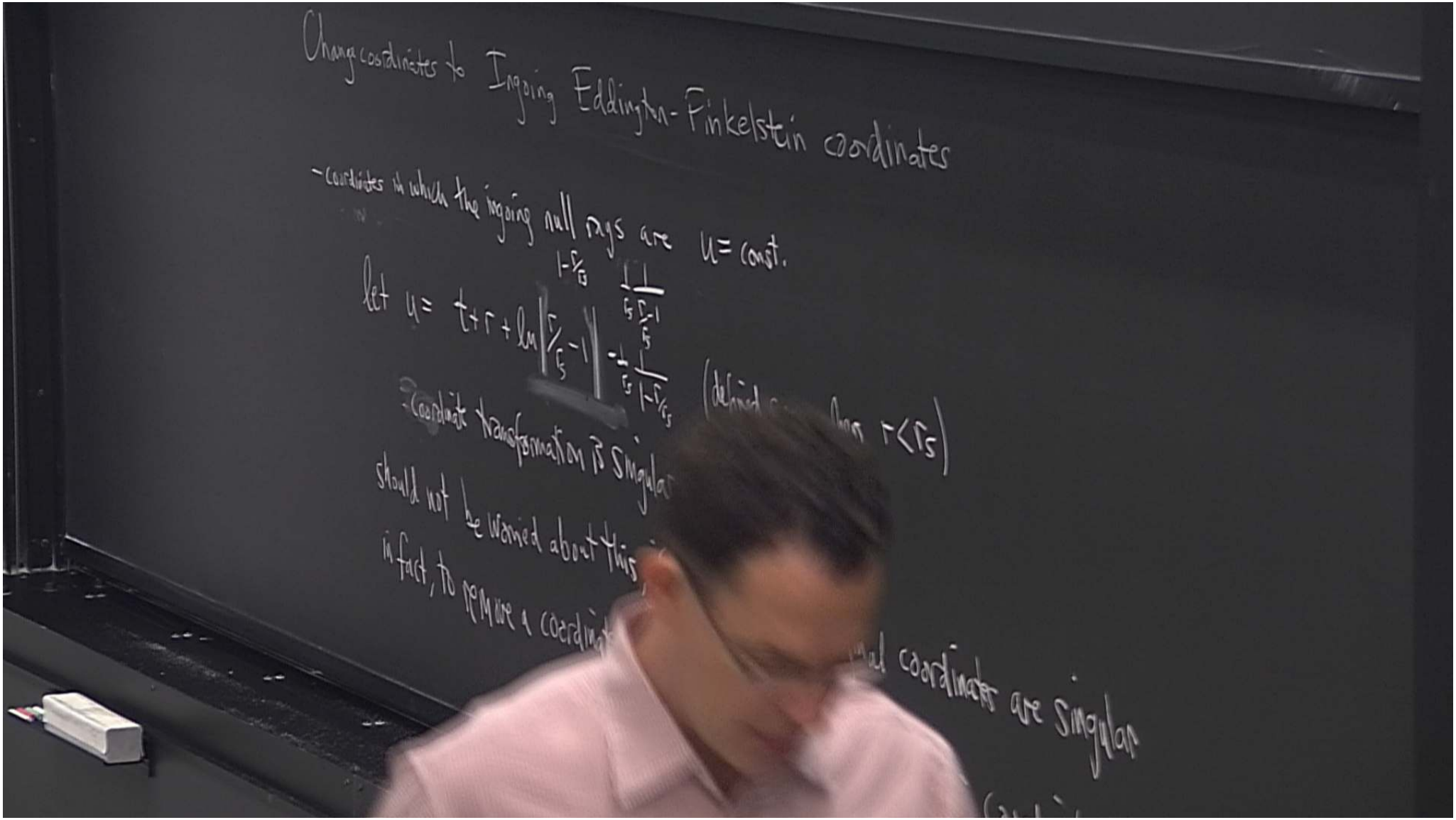
Change coordinates to Ingoing Eddington-Finkelstein coordinates

- coordinates in which the ingoing null rays are $u = \text{const.}$

let $u = t + r + \ln\left|\frac{r}{r_s} - 1\right|$ (defined even when $r < r_s$)

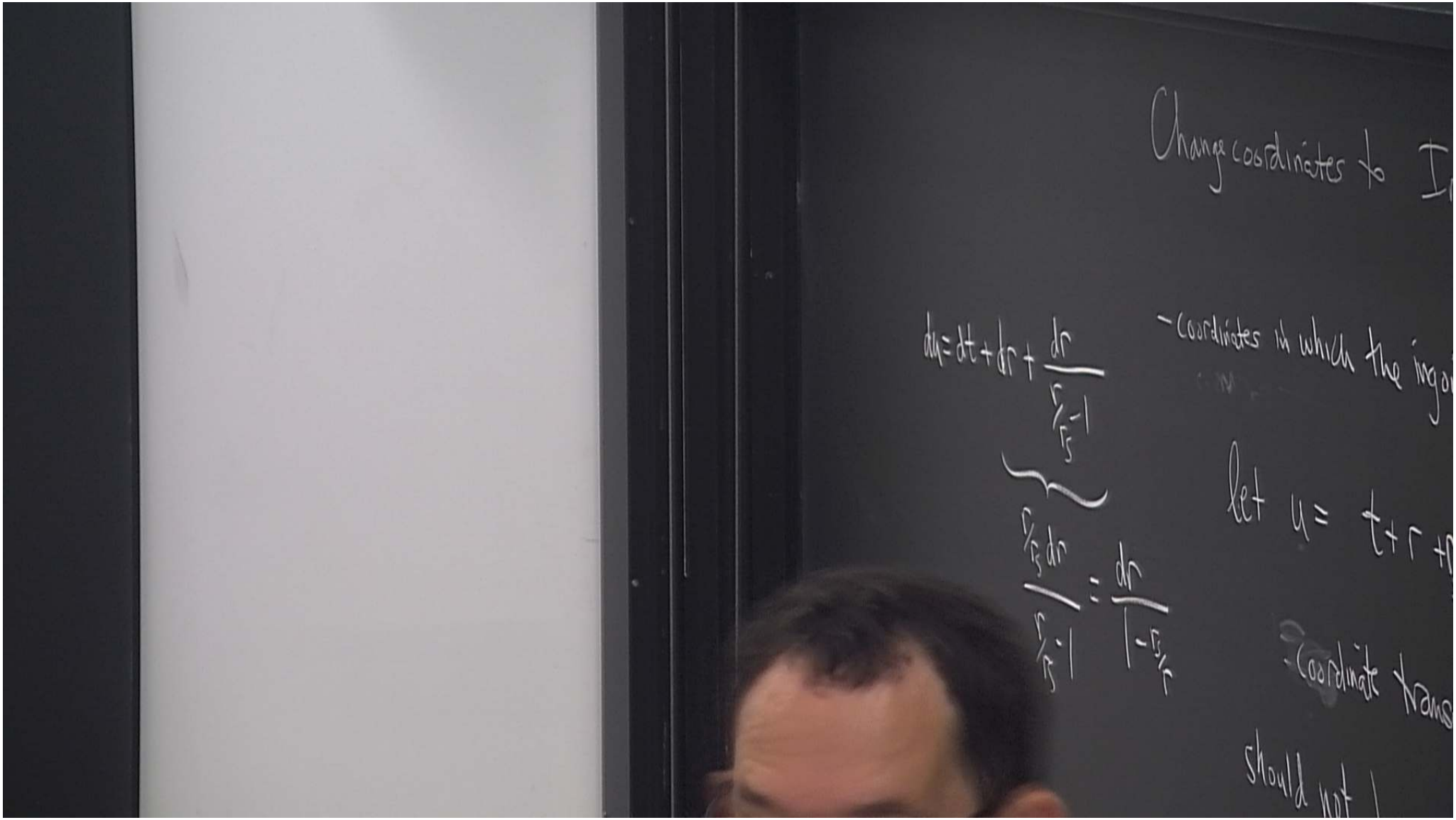
- coordinate transformation is singular at $r = r_s$


coordinates to Ingoing Eddington-Finkelstein coordinates
in which the ingoing null rays are u=
let u = t + r + ln|r/rs − 1|
when r<rs)
coordinate transformation is singular
should not be worried about this, if
original coordinates are singular
in fact, to remove a coordinate singula
do a singular coordinate transformation.


Change coordinates to Ingoing Eddington-Finkelstein coordinates
- coordinates in which the ingoing null rays are u = const.
let u = t + r + ln|r/r_s - 1| - 1/r_s 1/(1 - r/r_s)
coordinate transformation is singular
should not be worried about this
in fact, to remove a coordinate
coordinates are singular

$$du = dt + \frac{dr}{1-\frac{r_s}{r}} \quad \Rightarrow \quad dt = du - \frac{dr}{1-\frac{r_s}{r}}$$

$$\Rightarrow \quad ds^2 = -\left(1-\frac{r_s}{r}\right)du^2 + 2\,du\,dr + r^2 d\Omega_2^2$$


Change coordinates to
- coordinates in which the
let u = t + r +
- coordinate trans
should not

Change coordinates to Ingoing Eddington-Finkelstein c

- coordinates in which the ingoing null rays are $u = \text{const.}$

let $u = t + r + r_s \ln\left|\frac{r}{r_s} - 1\right|$ $\quad \therefore \frac{r_s}{r_s}\frac{t}{\frac{r}{r_s}-1}$ (defined even wh

$\frac{dr}{1 - r_s/r}$

- coordinate transformation is singular at $r = r_s$

should not be worried about this, if we believe the ori

in fact, to remove a coordinate singularity, you must d

$$du = dt + \frac{dr}{1-\frac{r_s}{r}} \Rightarrow dt = du - \frac{dr}{1-\frac{r_s}{r}}$$

$$\Rightarrow ds^2 = -\left(1-\frac{r_s}{r}\right)du^2 + 2\,du\,dr + r^2 d\Omega_2^2$$

$$\text{at } r=r_s,\ g = \begin{pmatrix} u & r & \theta & \varphi \\ 0 & 1 & & \\ 1 & 0 & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix}$$

nonsingular, evalues of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are just $+1, -1$

So these coordinates extend nicely across $r = r_s$

- we can follow an ingoing light ray ($u = \text{const}$) across r_s :

So these coordinates extend nicely across $r = r_s$

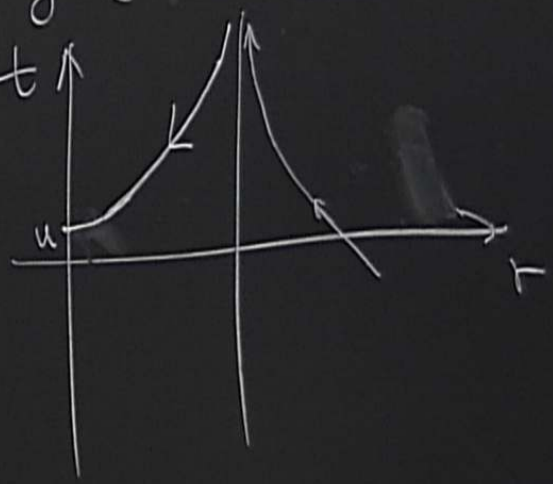
- we can follow an ingoing light ray ($u = \text{const}$) across r_s :

$$u = \text{const} \Rightarrow t = -r - r_s \ln\left(1 - \frac{r}{r_s}\right)$$

So these coordinates extend nicely across $r = r_s$

- we can follow an ingoing light ray ($u = \text{const}$) across r_s :

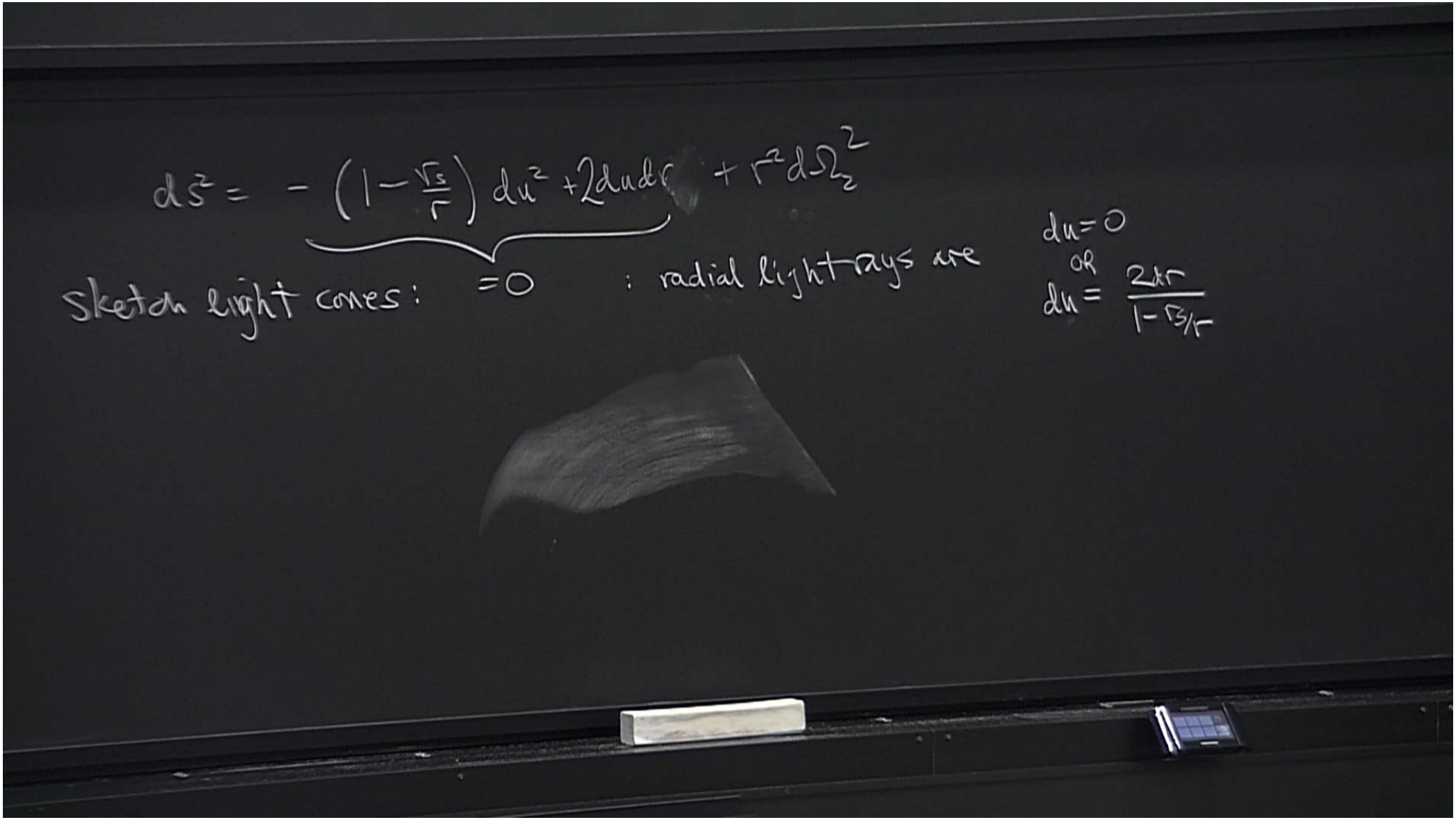
$$u = \text{const} \Rightarrow t = -r - r_s \ln\left|1 - \frac{r}{r_s}\right| + \text{const}$$



across $r = r_s$

($u = \text{const}$) across r_s :

Notice: direction of "time" r is decreasing r
= increasing time


$ds^2 = -\left(1-\frac{r_s}{r}\right)du^2 + 2du\,dr + r^2 d\Omega_2^2$
Sketch light cones: $=0$: radial light rays are
$du = 0$
OR
$du = \frac{2dr}{1-r_s/r}$

$$\frac{2dr}{1-r_s/r}$$

null rays will be $r = u - t'$ i.e. line at 45°.

$\ldots du dr + r^2 d\Omega_2^2$

: radial light-rays are $du = 0$ ingoing
OR
$du = \frac{2dr}{1 - 2M/r}$ outgoing

$\ldots$, s.t. $u = t' + r \Rightarrow$ ingoing null rays will be $r = u - t'$ i.e. line at 45°.

outgoing null rays

$$du = \frac{2\,dr}{1-\frac{r_s}{r}} = dt' + dr$$

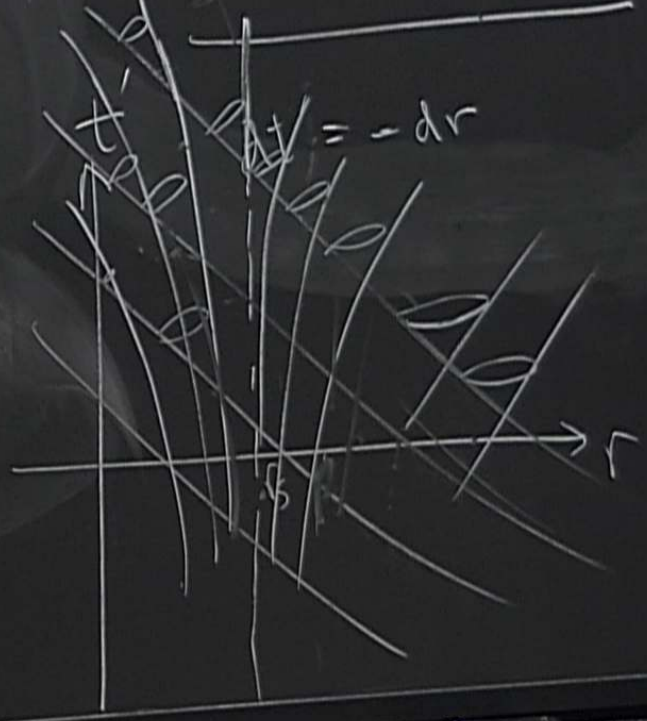
$$\left(\frac{2}{1-\frac{r_s}{r}} - 1\right) dr$$

$$dt' = \frac{r+r_s}{r-r_s}\,dr$$

outgoing

ingoing

$dt' = -dr$



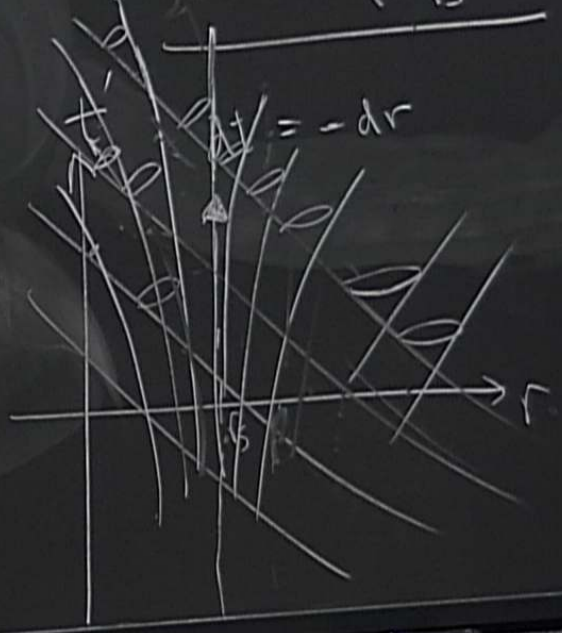
outgoing null rays

$$du = \frac{2dr}{1-\frac{r_s}{r}} = dt' + dr$$

$$\left(\frac{2}{1-\frac{r_s}{r}} - 1\right)dr$$

$$\therefore dt' = \frac{r+r_s}{r-r_s}dr \quad \text{outgoing}$$

$$dt' = -dr \quad \text{ingoing}$$



about outgoing null geodesic coordinates?

Let $v = t - r - r_s \ln\left|\frac{r}{r_s} - 1\right|$

$\Rightarrow ds^2 = -\left(1 - \frac{r_s}{r}\right) dv^2 - 2\,dv\,dr + r^2 d\Omega_2^2$ also regular at $r = r_s$.

Basic coordinates?

$\left|\frac{r}{r_s}-1\right|$

$-2\,dv\,dr + r^2 d\Omega_2^2$ also regular at $r=r_s$.

In these coordinates, direction of increasing time is increasing r.

$\| / -\infty$

– a blatant contradiction!

in ingoing null coord's, r decr. is incr. time

outgoing " " r incr. is incr. time

—— actually, each describes a different interior $r < r_s$!

decr. is incr time

incr. is incr. time

interior $r < r_s$!

Two versions, with opposite senses of increasing time, both connected to same exterior.

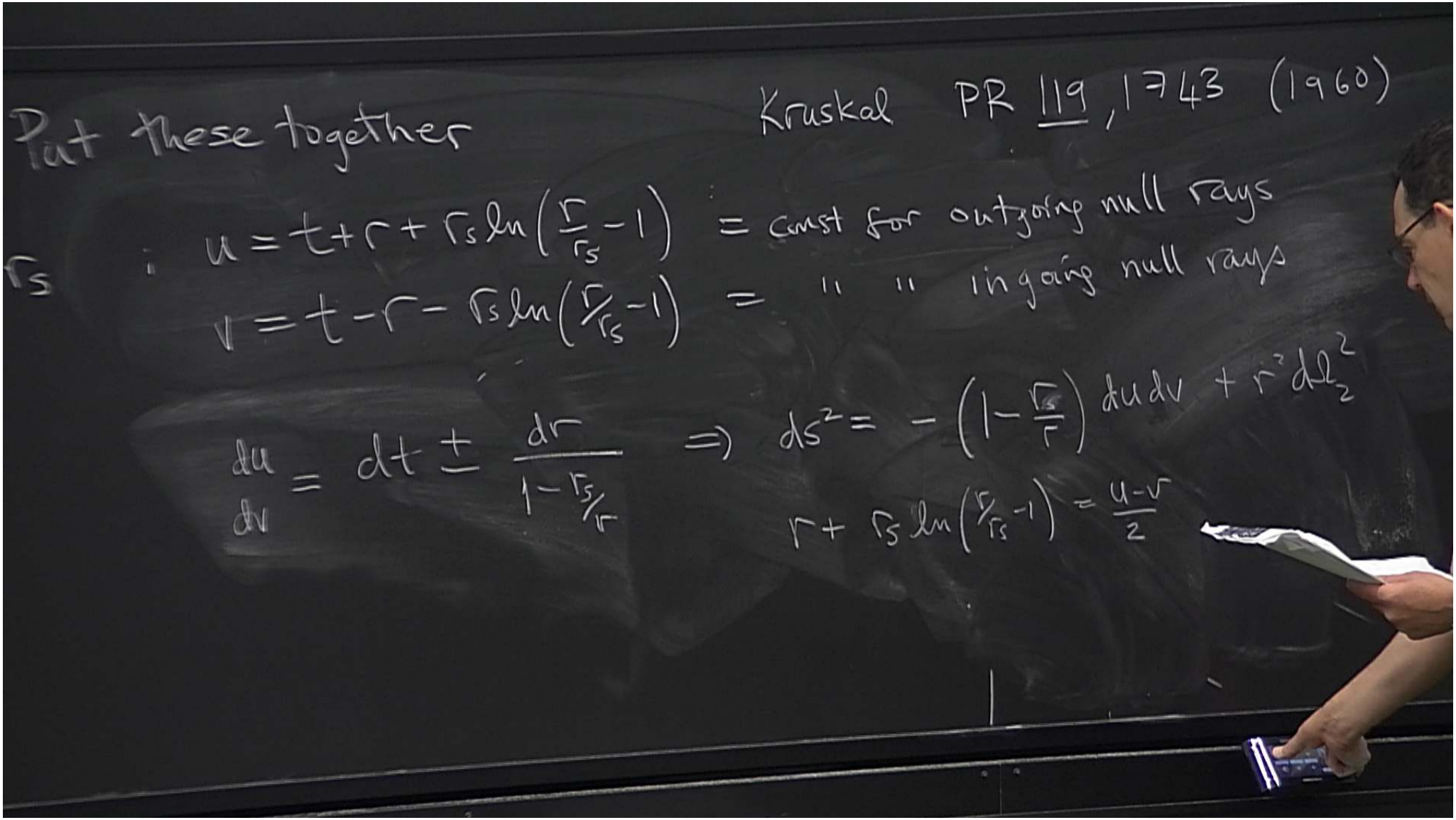
Put these together

Kruskal PR 119, 1743 (1960)

r_s : $u = t + r + r_s \ln\left(\frac{r}{r_s} - 1\right)$ = const for outgoing null rays

$v = t - r - r_s \ln\left(\frac{r}{r_s} - 1\right)$ = " " ingoing null rays

$$\frac{du}{dv} = dt \pm \frac{dr}{1 - r_s/r} \Rightarrow ds^2 = -\left(1 - \frac{r_s}{r}\right) du\,dv + r^2 d\Omega_2^2$$


Put these together
Kruskal PR 119, 1743 (1960)
r_s : $u = t + r + r_s \ln\left(\frac{r}{r_s} - 1\right)$ = const for outgoing null rays
$v = t - r - r_s \ln\left(\frac{r}{r_s} - 1\right)$ = " " ingoing null rays
$\frac{du}{dv} = dt \pm \frac{dr}{1 - r_s/r} \Rightarrow ds^2 = -\left(1 - \frac{r_s}{r}\right) du\,dv + r^2 d\Omega_2^2$
$r + r_s \ln\left(\frac{r}{r_s} - 1\right) = \frac{u - v}{2}$

Kruskal PR 119, 1743 (1960)

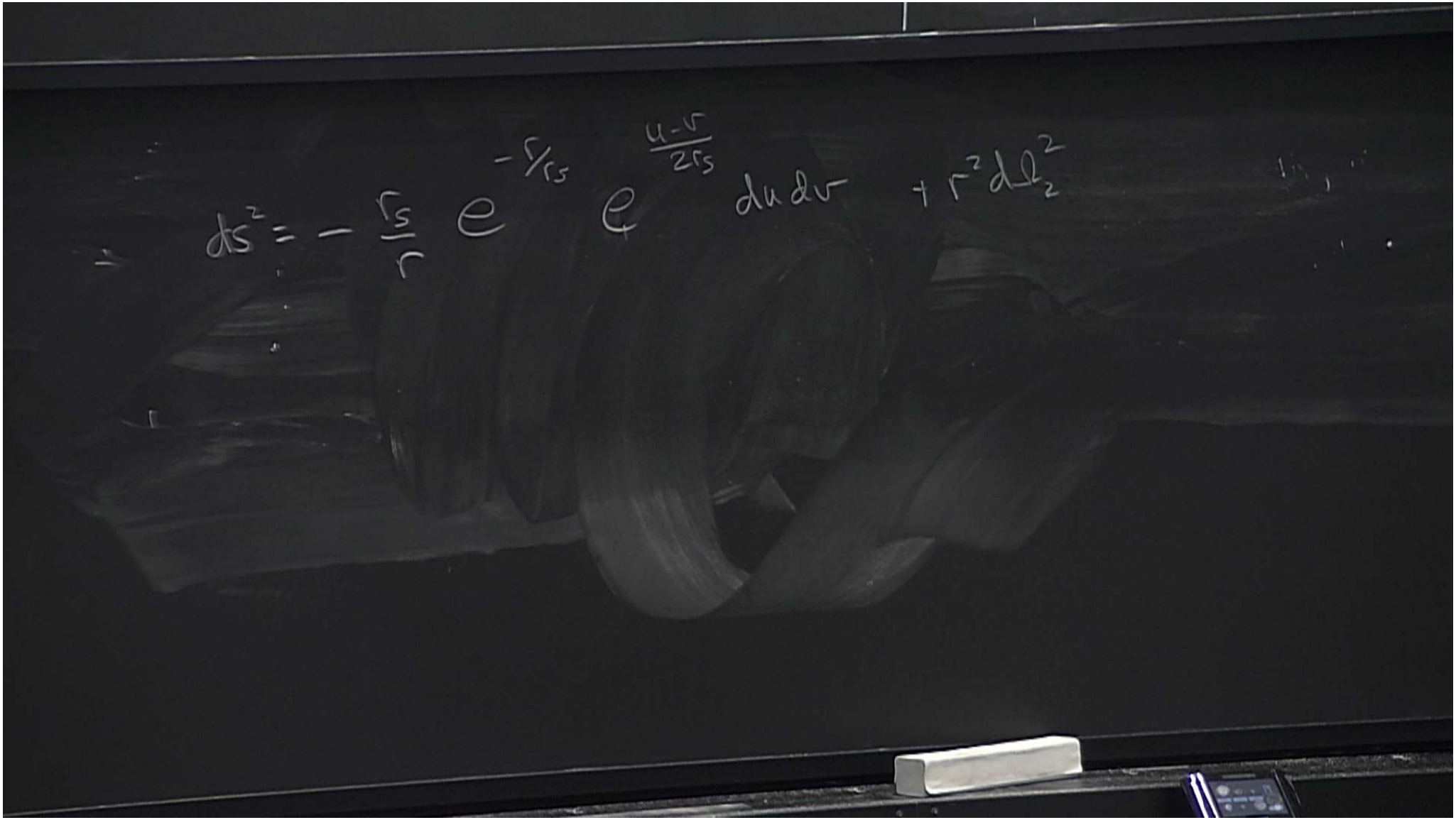
= const for outgoing null rays

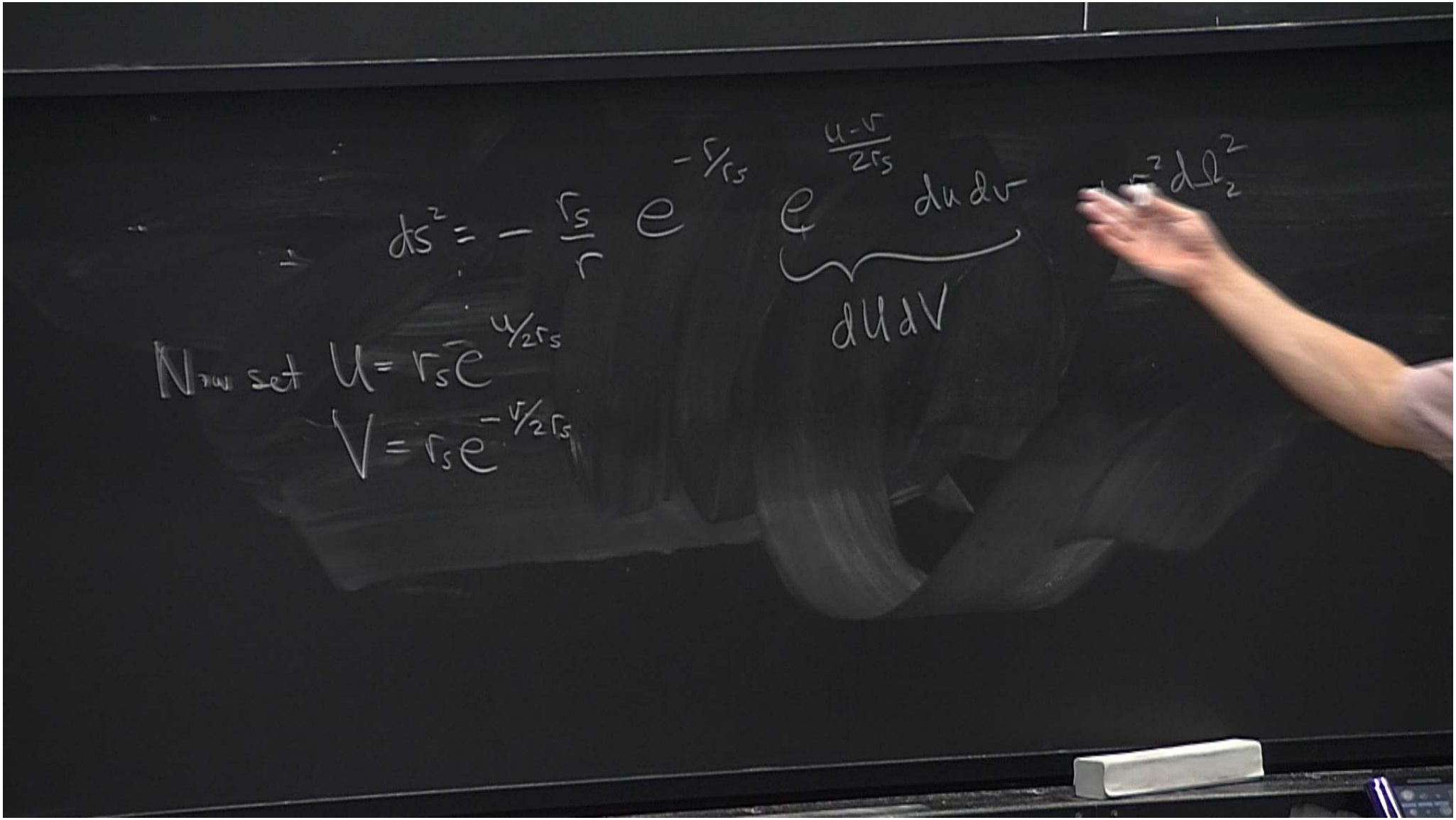
= " " ingoing null rays

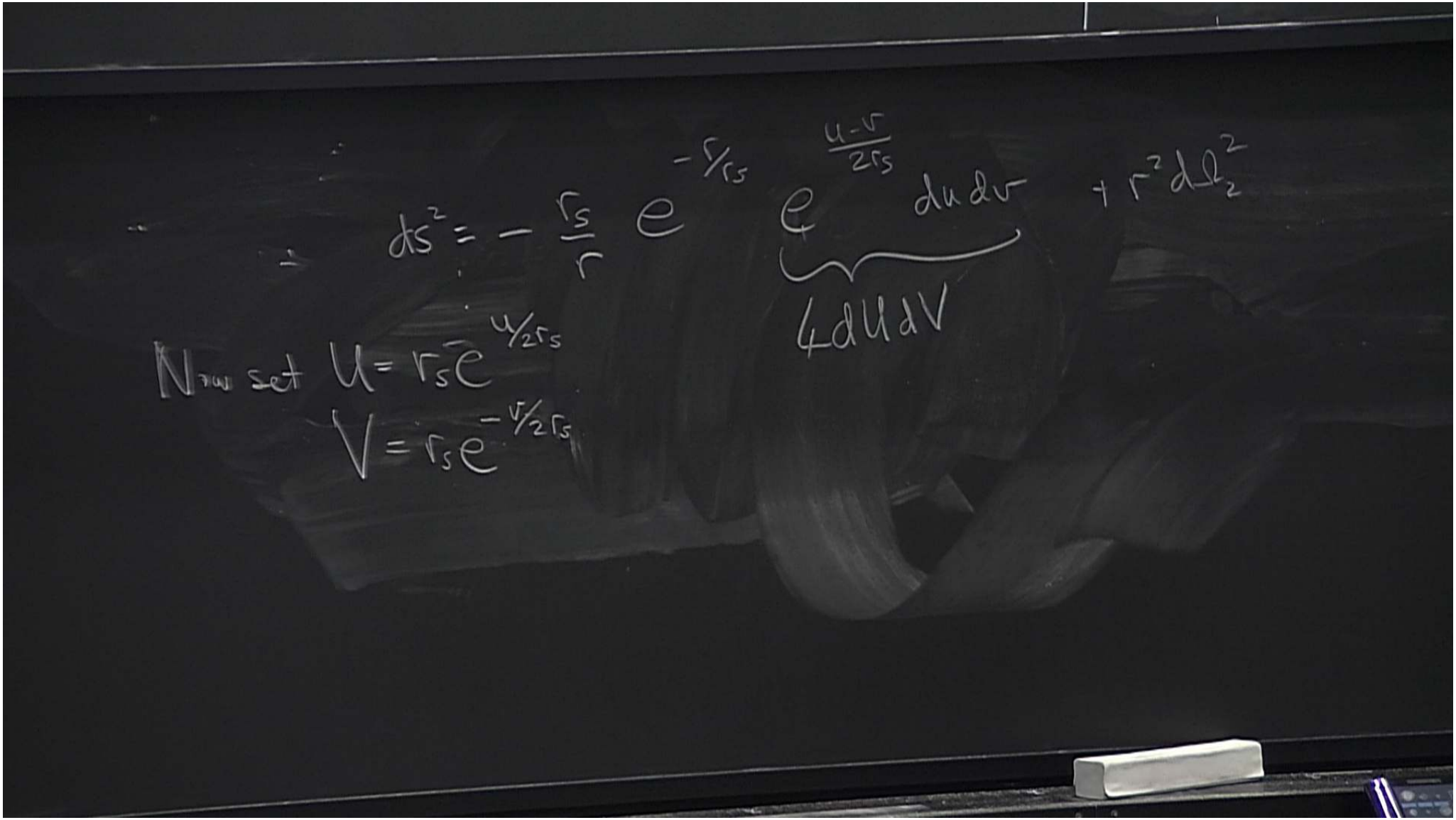
$$\Rightarrow ds^2 = -\left(1-\frac{r_s}{r}\right) du\,dv + r^2 d\Omega_2^2$$

$$r + r_s \ln\left(\frac{r}{r_s}-1\right) = \frac{u-v}{2} \Rightarrow \frac{r}{r_s} - 1 = e^{-r/r_s} e^{\frac{u-v}{2r_s}}$$

$$\Rightarrow 1 - \frac{r_s}{r} = \frac{r_s}{r} e^{-r/r_s} e^{\frac{u-v}{2r_s}}$$


$ds^2 = -\frac{r_s}{r} e^{-r/r_s} e^{\frac{u-v}{2r_s}} du\,dv + r^2 d\Omega_2^2$


$\therefore ds^2 = -\frac{r_s}{r} e^{-r/r_s} e^{\frac{u-v}{2r_s}} du\,dv$
$du\,dV$
Now set $u = r_s e^{u/2r_s}$
$V = r_s e^{-v/2r_s}$


$\therefore ds^2 = -\frac{r_s}{r} e^{-r/r_s} e^{\frac{u-v}{2r_s}} du\,dv + r^2 d\Omega_2^2$
$4 du\,dv$
Now set $u = r_s e^{u/2r_s}$
$v = r_s e^{-v/2r_s}$

$$\therefore \quad ds^2 = -\frac{r_s}{r} e^{-r/r_s} e^{\frac{u-v}{2r_s}} du\,dv + r^2 d\Omega_2^2$$

$$-4\,dU\,dV$$

Now set $U = -r_s e^{u/2r_s}$

$$V = r_s e^{-v/2r_s}$$

$$ds^2 = 4\frac{r_s}{r} e^{-r/r_s} dU\,dV + r^2 d\Omega_2^2$$

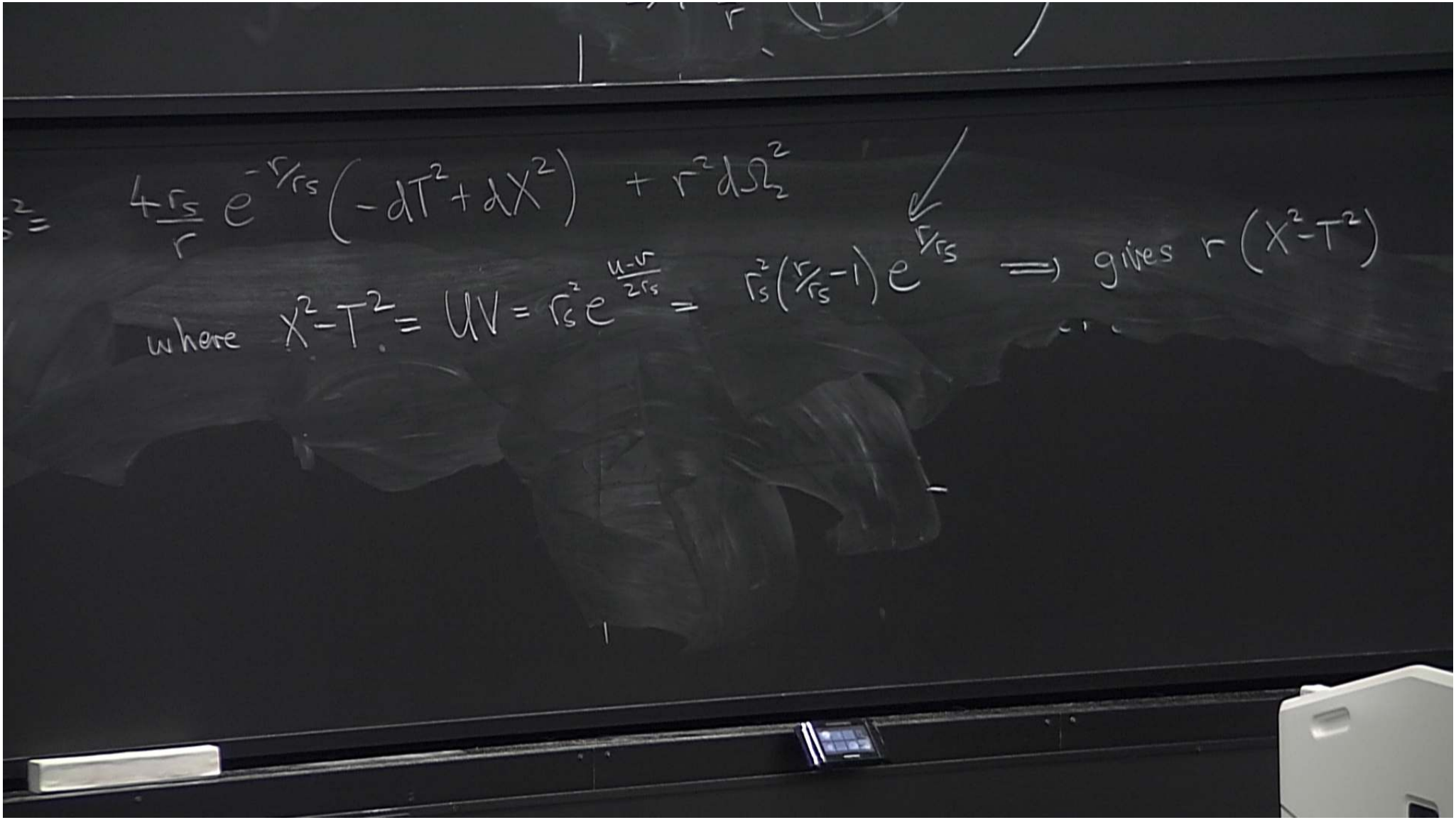
$$ds^2 = -\frac{r_s}{r} e^{-r/r_s} \underbrace{e^{\frac{u-v}{2r_s}} du\,dv}_{-4\,dU\,dV} + r^2 d\Omega_2^2$$

$$U = -r_s e^{u/2r_s}$$

$$V = r_s e^{-v/2r_s}$$

$$ds^2 = 4\frac{r_s}{r} e^{-r/r_s} dU\,dV + r^2 d\Omega_2^2$$

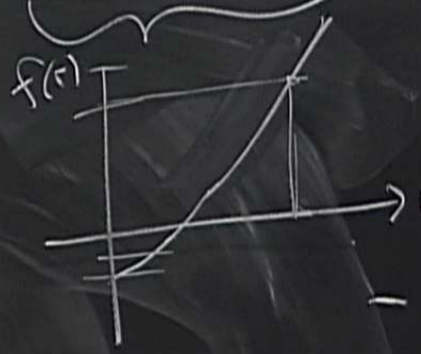
Finally, set $U = X + T$, $V = X - T$. X is a new "radius"

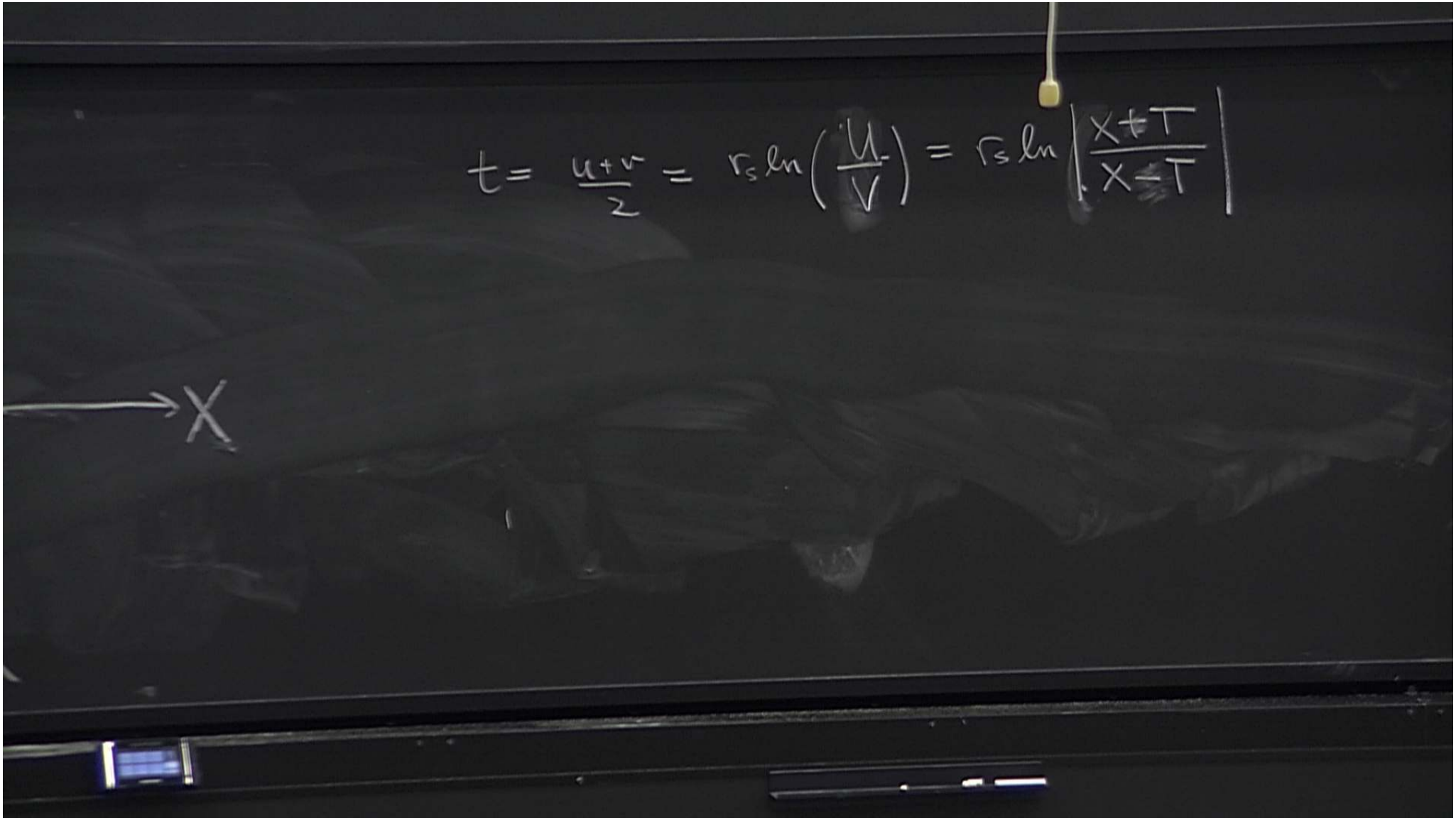

$= \frac{4r_s}{r} e^{-r/r_s}(-dT^2 + dX^2) + r^2 d\Omega_2^2$
where $X^2 - T^2 = UV = r_s^2 e^{\frac{U-V}{2r_s}} = r_s^2(r/r_s - 1)e^{r/r_s}$ $\Rightarrow$ gives $r(X^2 - T^2)$

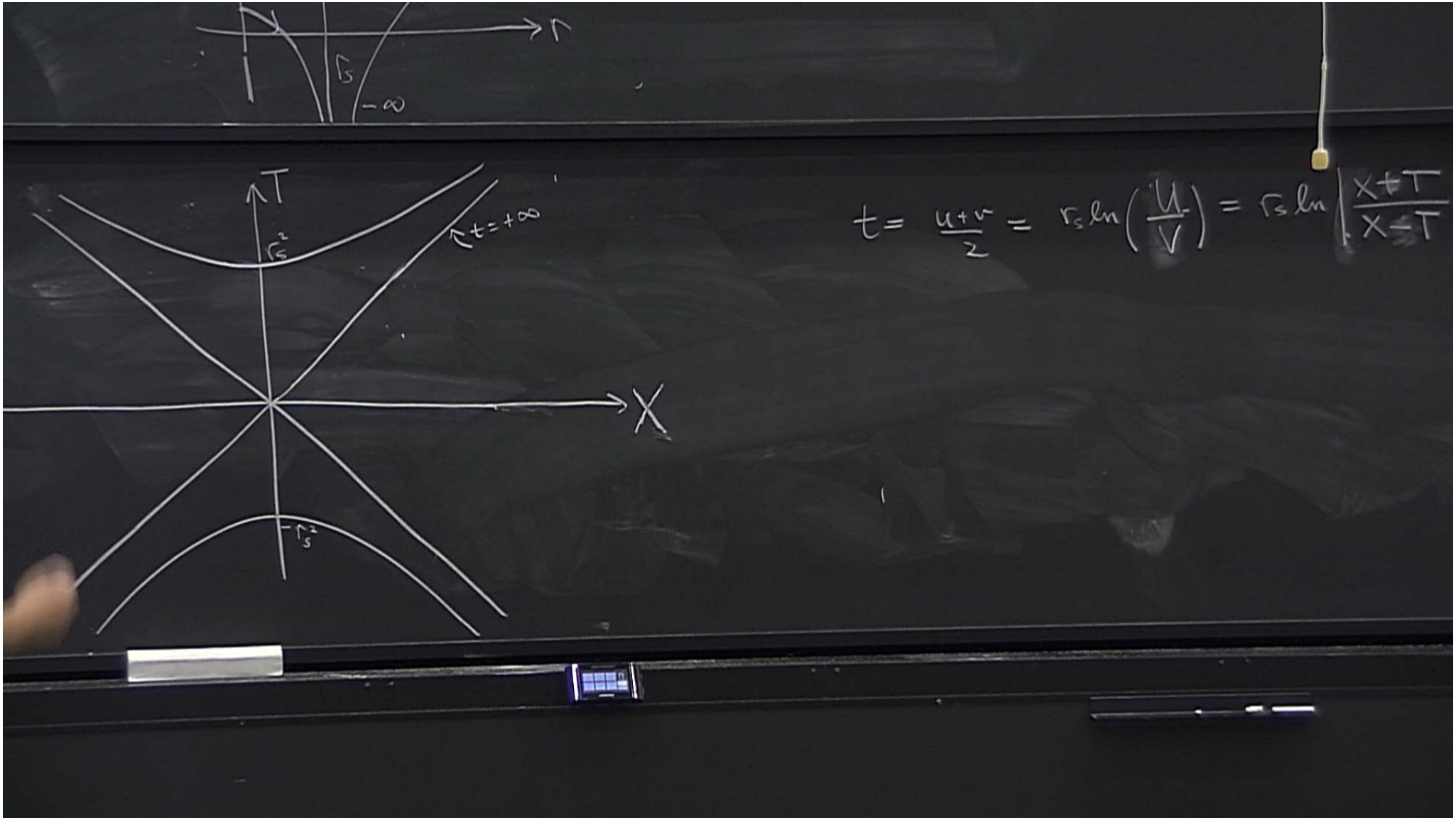
$$s^2 = \frac{4r_s}{r} e^{-r/r_s}\left(-dT^2 + dX^2\right) + r^2 d\Omega_2^2$$

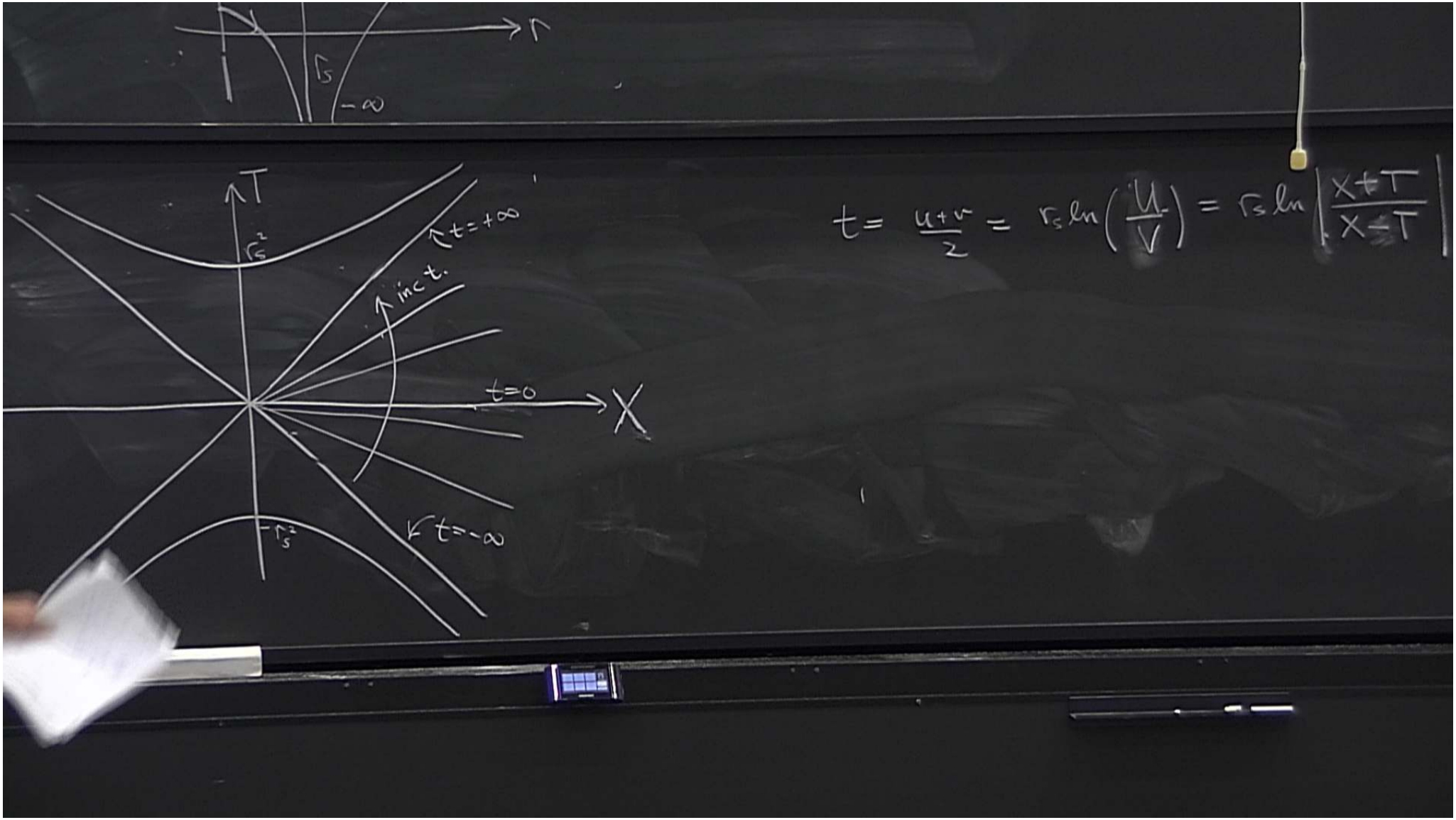
where $X^2 - T^2 = UV = r_s^2 e^{\frac{u-v}{2r_s}} = r_s^2\left(\frac{r}{r_s} - 1\right) e^{r/r_s}$ $\Longrightarrow$

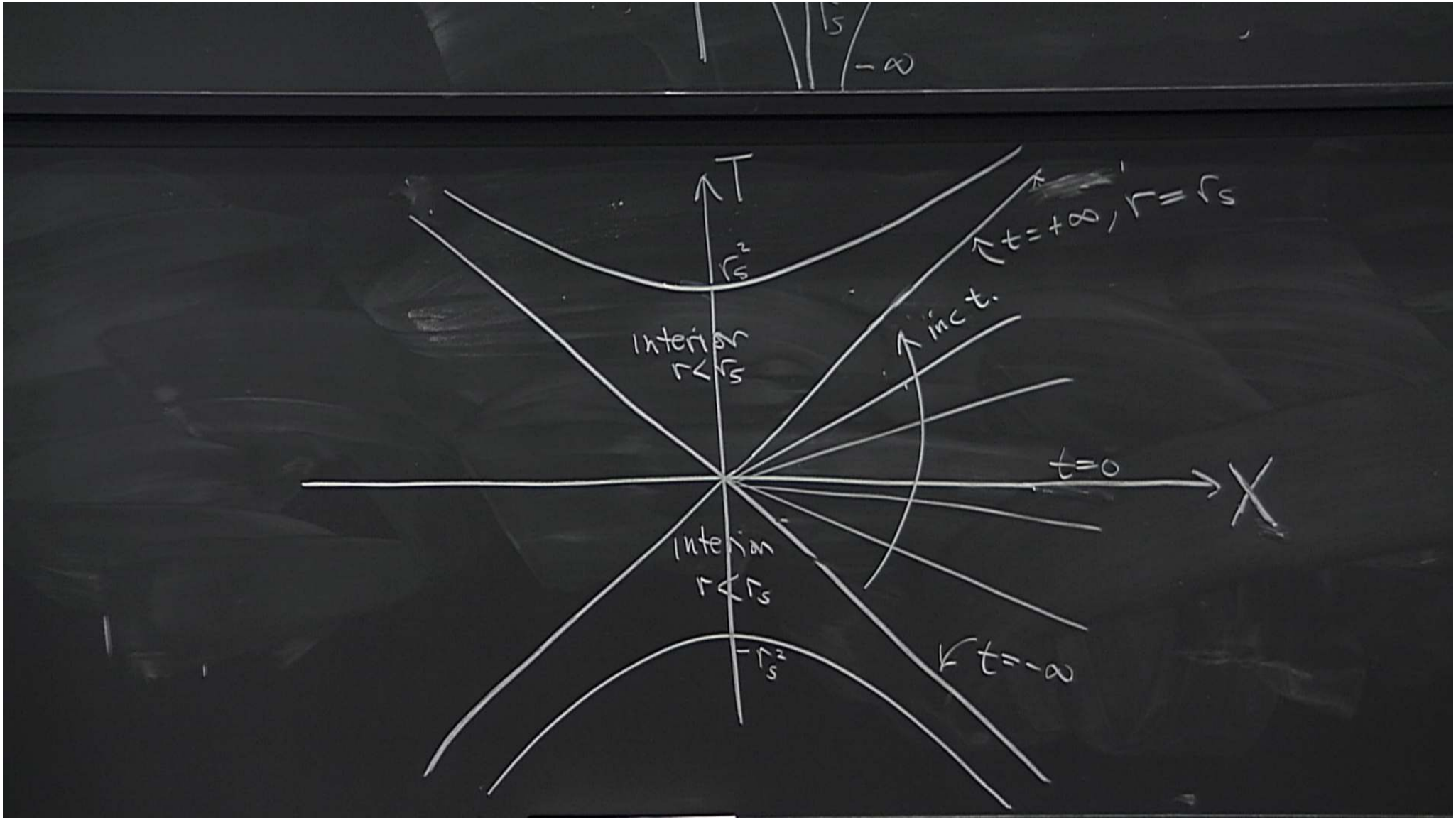
only exists an $r > 0$ for $X^2 - T^2 > -r_s^2$.

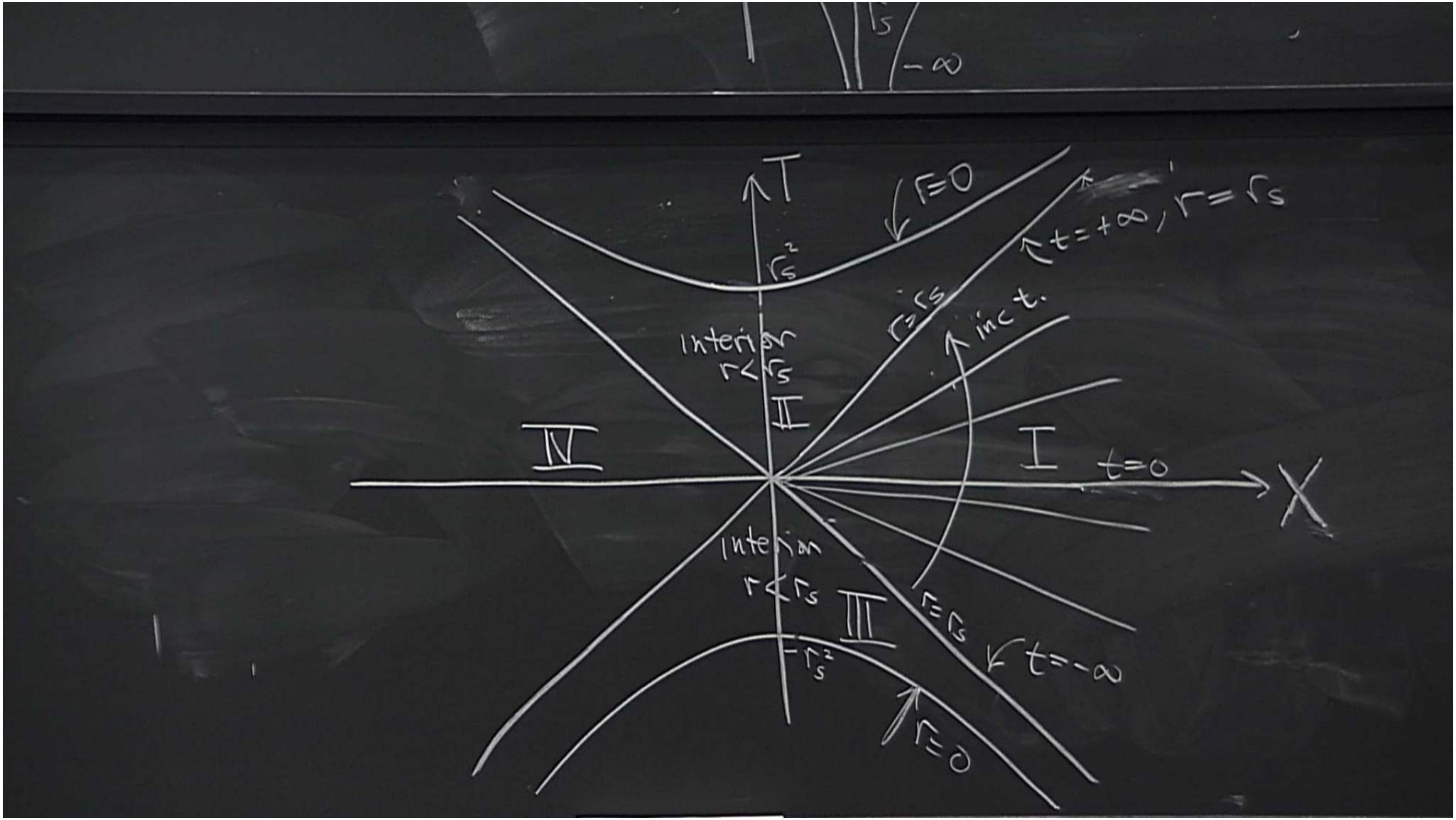



$t = \frac{u+v}{2} = r_s \ln\left(\frac{u}{v}\right) = r_s \ln\left|\frac{x+T}{x-T}\right|$
X


r
r_s
$-\infty$
T
r_s^2
$t=+\infty$
X
$-r_s^2$
$t = \frac{u+v}{2} = r_s \ln\left(\frac{U}{V}\right) = r_s \ln\left|\frac{X+T}{X-T}\right|$


r
r_s
$-\infty$
T
r_s^2
$t=+\infty$
inc t.
$t=0$
X
$-r_s^2$
$t=-\infty$
$t = \frac{u+v}{2} = r_s \ln\left(\frac{u}{v}\right) = r_s \ln\left|\frac{X+T}{X-T}\right|$


r_s
$-\infty$
T
r_s^2
$t=+\infty$, $r=r_s$
Interior
$r<r_s$
inc t.
$t=0$
X
Interior
$r<r_s$
$-r_s^2$
$t=-\infty$



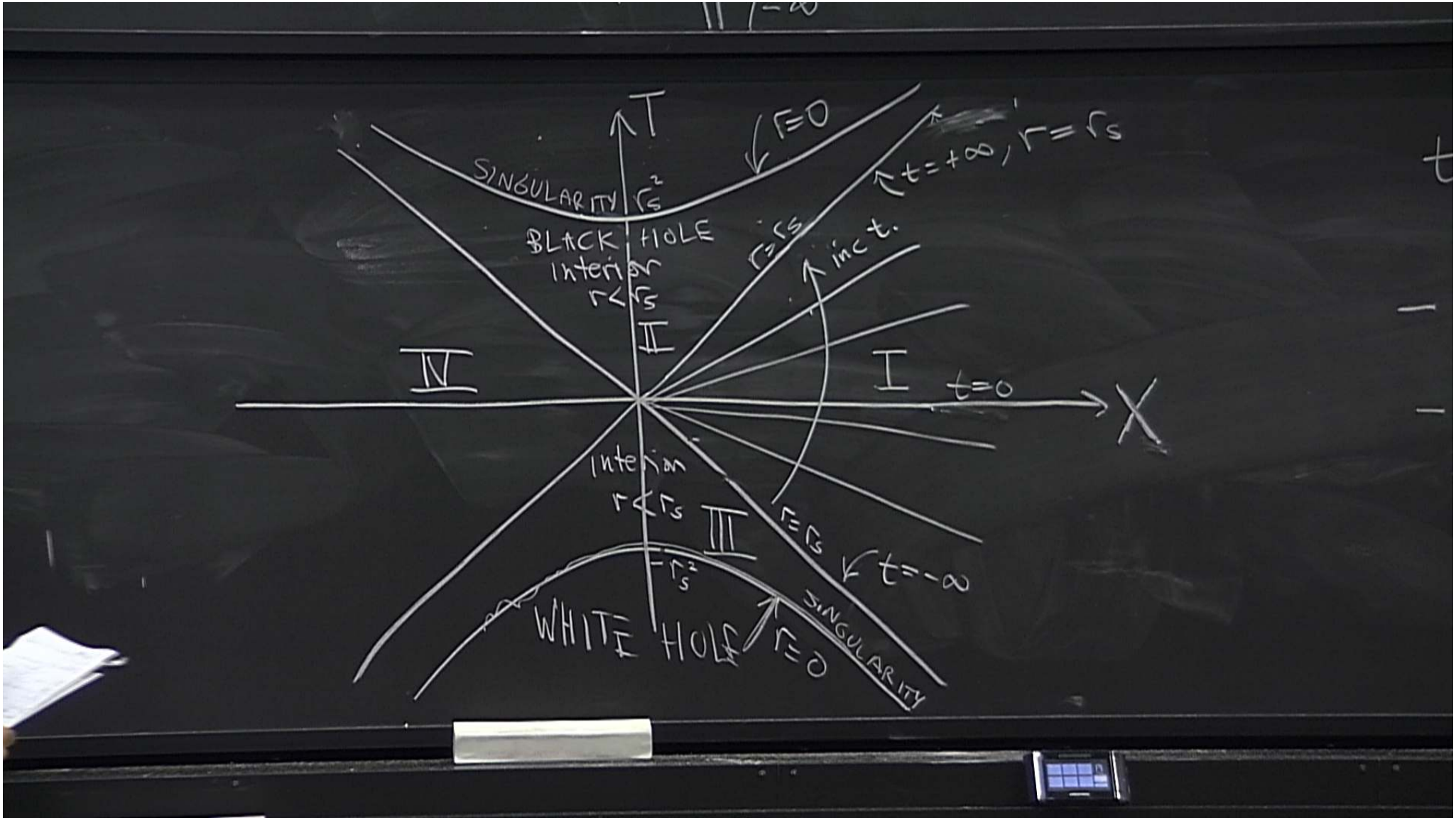
r_s
$-\infty$
T
$r=0$
$t=+\infty, r=r_s$
r_s^2
$r=r_s$
inc t.
interior
$r<r_s$
II
IV
I
$t=0$
X
interior
$r<r_s$
III
$r=r_s$
$t=-\infty$
$-r_s^2$
$r=0$

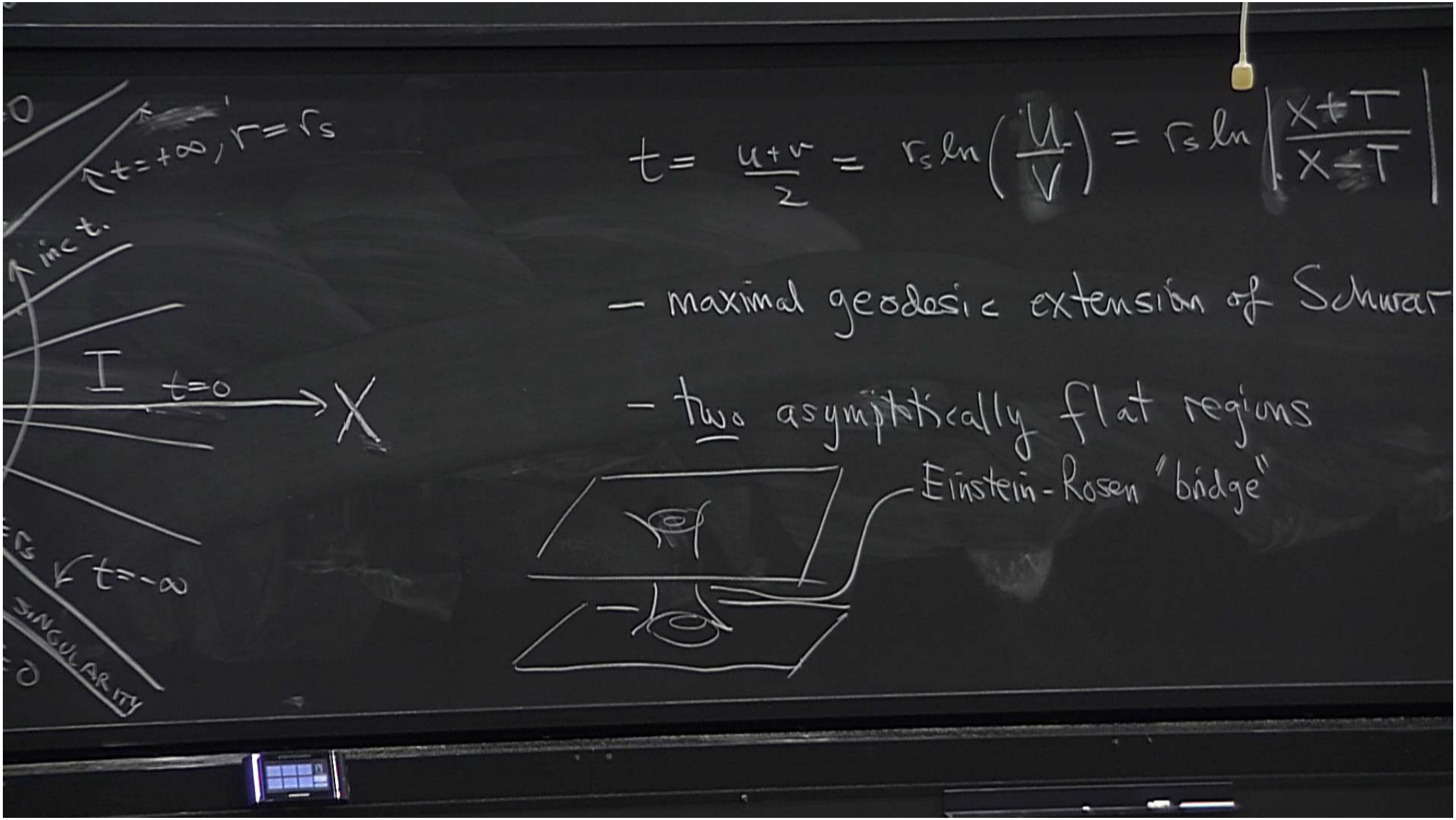
$$t = \frac{u+v}{2} = r_s \ln\left(\frac{U}{V}\right) = r_s \ln\left|\frac{X+T}{X-T}\right|$$

– maximal geodesic extension of Schwarzschild

$$t = \frac{u+v}{2} = r_s \ln\left(\frac{U}{V}\right) = r_s \ln\left|\frac{X+T}{X-T}\right|$$

- maximal geodesic extension of Schwarzschild
- two asymptotically flat regions

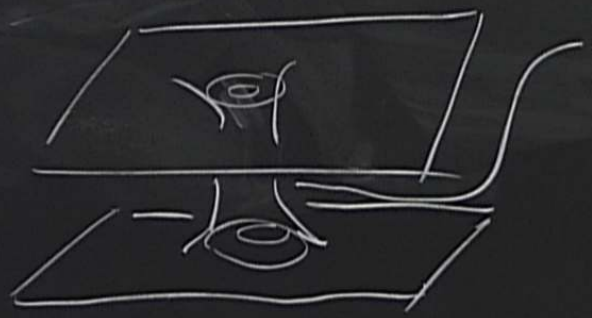

T
r=0
SINGULARITY r_s^2
BLACK HOLE
interior
$r<r_s$
II
IV
I
t=0
X
$t=+\infty$, $r=r_s$
$r=r_s$
inc t.
interior
$r<r_s$
III
$-r_s^2$
$r=r_s$
$t=-\infty$
WHITE HOLE
r=0
SINGULARITY


$t = \frac{u+v}{2} = r_s \ln\left(\frac{u}{v}\right) = r_s \ln\left|\frac{X+T}{X-T}\right|$
$t=+\infty,\ r=r_s$
inc t.
I
$t=0$
X
$t=-\infty$
SINGULARITY
- maximal geodesic extension of Schwar
- two asymptotically flat regions
Einstein-Rosen "bridge"

$$t = \frac{u+v}{2} = r_s \ln\left(\frac{u}{v}\right) = r_s \ln\left|\frac{X+T}{X-T}\right|$$

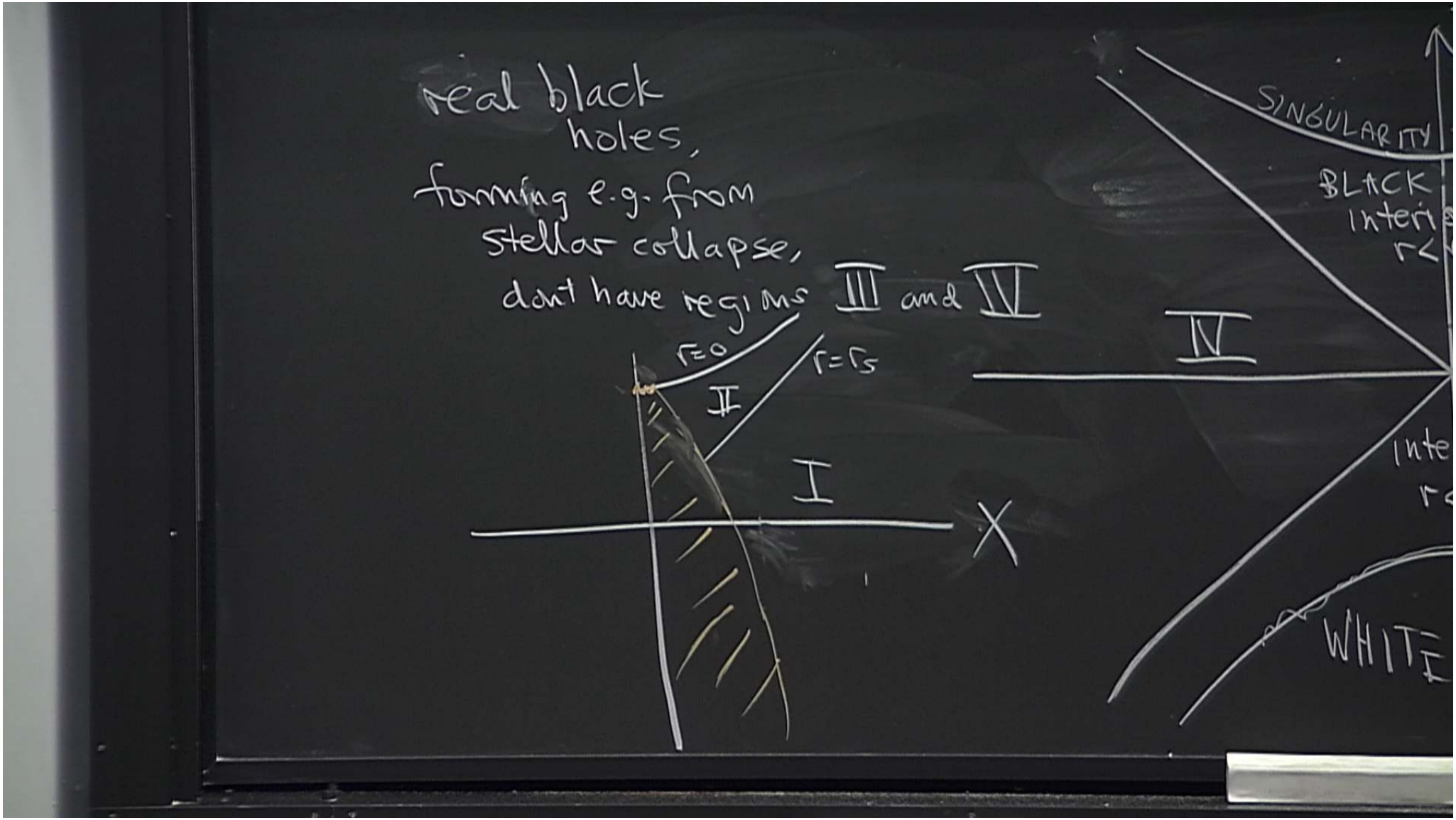
- maximal geodesic extension of Schwarzchild

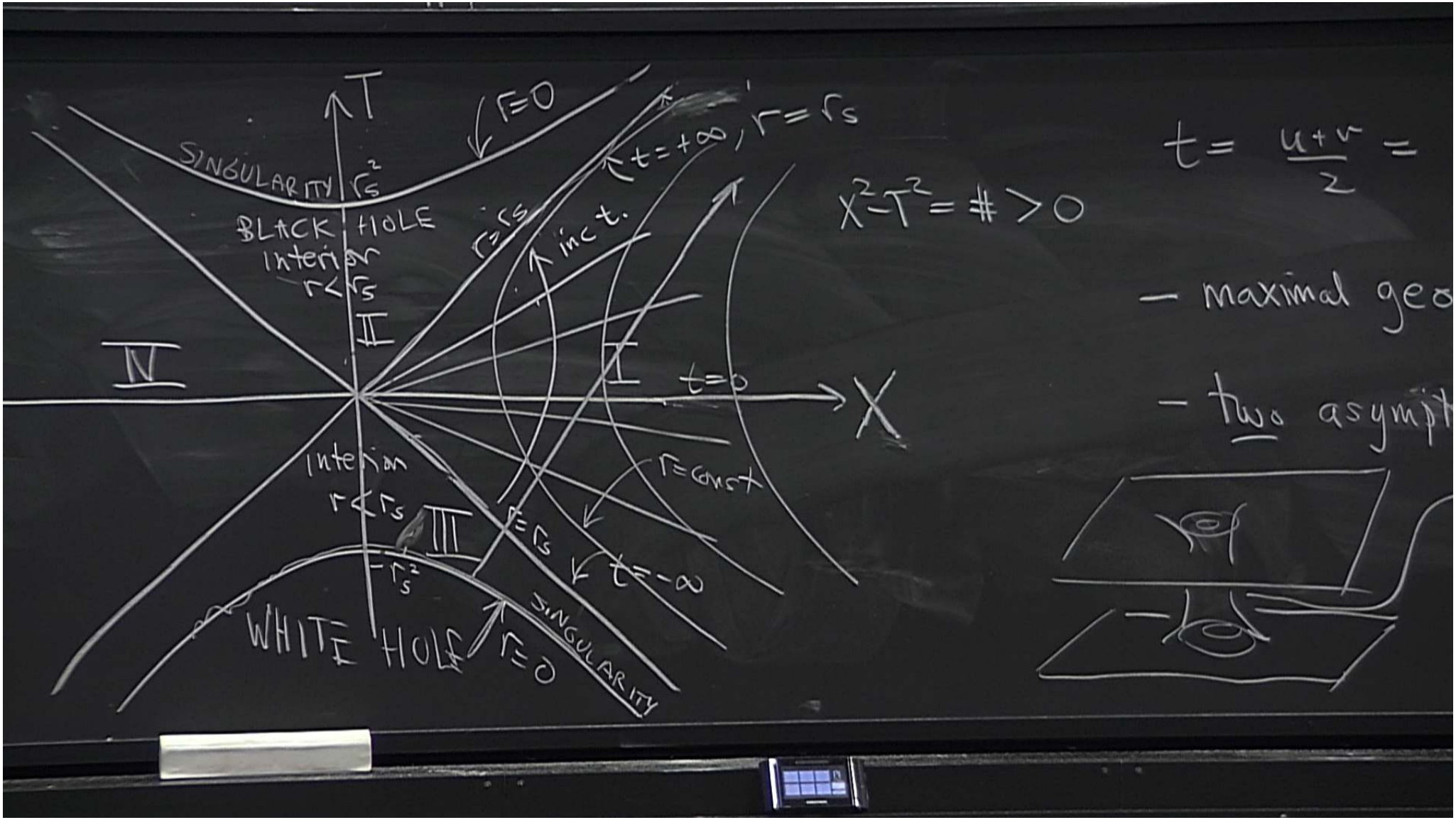
- two asymptotically flat regions

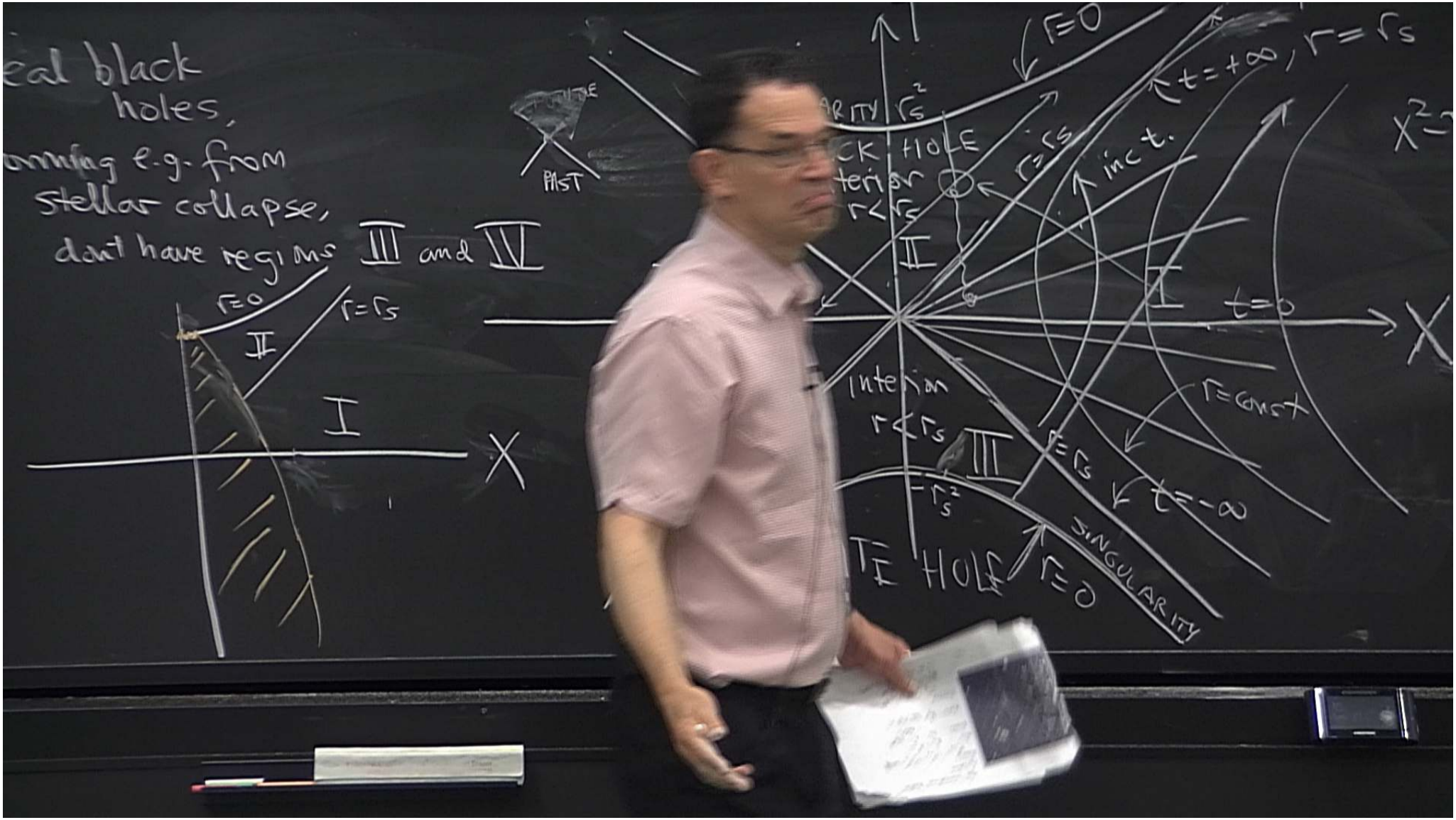


Einstein-Rosen "bridge"

Cannot communicate with other universe. :(


real black holes,
forming e.g. from stellar collapse,
don't have regions III and IV
r=0
r=rs
II
I
X
IV
SINGULARITY
BLACK
inter
r<
inte
r<
WHITE


T
r=0
SINGULARITY
r_s^2
BLACK HOLE
interior
$r<r_s$
II
IV
I
t=0
X
$r=r_s$
$t=+\infty$
inc t.
$X^2-T^2=\#>0$
$t=\frac{u+v}{2}=$
– maximal geo
– two asympt
interior
$r<r_s$
III
$r=r_s$
$t=-\infty$
r=const
$-r_s^2$
WHITE HOLE
r=0
SINGULARITY


eal black
holes,
orming e.g. from
stellar collapse,
dont have regions III and IV
r=0
r=rs
II
I
X
PAST
r=0
t=+∞, r=rs
inc t.
r=rs
r<rs
II
I
t=0
X
interior
r<rs
III
r=const
r=rs
t=-∞
SINGULARITY
HOLE
r=0