

Title: 13/14 PSI - Relativity - Lecture 14

Date: Sep 26, 2013 09:00 AM

URL: <http://pirsa.org/13090028>

Abstract:

Schwarzschild

$$ds^2 = - \left(1 - \frac{2GM}{rc^2}\right) dt^2 + \frac{1}{\left(1 - \frac{2GM}{rc^2}\right)} dr^2 + r^2 d\Omega^2$$

all

$$S = -m \int d\tau = -m \int \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\lambda$$

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

recall

$$S = -m \int d\tau = -m \int \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\lambda$$

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

improved action (no square root)

$$\frac{m}{2} \int \left(\frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e} - c^2 e \right) d\lambda$$

$$S = -m \int d\tau = -mc \int \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\lambda$$

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

dim (no square root)

$$\frac{m}{2} \int \left(\frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e} - c^2 e \right) d\lambda$$

$$-mc^2 \int dt \sqrt{1 - \left(\frac{dx}{dt} \right)^2 \frac{1}{c^2}}$$

$$= -mc \int \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\lambda$$

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

$$g = \eta$$

$$-mc^2 \int dt \sqrt{1 - \left(\frac{dx}{dt}\right)^2 \frac{1}{c^2}}$$

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$$\frac{m}{2} \int \left(\frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e} - c^2 e \right) d\lambda$$

$$\bar{x}^\nu d\lambda$$

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

$$g = \eta$$

$$-mc^2 \int dt \sqrt{1 - \left(\frac{dx}{dt}\right)^2 \frac{1}{c^2}}$$

$$c^2 e) d\lambda$$

metric on the world line

$$\lambda \rightarrow \bar{\lambda}(\lambda)$$

$$e \rightarrow \frac{d\lambda}{d\bar{\lambda}} e$$

$e d\lambda$ is invariant
under reparametrisations
of world line

$$S = -m \int d\tau = -mc \int \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\lambda \quad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

$S[x^\mu(\lambda), e(\lambda)]$ $g = \eta$ $-mc^2 \int dt \sqrt{1 - \left(\frac{dx}{dt}\right)^2 \frac{1}{c^2}}$

(no square root)

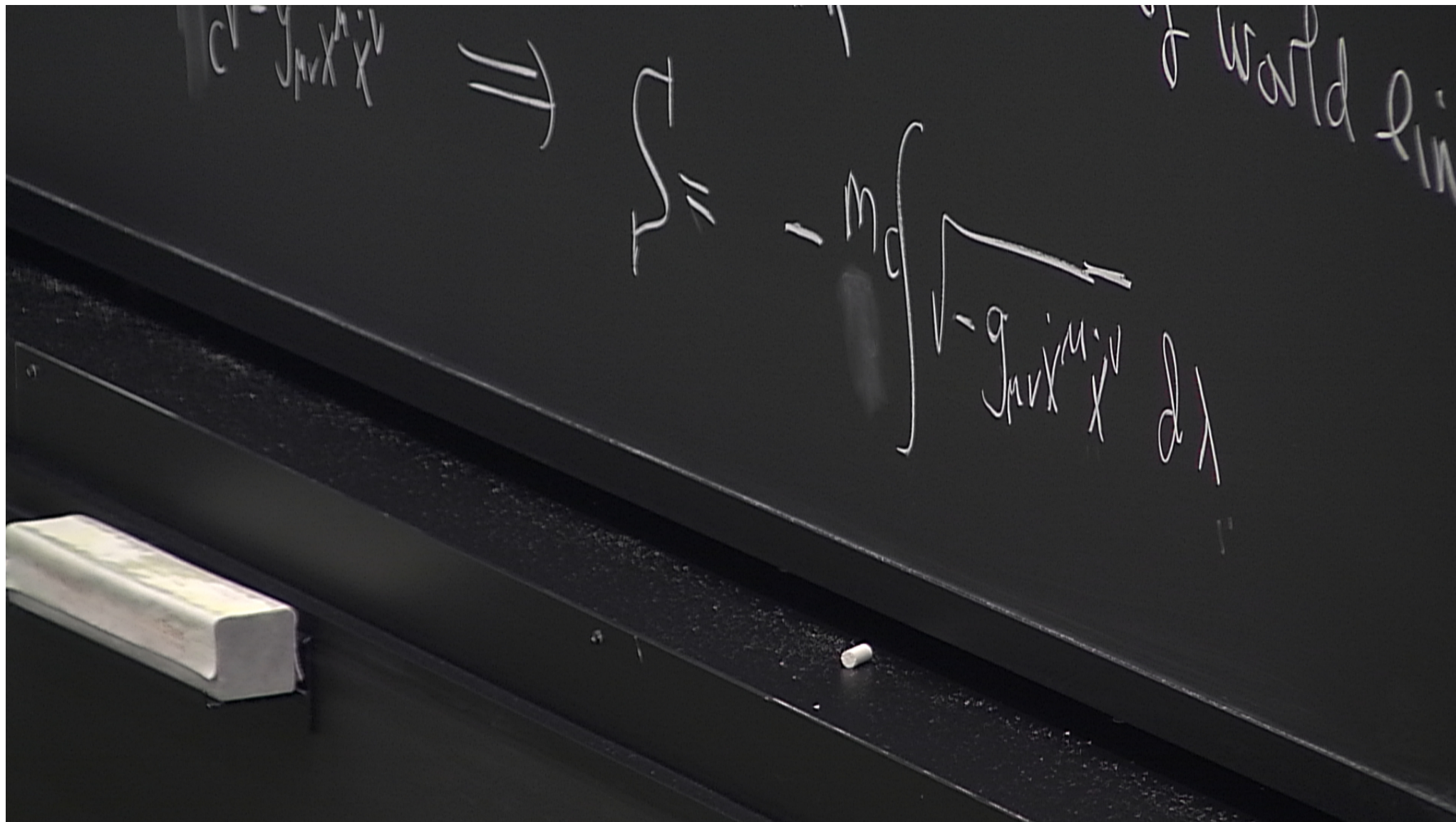
$$\frac{m}{2} \int \left(\frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e} - c^2 e \right) d\lambda$$

"einbein": square root of the 1-d metric on the world line : $\lambda \rightarrow \tilde{\lambda}(\lambda)$
 $e \rightarrow \frac{d\lambda}{d\tilde{\lambda}} e$ $e d\lambda$ is invariant under reparam of λ

no square root, easier to quantise

vary w.r.t. e $-\frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e^2} - c^2 \Rightarrow e = \frac{1}{c} \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$ Subst back into action

$$\Rightarrow S = -mc \int \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$$



$$S = -m \int d\tau = -mc \int \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\lambda \quad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

$\underbrace{\hspace{10em}}_{g=\eta}$

$$-mc^2 \int dt \sqrt{1 - \left(\frac{dx}{dt}\right)^2 \frac{1}{c^2}}$$

(no square root)

$$\frac{m}{2} \int \left(\frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e} - c^2 e \right) d\lambda$$

"einbein"

1-d metric on the world line

$$\lambda \rightarrow \tilde{\lambda}(\lambda)$$

$$e \rightarrow \frac{d\lambda}{d\tilde{\lambda}} e$$

$e d\lambda$ is invariant under reparametrization of world line

no square root, easier to quantise

Subst back into action

$$\frac{m}{2} \int \left(\frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e^2} - c^2 \right) d\lambda \Rightarrow e = \frac{1}{c} \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \Rightarrow S = -mc \int \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\lambda$$

not)

$$\frac{m}{2} \int \left(\frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e} - c^2 e \right) d\lambda$$

: square root of the 1-d metric on the world line

$$\lambda \rightarrow \tilde{\lambda}(\lambda)$$

$$e \rightarrow \frac{d\lambda}{d\tilde{\lambda}} e$$

$e d\lambda$ is invariant under reparam of world line

not, easier to quantise

Subst back into action

$$e \left(- \frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e^2} - c^2 \right) \Rightarrow e = \frac{1}{c} \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \Rightarrow S = -mc \int \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\lambda$$

↓ this eqⁿ is

$$u^\mu u^\nu = -c^2$$

$$u^\mu = \frac{\dot{x}^\mu}{e}$$

$$\frac{m}{2} \int \left(\frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e} - c^2 e \right) d\lambda$$

: square root of the 1-d metric on the world line

$$\lambda \rightarrow \tilde{\lambda}(\lambda)$$

$$e \rightarrow \frac{d\lambda}{d\tilde{\lambda}} e$$

$e d\lambda$ is invariant under reparam of world

not, easier to quantise

Subst back into action

$$\underbrace{-\frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e^2} - c^2}_{=0} \Rightarrow e = \frac{1}{c} \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$$

$$\Rightarrow S = -mc \int \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\lambda$$

↓ this eqⁿ is $u^\mu u^\nu = -c^2$

$$u^\mu = \frac{\dot{x}^\mu}{e}$$

(invariant under reparam)

↓ this eqⁿ is

$$u^\mu = \frac{\dot{x}^\mu}{c}$$

(invariant under reparam)

$$\underline{u^\mu u^\nu = -c^2}$$

$$(u^\mu u^\nu = -1)$$

$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$, where $d\tau = \underbrace{e d\lambda}_{\text{rep. dr. inv. x.}}$

(invariant under reparam)

the constraint on the 4-velocity and the geodesic

$$u^\mu u^\nu g_{\mu\nu} = -c^2$$

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0, \text{ where } d\tau = \frac{cd\lambda}{\text{reparam. invariant.}}$$

For a photon, $m=0$, use instead
the action $\frac{C}{2} \int \frac{\dot{x}^\mu \dot{x}^\nu}{e} g_{\mu\nu} d\lambda$, where C is identified

For a photon, $m=0$, use instead

the action $\frac{c}{2} \int \frac{\dot{x}^\mu \dot{x}^\nu}{e} g_{\mu\nu} d\lambda$, where c is

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu}$$

For a photon, $m=0$, use instead

the action $\frac{G'}{2} \int \frac{\dot{x}^\mu \dot{x}^\nu}{c} g_{\mu\nu} d\lambda$, where

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{G' g_{\mu\nu} \dot{x}^\nu}{c}$$

For a photon, $m=0$, use instead

the action $\frac{C}{2} \int \frac{\dot{x}^\mu \dot{x}^\nu}{e} g_{\mu\nu} d\lambda$, where C is

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{C g_{\mu\nu} \dot{x}^\nu}{e}$$

$$S(x^\mu(\lambda), e(\lambda)) = \frac{m}{2} \int \left(\underbrace{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}_{E_{\text{for photon}}^2} - \underbrace{c^2 e^2}_{0 \text{ for photon, } +c^2 e^2 \text{ tachyon}} \right) d\lambda$$

square root of the 1-d metric on the world line

... easier to quantise

Subst back into action

$$\underbrace{- \frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e^2} - c^2}_{=0} \Rightarrow e = \frac{1}{c} \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$$

$$\Rightarrow \int =$$

For a photon, $m=0$, use instead

the action $\frac{C}{2} \int \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} d\lambda$, where C is identified with $|\underline{p}| = p^0$ for photon.

$$\underline{p}_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{C g_{\mu\nu} \dot{x}^\nu}{c}$$

$$\frac{\delta}{\delta c} \Rightarrow \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} = 0 \Rightarrow \text{null.}$$

Conserved quantities: if $g_{\mu\nu}(x)$ is independent of some coordinate, e.g. Schw. t and ϕ , then the corresponding momenta are conserved

S

Conserved quantities: if $g_{\mu\nu}(x)$ is independent of some coordinate, e.g. Schw.
 must to work in gauge $e=1$. Momenta are conserved

$$S = \frac{m}{2} \int d\tau \left[- \left(1 - \frac{2GM}{rc^2} \right) c^2 \dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2GM}{rc^2} \right)} + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right] \quad \text{where } \dot{} = \frac{d}{d\tau}$$

gurt. e $\underbrace{-\frac{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{e^2}}_{-c^2} \Rightarrow e = \frac{1}{c} \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \Rightarrow \int = -m_0 \int \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} d\lambda$

gives both the constraint on the 4-velocity and the geodesic equation

at $e \Rightarrow u^\mu u^\nu g_{\mu\nu} = -c^2 \quad \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} g_{\mu\nu} = -c^2 e^2$

at $x^\mu \rightarrow \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$, where $d\tau = \frac{e d\lambda}{c}$, rep. invar.

Convenient to work in gauge $e=1$, momenta are conserved

$$S = \frac{m}{2} \int d\tau \left[- \left(1 - \frac{2GM}{rc^2} \right) c^2 \dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2GM}{rc^2} \right)} + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right]$$

Lagrange eqⁿs

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^a} \right) = \frac{\partial L}{\partial x^a}$$

$$x^a = \theta$$

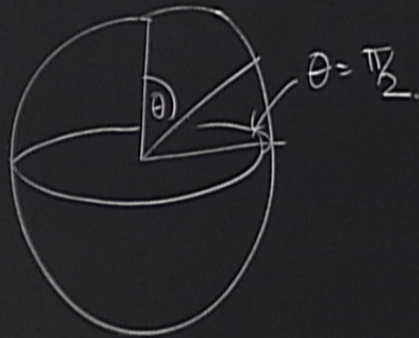
$$(r^2 \dot{\theta})' = r^2 \sin \theta \cos \theta \dot{\phi}^2$$

momenta are conserved

$$\left[-\frac{\dot{r}^2}{\left(1 - \frac{2GM}{rc^2}\right)} + r^2(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2) \right]$$

where $\dot{} = \frac{d}{d\tau}$

$$= r^2 \sin\theta \cos\theta \dot{\phi}^2$$

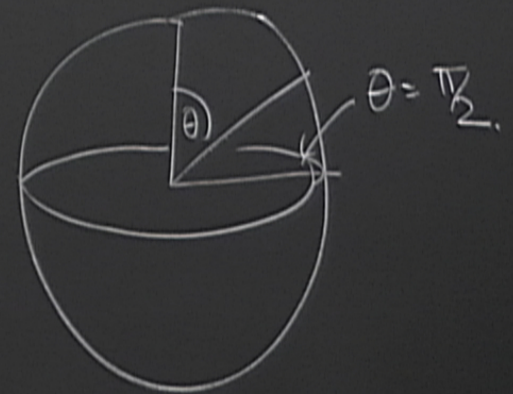


conserved quantities: if $g_{\mu\nu}(x)$ is independent of some coordinate,
 work in gauge $e=1$. Momenta are conserved

$$= \frac{m}{2} \int d\tau \left[- \left(1 - \frac{2GM}{rc^2} \right) c^2 \dot{t}^2 + \frac{\dot{r}^2}{\left(1 - \frac{2GM}{rc^2} \right)} + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right] \quad \text{where}$$

$$x^\mu = \theta \quad (r^2 \dot{\theta})' = r^2 \sin \theta \cos \theta \dot{\phi}^2$$

$\theta = \pi/2$ is a solⁿ, choose this from now on.



$$x^M = \varphi \quad \frac{d}{d\tau}(r^2 \sin^2 \theta \dot{\varphi}) = 0 \quad \Rightarrow \quad r^2 \dot{\varphi} \equiv L_z = \text{constant}$$

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$$x^\mu = t \quad \frac{d}{d\tau} \left(c \dot{t} \left(1 - \frac{2GM}{rc^2} \right) \right) = 0$$

$$x^\mu = \varphi \quad \frac{d}{d\tau} \left(r^2 \sin^2 \theta \dot{\varphi} \right) = 0 \quad \Rightarrow \quad r^2 \dot{\varphi} \equiv L_z = \text{constant}$$

$$x^\mu = t \quad \frac{d}{d\tau} \left(c \dot{t} \left(1 - \frac{2GM}{rc^2} \right) \right) = 0 \quad c \dot{t} \left(1 - \frac{2GM}{rc^2} \right) = E = \text{const.}$$

$$x^M = \varphi \quad \frac{d}{d\tau} (r^2 \sin^2 \theta \dot{\varphi}) = 0 \Rightarrow r^2 \dot{\varphi} = L_z = \text{const}$$

$$x^M = t \quad \frac{d}{d\tau} \left(c \dot{t} \left(1 - \frac{2GM}{rc^2} \right) \right) = 0 \quad c \dot{t} \left(1 - \frac{2GM}{rc^2} \right) =$$

Constraint

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -c^2 \Rightarrow \quad - \left(1 - \frac{2GM}{rc^2} \right)^{-1} \dot{t}^2 + \left(1 - \frac{2GM}{rc^2} \right)^{-1} \dot{r}^2 + \frac{L_z^2}{r^2} = -c^2$$

$$x^M = \varphi \quad \frac{d}{d\tau} (r^2 \sin^2 \theta \dot{\varphi}) = 0 \quad \Rightarrow \quad r^2 \dot{\varphi} \equiv L_z = \text{constant}$$

$$x^M = t \quad \frac{d}{d\tau} \left(c \dot{t} \left(1 - \frac{2GM}{rc^2} \right) \right) = 0 \quad c \dot{t} \left(1 - \frac{2GM}{rc^2} \right) = E = \text{const.}$$

$$\Rightarrow \quad - \left(1 - \frac{2GM}{rc^2} \right)^{-1} E^2 + \left(1 - \frac{2GM}{rc^2} \right)^{-1} \dot{r}^2 + \frac{L_z^2}{r^2} = -c^2 \quad (\text{massive particle})$$

$$\dot{r}^2 = 0 \quad (\text{photon})$$

photon $\swarrow$ affine parameter
 $x^\mu(\lambda)$

$$p_\mu = g_{\mu\nu} \dot{x}^\nu$$

scale)

photon

$x^\mu(\lambda)$

affine parameter

$$p_\mu = g_{\mu\nu} \dot{x}^\nu$$

u^μ 4-vel of atom

$\downarrow$
 p_μ

Energy of photon
as seen by atom

$$= u^\mu p_\mu$$

$$\Rightarrow \dot{r}^2 + V(r) = E^2$$

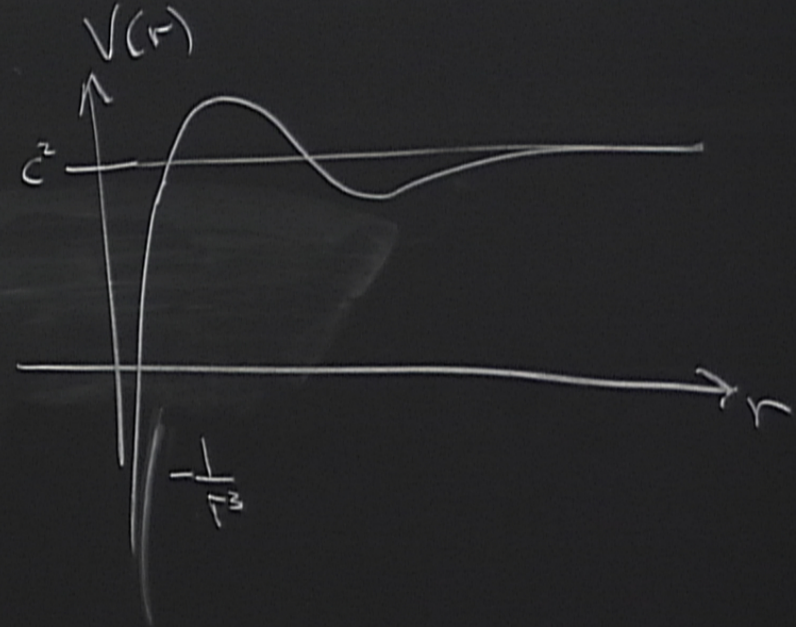
$$V(r) = c^2 \left(1 + \frac{L^2}{r^2 c^2} \right) \left(1 - 2 \frac{GM}{rc^2} \right)$$

↑
for photon, 0

$$+ V(r) = E^2$$

$$r) = c^2 \left(1 + \frac{L^2}{r^2 c^2} \right) \left(1 - 2 \frac{GM}{rc^2} \right)$$

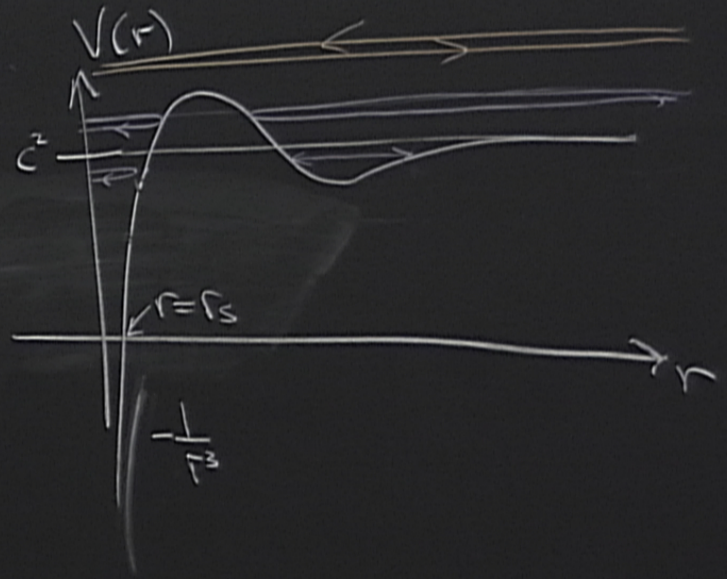
for photon, 0



$$c^2 + V(r) = E^2$$

$$V(r) = c^2 \left(1 + \frac{L^2}{r^2 c^2} \right) \left(1 - 2 \frac{GM}{rc^2} \right)$$

for photon, 0

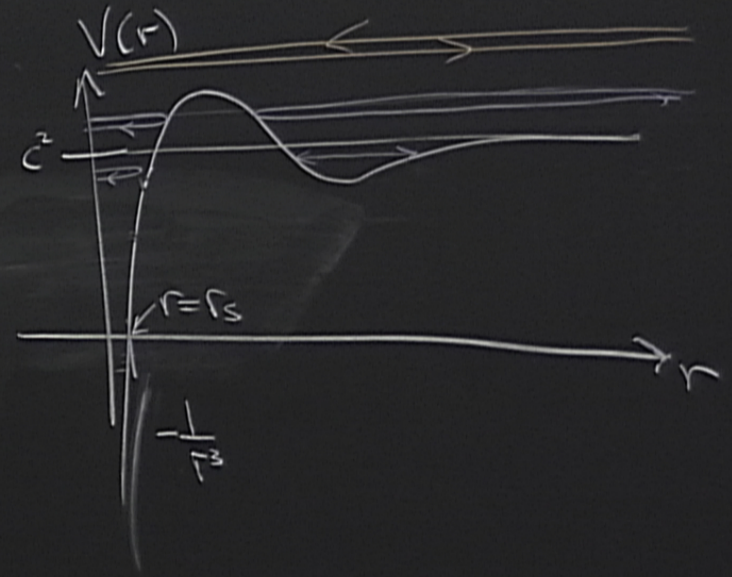


$$\dot{r}^2 + V(r) = E^2$$

$$V(r) = c^2 \left(1 + \frac{L^2}{r^2 c^2} \right) \left(1 - 2 \frac{GM}{rc^2} \right)$$

for photon, 0

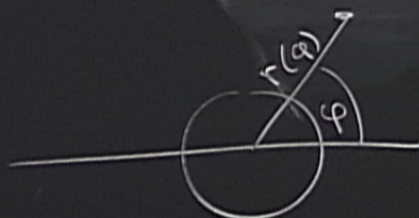
extrapolation to: $r < r_s$ looks fine!



$$\frac{d}{dt}(\overline{\partial \dot{x}^n}) = \overline{\partial x^n}$$

$\theta = \pi/2$ is a solⁿ, choose this from now on.

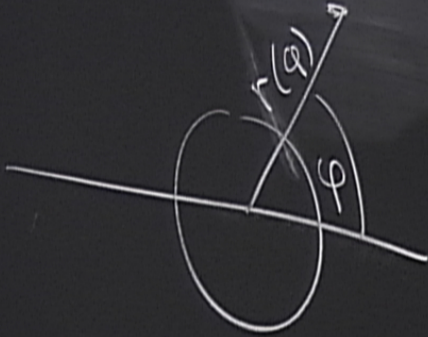
Solution: $r(t) \Rightarrow r(\varphi)$



Solution:

$$r(t) \Rightarrow r(\varphi)$$

$$' = \frac{d}{d\varphi}$$



$$L_z = r^2 \dot{\varphi}$$

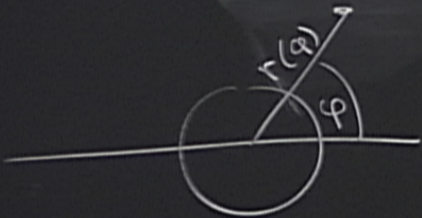
$$\dot{r}^2 = r^{1/2} \dot{\varphi}^2 = r^{1/2} \frac{L_z^2}{r^4}$$

$$d(\overline{\partial \dot{x}^n}) = \overline{\partial x^n}$$

$\theta = \pi/2$ is a solⁿ, choose this from now on.

Solution: ① $r(t) \Rightarrow r(\varphi)$

$$' = \frac{d}{d\varphi}$$



$$L_z = r^2 \dot{\varphi}$$

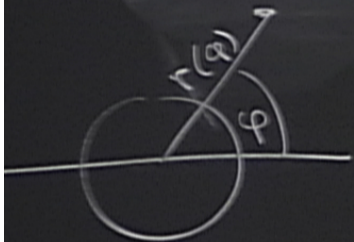
$$\dot{r}^2 = r'^2 \dot{\varphi}^2 = r'^2 \frac{L_z^2}{r^4} = L_z^2 u'^2$$

② $u = \frac{1}{r}$

$\theta = \pi/2$ is a solⁿ. choose this from now on.

Solution: ① $r(t) \Rightarrow r(\varphi)$

$$' = \frac{d}{d\varphi}$$



$$L_z = r^2 \dot{\varphi}$$

$$\dot{r}^2 = r'^2 \dot{\varphi}^2 = r'^2 \frac{L_z^2}{r^4} = L_z^2 u'^2$$

$$\textcircled{2} \quad u = \frac{1}{r}$$

$$(*) \Rightarrow L_z^2 u'^2 + c^2 \left(1 - \frac{2GMu}{c^2}\right) \left(1 + \frac{L_z^2 u^2}{c^2}\right) \Rightarrow u'^2 = \frac{E^2 - c^2}{L_z^2} - u^2 + \frac{2GMu}{L_z^2} + \frac{2GMu^3}{c^2}$$



$$\frac{d}{dq} : \Rightarrow u'' = -u + \frac{GM}{L^2} + \frac{3GM}{c^2} u^2$$

$$\frac{d}{d\varphi} : \Rightarrow u'' = -u + \frac{GM}{L^2} + \frac{3GM}{c^2} u^2$$

Solve as expⁿ in
 $\frac{1}{c^2}$

$$u = u_0 + \frac{1}{c^2} u_1 + \dots$$

$$\frac{d}{d\varphi} : \Rightarrow \underline{u'' = -u + \frac{GM}{L^2} + \frac{3GM}{c^2} u^2}$$

Solve as expⁿ in
 $\frac{1}{c^2}$

$$u = u_0 + \frac{1}{c^2} u_1 + \dots$$

$$u = \frac{GM}{L^2} + \text{const.} \cos \varphi$$

$$\frac{d}{d\varphi} : \Rightarrow \underline{u'' = -u + \frac{GM}{L^2} + \frac{3GM}{c^2} u^2}$$

solve as expⁿ in $\frac{1}{c^2}$

$$u = u_0 + \frac{1}{c^2} u_1 + \dots$$

$$u = \frac{GM}{L^2} + \text{const.} \cos \varphi \equiv \frac{GM}{L^2} (1 + e \cos \varphi)$$

ellipticity of orbit

$$\frac{d}{d\varphi} : \Rightarrow \underline{u'' = -u + \frac{GM}{L^2} + \frac{3GM}{c^2} u^2}$$

solve as expⁿ in

$$\frac{1}{c^2}$$

$$u = u_0 + \frac{1}{c^2} u_1 + \dots$$

$$u = \frac{GM}{L^2} + \text{const.} \cos \varphi \equiv \frac{GM}{L^2} (1 + e \cos \varphi)$$

circular orbit
ellipticity of orbit

$$\frac{GM}{L^2} + \frac{3GM}{c^2} u^2$$

circular orbit
 ellipticity of orbit

$$r = \frac{A}{(1 + e \cos \varphi)}$$

is an ellipse.

$$\equiv \frac{GM}{L^2} (1 + e \cos \varphi)$$

$$\frac{d}{d\varphi} : \Rightarrow u'' = -u + \frac{GM}{L^2} + \frac{3GM}{c^2} u^2$$

as expⁿ in $\frac{1}{c^2}$

$$u = u_0 + \frac{1}{c^2} u_1 + \dots$$

$$u = \frac{GM}{L^2} + \text{const.} \cos \varphi \equiv \frac{GM}{L^2} (1 + e \cos \varphi)$$

$$u'' = -u + \frac{3GM}{c^2} \frac{(GM)^2}{L^4} (1 + e \cos \varphi)^2$$

circular orbit
ellipticity of orbit

$$r = \frac{r_0}{(1 + e \cos \varphi)}$$

$$\frac{GM}{L^2} + \frac{3GM}{c^2} u^2$$

circle orbit
ellipticity of orbit

$$r = \frac{r_0}{(1 + e \cos \varphi)}$$

is an ellipse.

$$= \frac{GM}{L^2} (1 + e \cos \varphi)$$

: term that matters is $\cos \varphi$. : driving oscillator at frequency of oscillator $\Rightarrow$ resonance.

$$u_1'' = -u_1 + \epsilon \cos \varphi$$

$$\Rightarrow u_1 = \epsilon/2 \varphi \sin \varphi$$

$$u_1'' = -u_1 + \epsilon \cos \varphi$$

$$\Rightarrow u_1 = \epsilon/2 \varphi \sin \varphi$$

$$\Rightarrow u \approx \frac{GM}{L^2} \left(1 + e \cos \varphi + \frac{3(GM)^2}{L^2 c^2} e \varphi \sin \varphi \right)$$

$$\underbrace{\left(1 + e \cos \left(\varphi \left(1 - \frac{3(GM)^2}{L^2 c^2} \right) \right) \right)}$$

$$u_1'' = -u_1 + \epsilon \cos \varphi$$

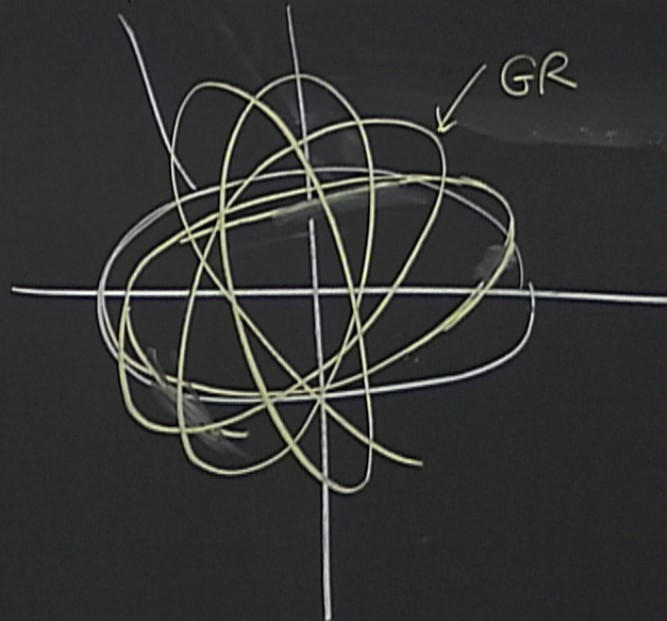
$$\Rightarrow u_1 = \epsilon/2 \varphi \sin \varphi$$

$$u \approx \frac{GM}{L^2} \left(1 + e \cos \varphi + \frac{3(GM)^2}{L^2 c^2} \varphi \sin \varphi \right)$$

$$\approx 1 + e \cos \left(\varphi \left(1 - \frac{3(GM)^2}{L^2 c^2} \right) \right)$$

as long as $\frac{3(GM)^2 e \varphi}{L^2 c^2}$

Newton $r = \frac{r_0}{1 + e \cos \phi}$



GR

u_1''

$\Rightarrow$

$\Rightarrow$

in φ of the maximum radius per orbit

$$\Delta\varphi = 6\pi \left(\frac{GM}{L^2} \right)^2 \frac{1}{c^2} = 6\pi \frac{GM}{r_0 c^2} \sim \frac{v_{\text{orbit}}^2}{c^2} \text{ largest for smallest } r_0$$

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$$\Rightarrow \text{Mercury} \Rightarrow \Delta\varphi = 42.98''/\text{century}$$

per orbit

$$= 6\pi \frac{GM}{r_0 c^2} \sim \frac{v_{\text{orbit}}^2}{c^2} \quad \text{largest for smallest } r_0$$

Mercury $\Rightarrow \Delta\phi = 42.98''/\text{century}$

c.f. observations $\Rightarrow$ anomaly $43''/\text{century}$ discovered in
the 19th century.

an radius per orbit

$$= 6\pi \frac{GM}{r_0 c^2} \sim \frac{v_{\text{orbit}}^2}{c^2} \quad \text{largest for smallest } r_0$$

Mercury $\Rightarrow \Delta\phi = 42.98''/\text{century}$

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c.f. observations $\Rightarrow$ anomaly 43''/century discovered in the 19th century.

Photons: $V(r) = \frac{L_z^2}{r^2} \left(1 - 2\frac{GM}{rc^2} \right)$

thus: $V(r) = \frac{L_z^2}{r^2} \left(1 - 2 \frac{GM}{rc^2} \right)$

$$= \frac{L_z^2}{r^2} \left(1 - \frac{r_s}{r} \right) = L_z^2 \left(\frac{1}{r^2} - \frac{r_s}{r^3} \right)$$



$$= \frac{L^2}{r^2} \left(1 - \frac{2GM}{rc^2} \right)$$

$$= \frac{L^2}{r^2} \left(1 - \frac{r_s}{r} \right) = L^2 \left(\frac{1}{r^2} - \frac{r_s}{r^3} \right)$$

$$-\frac{2}{r^3} + \frac{3r_s}{r^4} = 0$$

$$\Rightarrow r = \frac{3}{2} r_s$$

$$= \frac{3GM}{c^2}$$



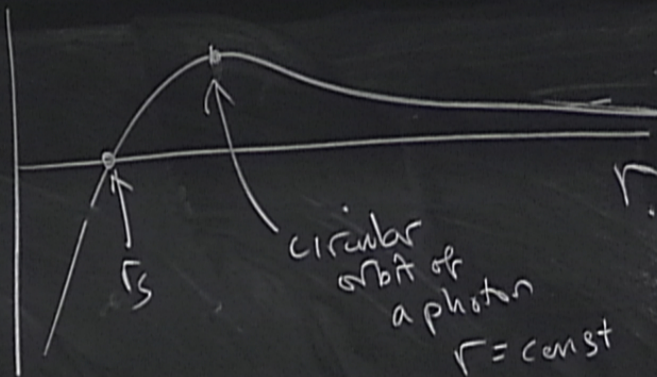
$$\frac{L^2}{r^2} \left(1 - \frac{2GM}{rc^2} \right)$$

$$\frac{L^2}{r^2} \left(1 - \frac{r_s}{r} \right) = \frac{L^2}{r^2} \left(\frac{1}{r^2} - \frac{r_s}{r^3} \right)$$

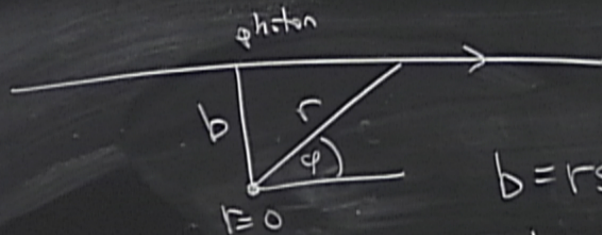
$$-\frac{2}{r^3} + \frac{3r_s}{r^4} = 0$$

$$\Rightarrow r = \frac{3}{2} r_s$$

$$= \frac{3GM}{c^2}$$



Photon: same procedure



$$b = r \sin \varphi$$
$$\Rightarrow u = \frac{1}{r} = \frac{\sin \varphi}{b}$$

$$u'' = -u + \frac{3GM}{c^2} u^2$$

$$u_0 = \frac{1}{b} \sin \varphi$$

$$-u + \frac{3GM}{c^2} u^2$$

$$\frac{1}{b} \sin \varphi \quad ; \quad u_1'' + u_1 = \frac{3GM}{b^2 c^2} \sin^2 \varphi = \frac{3GM}{2b^2 c^2} (1 - \cos 2\varphi)$$

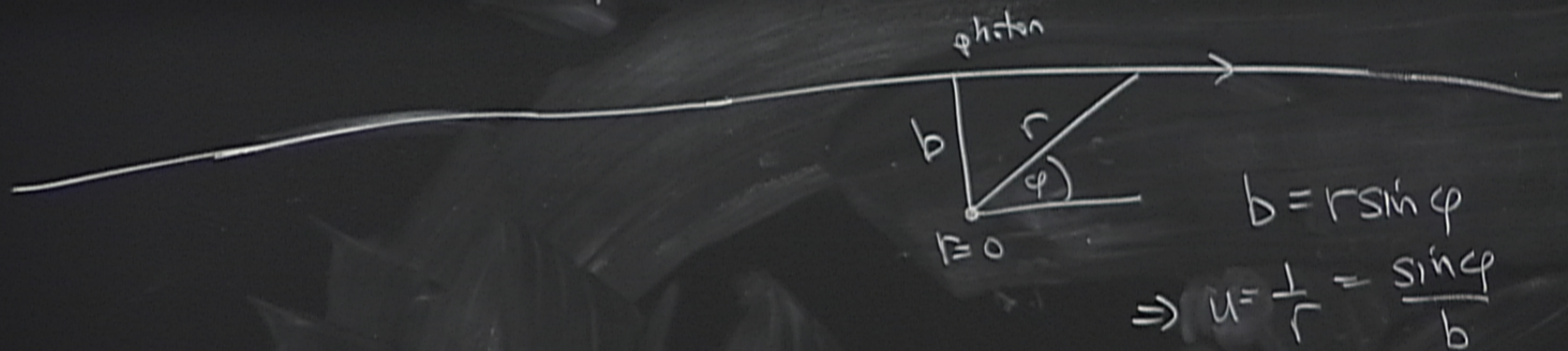
$$\Rightarrow \text{P.T.} \quad u_1 = \frac{3GM}{2b^2 c^2} (1 + \frac{1}{3} \cos 2\varphi)$$

$$u'' = -u + \frac{3GM}{c^2} u^2$$

$$u = \frac{1}{b} \sin \varphi \quad ; \quad u_1'' + u_1 = \frac{3GM}{b^2 c^2} \sin^2 \varphi = \frac{3GM}{2b^2 c^2} (1 - \cos 2\varphi)$$

$$\Rightarrow \text{P.T.} \quad u_1 = \frac{3GM}{2b^2 c^2} \left(1 + \frac{1}{3} \cos 2\varphi \right)$$

Photon: same procedure



$$u'' = -u + \frac{3GM}{c^2} u^2$$

$$u_0 = \frac{1}{b} \sin \varphi \quad ; \quad u_1'' + u_1 = \frac{3GM}{b^2 c^2} \sin^2 \varphi = \frac{3GM}{2b^2 c^2} (1 - \cos 2\varphi)$$

$$\Rightarrow \text{P.T. } u_1 = \frac{3GM}{2b^2 c^2} (1 + \frac{1}{3} \cos 2\varphi) \quad \varphi \sim \epsilon \ll 1$$

$$\text{Solution is: } u = \frac{1}{b} \sin \varphi + u_1$$

$$r = \infty : u \rightarrow 0$$

$$+u_1 = \frac{3GM}{b^2 c^2} \sin^2 \varphi = \frac{3GM}{2b^2 c^2} (1 - \cos 2\varphi)$$

$$u_1 = \frac{3GM}{2b^2 c^2} \left(1 + \underbrace{\frac{1}{3} \cos 2\varphi}_{4/3} \right) \quad \varphi \sim \epsilon \ll 1$$

$$u = \frac{1}{b} \sin \varphi + u_1 \Rightarrow 0 = \frac{\epsilon}{b} + \frac{3GM}{2b^2 c^2} \cdot \frac{4}{3}$$

$$\epsilon = - \frac{2GM}{c^2 b}$$

$$u'' = -u + \frac{3GM}{c^2} u^2$$

$$u_0 = \frac{1}{b} \sin \varphi \quad ; \quad u_1'' + u_1 = \frac{3GM}{b^2 c^2} \sin^2 \varphi = \frac{3GM}{2b^2 c^2} (1 - \cos 2\varphi)$$

$$\Rightarrow \text{P.I. } u_1 = \frac{3GM}{2b^2 c^2} \left(1 + \frac{1}{3} \cos 2\varphi \right)$$

$$\text{Solution is: } u = \frac{1}{b} \sin \varphi + u_1 \Rightarrow$$

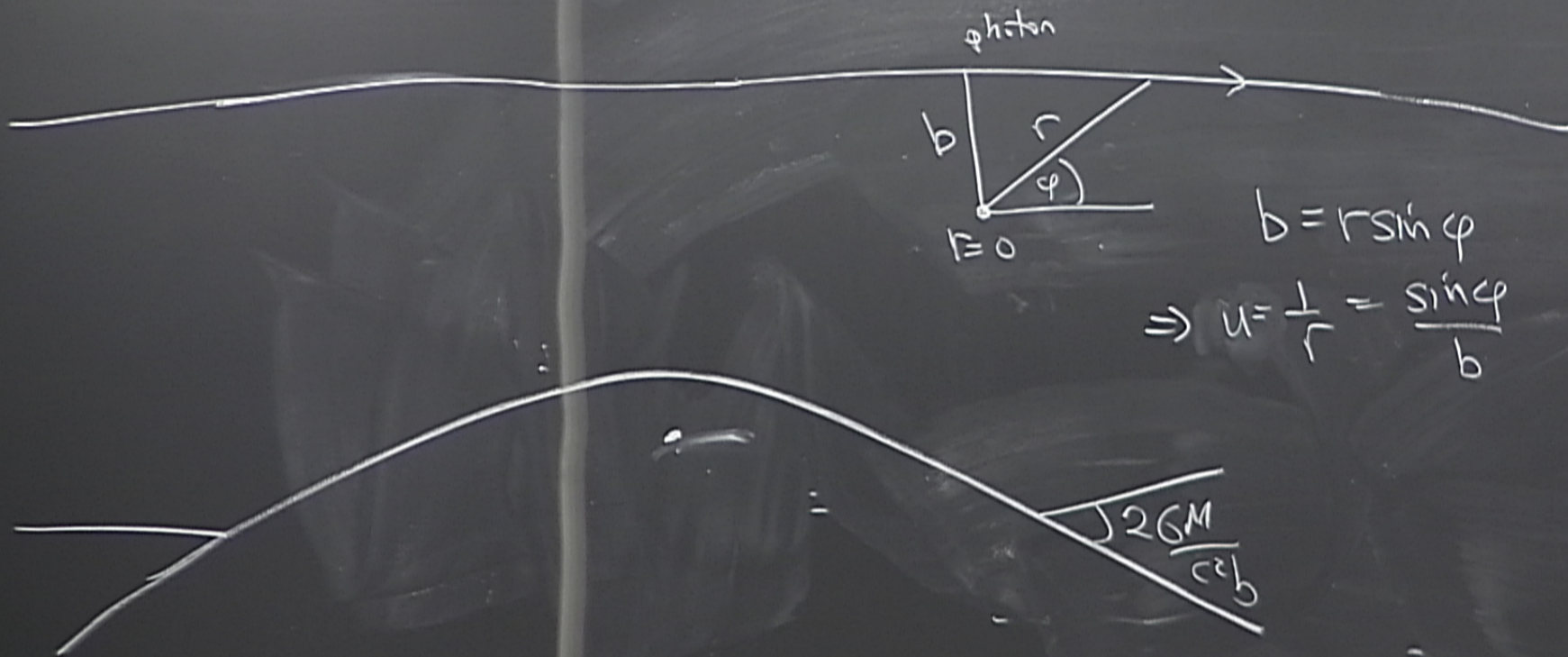
$$r = \infty : u \rightarrow 0 \quad \varphi = -\frac{2GM}{c^2 b}$$

$$\text{or } \varphi = \pi + \frac{2GM}{c^2 b}$$

Photon: same procedure

$$u'' = -$$

$$u_0 = \frac{1}{r}$$



$$b = r \sin \varphi$$

$$\Rightarrow u = \frac{1}{r} = \frac{\sin \varphi}{b}$$

$$r = \infty$$

$\Rightarrow$ bending angle $\Delta\varphi = \frac{4GM_{\odot}}{c^2 b} = 1.75''$
(assuming $b = R_{\odot}$)

$\frac{GM}{r_0 c^2} \sim \frac{1}{c^2}$ largest for smallest r_0
 $\Rightarrow \Delta\phi = 42.98''/\text{century}$
 cf. observations $\Rightarrow$ anomaly $43''/\text{century}$ discovered the 19th century.
 Nookham

