

Title: 13/14 PSI - Relativity - Lecture 12

Date: Sep 24, 2013 09:00 AM

URL: <http://pirsa.org/13090026>

Abstract:

$$-dt^2 + a^2(t) \gamma_{ij} dx^i dx^j$$

$$\left\{ \begin{array}{l} \frac{\dot{H}}{H^3} \\ \frac{\ddot{H}}{H^3} \end{array} \right\}$$

$$k = \begin{array}{l} +1 \\ 0 \\ -1 \end{array}$$

inverse
curvature radius
squared

$$\frac{1}{R_c^2} = \frac{k}{a^2}$$

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} \quad \text{Friedmann}$$

$$\left(\frac{\dot{a}}{a} \right)^2 + 2 \frac{\ddot{a}}{a} + \frac{k}{a^2} = -8\pi G p$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p)$$

Raychaudhuri

$$\dot{p} = -3 \frac{\dot{a}}{a} (p + \rho)$$

nology

$$-dt^2 + a^2(t) \delta_{ij} dx^i dx^j \quad \left. \begin{matrix} \frac{3}{H^3} \\ \frac{1}{H^3} \end{matrix} \right\} k = \begin{matrix} +1 \\ 0 \\ -1 \end{matrix} \quad \left. \begin{matrix} \text{inverse} \\ \text{curvature radius} \\ \text{squared} \end{matrix} \right\} \frac{1}{R_c^2} = \frac{k}{a^2}$$

$$G_{00} \rightarrow \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} \quad \left. \begin{matrix} \\ \end{matrix} \right\} \text{Friedmann}$$

$$G_{ij} \rightarrow \left(\frac{\dot{a}}{a} \right)^2 + 2 \frac{\ddot{a}}{a} + \frac{k}{a^2} = -8\pi G p \quad \rightarrow \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) \quad \text{Raychaudhuri}$$

$$\nabla_\mu T^{\mu\nu} = 0 \rightarrow \dot{\rho} = -3 \frac{\dot{a}}{a} (\rho + p)$$

$$-dt^2 + a^2(t) \gamma_{ij} dx^i dx^j \quad \left. \begin{matrix} S^3 \\ E^3 \\ H^3 \end{matrix} \right\} k = \begin{matrix} +1 \\ 0 \\ -1 \end{matrix} \quad \left. \begin{matrix} \text{inverse} \\ \text{curvature radius} \\ \text{squared} \end{matrix} \right\} \frac{1}{R_c^2} = \frac{k}{a^2}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad \left. \begin{matrix} \text{Friedmann} \\ \text{Raychaudhuri} \end{matrix} \right\} \text{(only single time derivatives: constraint)}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} + \frac{k}{a^2} = -8\pi G p \quad \Rightarrow \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$$\Rightarrow \dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$$

Cosmology

$$-dt^2 + a^2(t) \underbrace{\gamma_{ij} dx^i dx^j}_{\left\{ \frac{3}{H^3} \right\} k =}$$

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$$

:

$$\begin{aligned} G_{00} &\rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} \\ ij &\rightarrow \left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} + \frac{k}{a^2} = -8\pi G p \end{aligned} \left. \vphantom{\begin{aligned} G_{00} &\rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} \\ ij &\rightarrow \left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} + \frac{k}{a^2} = -8\pi G p \end{aligned}} \right\} \text{Friedmann} \rightarrow \frac{\ddot{a}}{a} =$$

$$\nabla_\mu T^{\mu\nu} = 0 \rightarrow \dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$$

$$\int (t) \delta_{ij} dx^i dx^j \quad \left. \begin{matrix} \frac{3}{H^3} \\ \frac{E^3}{H^3} \end{matrix} \right\} k = \begin{matrix} +1 \\ 0 \\ -1 \end{matrix} \quad \left. \begin{matrix} \text{inverse} \\ \text{curvature radius} \\ \text{squared} \end{matrix} \right\} \frac{1}{R_c^2} = \frac{k}{a^2}$$

$$\rho - \frac{k}{a^2} \quad \left. \begin{matrix} \text{Friedmann} \\ + \frac{k}{a^2} = -8\pi G\rho \end{matrix} \right\} \text{ (only single time derivatives: constraint)}$$

$$\Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \quad \left. \begin{matrix} \text{Raychaudhuri} \\ \text{(evolution equation: second time derivatives)} \end{matrix} \right\}$$

$$3\frac{\dot{a}}{a}(\rho + p)$$

Cosmology

$$-dt^2 + a^2(t) \delta_{ij} dx^i dx^j$$

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$$

$$G_{00} \rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2}$$

$$ij \rightarrow \left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} + \frac{k}{a^2} = -8\pi p$$

G_{00} eqⁿs constraints
 G_{0i}

G_{ij} eqⁿs evolution
 eqⁿs

$$\nabla_\mu T^{\mu\nu} = 0 \rightarrow \dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$$

For simple types of perfect fluids, $p = p(\rho)$

types of perfect fluids, $p = p(\rho)$

"dust" (pressureless fluid) $p = 0$: $\dot{\rho} = -3\frac{\dot{a}}{a}\rho \Rightarrow \rho = \frac{C_m}{a^3}$

$$\rho = \rho(p)$$

$$p = 0$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho \Rightarrow \rho = \frac{C_m}{a^3}$$

non-relativistic
matter

i.e. gas of particles w/ $|v| \ll c$.

$$\rho = \rho(p)$$

$$p = 0$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho \Rightarrow \rho = \frac{C_m}{a^3}$$

non-relativistic
matter

i.e. gas of particles w/ $|v| \ll c$.

total mass in a comoving volume
in x

$$\rho = \rho(p)$$

$$p = 0$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho \Rightarrow \rho = \frac{C_m}{a^3}$$

non-relativistic
matter

i.e. gas of particles w/ $|v| \ll c$.

total mass in a comoving volume $\propto \rho a^3 = \text{const.}$
in x

For simple types of perfect fluids, $p = p(\rho)$

e.g. "dust" (pressureless fluid) $p = 0$

e.g. "cold dark matter"

For simple types of perfect fluids, $p = p(\rho)$

e.g. "dust" (pressureless fluid) $p = 0$: $\dot{\rho} = -3\frac{\dot{a}}{a}\rho \Rightarrow$

e.g. "cold dark matter"

total mass in a

"radiation" e.g. photons or massless neutrinos, $p = \frac{1}{3}\rho$

at fluids, $p = p(\rho)$

massless fluid) $p = 0$

dark matter

photons or massless neutrinos, $p = \frac{1}{3} \rho$

$$\dot{\rho} = -3 \frac{\dot{a}}{a} \rho \Rightarrow \rho = \frac{C_m}{a^3}$$

non-relativistic matter
i.e. gas

total mass in a comoving volume $\propto \rho$
stat. mech.
 $\rho \propto T^4$
 $p = \frac{1}{3} \rho$
in $\underline{x}$

perfect fluids, $p = p(\rho)$

(pressureless fluid) $p = 0$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho \Rightarrow \rho = \frac{C_m}{a^3}$$

e.g. "cold dark matter"

"relativistic" e.g. photons or massless neutrinos, $p = \frac{1}{3}\rho$

total mass in a comoving volume
statically in x

$$\rho \propto T^4$$
$$p = \frac{1}{3}\rho$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{1}{3}\rho\right) = -4\frac{\dot{a}}{a}\rho \Rightarrow \rho = \frac{C_r}{a^4}$$

extra $\frac{1}{a}$: Energy of each photon

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho \Rightarrow \rho = \frac{C_m}{a^3}$$

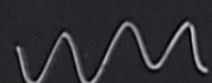
non-relativistic matter
i.e. gas of particles w/ $|v| \ll c$.

total mass in a comoving volume $\propto \rho a^3 = \text{const.}$
 stat. mech.
 in x

$$\rho \propto T^4$$

$$P = \frac{1}{3}\rho$$

$\frac{1}{a}$: Energy of each photon decreases as $\frac{1}{a}$ as universe expands

$p = \frac{1}{3} \rho$
Energy of each photon decreases as $\frac{1}{a}$ as universe expands
 $E = h\nu = \frac{hc}{\lambda}$  wavelength stretched by expansion
 $\lambda \propto a$.

$$g_{\mu\nu} \rightarrow \Omega^2(x) g_{\mu\nu}$$


 earth


 star


 $\propto \frac{1}{a}$

$$E = h\nu = h \frac{c}{\lambda}$$

VVVV

wavelength stre
 $\lambda \propto a$.

Lorentz invariant: $p = -p$

$$\Rightarrow \dot{p} = 0 \Rightarrow p = c_A, \text{ a constant.}$$

$$E = h\nu = h\frac{c}{\lambda} \quad \lambda \propto a.$$

Lorentz invariant: $\dot{p} = -\rho$

$$\Rightarrow \dot{\rho} = 0 \Rightarrow \rho = c_\Lambda, \text{ a constant.}$$

$$\Rightarrow \text{Friedmann: assume } k=0 \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} c_\Lambda \Rightarrow \underline{a \propto e^{H_\Lambda t}}, \quad H_\Lambda^2 = \frac{8\pi G}{3} c_\Lambda$$

dark energy¹⁾
cosmological constant¹⁾

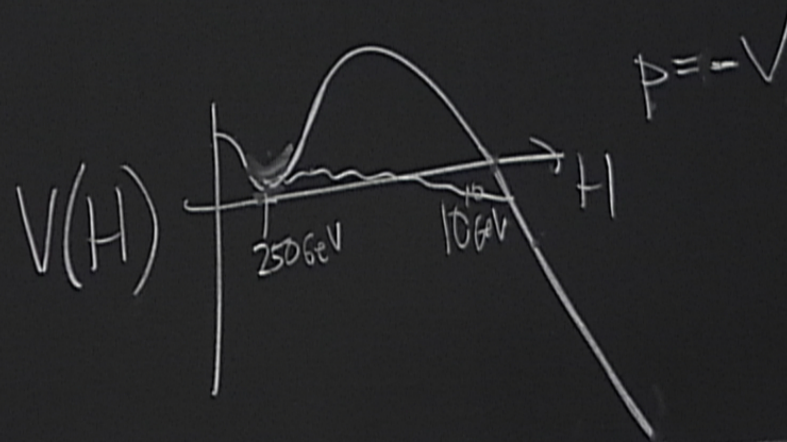
Lorentz invariant: $p = -\rho$

$$\Rightarrow \dot{\rho} = 0 \Rightarrow \rho = \underline{\underline{c_\Lambda}}, \text{ a const}$$

* many contributions

vac flucⁿs
gluon condensate
Higgs field H ,

$$\Rightarrow \text{Friedmann: assume } k=0 \quad \left(\frac{\ddot{a}}{a}\right)^2 = \frac{8\pi G}{3} c_\Lambda \Rightarrow \underline{\underline{a \propto}}$$



$$g_{\mu\nu} \rightarrow \Omega^2(x) g_{\mu\nu}$$

~ (C) star

$$\propto \frac{1}{a}$$

"dark energy"

"cosmological constant"

$$T_{\alpha\beta} = \Lambda g_{\alpha\beta}$$

* many contributions

vac flucⁿs

gluon condensate

Higgs field H , $V(H)$

Lorentz inv

$\Rightarrow$ Friedmann

250G

General solution with matter, radiation and Λ

Friedmann
$$\ddot{a}^2 - \frac{8\pi G}{3} \underbrace{(\rho_m + \rho_r + \rho_\Lambda)}_{\frac{C_m}{a} + \frac{C_r}{a^2} + \rho_\Lambda a^2} a^2 = -k$$

cf.
$$\frac{1}{2} m \dot{x}^2 + V(x) = E$$

General solution with matter, radiation and Λ

Friedmann
$$\dot{a}^2 - \frac{8\pi G}{3} \underbrace{\left(\rho_m + \rho_r + \rho_\Lambda \right)}_{\frac{C_m}{a} + \frac{C_r}{a^2} + \rho_\Lambda a^2} a^2 = -k$$

c.f.
$$\frac{1}{2} m \dot{x}^2 + V(x) = E$$

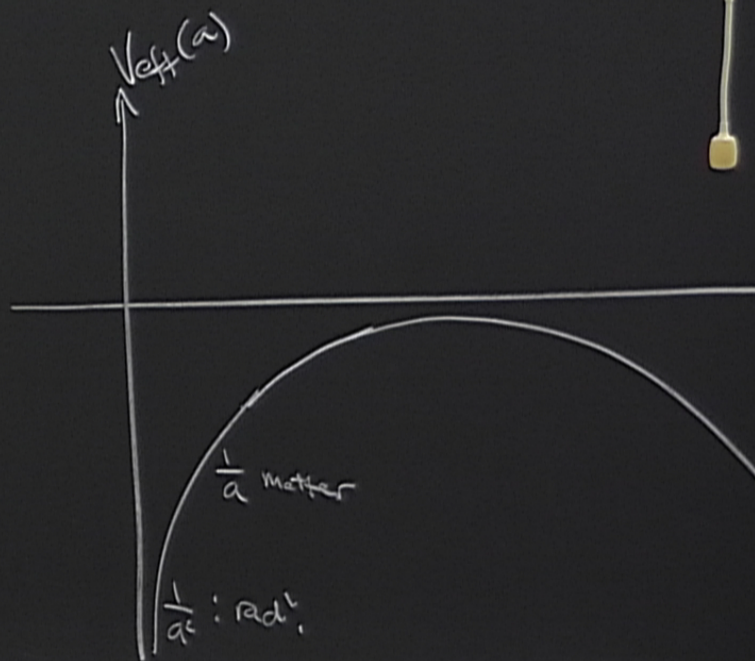
Here, $m=2$.
$$V_{\text{eff}}(a) = -\frac{8\pi G}{3} \left(\frac{C_m}{a} + \frac{C_r}{a^2} + C_\Lambda a^2 \right)$$

solution with matter, radiation and Λ

$$-\frac{8\pi G}{3} \underbrace{(\rho_m + \rho_r + \rho_\Lambda)}_{\frac{C_m}{a} + \frac{C_r}{a^2} + \rho_\Lambda a^2} a^2 = -k$$

$$\frac{1}{2} m \dot{x}^2 + V(x) = E$$

Here, $m=2$.
$$V_{\text{eff}}(a) = -\frac{8\pi G}{3} \left(\frac{C_m}{a} + \frac{C_r}{a^2} + C_\Lambda a^2 \right)$$

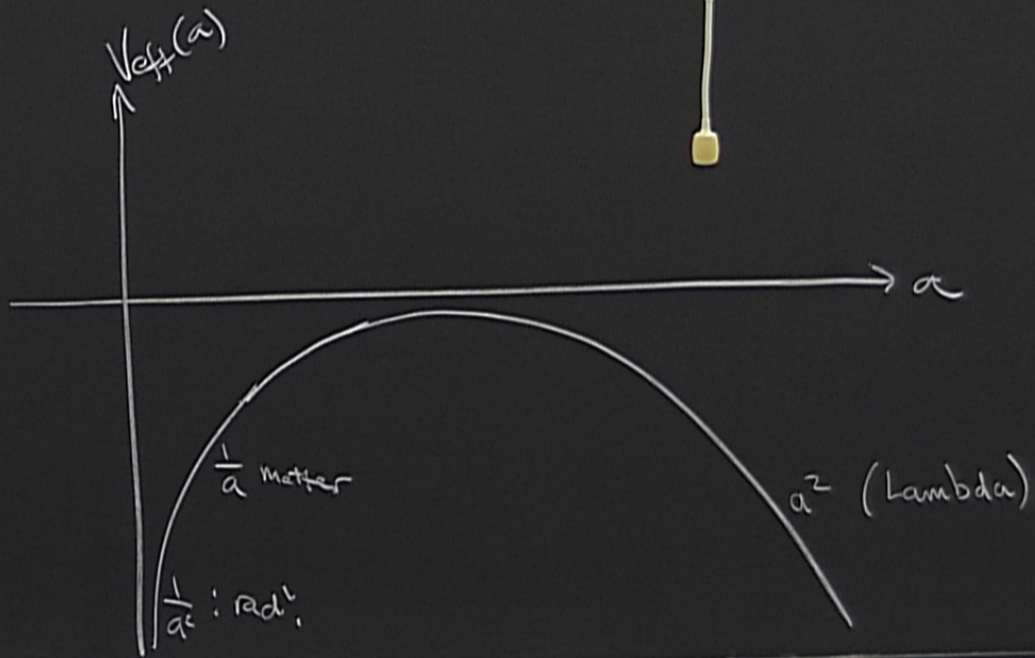


er, radiation and Λ

$$a^2 = -k$$

$$a^2$$

$$-\frac{8\pi G}{3} \left(\frac{c_m}{a} + \frac{c_r}{a^2} + c_\Lambda a^2 \right)$$



invariant: $p = -\rho$

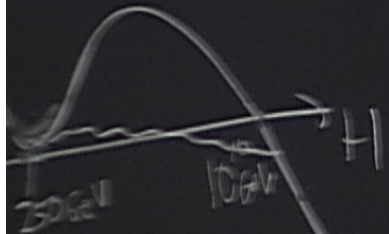
$$\Rightarrow \dot{\rho} = 0 \Rightarrow \rho = \underline{\underline{c_\Lambda}}, \text{ a constant.}$$

de Sitter spacetime

assum: assume $k=0$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} c_\Lambda \Rightarrow \underline{a \propto e^{H_\Lambda t}}, \quad H_\Lambda^2 = \frac{8\pi G}{3} c_\Lambda$$

$$p = -V$$



invariant: $\dot{p} = -\rho$

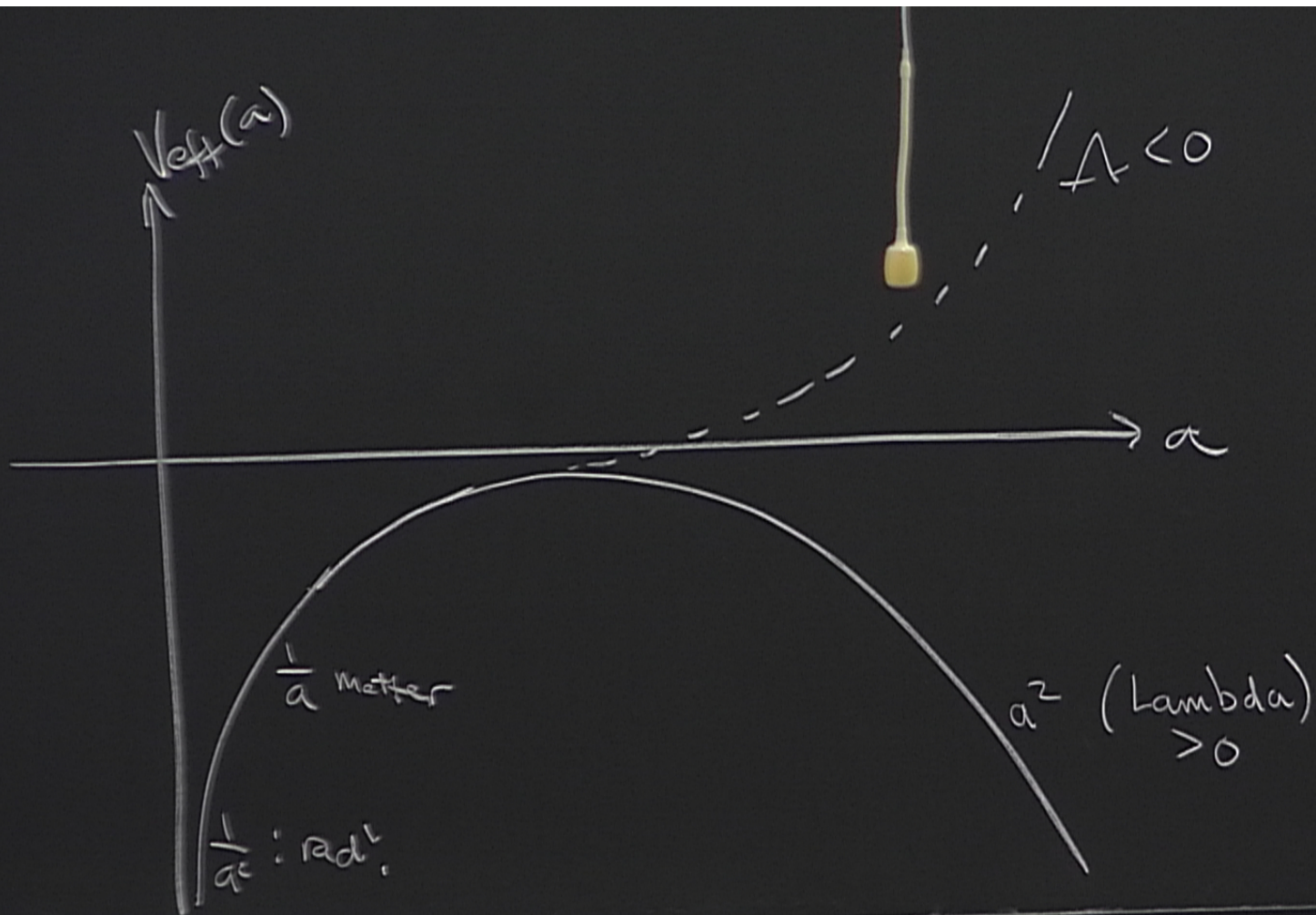
$$\Rightarrow \dot{\rho} = 0 \Rightarrow \rho = \underline{\underline{c_\Lambda}}, \text{ a constant.}$$

$\Lambda > 0$
de Sitter spacetime
 $\Lambda < 0$
anti-de Sitter spacetime

ann: assume $k=0$ $\left(\frac{\ddot{a}}{a}\right)^2 = \frac{8\pi G}{3} c_\Lambda \Rightarrow \underline{a \propto e^{H_\Lambda t}}, H_\Lambda^2 = \frac{8\pi G}{3} c_\Lambda$

$$p = -V$$





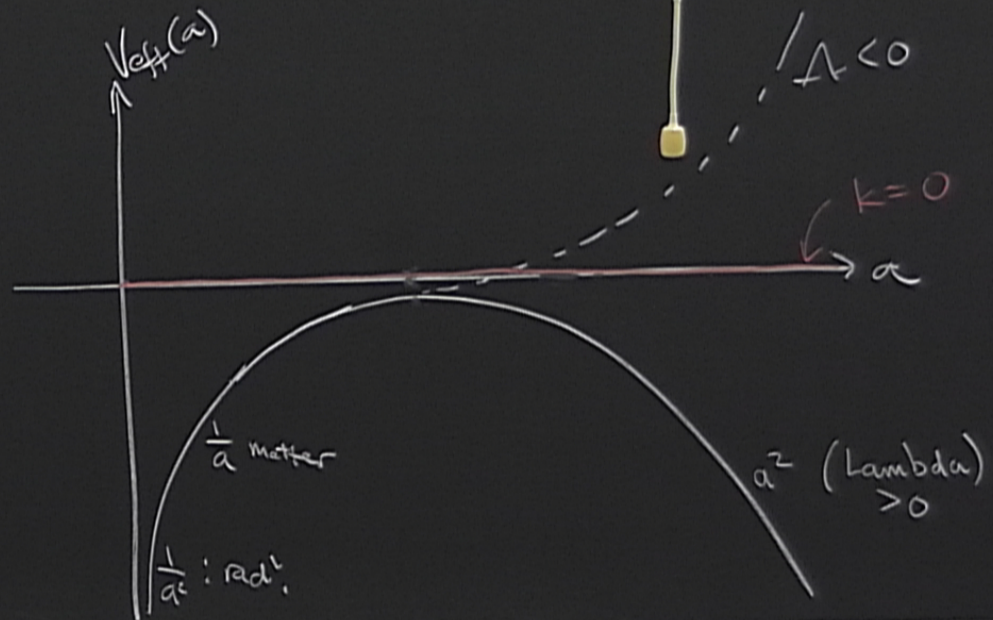
with matter, radiation and Λ

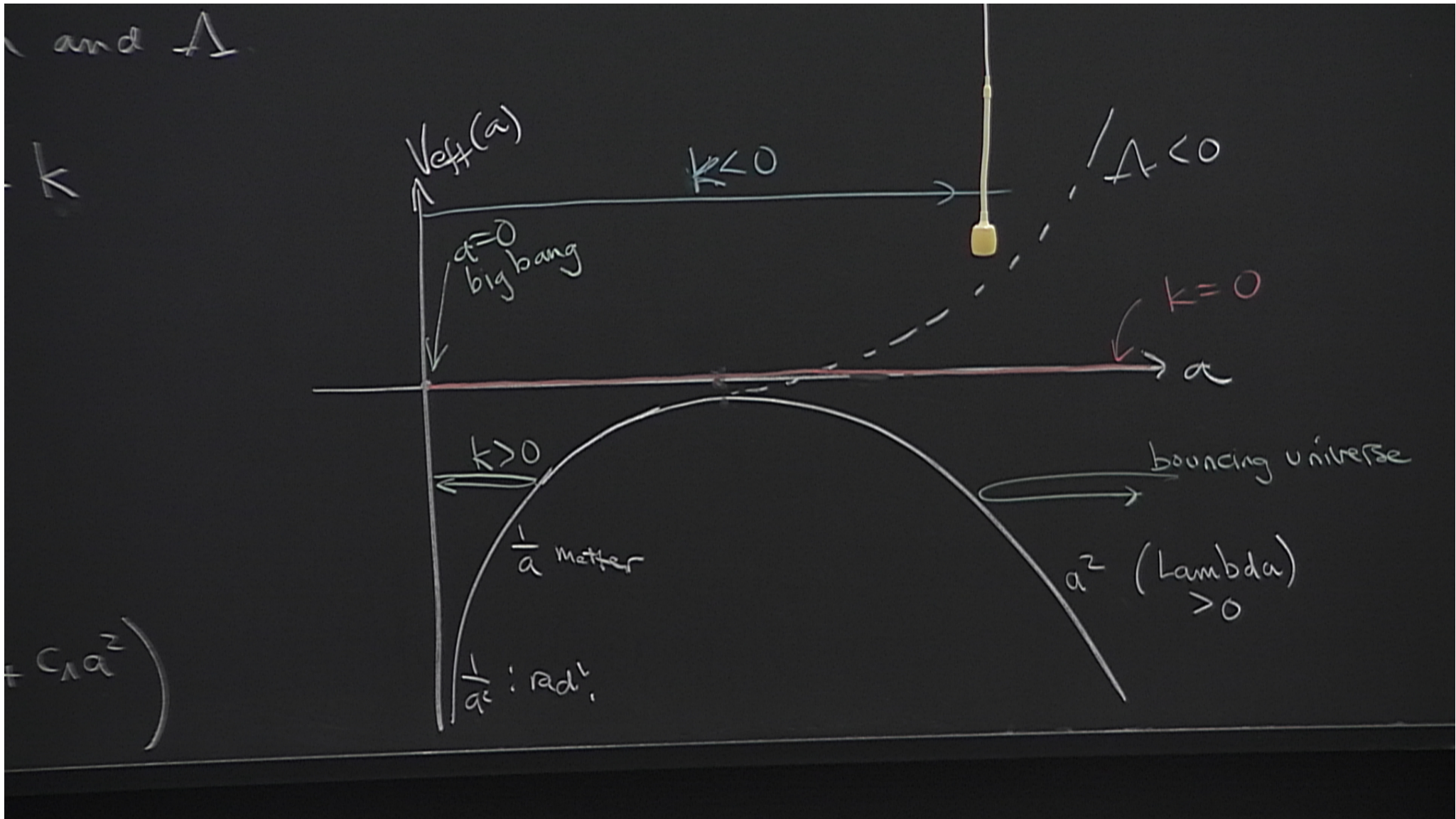
$$\left(\frac{\rho_m}{a^3} + \frac{\rho_r}{a^4} + \rho_\Lambda \right) a^2 = -k$$

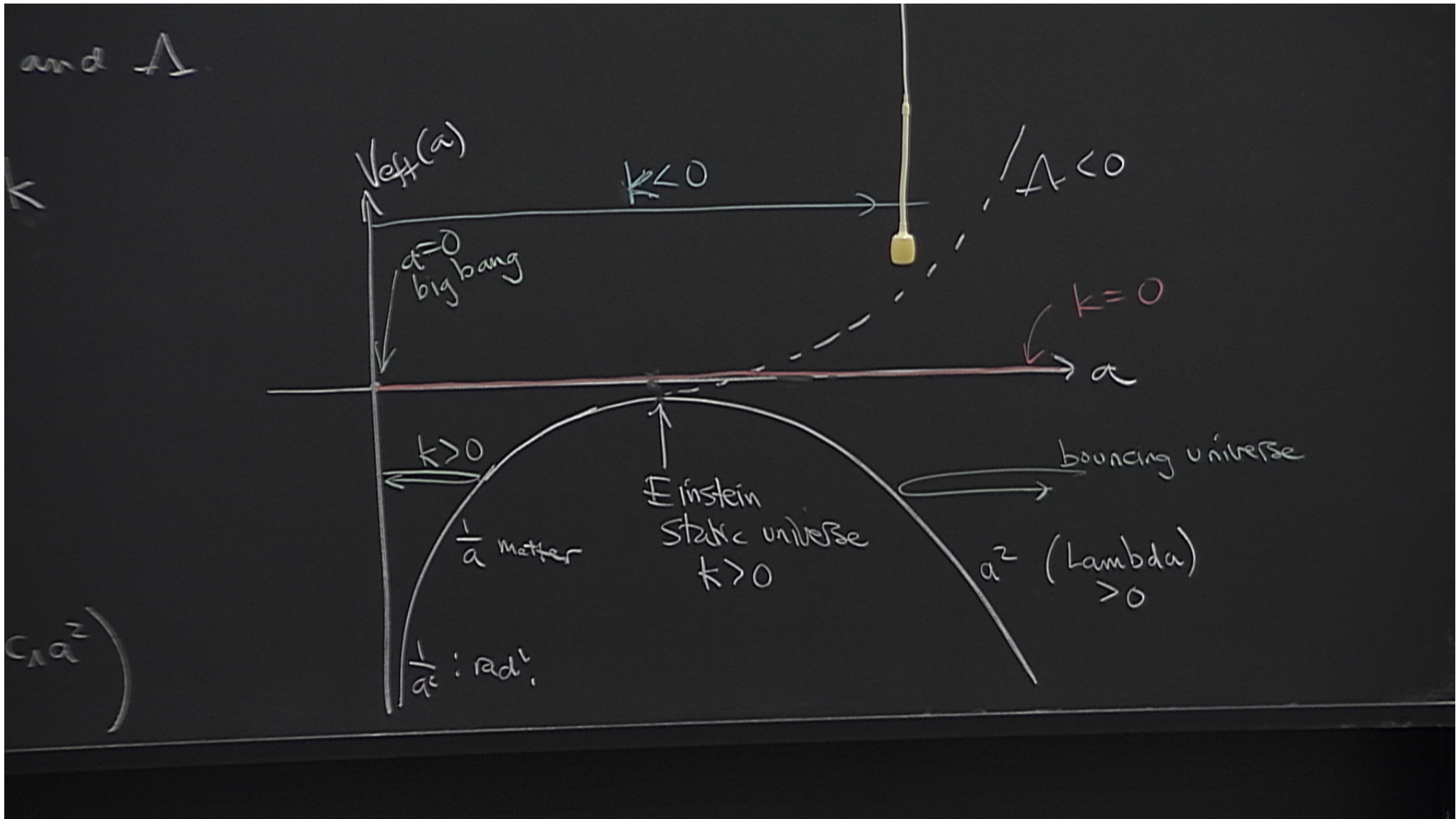
$$\frac{\rho_m}{a^3} + \frac{\rho_r}{a^4} + \rho_\Lambda a^2$$

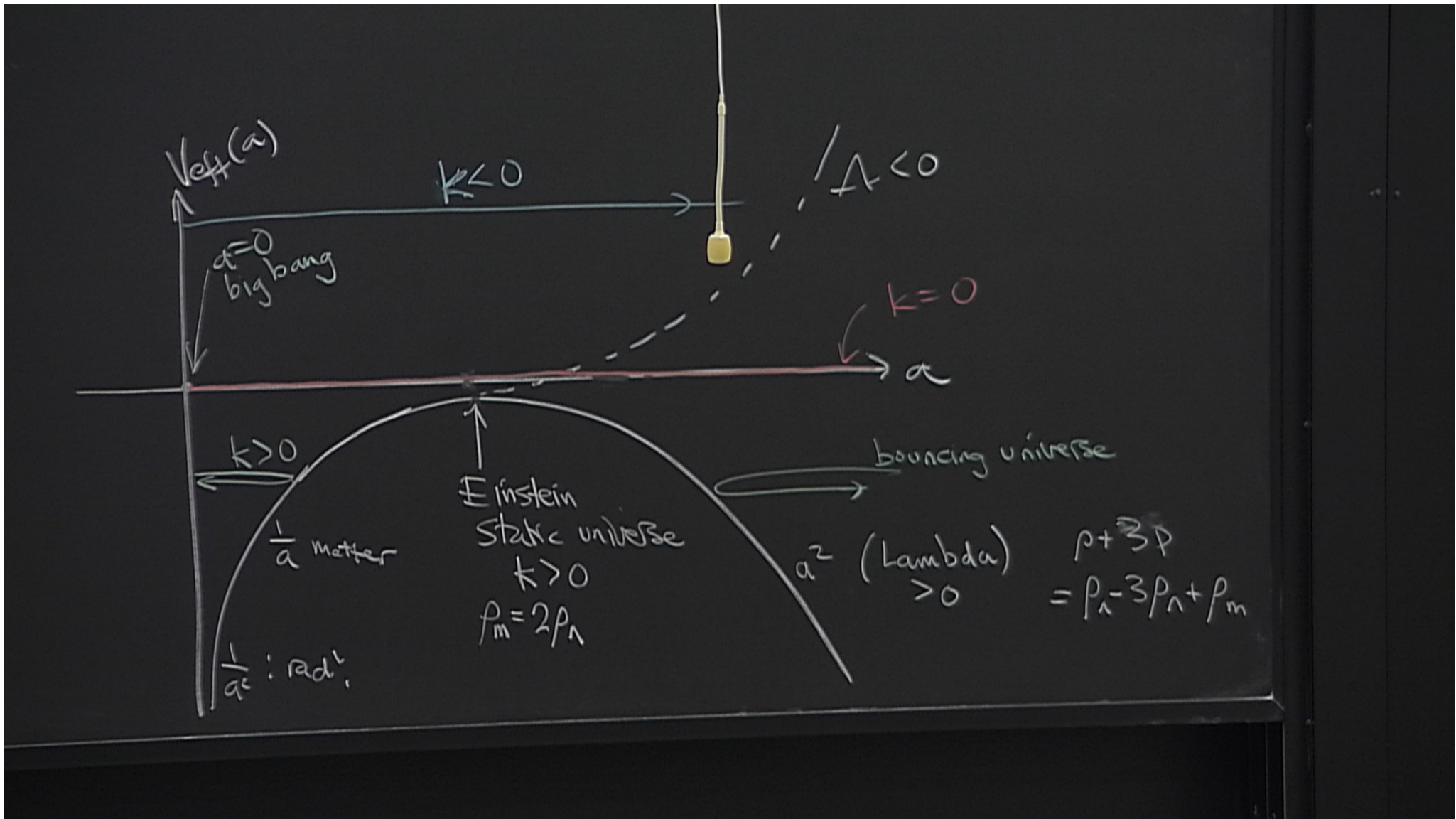
$$= E$$

$$V_{\text{eff}}(a) = -\frac{8\pi G}{3} \left(\frac{c_m}{a} + \frac{c_r}{a^2} + c_\Lambda a^2 \right)$$









Static universe

$$k > 0$$

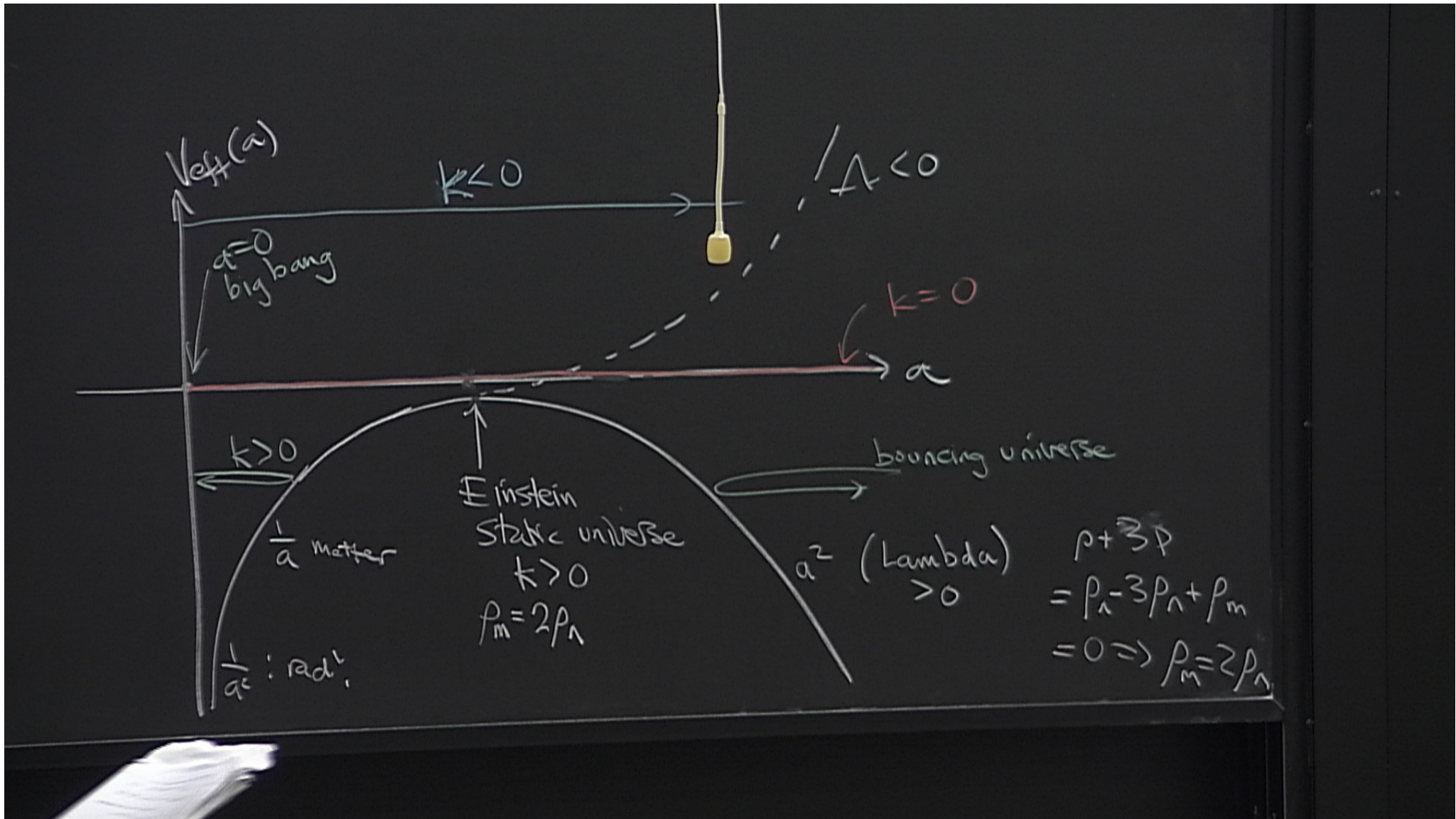
$$\rho_m = 2\rho_\Lambda$$

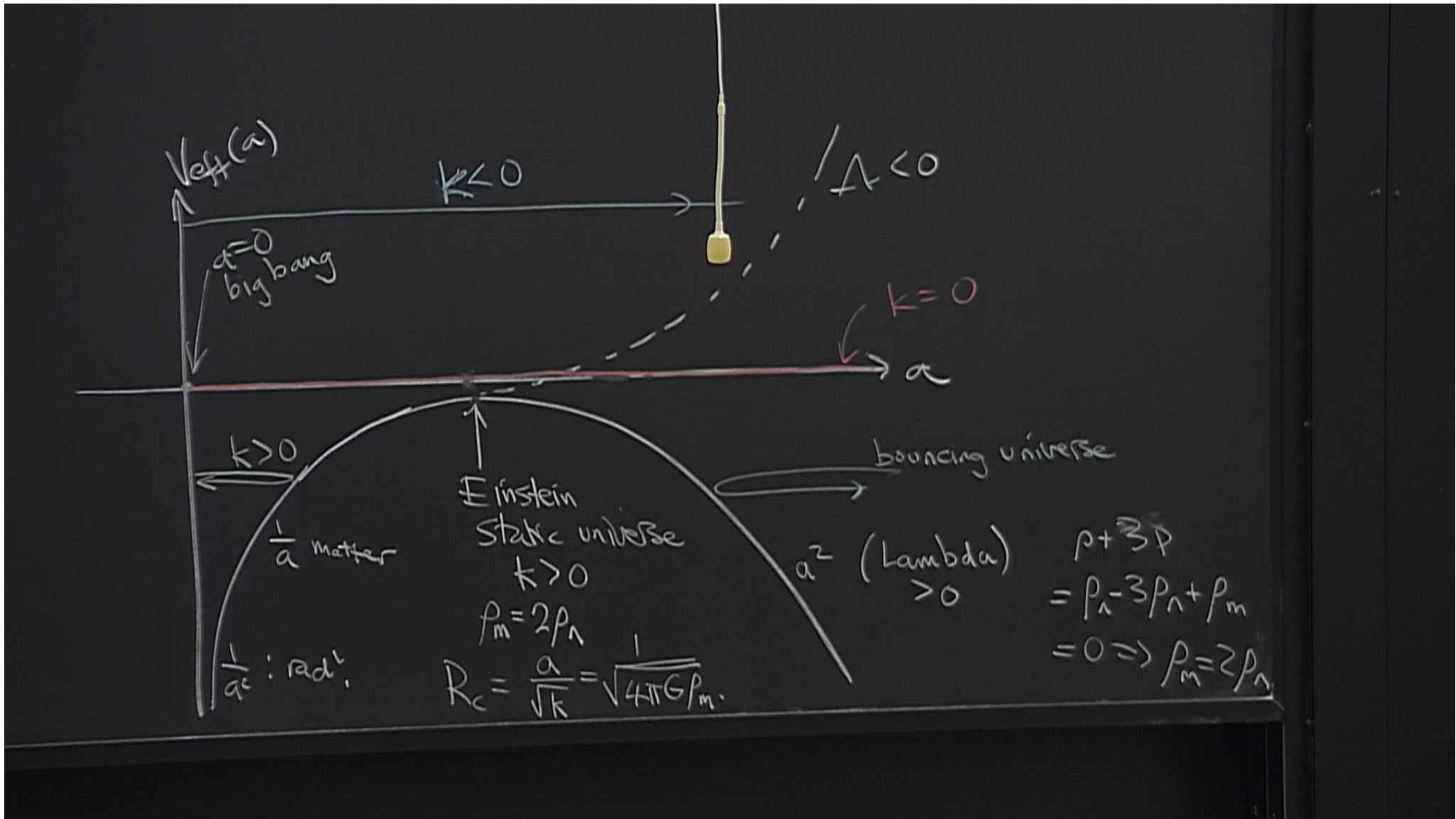
a^2 (Lambda)
 > 0

$$\rho + 3p$$

$$= \rho_\Lambda - 3\rho_\Lambda + \rho_m$$

$$= 0 \Rightarrow \rho_m = 2\rho_\Lambda$$

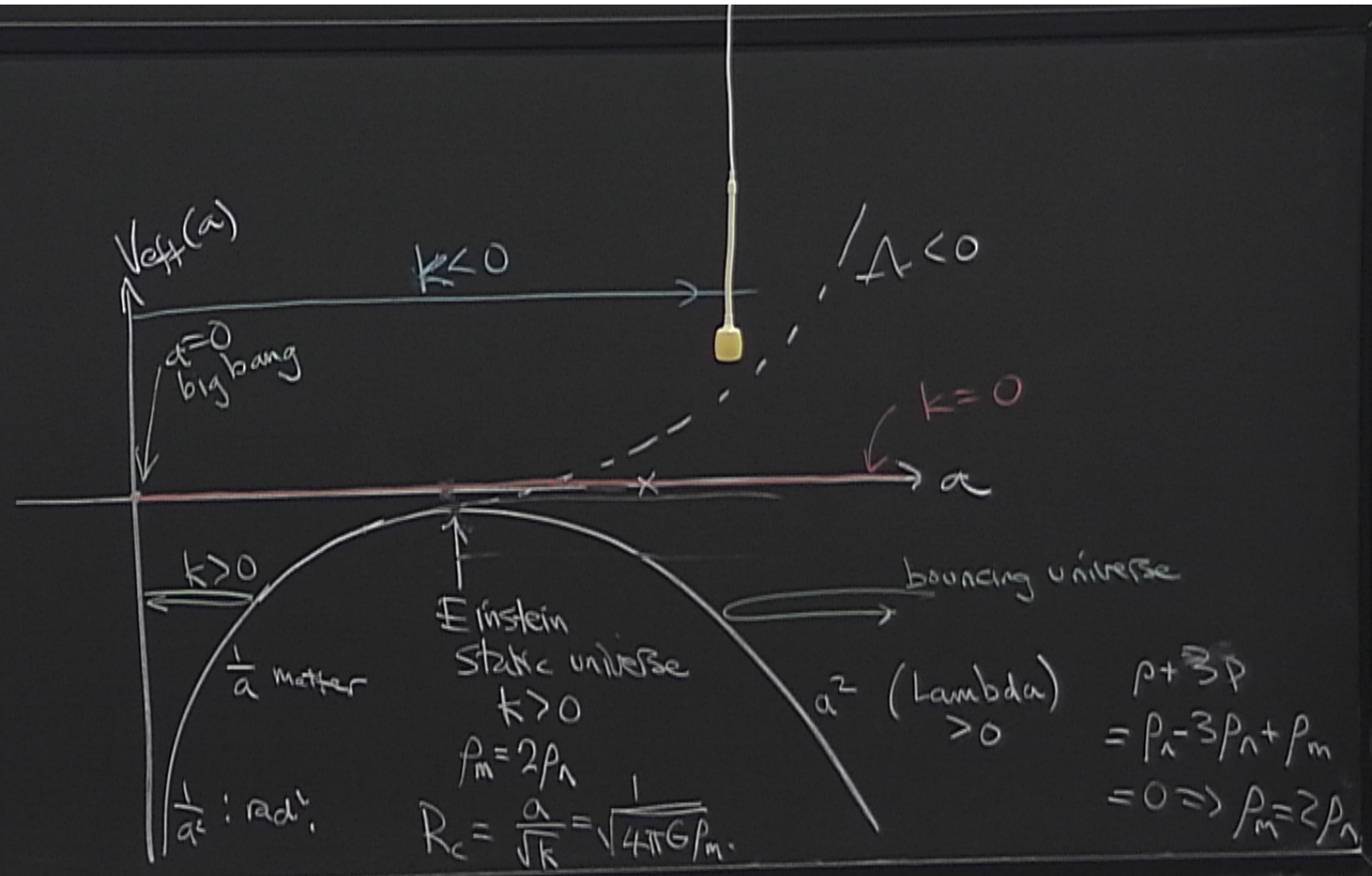




radiation and Λ

$$= -k$$

$$\left(\frac{\rho}{a} + \frac{C_1}{a^2} + C_2 a^2 \right)$$



Compare with observations:

$$\underbrace{\left(\frac{\ddot{a}}{a}\right)^2}_{H^2} - \frac{8\pi G}{3} (\rho_m + \rho_r + \rho_\Lambda) + \frac{k}{a^2} = 0$$

H = Hubble const ; consider particles / galaxies / stars at fixed x .

$$d^2 = \vec{a}^2(t) (\Delta x)^2$$

geodesic eqⁿ) Separation of points at fixed x grows like $a(t)$

compare with observations:

$$\underbrace{\left(\frac{\dot{a}}{a}\right)^2}_{H^2} - \frac{8\pi G}{3} (\rho_m + \rho_r + \rho_\Lambda) + \frac{k}{a^2} = 0$$

ble const, : consider particles / galaxies / stars at fixed $\underline{x}$. (s

$$\frac{\dot{d}}{d} = \frac{\dot{a}}{a} \Rightarrow \dot{d} = H d$$

ations:

$$\frac{8\pi G}{3}(\rho_m + \rho_r + \rho_\Lambda) + \frac{k}{a^2} = 0$$

consider particles/galaxies/stars at fixed $\underline{x}$. (solⁿ of geodesic eqⁿ) Separat

$$\frac{\dot{d}}{d} = \frac{\dot{a}}{a} \Rightarrow \dot{d} = H d \quad \text{recession speed is proportional to distance}$$

$$\frac{\pi G}{3} (\rho_m + \rho_r + \rho_\Lambda) + \frac{k}{a^2} = 0$$

consider particles / galaxies / stars at fixed x . (solⁿ of geodesic eqⁿ) Separat

$$\frac{\dot{d}}{d} = \frac{\dot{a}}{a} \Rightarrow \dot{d} = H d \quad \text{recession speed is proportional to distance}$$

Hubble's law

H^2

H = Hubble constant ; consider particles / galaxies / stars at fixed
parameter

- function of time !

$$\frac{\dot{d}}{d} = \frac{\dot{a}(t)}{a} \Rightarrow \dot{d} = H d$$

$d(t), a(t)$

$d(t), a(t)$

Hubble's law

Hubble measured $H = 550 \text{ km s}^{-1} \text{ Mpc}^{-1}$

$1 \text{ Mpc} = 10^6 \text{ pc}$, 1 pc

actual value measured today $\approx 65 \pm 10 \text{ km s}^{-1} \text{ Mpc}^{-1}$

Compare with observations:

Defn
 $H = \frac{\dot{a}}{a}$
can be pos or negative

$$\underbrace{\left(\frac{\dot{a}}{a}\right)^2}_{H^2}$$

$$- \frac{8\pi G}{3} (\rho_m + \rho_r + \dots)$$

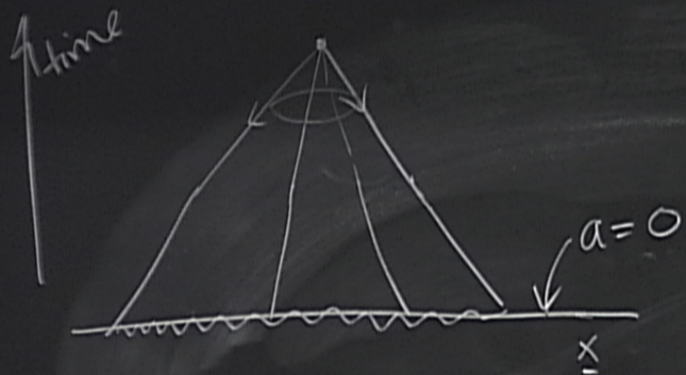
H = Hubble constant ;
parameter

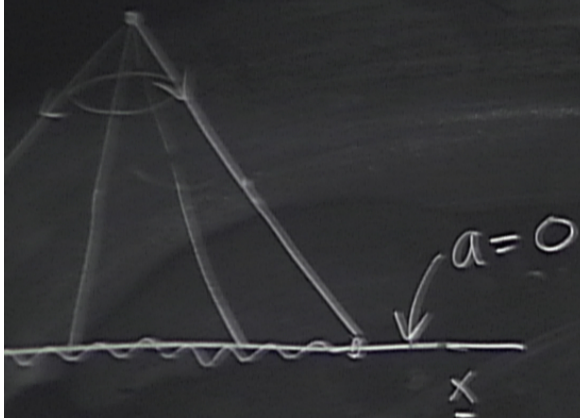
consider particles /

- function of time !

$$\frac{\dot{d}}{d} = \frac{\dot{a}(t)}{a(t)} = H(t)$$

d





$$-dt^2 + a^2(t) \gamma_{ij} dx^i dx^j$$

define new time
 $dt = a(t) d\eta$

$$\eta = \int \frac{dt}{a(t)}$$

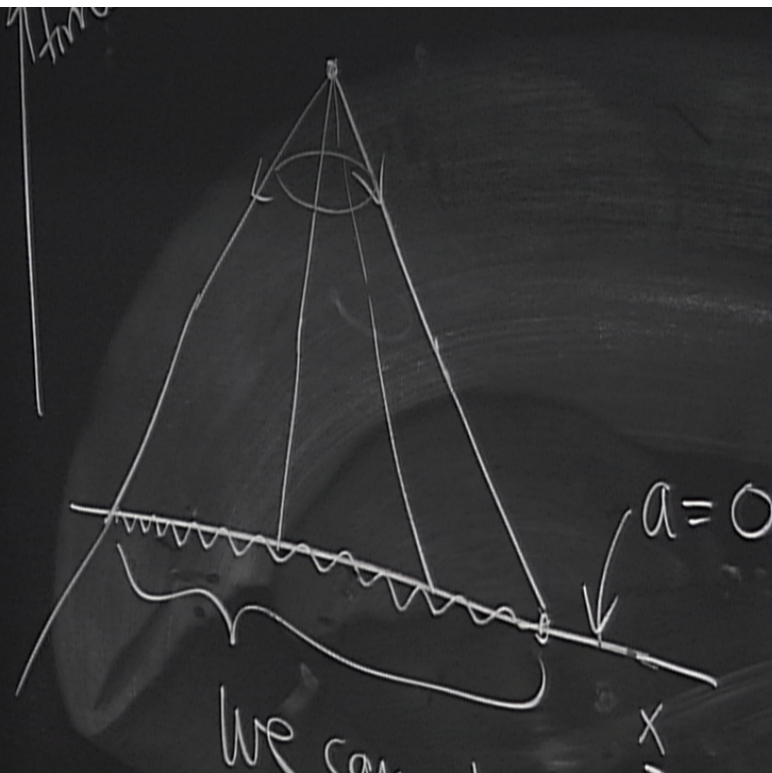
$$a^2(\eta) (-d\eta^2 + \gamma_{ij} dx^i dx^j)$$

invisible
to photons

no singularity!

g universe

$$\begin{aligned} \text{da)} \quad & \rho + 3p \\ &= \rho_n - 3\rho_n + \rho_m \\ &= 0 \Rightarrow \rho = 2\rho_n \end{aligned}$$



We can only
see this part
of the universe

define

$$dt =$$

$$\eta =$$

Hubble measured $H = +550 \text{ km s}^{-1} \text{ Mpc}^{-1}$

$$1 \text{ Mpc} = 10^6 \text{ pc}, \quad 1 \text{ pc} \approx 3.26 \times 10^3 \text{ ly}$$

actual value measured today $\approx 65 \pm 10 \text{ km s}^{-1} \text{ Mpc}^{-1}$

conventional to rewrite (*) as follows: divide by H^2 , define $\frac{8\pi G \rho_m}{3H^2} \equiv \Omega_m$

Hubble measured $H = +550 \text{ km s}^{-1} \text{ Mpc}^{-1}$

$$1 \text{ Mpc} = 10^6 \text{ pc}, \quad 1 \text{ pc} \approx 3.26 \times 10^3 \text{ ly}$$

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$$1 - (\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k) = 0$$

Hubble measured $H = +550 \text{ km s}^{-1} \text{ Mpc}^{-1}$ $1 \text{ Mpc} = 10^6 \text{ pc}$, 1 pc

actual value measured today $\approx 65 \pm 10 \text{ km s}^{-1} \text{ Mpc}^{-1}$

conventional to rewrite (*) as follows: divide by H^2 , define $\frac{8\pi G \rho_m}{3H^2} =$

$$1 - (\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k) = 0$$

Obsⁿ: $.28 \quad .0003 \quad .72 \quad 0 \pm .01$

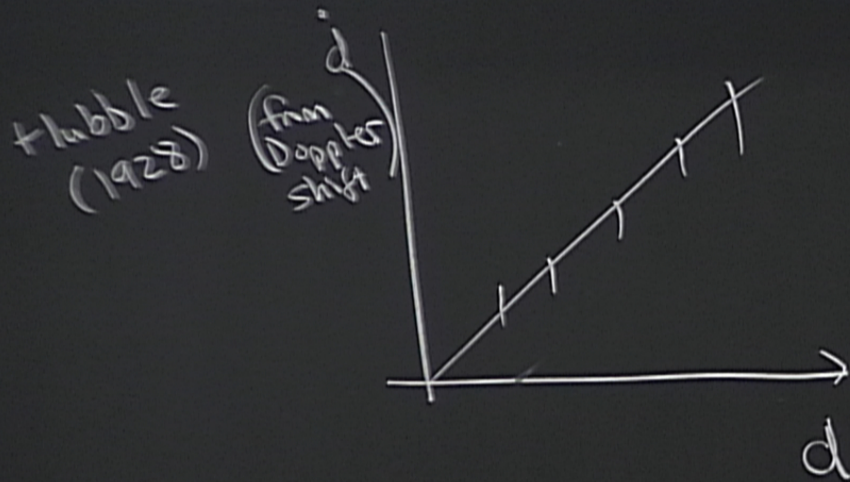
$$1 \text{ Mpc} = 10^6 \text{ pc} \quad , \quad 1 \text{ pc} \approx 3 \text{ light years.}$$

$$65 \pm 10 \text{ km s}^{-1} \text{ Mpc}^{-1}$$

divide by H^2 , define $\frac{8\pi G \rho_m}{3H^2} \equiv \Omega_m$

$$= 0$$

Ω_m includes cold dark matter
baryons, atoms etc
massive neutrinos



Hubble measured $H = +55$

actual value measured

conventional to rewrite (*) as

$$\Omega_k \equiv \frac{k}{a^2 H^2}$$

obs? $\rightarrow |\Omega_k| \lesssim 0.01$

Obsⁿ:

$$1 - (\Omega_m + \Omega_k)$$

-28 .0

today $\approx 65 \pm 10 \text{ km s}^{-1} \text{ Mpc}^{-1}$

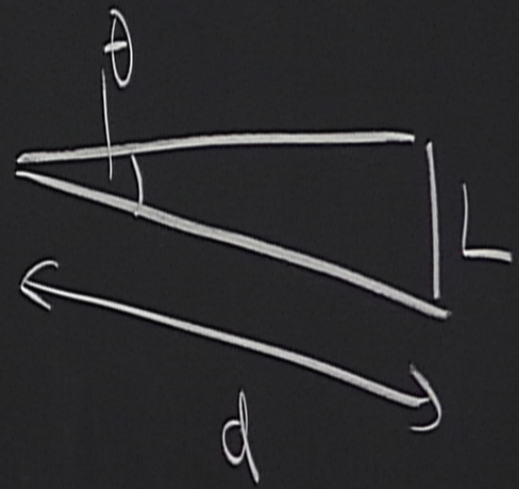
follows: divide by H^2 , define

$$\Omega_m + \Omega_k = 0$$

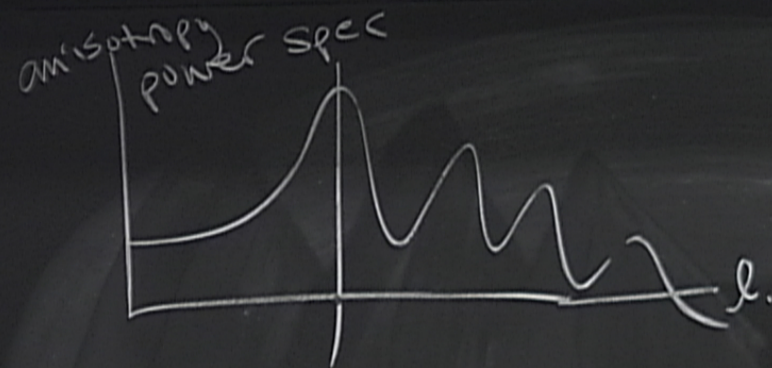
$$\Omega_m \approx 0.3$$

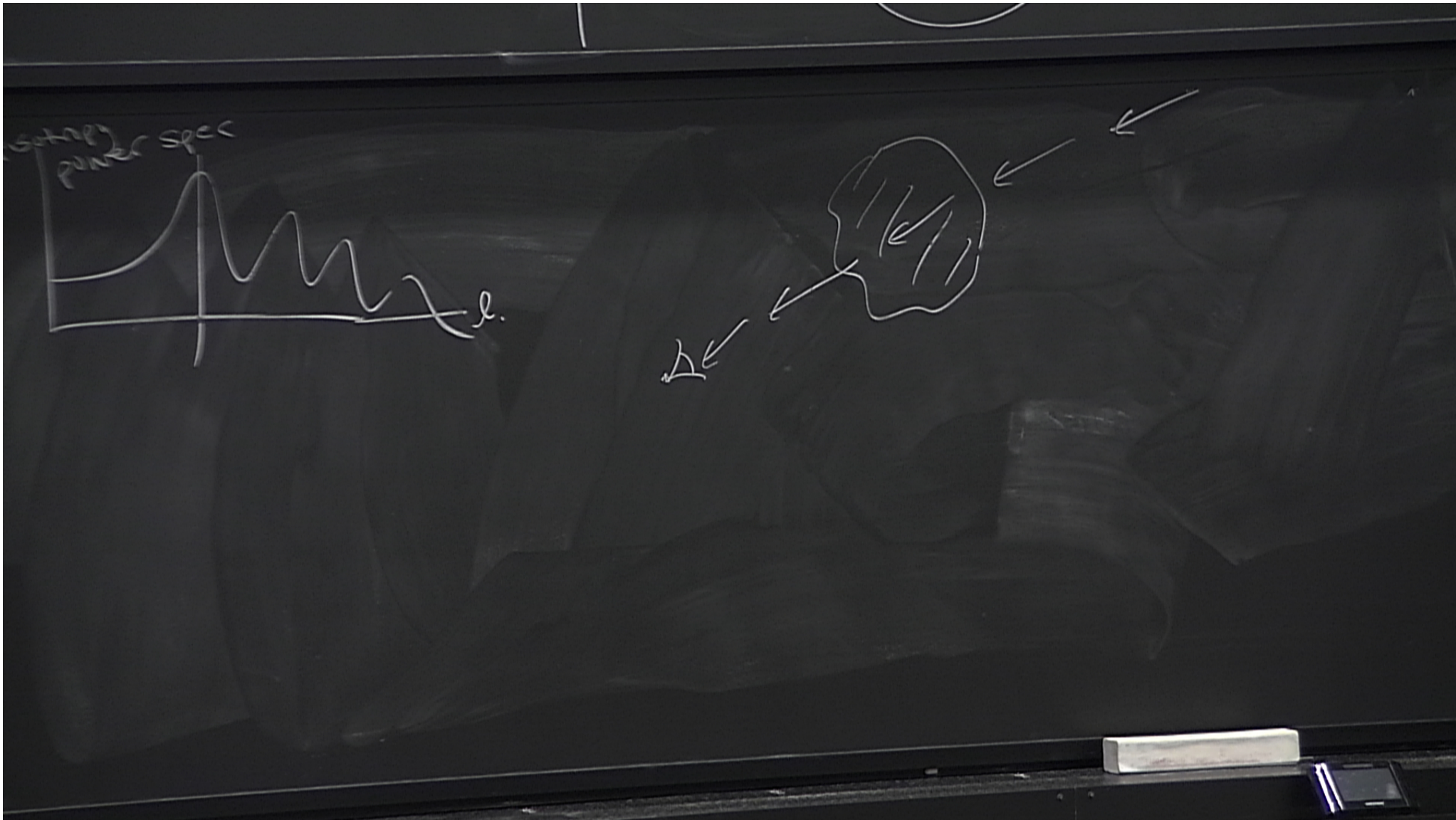
$$\frac{8\pi G \rho_m}{3H^2} \equiv \Omega_m$$

$$\theta \approx L/d$$



Ω_m includes all matter





$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \frac{c_m}{a^3}$$

$$\Rightarrow a \dot{a}^2 = \text{const}$$

$$a^{1/2} \dot{a} = \text{const}$$

$$a^{3/2} = \text{const} \cdot t$$

$$\Rightarrow a \propto t^{2/3}$$

$$\text{ly, } k=0 \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \frac{c_m}{a^3} \Rightarrow a \dot{a}^2 = \text{const}$$

$$a^{1/2} \dot{a} = \text{const.}$$

$$a^{3/2} = \text{const. } t$$

$$\Rightarrow a \propto t^{2/3} \Rightarrow \rho_m = \frac{1}{6\pi G t^2}$$

$$H = \frac{\dot{a}}{a} = \frac{2}{3t} \Rightarrow \boxed{t = \frac{2}{3} H^{-1}}$$

$$\frac{c_m}{a^3} \Rightarrow a \dot{a}^2 = \text{const}$$

$$a^{1/2} \dot{a} = \text{const}$$

$$a^{3/2} = \text{const} \cdot t$$

$$\Rightarrow a \propto t^{2/3} \Rightarrow \rho_m = \frac{1}{6\pi G t^2}$$

$$H = \frac{\dot{a}}{a} = \frac{2}{3t} \Rightarrow \boxed{t = \frac{2}{3} H^{-1}}$$

$$H^{-1} = \text{time}$$

$$\simeq 15 \times 10^9 \text{ yrs.}$$

$\Rightarrow$ m.d.u. would be 10 billion years old.

er and dark energy, (and $k \rightarrow 0$)

$$t = \int_0^a \frac{da}{\sqrt{\frac{8\pi G}{3} \left(\frac{C_M}{a} + C_\Lambda a^2 \right)}} = \frac{1}{H}$$

$$a = x^{2/3}$$

$$\overline{\Omega_m a^3} = \frac{2}{3H} \int_0^1 \frac{dx}{\sqrt{\Omega_m + \Omega_\Lambda x^2}} = \frac{2}{3H} \frac{1}{\Omega_\Lambda^{1/2}} \sinh^{-1} \left(\left(\frac{\Omega_\Lambda}{\Omega_m} \right)^{1/2} \right) \approx \frac{1}{H}$$

$x = X$

$$\frac{2}{3H} \int_0^1 \frac{dx}{\sqrt{\Omega_m + \Omega_\Lambda x^2}} = \frac{2}{3H} \frac{1}{\Omega_\Lambda^{1/2}} \sinh^{-1} \left(\left(\frac{\Omega_\Lambda}{\Omega_m} \right)^{1/2} \right) \approx \frac{1}{H} \text{ for observed values}$$