Title: 13/14 PSI - Relativity - Lecture 12

Date: Sep 24, 2013 09:00 AM

URL: http://pirsa.org/13090026

Abstract:

$-dt^2 + a^2(t)\gamma_{ij}dx^i dx^j$

S^3, E^3, H^3 $\left\{ k = \begin{matrix} +1 \\ 0 \\ -1 \end{matrix} \right.$ $\left\{ \begin{matrix} \text{inverse} \\ \text{curvature radius} \\ \text{squared} \end{matrix} \right.$ $\frac{1}{R_c^2} = \frac{k}{a^2}$

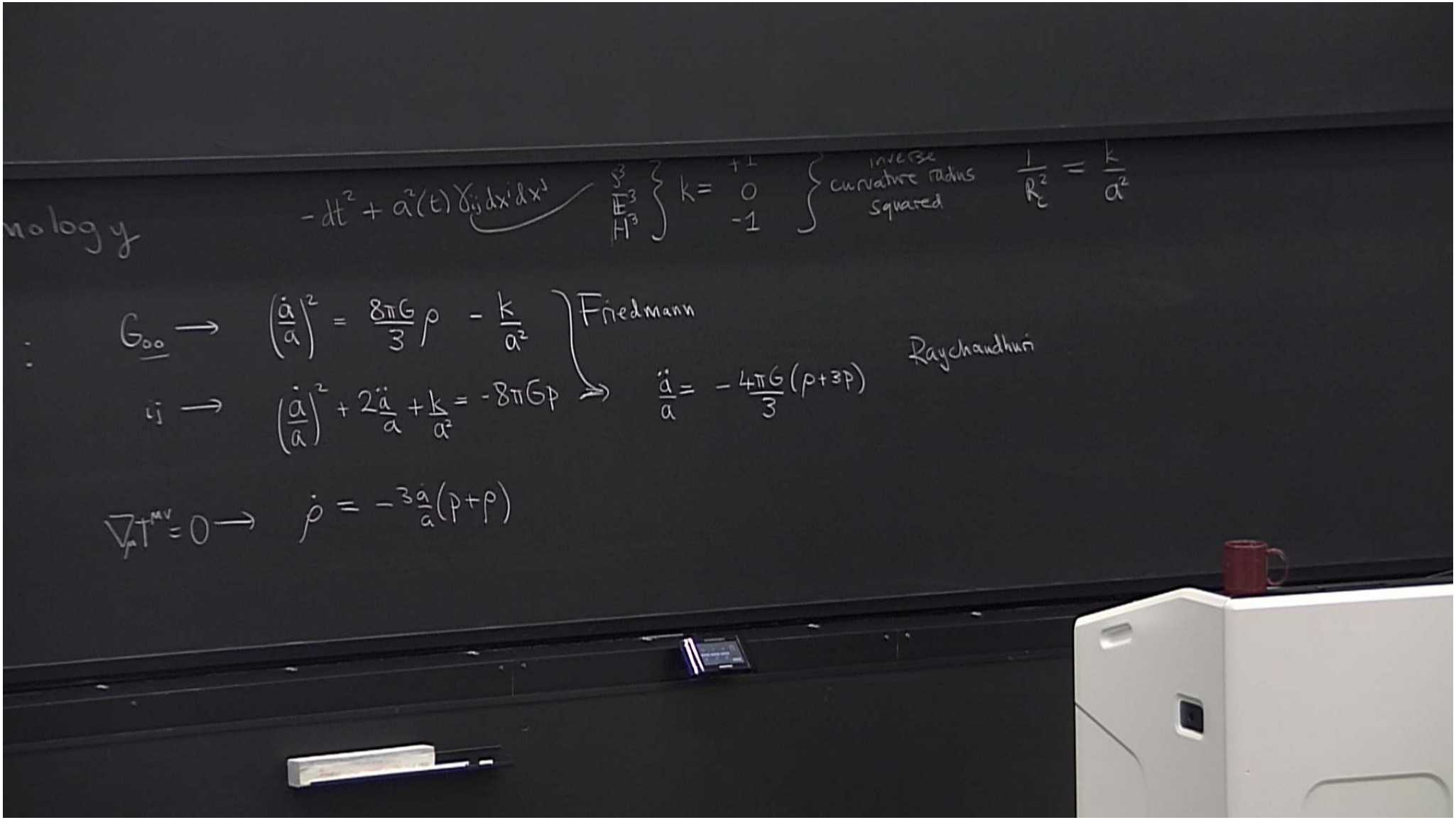
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} + \frac{k}{a^2} = -8\pi G p$$

Friedmann

$$\Rightarrow \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$ Raychaudhuri

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$$


nology
$-dt^2 + a^2(t)\gamma_{ij}dx^i dx^j$
S^3 E^3 H^3
$k = +1, 0, -1$
inverse curvature radius squared
$\frac{1}{R_c^2} = \frac{k}{a^2}$
$G_{00} \rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$
Friedmann
$ij \rightarrow \left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} + \frac{k}{a^2} = -8\pi G p \Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$
Raychaudhuri
$\nabla_\mu T^{\mu\nu} = 0 \rightarrow \dot{\rho} = -3\frac{\dot{a}}{a}(p + \rho)$

$$-dt^2 + a^2(t)\,\gamma_{ij}\,dx^i dx^j$$

S^3, E^3, H^3 $\Big\}$ $k = \begin{matrix} +1 \\ 0 \\ -1 \end{matrix}$ $\Big\}$ inverse curvature radius squared $\quad \frac{1}{R_c^2} = \frac{k}{a^2}$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

$$\left(\frac{\dot a}{a}\right)^2 + 2\frac{\ddot a}{a} + \frac{k}{a^2} = -8\pi G p$$

Friedmann (only single time derivatives: constraint)

Raychaudhuri

$$\Rightarrow \quad \frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$$\Rightarrow \quad \dot\rho = -3\frac{\dot a}{a}(p+\rho)$$

Cosmology

$-dt^2 + a^2(t)\gamma_{ij}dx^i dx^j$ $\left.\begin{matrix} S^3 \\ E^3 \\ H^3 \end{matrix}\right\}$ $k=$

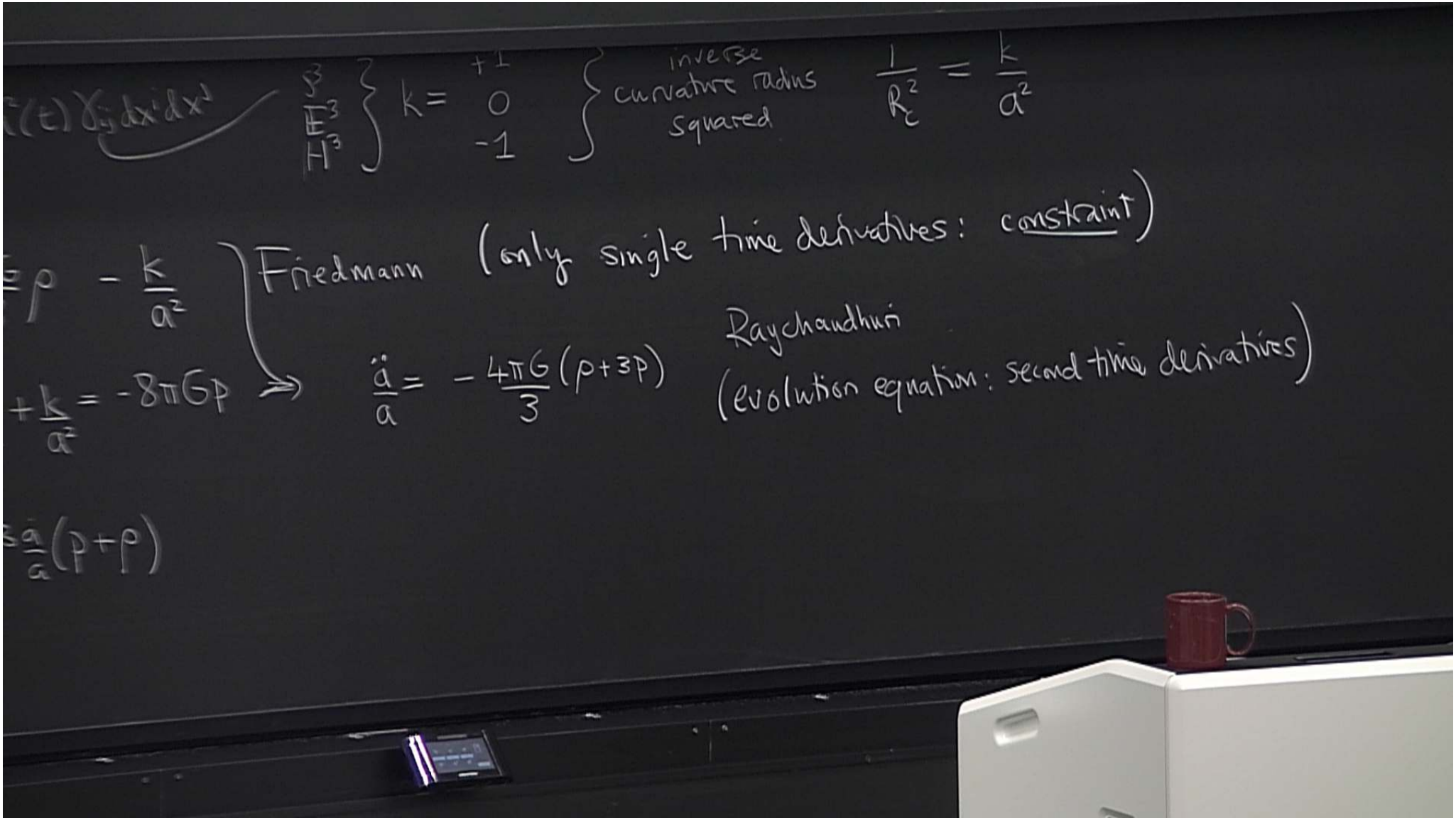
$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$$

$$G_{00} \longrightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

$$ij \longrightarrow \left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} + \frac{k}{a^2} = -8\pi G p$$

Friedmann $\Rightarrow \quad \frac{\ddot{a}}{a} =$

$$\nabla_\mu T^{\mu\nu} = 0 \longrightarrow \dot{\rho} = -3\frac{\dot{a}}{a}(p+\rho)$$


$a^2(t)\gamma_{ij}dx^i dx^j$
S^3 E^3 H^3
$k = +1, 0, -1$
inverse curvature radius squared
$\frac{1}{R_c^2} = \frac{k}{a^2}$
$\rho - \frac{k}{a^2}$
Friedmann (only single time derivatives: constraint)
$+\frac{k}{a^2} = -8\pi G\rho$
$\Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$
Raychaudhuri
(evolution equation: second time derivatives)
$\frac{\dot{a}}{a}(p+\rho)$

Cosmology

$-dt^2 + a^2(t)\gamma_{ij}dx^i d$

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$$

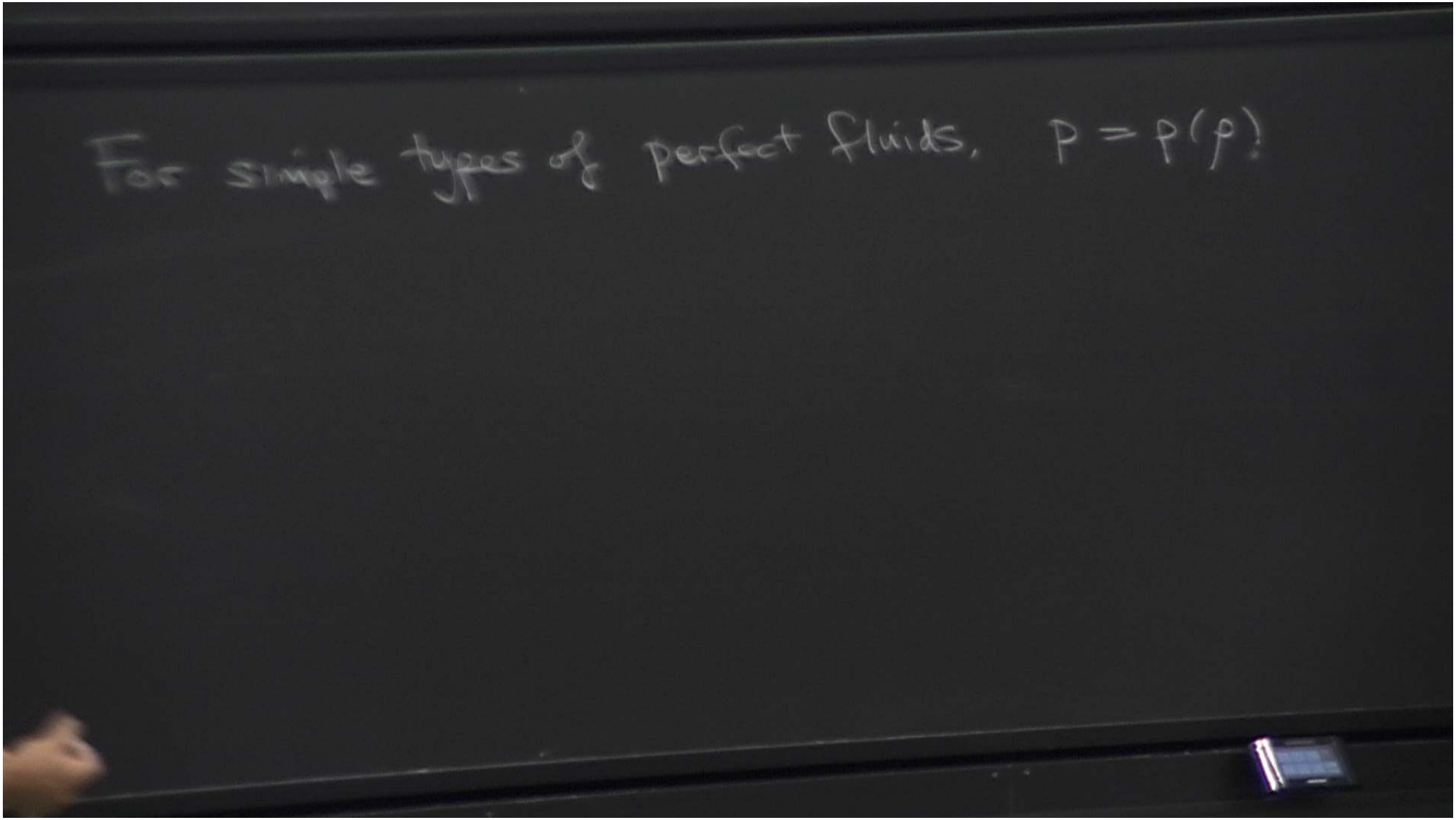
$G_{00} \rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$

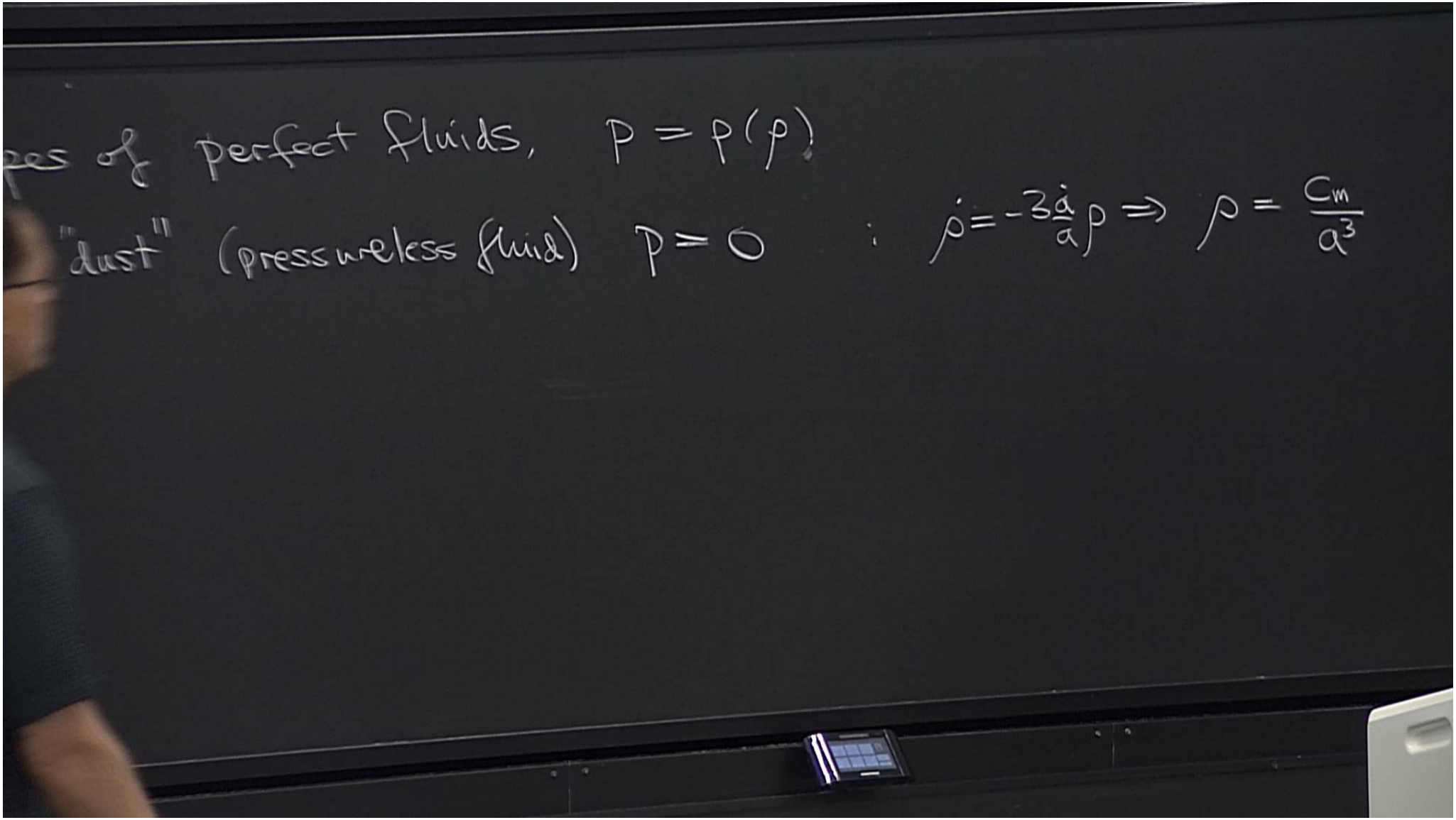
$ij \rightarrow \left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} + \frac{k}{a^2} = -8\pi$

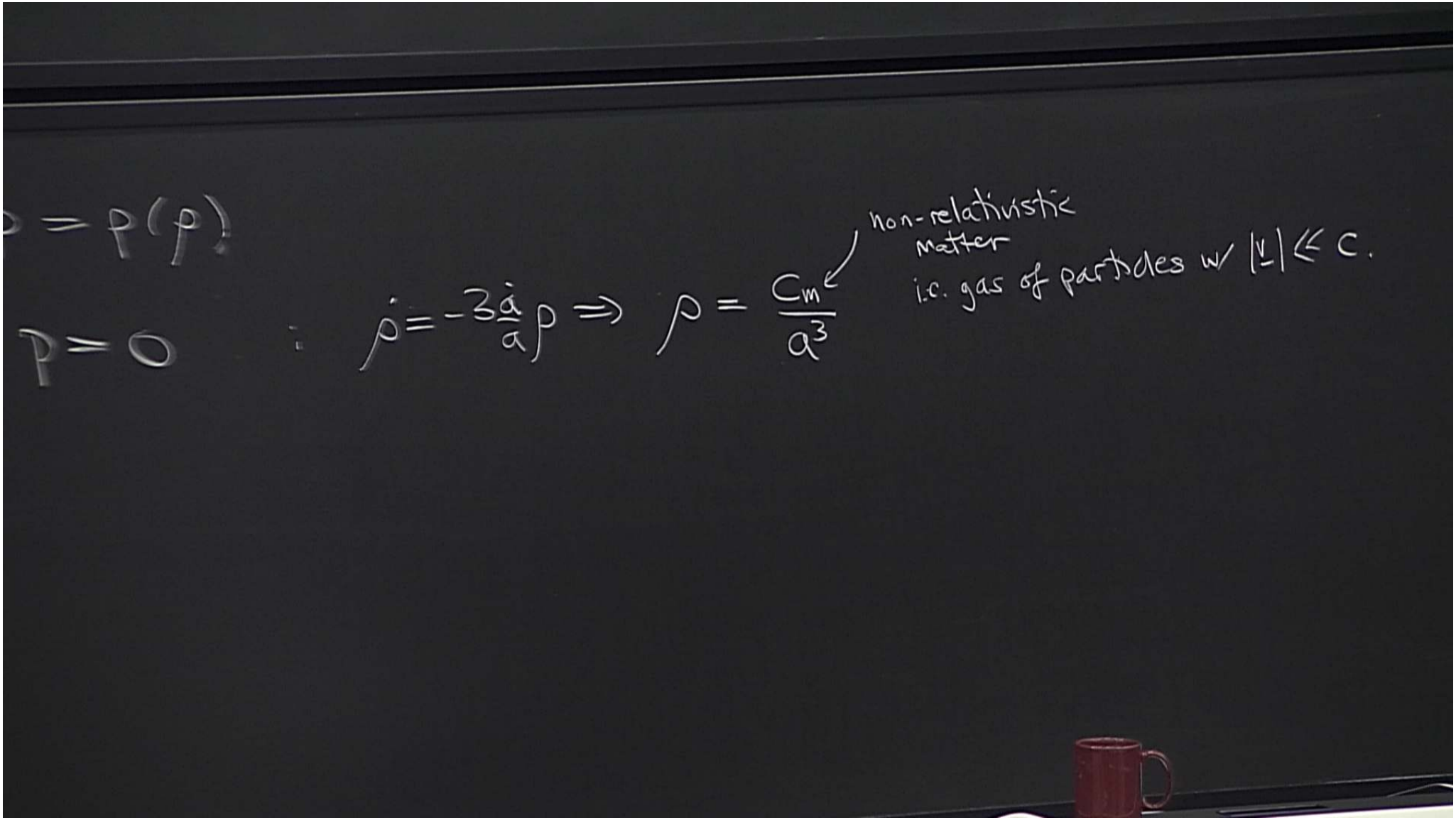
G_{00}, G_{0i} } eq^{n}s constraints

G_{ij} eq^{n}s evolution eq^{n}s

$\nabla_\mu T^{\mu\nu} = 0 \rightarrow \dot{\rho} = -3\frac{\dot{a}}{a}(p+\rho)$


For simple types of perfect fluids, $p = p(\rho)$


pes of perfect fluids, $p = p(\rho)$
"dust" (pressureless fluid) $p = 0$: $\dot{\rho} = -3\frac{\dot{a}}{a}\rho \Rightarrow \rho = \frac{C_m}{a^3}$


$p = p(\rho)$
$p = 0$: $\dot{\rho} = -3\frac{\dot{a}}{a}\rho \Rightarrow \rho = \frac{C_m}{a^3}$
non-relativistic matter
i.e. gas of particles w/ $|\underline{v}| \ll c$.

$p = p(\rho)$

$p = 0$: $\dot{\rho} = -3\frac{\dot{a}}{a}\rho \Rightarrow \rho = \frac{C_m}{a^3}$

non-relativistic matter
i.e. gas of particles w/ $|\underline{v}| \ll c$.

total mass in a comoving volume in $\underline{x}$

$p = p(\rho)$

$p = 0$: $\dot{\rho} = -3\frac{\dot{a}}{a}\rho \Rightarrow \rho = \frac{C_m}{a^3}$ ← non-relativistic matter i.e. gas of particles w/ $|\underline{v}| \ll c$.

total mass in a comoving volume in $\underline{x}$ $\propto \rho a^3 = \text{const.}$

For simple types of perfect fluids, $p = p(\rho)$

e.g. "dust" (pressureless fluid) $p = 0$

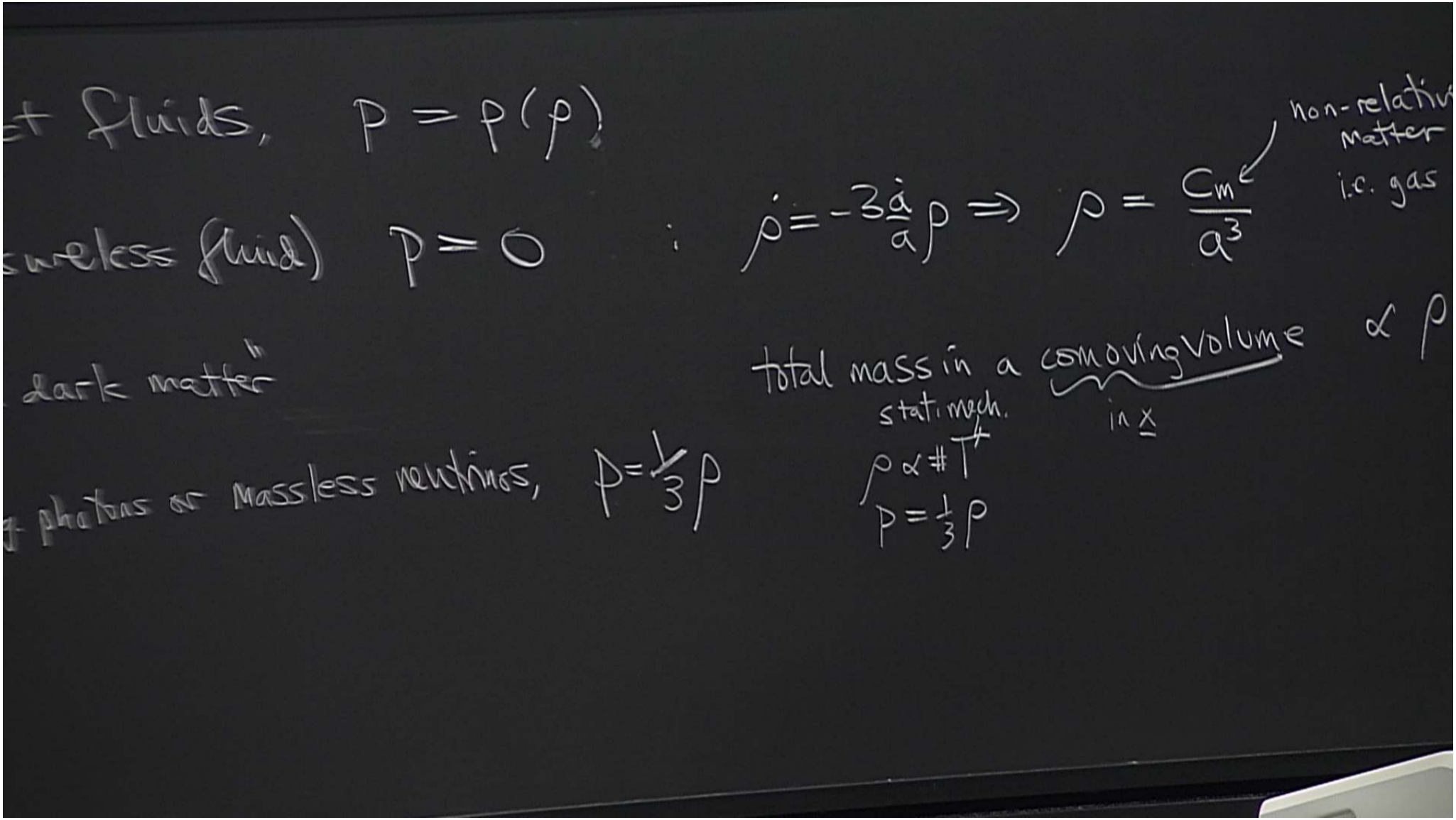
e.g. "cold dark matter"

For simple types of perfect fluids, $p = p(\rho)$

e.g. "dust" (pressureless fluid) $p = 0$: $\dot{\rho} = -3\frac{\dot{a}}{a}\rho \Rightarrow$ total mass in a

e.g. "cold dark matter"

"radiation" e.g. photons or massless neutrinos, $p = \frac{1}{3}\rho$


et fluids, $p = p(\rho)$
sureless fluid $p = 0$: $\dot{\rho} = -3\frac{\dot{a}}{a}\rho \Rightarrow \rho = \frac{c_m}{a^3}$
non-relativ
matter
i.e. gas
dark matter"
total mass in a comoving volume $\propto \rho$
in Σ
stat. mech.
$\rho \propto \# T^4$
$p = \frac{1}{3}\rho$
t photons or massless neutrinos, $p = \frac{1}{3}\rho$

perfect fluids, $p = p(\rho)$

(pressureless fluid) $p = 0$: $\dot{\rho} = -3\frac{\dot{a}}{a}\rho \Rightarrow \rho = \frac{C_m}{a^3}$

e.g. "cold dark matter"

total mass in a comoving volum in $\underline{x}$

stat. mech.

$\rho \propto \# T^4$

$p = \frac{1}{3}\rho$

ation" e.g. photons or massless neutrinos, $p = \frac{1}{3}\rho$

$\Rightarrow \dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{1}{3}\rho\right) = -4\frac{\dot{a}}{a}\rho \Rightarrow \rho = \frac{C_r}{a^4}$ extra $1/a$: Energy of each photon

$\dot{\rho} = -3\frac{\dot{a}}{a}p \Rightarrow \rho = \frac{C_m}{a^3}$ ← non-relativistic matter i.e. gas of particles w/ $|\underline{v}| \ll c$.

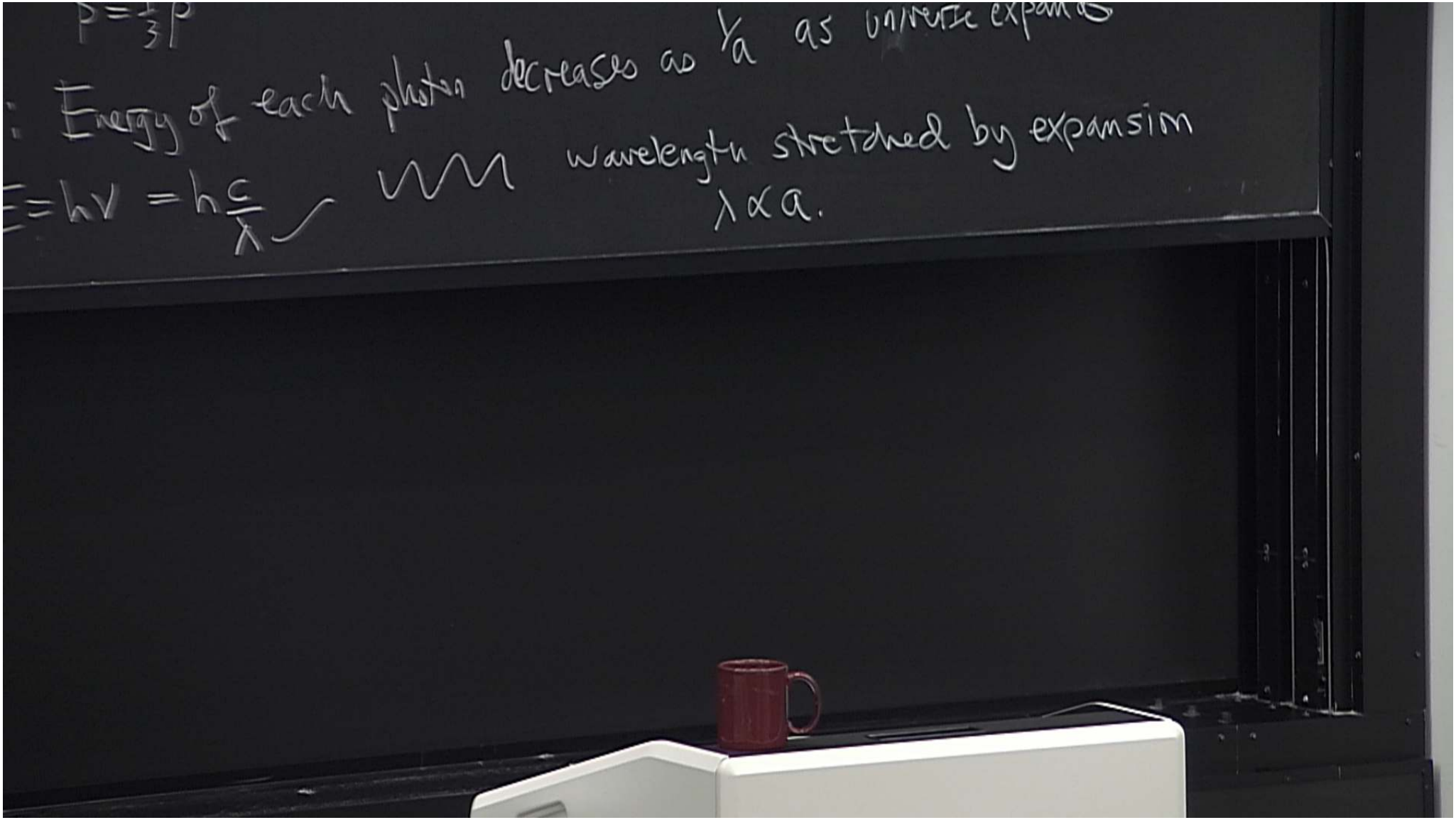
Total mass in a comoving volume $\propto \rho a^3 = \text{const.}$

stat. mech. in $\underline{x}$

$\rho \propto \#\, T^4$

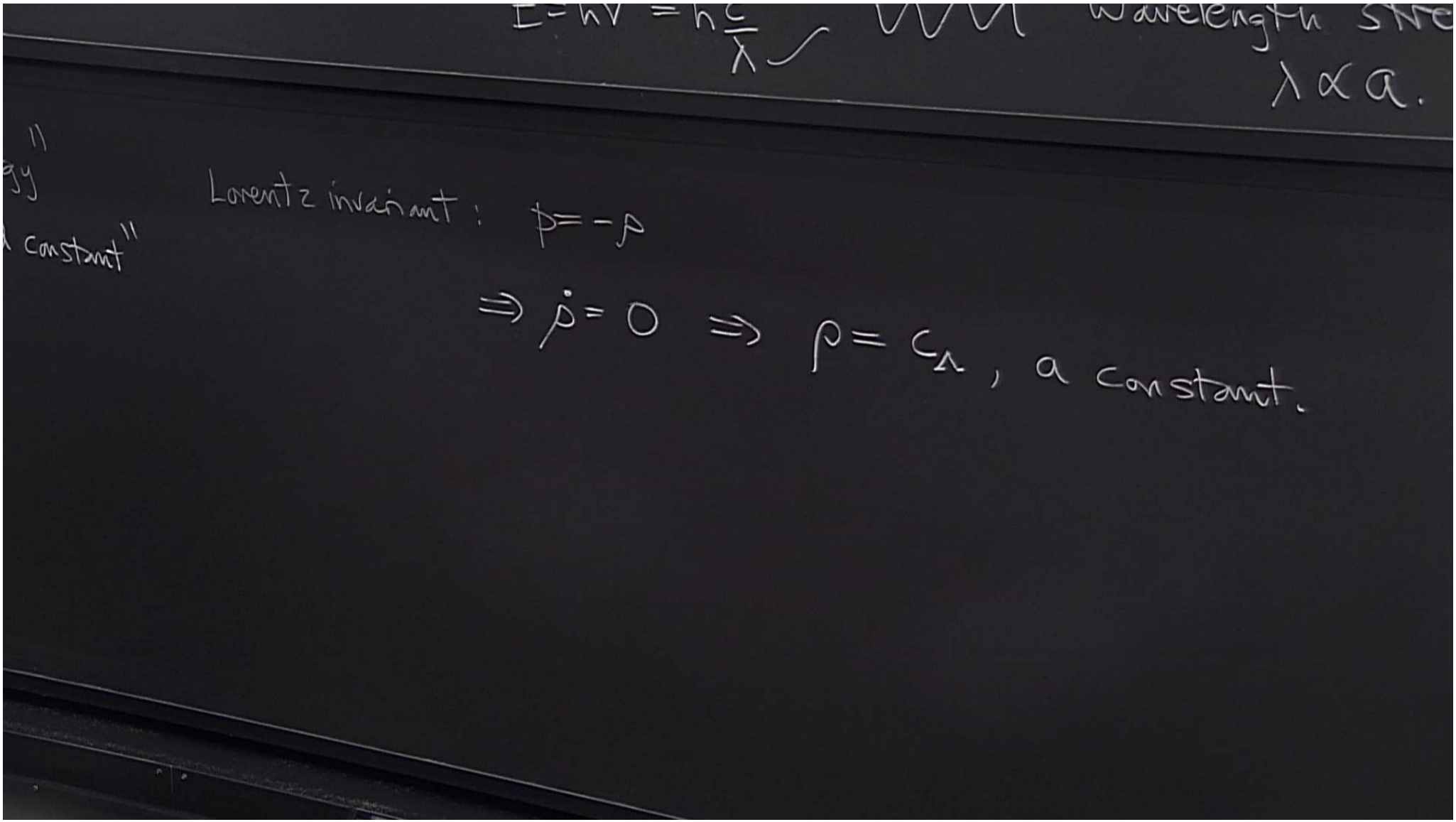
$p = \frac{1}{3}\rho$

$1/a$: Energy of each photon decreases as $1/a$ as universe expands


$p = \frac{1}{3}\rho$
Energy of each photon decreases as $1/a$ as universe expands
$E = h\nu = h\frac{c}{\lambda}$
wavelength stretched by expansion
$\lambda \propto a$.

$$g_{\mu\nu} \to \Omega^2(x)\, g_{\mu\nu}$$

$$V_{photon} \propto \frac{1}{a}$$


$E = h\nu = h\frac{c}{\lambda}$
wavelength
$\lambda \propto a$.
"
gy
constant"
Lorentz invariant: $p = -\rho$
$\Rightarrow \dot{\rho} = 0 \Rightarrow \rho = c_\Lambda$, a constant.

$E = h\nu = h\frac{c}{\lambda}$ ~ $\lambda \propto a$.

Lorentz invariant: $p = -\rho$

$\Rightarrow \dot{\rho} = 0 \Rightarrow \rho = c_\Lambda$, a constant.

$\Rightarrow$ Friedmann: assume $k = 0$ $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} c_\Lambda \Rightarrow \underline{a \propto e^{H_\Lambda t}}$, $H_\Lambda^2 = \frac{8\pi G}{3} c_\Lambda$

"dark energy"

"cosmological constant"

Lorentz invariant: $p = -\rho$

$\Rightarrow \dot{\rho} = 0 \Rightarrow \rho = c_\Lambda$, a const

$\Rightarrow$ Friedmann; assume $k = 0$ $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} c_\Lambda \Rightarrow a \propto$

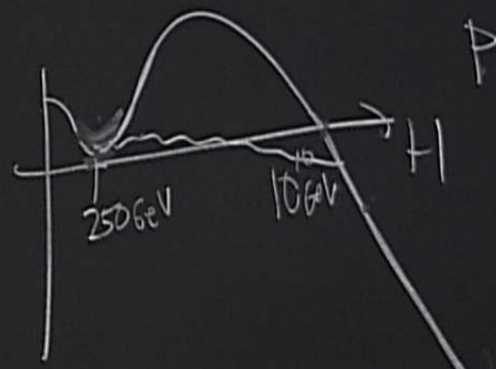
* many contributions

vac fluc^{n}s

gluon condensate

Higgs field H, V(H)

$p = -V$



$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$

$\propto \frac{1}{a}$

"dark energy"

"cosmological constant"

$T_{\alpha\beta} = \Lambda g_{\alpha\beta}$

many contributions

vac fluc^{n}s

gluon condensate

Higgs field H, V(H)

Lorentz inv

$\Rightarrow$ Friedmann

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General solution with matter, radiation and Λ

Friedmann $\dot{a}^2 - \frac{8\pi G}{3}\underbrace{\left(\rho_m + \rho_r + \rho_\Lambda\right)a^2}_{\frac{C_m}{a} + \frac{C_r}{a^2} + \rho_\Lambda a^2} = -k$

c.f. $\frac{1}{2}m\dot{x}^2 + V(x) = E$

General solution with matter, radiation and Λ

Friedmann $\dot{a}^2 - \frac{8\pi G}{3}\underbrace{\left(\rho_m + \rho_r + \rho_\Lambda\right)a^2}_{\frac{C_m}{a} + \frac{C_r}{a^2} + \rho_\Lambda a^2} = -k$

c.f. $\frac{1}{2}m\dot{x}^2 + V(x) = E$

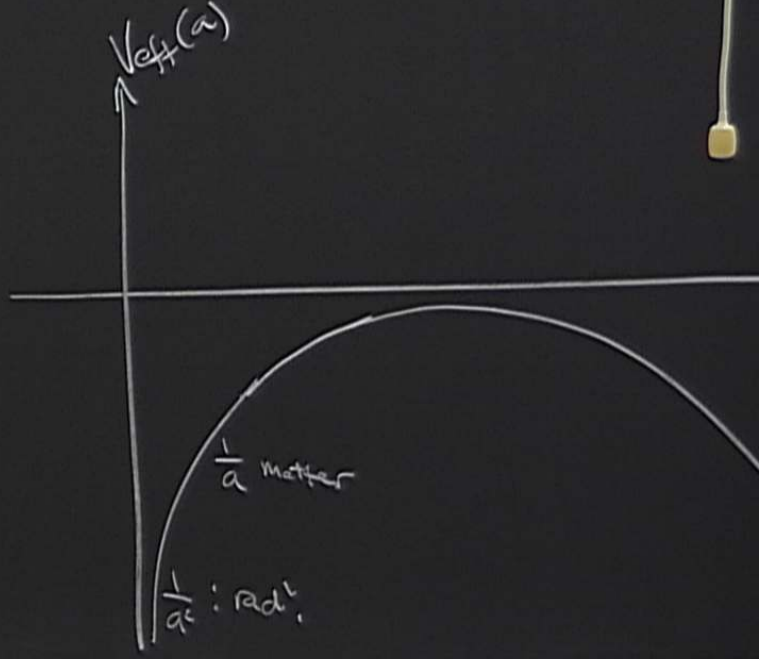
Here, $n=2$. $V_{eff}(a) = -\frac{8\pi G}{3}\left(\frac{C_m}{a} + \frac{C_r}{a^2} + C_\Lambda a^2\right)$

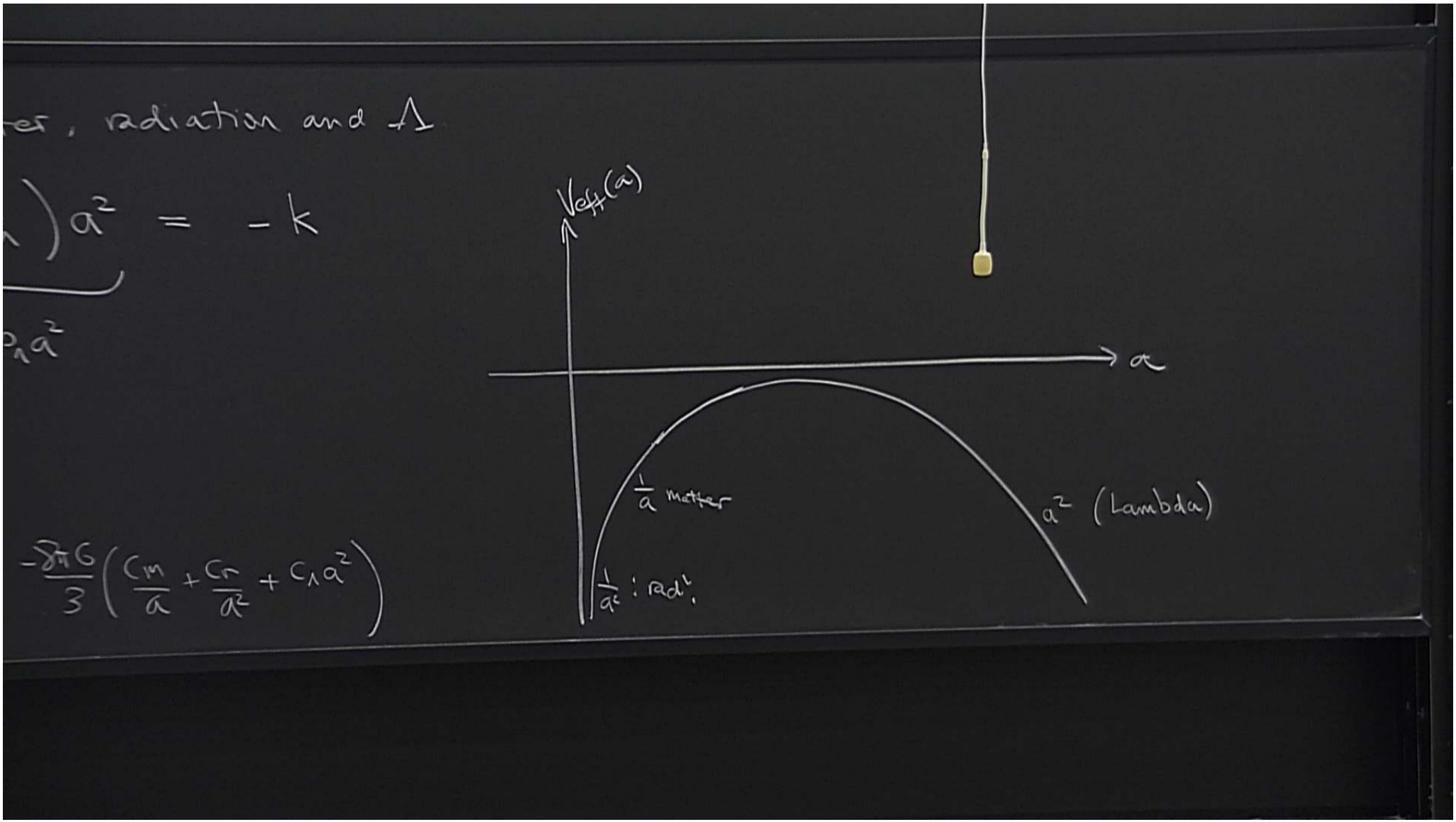
Solution with matter, radiation and Λ

$$-\frac{8\pi G}{3}\underbrace{\left(\rho_m + \rho_r + \rho_\Lambda\right)a^2}_{\frac{C_m}{a} + \frac{C_r}{a^2} + \rho_\Lambda a^2} = -k$$

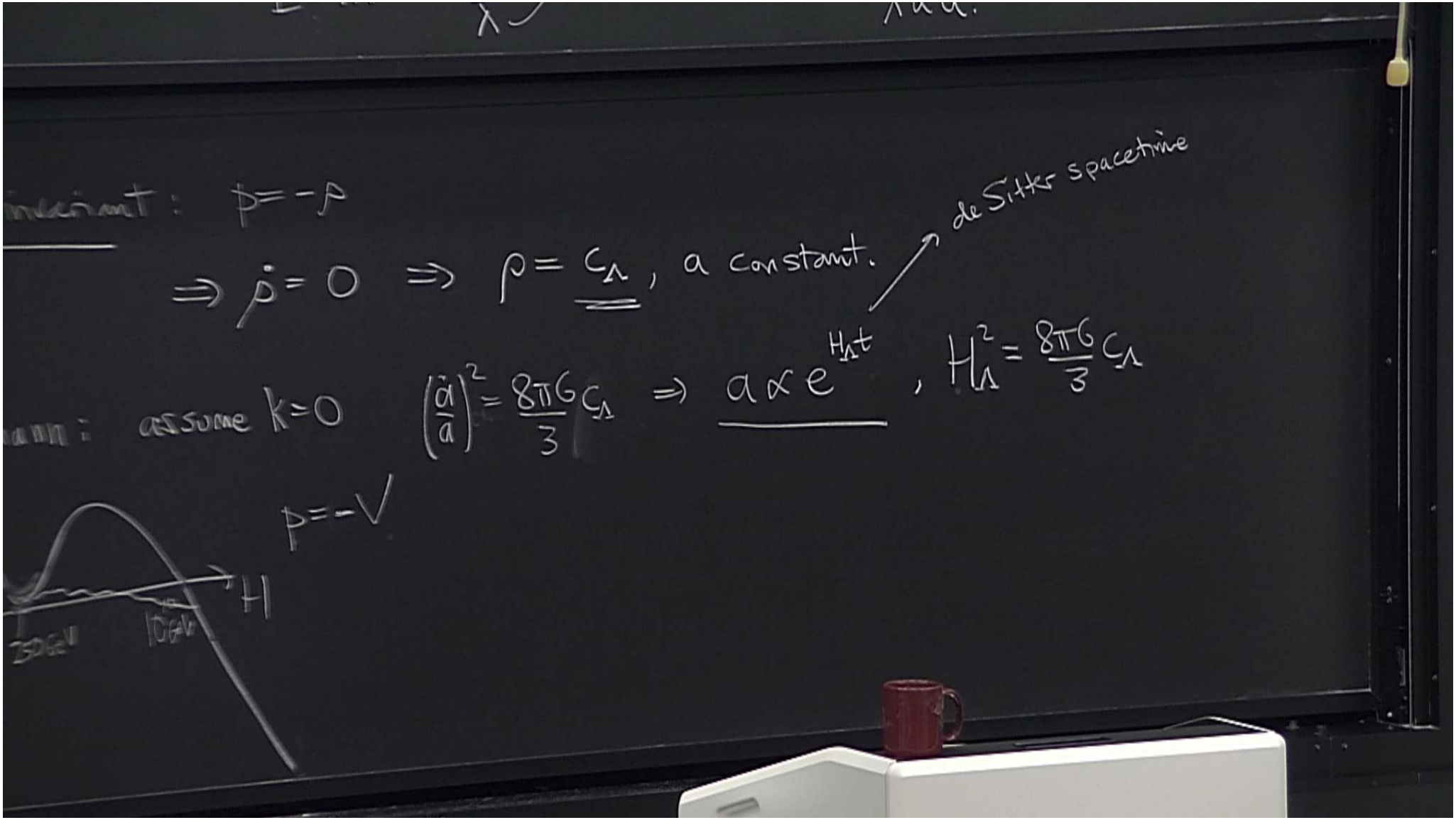
$$\frac{1}{2}m\dot{x}^2 + V(x) = E$$

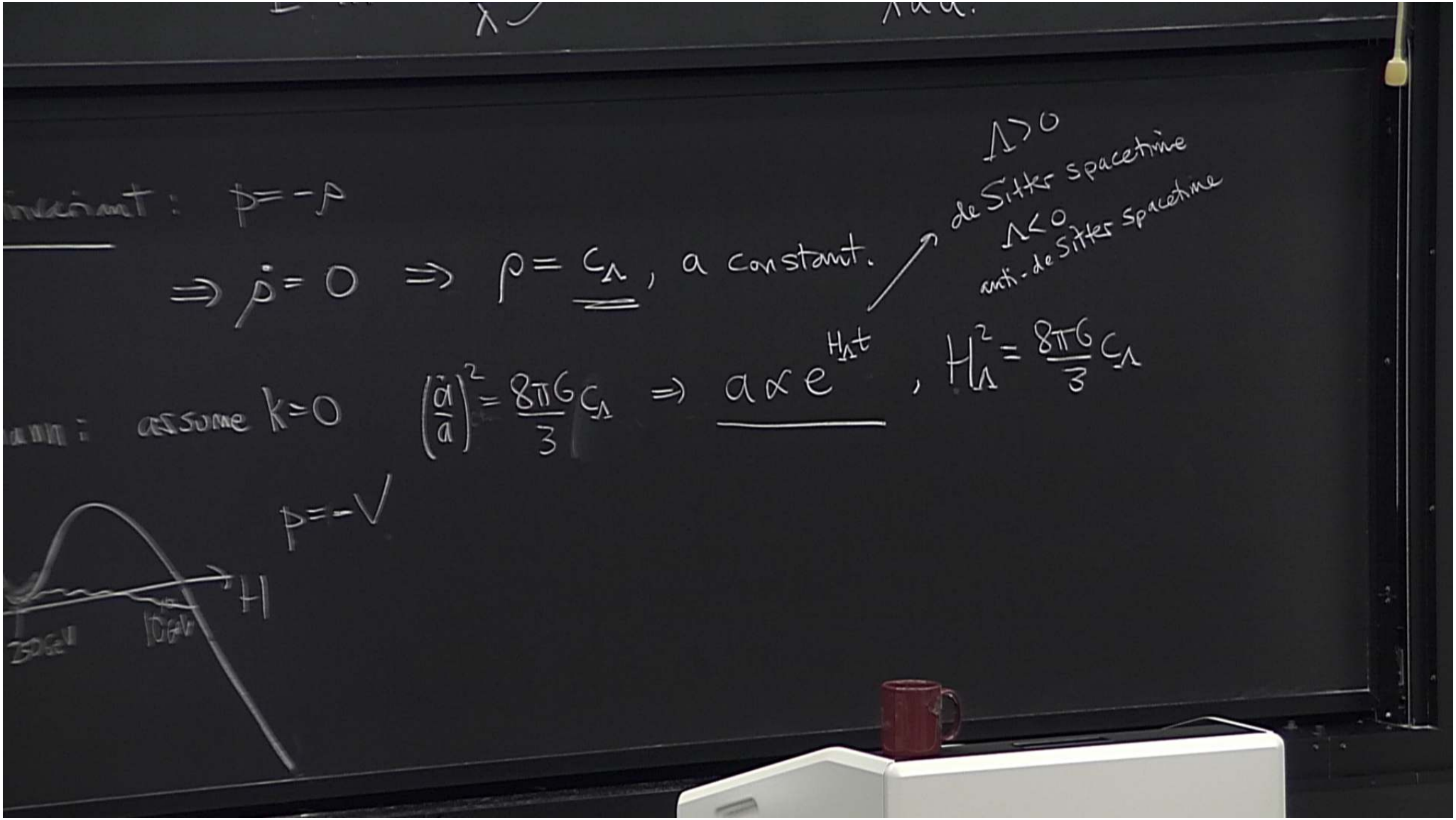
Here, $n=2$. $\quad V_{eff}(a) = -\frac{8\pi G}{3}\left(\frac{C_m}{a} + \frac{C_r}{a^2} + C_\Lambda a^2\right)$

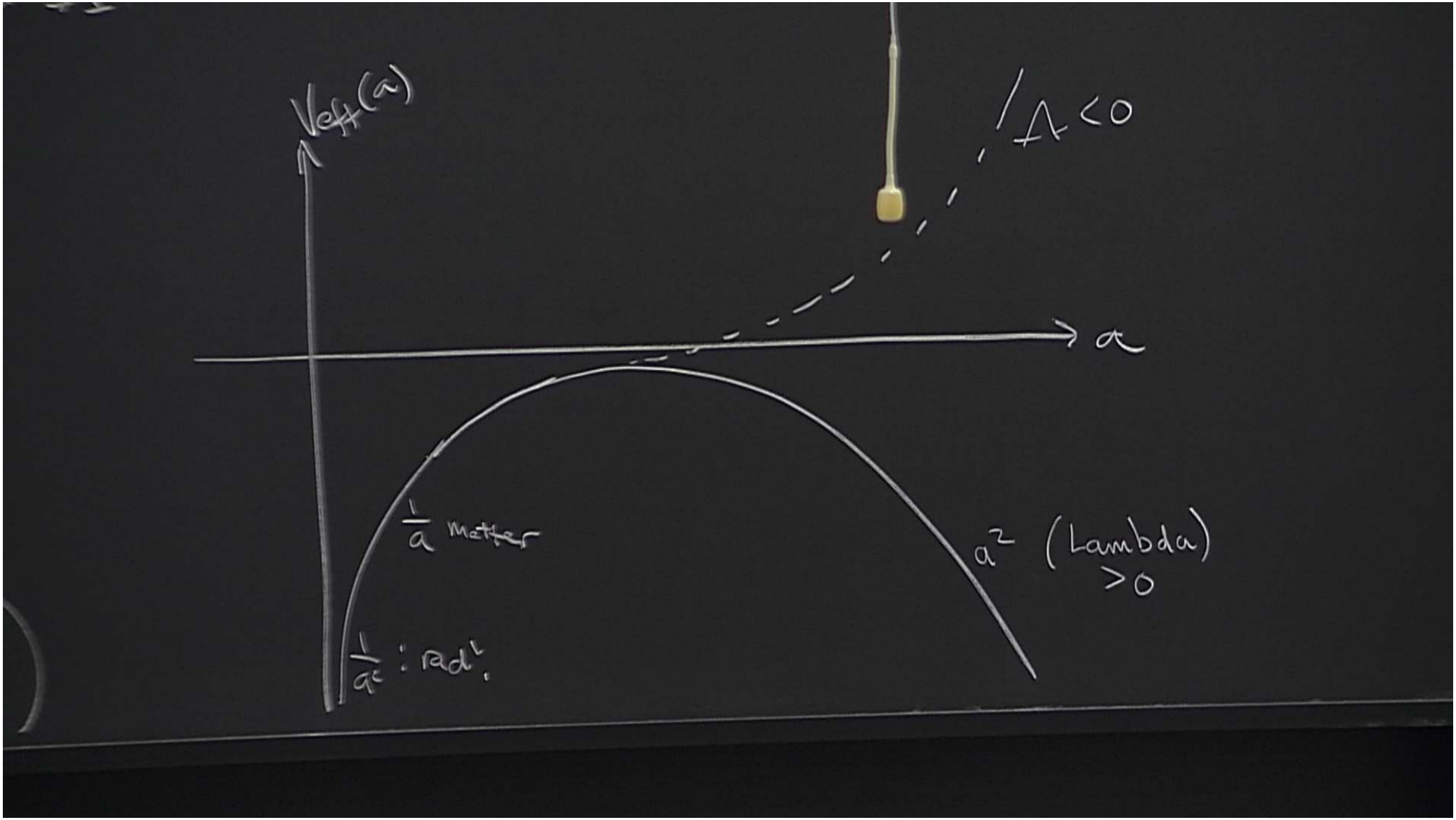




er, radiation and Λ
$)a^2 = -k$
$\rho_\Lambda a^2$
$-\frac{8\pi G}{3}\left(\frac{C_m}{a} + \frac{C_r}{a^2} + C_\Lambda a^2\right)$
$V_{eff}(a)$
a
$\frac{1}{a}$ matter
a^2 (Lambda)
$\frac{1}{a^2}$: rad^2.


inflation: $p=-\rho$
$\Rightarrow \dot{\rho}=0 \Rightarrow \rho=C_\Lambda$, a constant.
de Sitter spacetime
assume $k=0$
$\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G}{3}C_\Lambda \Rightarrow a \propto e^{H_\Lambda t}$, $H_\Lambda^2=\frac{8\pi G}{3}C_\Lambda$
$p=-V$
H
300eV
10GeV


invariant: $p=-\rho$
$\Rightarrow \dot{\rho}=0 \Rightarrow \rho=C_\Lambda$, a constant.
$\Lambda>0$ de Sitter spacetime
$\Lambda<0$ anti-de Sitter spacetime
assume $k=0$ $\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G}{3}C_\Lambda \Rightarrow a \propto e^{H_\Lambda t}$, $H_\Lambda^2=\frac{8\pi G}{3}C_\Lambda$
$p=-V$
H
300eV
10eV


$V_{eff}(a)$
$\Lambda < 0$
a
$\frac{1}{a}$ matter
a^2 (Lambda) > 0
$\frac{1}{a^2}$: rad.

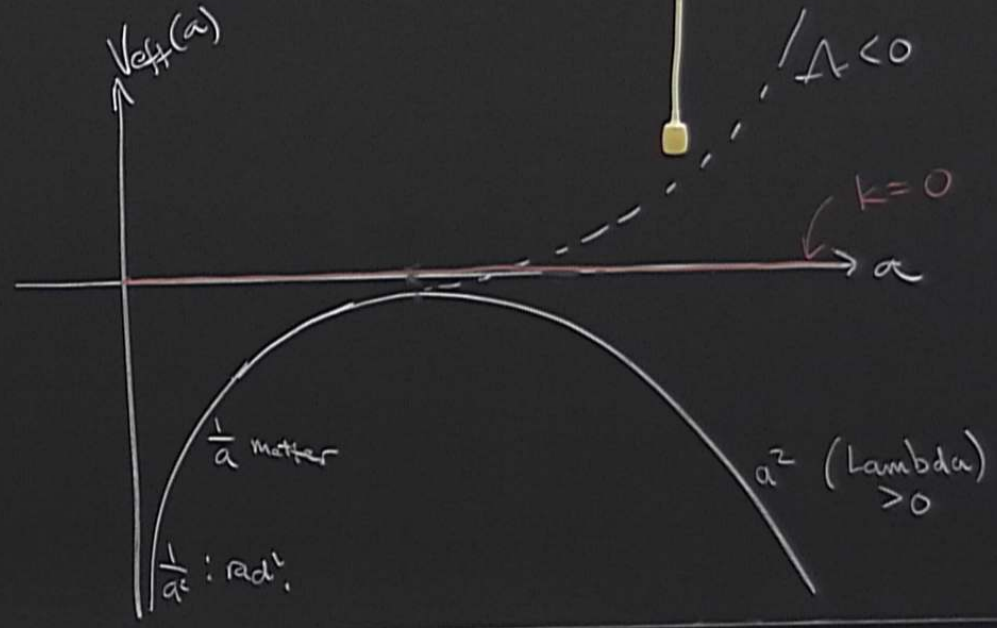
with matter, radiation and Λ

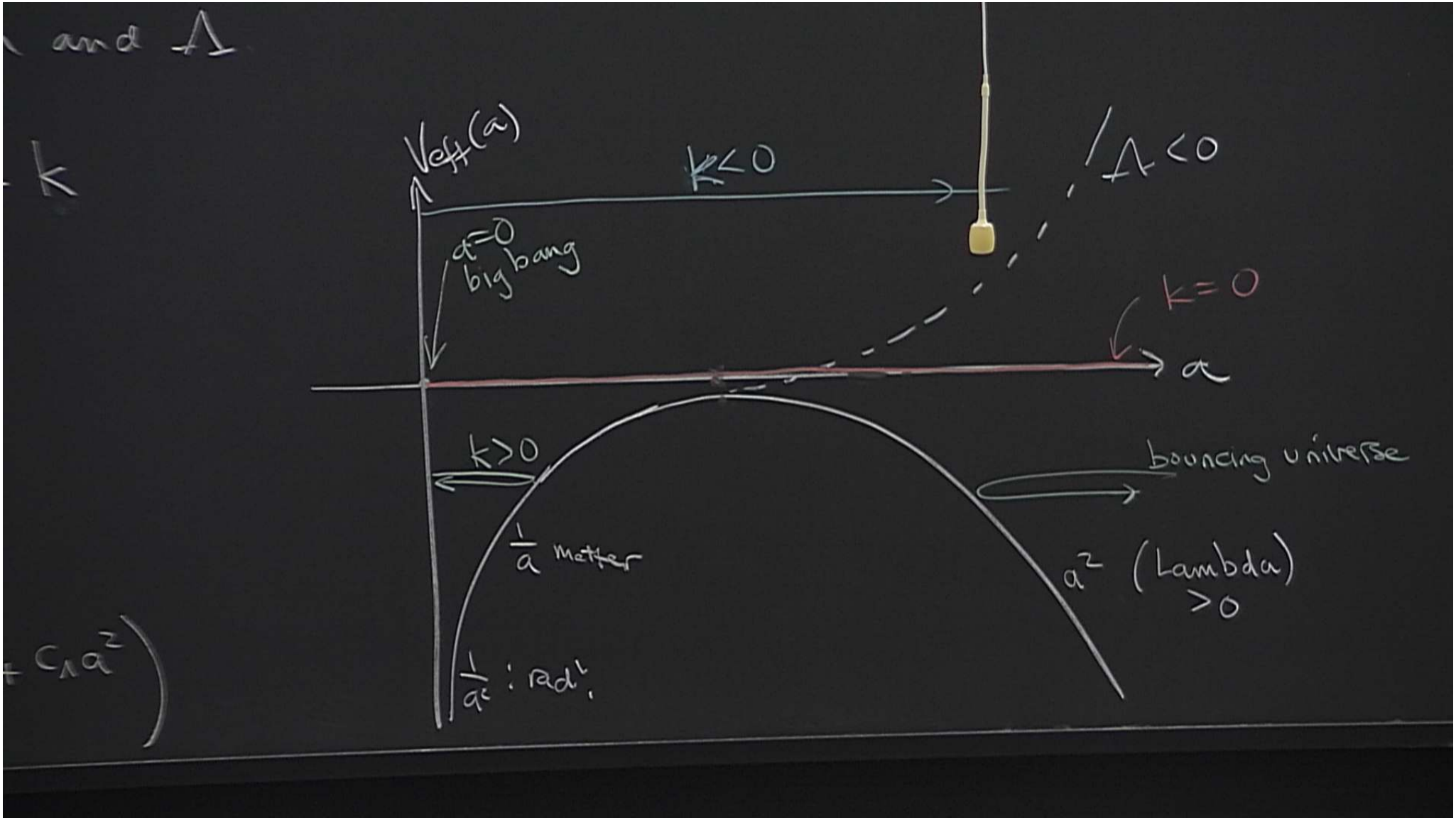
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho_m + \rho_r + \rho_\Lambda\right) - \frac{k}{a^2}$$

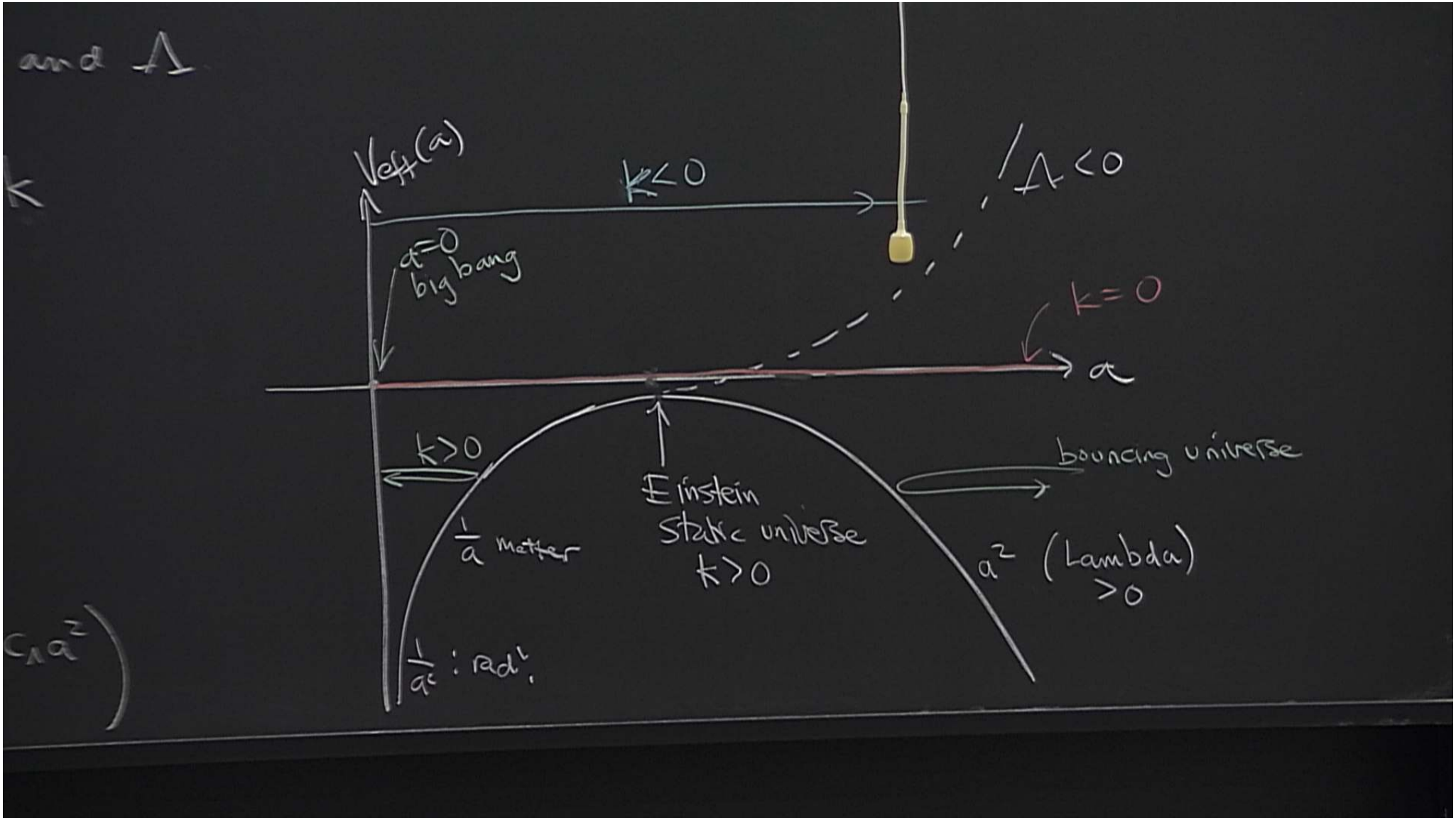
$$\frac{C_m}{a} + \frac{C_r}{a^2} + \rho_\Lambda a^2$$

$$= E$$

$$V_{eff}(a) = -\frac{8\pi G}{3}\left(\frac{C_m}{a} + \frac{C_r}{a^2} + C_\Lambda a^2\right)$$




and Λ
k
$V_{eff}(a)$
k<0
Λ<0
a=0
big bang
k=0
a
k>0
bouncing universe
$\frac{1}{a}$ matter
a^2 (Lambda) >0
$\frac{1}{a^2}$: rad.
$+ C_\Lambda a^2$)


and Λ
k
$V_{eff}(a)$
k<0
Λ<0
t=0
big bang
k=0
a
k>0
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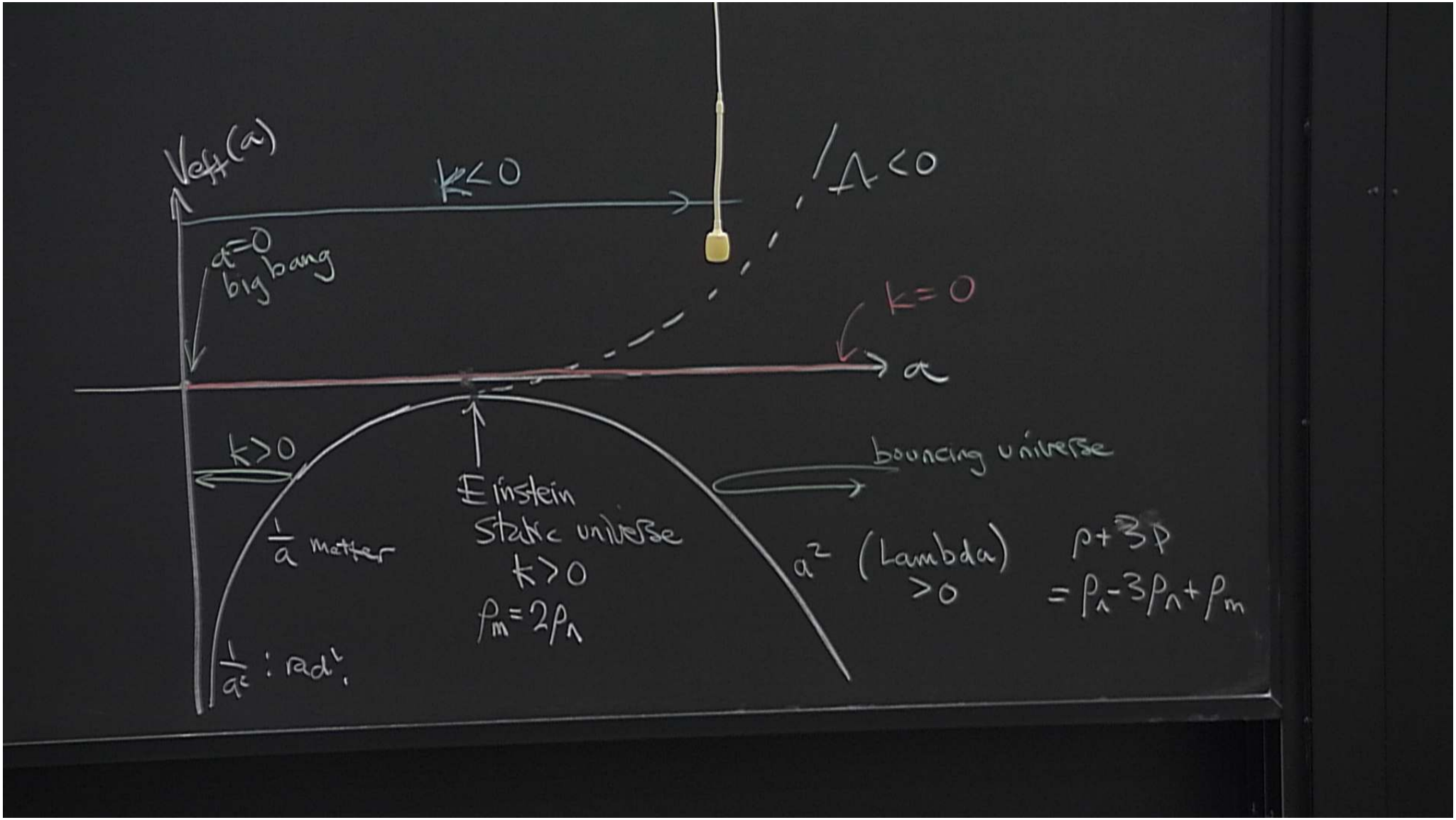
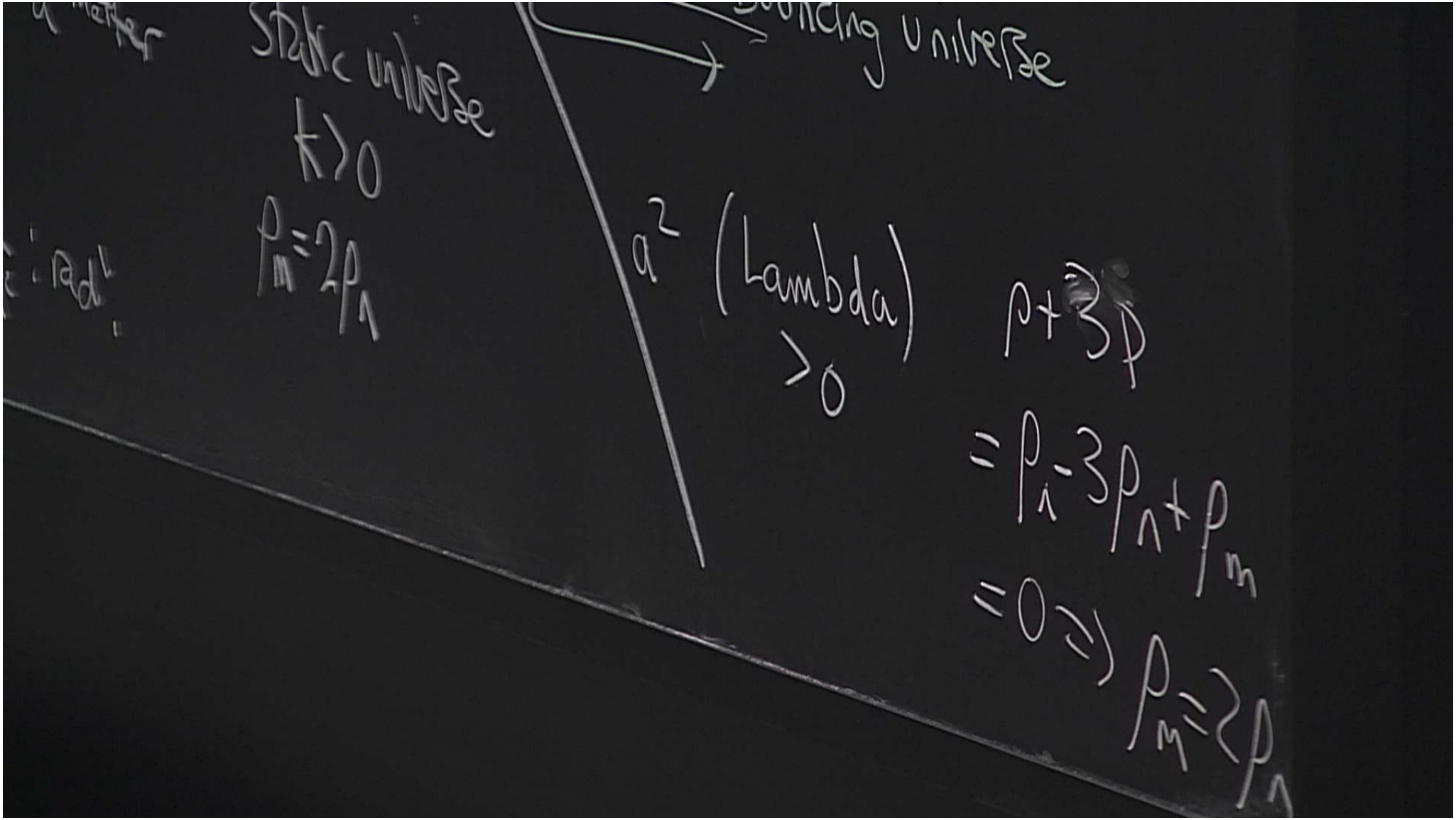
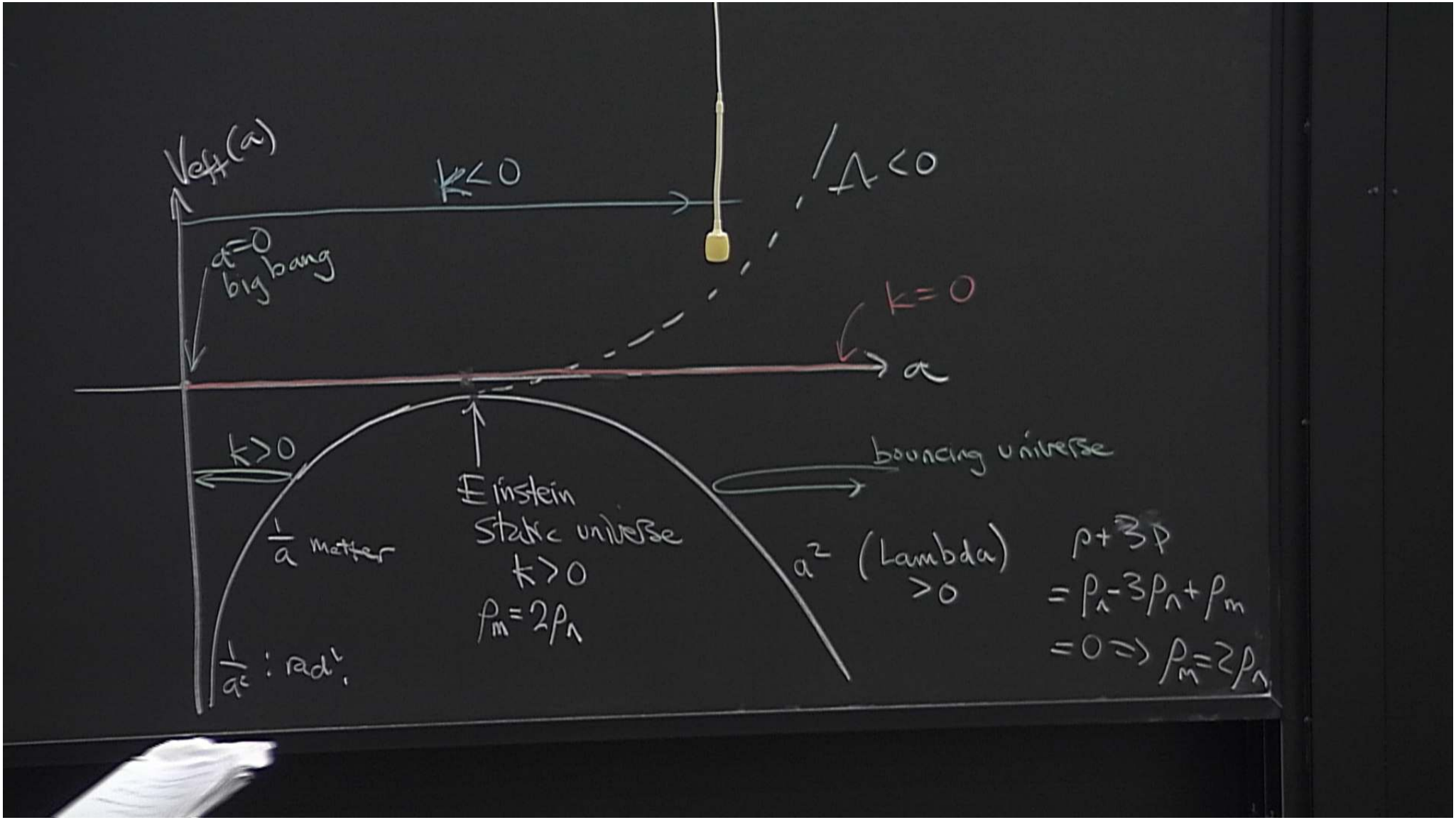
Einstein
static universe
k>0
$\frac{1}{a}$ matter
a^2 (Lambda)
>0
$\frac{1}{a^2}$: rad.
$c_1 a^2$)


$V_{eff}(a)$
$k<0$
$\Lambda<0$
$a=0$
big bang
$k=0$
a
$k>0$
bouncing universe
Einstein
Static universe
$k>0$
$\rho_m = 2\rho_\Lambda$
$\frac{1}{a}$ matter
$\frac{1}{a^2}$: rad^{n}.
a^2 (Lambda)
>0
$\rho + 3p$
$= \rho_\Lambda - 3\rho_\Lambda + \rho_m$


Static universe
k>0
$\rho_m = 2\rho_\Lambda$
a^2 (Lambda)
>0
$\rho + 3p$
$= \rho_\Lambda - 3p_\Lambda + \rho_m$
$= 0 \Rightarrow \rho_m = 2\rho_\Lambda$


$V_{eff}(a)$
$k<0$
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big bang
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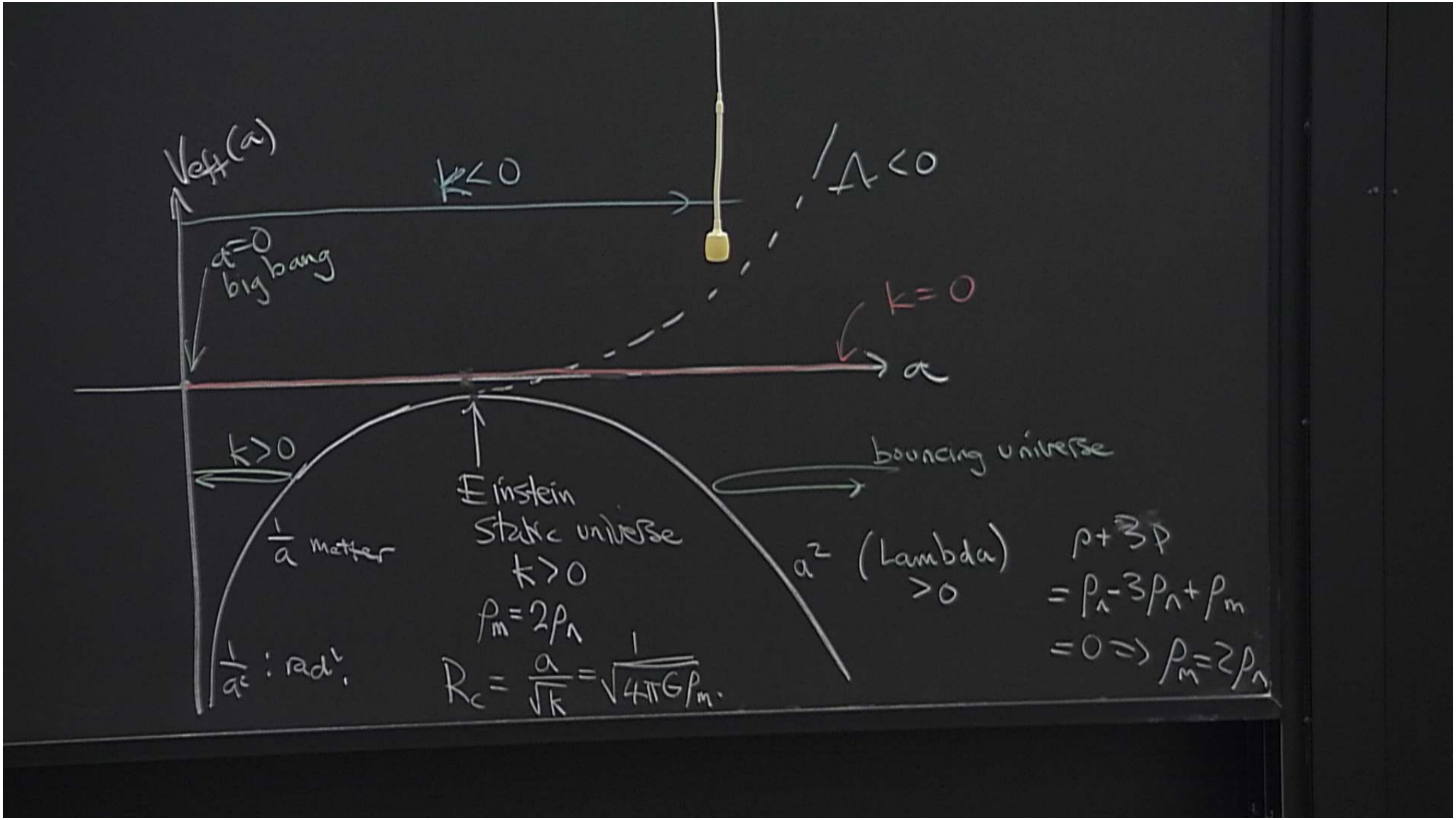
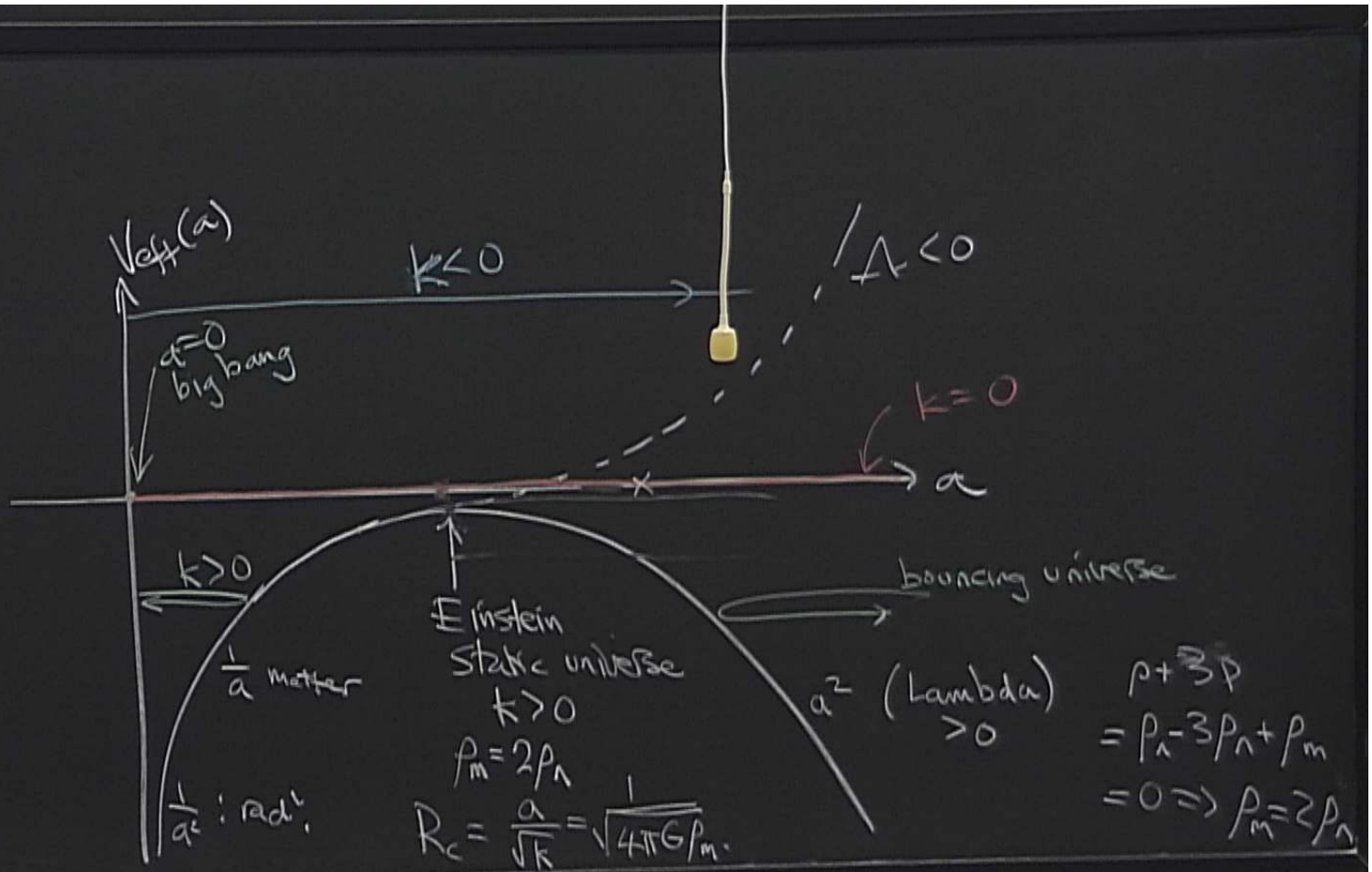
$k>0$
$\rho_m = 2\rho_\Lambda$
$\frac{1}{a}$ matter
$\frac{1}{a^2}$: rad.
a^2 (Lambda)
>0
$\rho + 3p$
$= \rho_\Lambda - 3\rho_\Lambda + \rho_m$
$= 0 \Rightarrow \rho_m = 2\rho_\Lambda$


$V_{eff}(a)$
$k<0$
$\Lambda<0$
$a=0$
big bang
$k=0$
a
$k>0$
bouncing universe
Einstein
Static universe
$k>0$
$\rho_m = 2\rho_\Lambda$
$\frac{1}{a}$ matter
$\frac{1}{a^2}$: rad.
a^2 (Lambda)
>0
$\rho + 3p$
$= \rho_\Lambda - 3\rho_\Lambda + \rho_m$
$= 0 \Rightarrow \rho_m = 2\rho_\Lambda$
$R_c = \frac{a}{\sqrt{k}} = \frac{1}{\sqrt{4\pi G \rho_m}}$.

diation and Λ

$= -k$

$\left(\frac{m}{a} + \frac{C_r}{a^2} + C_\Lambda a^2\right)$



Compare with observations:

$$\underbrace{\left(\frac{\dot{a}}{a}\right)^2}_{H^2} - \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_\Lambda) + \frac{k}{a^2} = 0$$

H = Hubble const : consider particles / galaxies / stars at fixed $\underline{x}$.

$$d^2 = a^2(t)(\Delta x)^2$$

geodesic eq^n:) Separation of points at fixed $\underline{x}$ grows like $a(t)$

pare with observations:

$$\underbrace{\left(\frac{\dot{a}}{a}\right)^2}_{H^2} - \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_\Lambda) + \frac{k}{a^2} = 0$$

ble const : consider particles / galaxies / stars at fixed $\underline{x}$. (s

$$\frac{\dot{d}}{d} = \frac{\dot{a}}{a} \Rightarrow \dot{d} = H d$$

ations:

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Hubble's law

H^2

H = Hubble constant : Consider particles / galaxies / stars at fix
parameter

- function of time !

$\frac{\dot{d}}{d} = \frac{\dot{a}(t)}{a} \Rightarrow \dot{d} = H d$

$d(t)$, $a(t)$

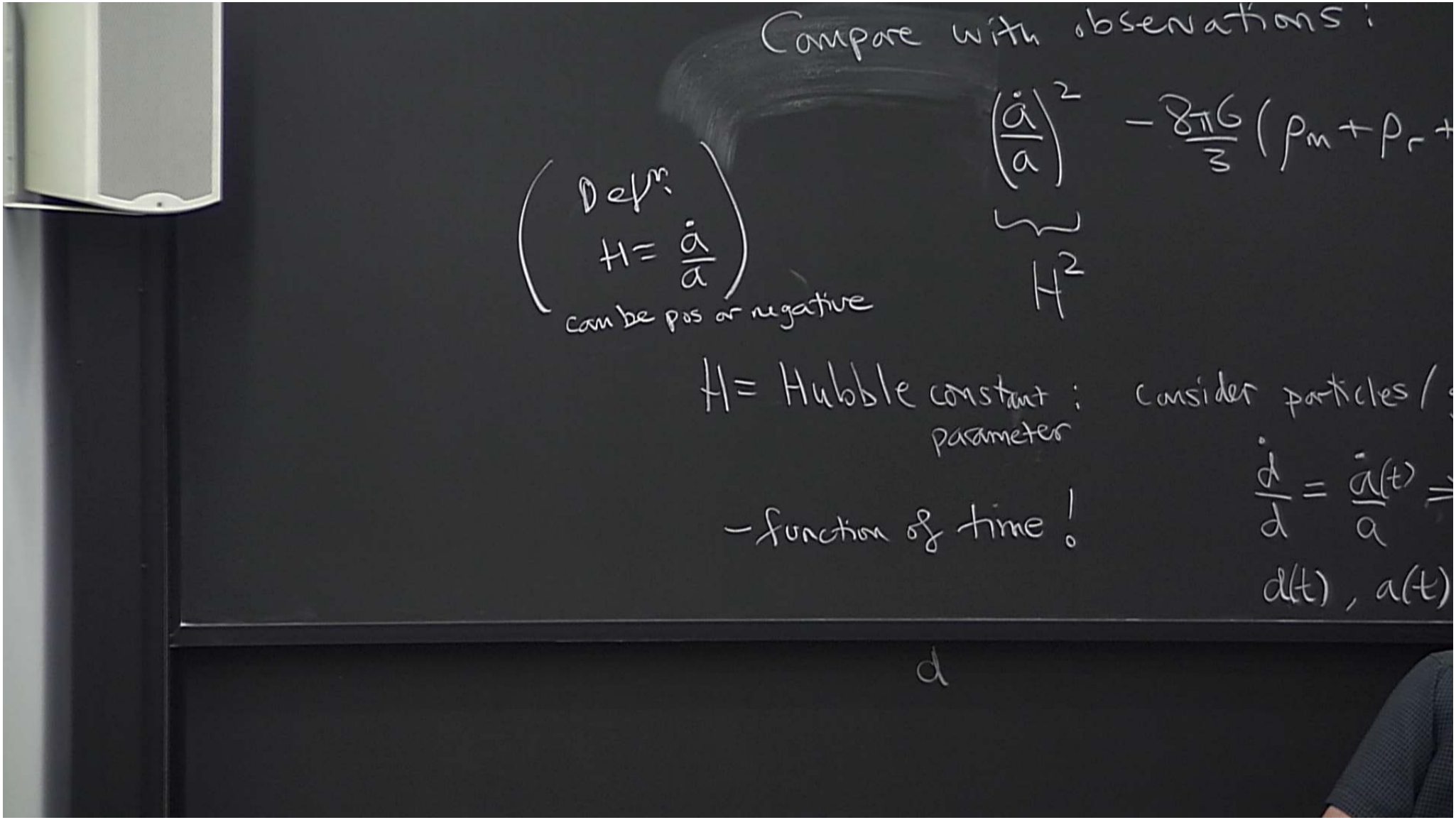
$d(t)$, $a(t)$

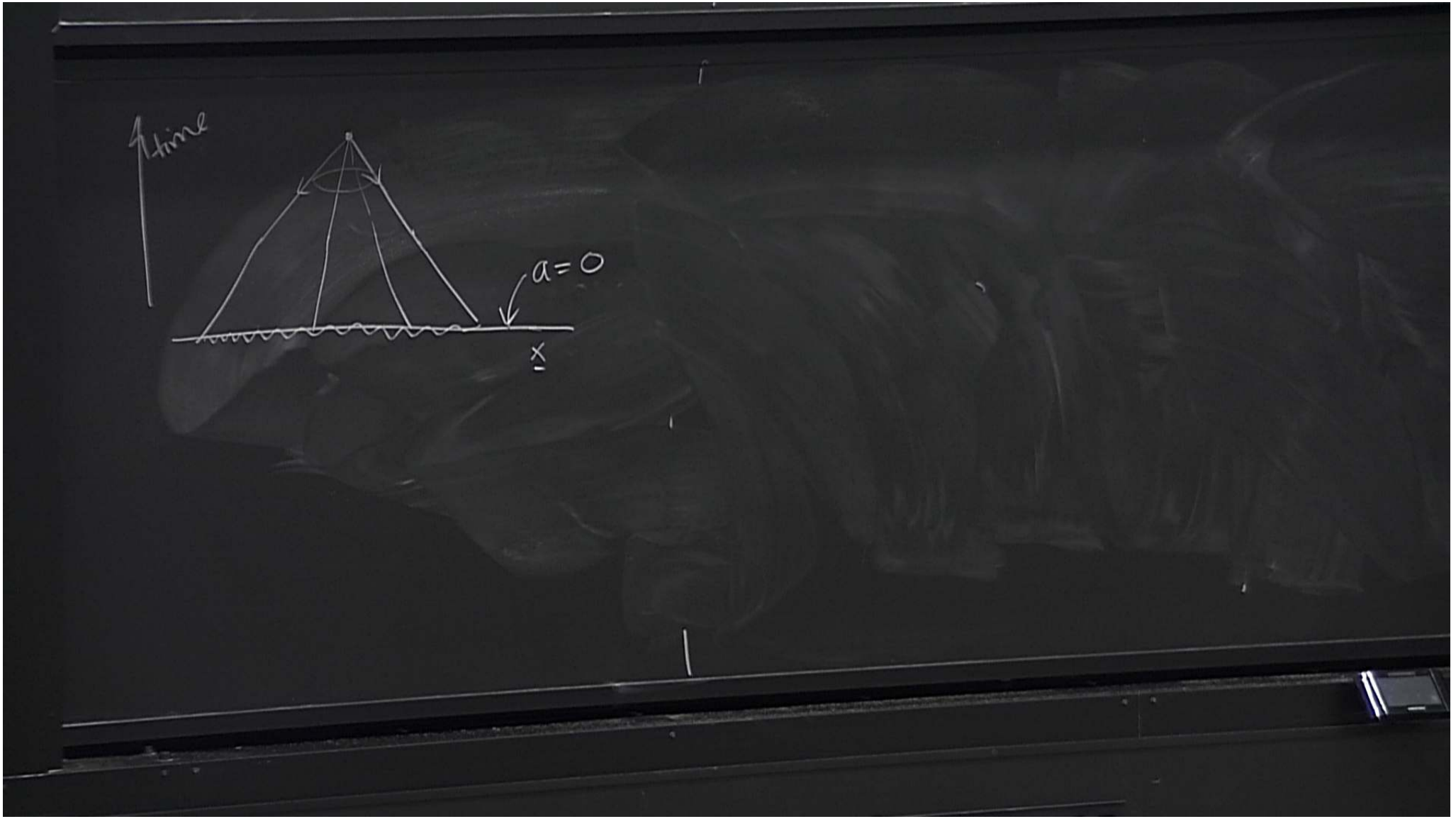
Hubble's law

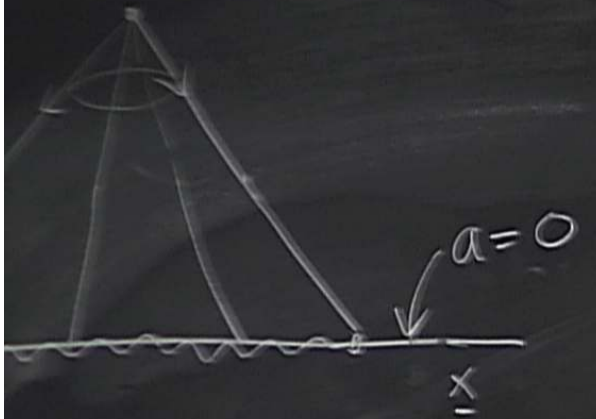
Hubble measured $H = 550 \text{ km s}^{-1} \text{ Mpc}^{-1}$

$1 \text{ Mpc} = 10^6 \text{ pc}$, 1pc

actual value measured today $\approx 65 \pm 10 \text{ km s}^{-1} \text{ Mpc}^{-1}$!


Compare with observations:
$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_r +$
Defn: $H = \frac{\dot{a}}{a}$
can be pos or negative
H^2
H = Hubble constant parameter: consider particles /
$\frac{\dot{d}}{d} = \frac{\dot{a}(t)}{a}$
- function of time!
$d(t)$, $a(t)$
d


time
a=0
x



$$-dt^2 + a^2(t)\,\gamma_{ij}dx^i dx^j$$

define new time

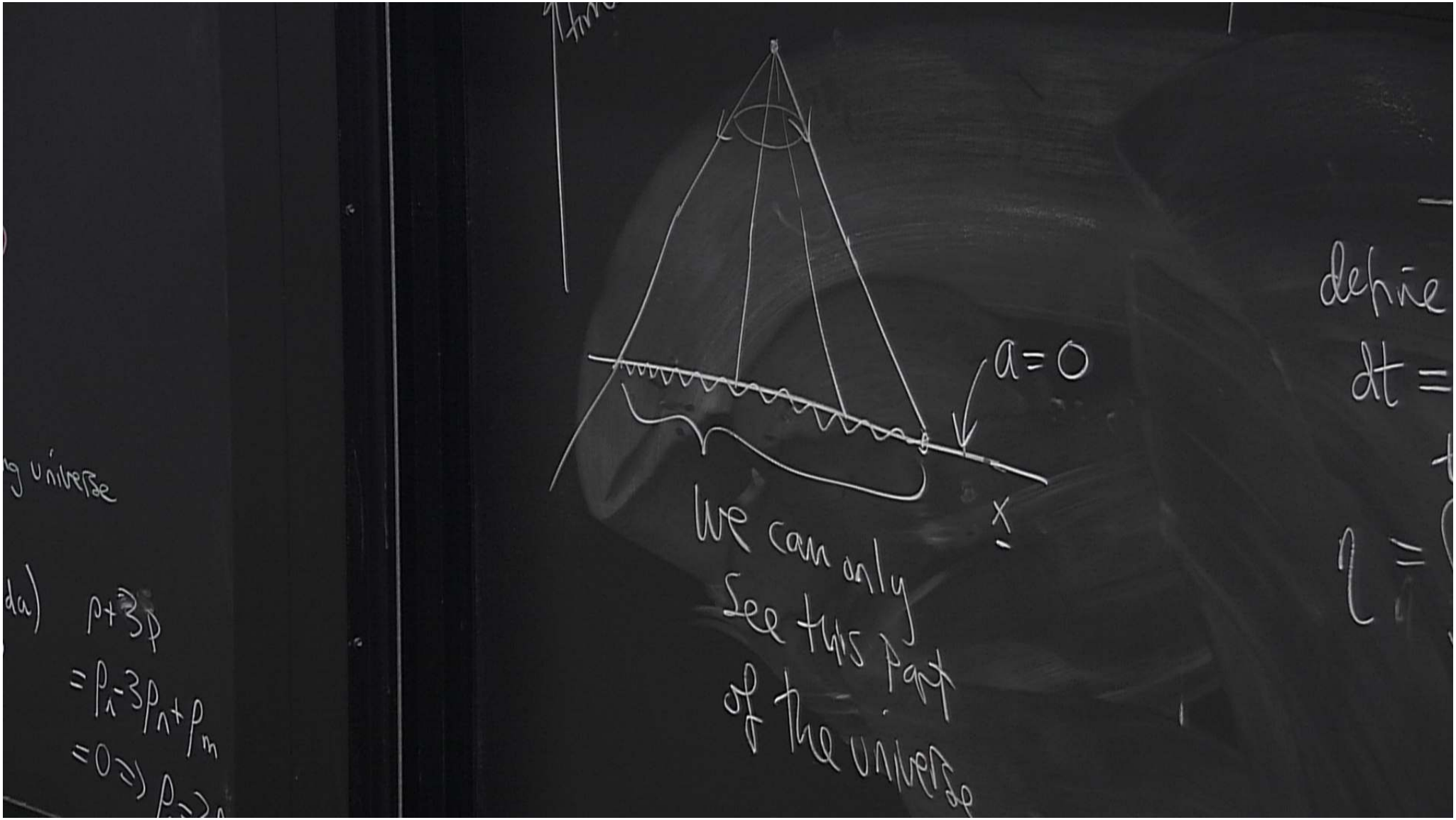
$$dt = a(t)\,d\eta$$

$$\eta = \int^t \frac{dt}{a(t)}$$

$$a^2(\eta)\left(-d\eta^2 + \gamma_{ij}dx^i dx^j\right)$$

$a^2(\eta)$: invisible to photons

$-d\eta^2 + \gamma_{ij}dx^i dx^j$: no singly!


a=0
x
We can only
See this part
of the universe
define
dt =
ρ+3p
= ρ_Λ - 3ρ_Λ + ρ_m
= 0 ⇒

Hubble measured $H = 550\ \text{km s}^{-1}\,\text{Mpc}^{-1}$ $\quad 1\,\text{Mpc} = 10^6\,\text{pc}$, $1\,\text{pc} \approx 3\,\ell$

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$$1 - (\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k) = 0$$

Obs^{n}!: $\quad .28 \quad .0003 \quad .72 \quad 0 \pm .01$

$1\,\mathrm{Mpc} = 10^6\,\mathrm{pc}$, $1\,\mathrm{pc} \approx 3$ light years.

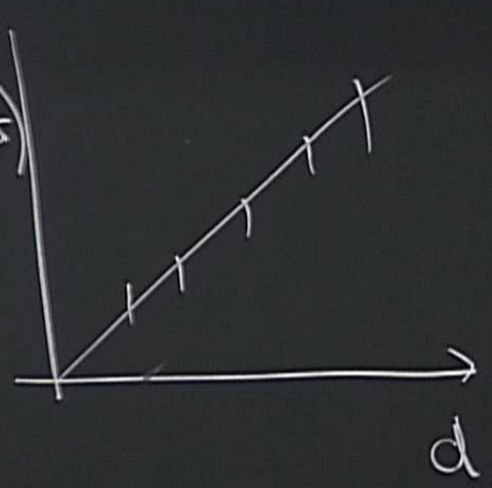
$65 \pm 10\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$.

ivide by H^2, define $\dfrac{8\pi G \rho_m}{3H^2} \equiv \Omega_m$

$= 0$

Ω_m includes cold dark matter
baryons, atoms etc
massive neutrinos

Hubble (1928)



Hubble measured $H = +55$

actual value measu

conventional to rewrite (*) a

$$\Omega_k \equiv \frac{k}{a^2}\frac{1}{H^2}$$

$$1 - (\Omega_m + \Omega$$

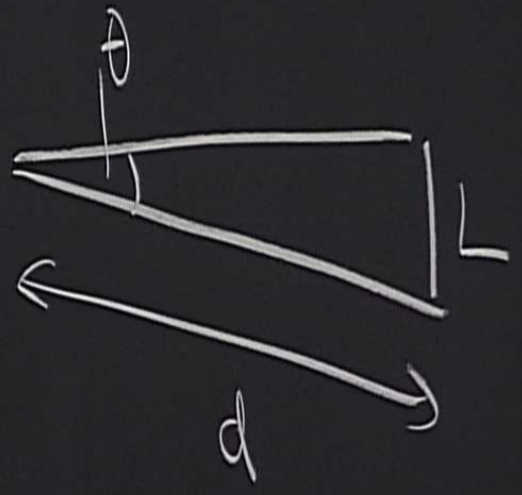
obs's → $|\Omega_k| \lesssim 0.01$ Obs'n: $\cdot 28 \cdot 0$

today $\approx 65 \pm 10$ km s^{-1} Mpc^{-1}.

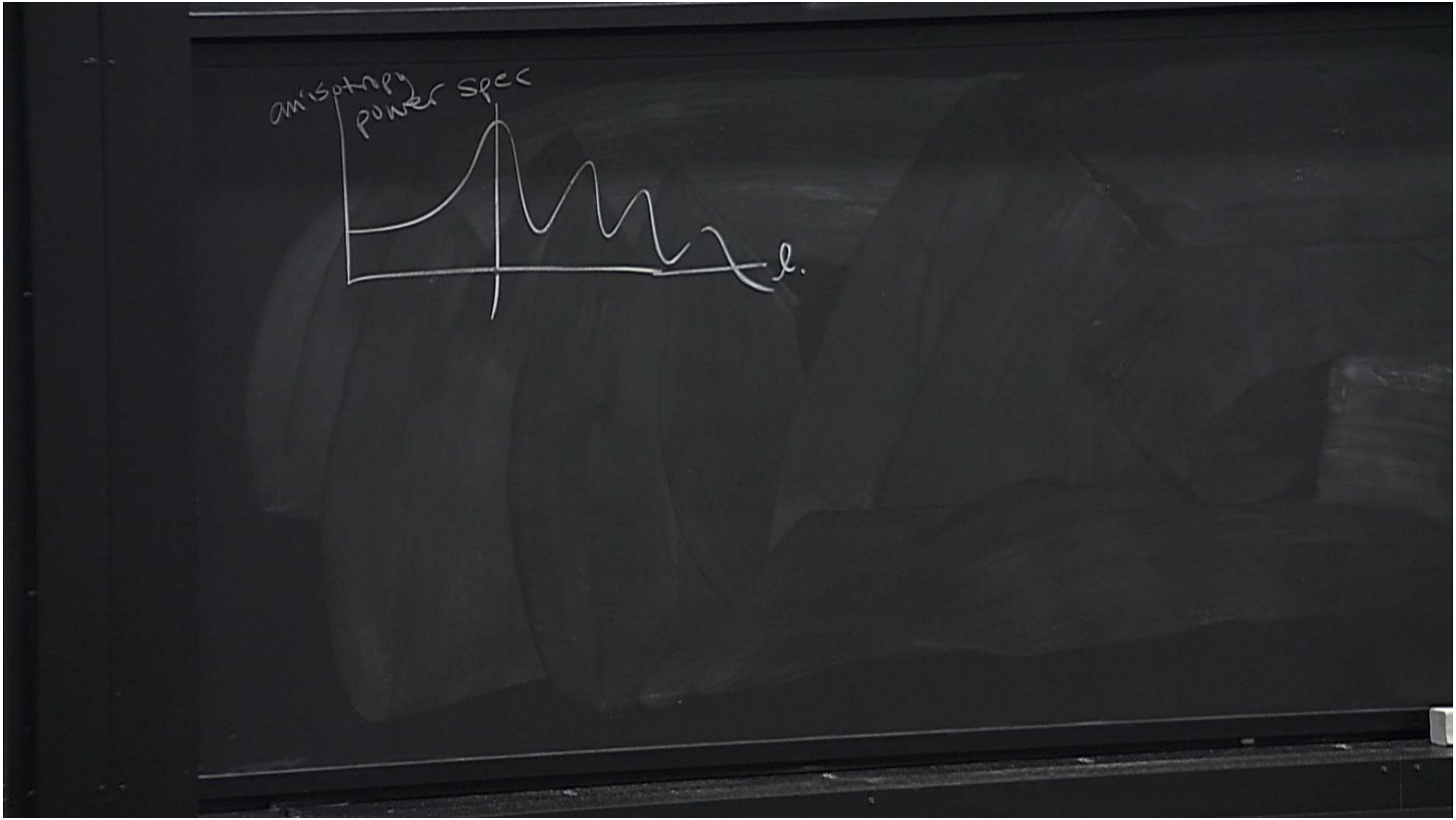
follows: divide by H^2, define $\frac{8\pi G \rho_m}{3H^2} \equiv \Omega_m$

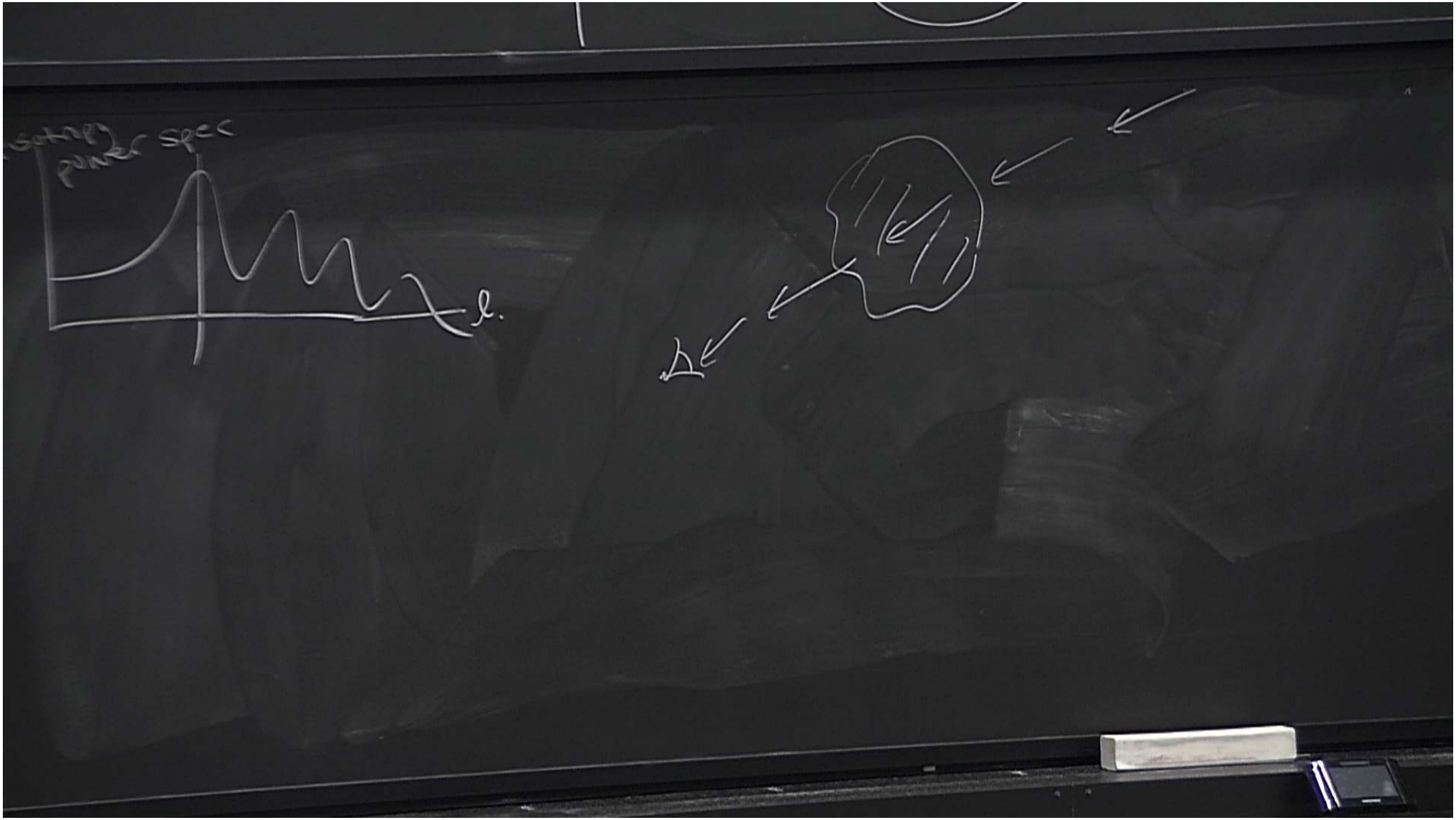
$+\Omega_\Lambda + \Omega_k) = 0$

Ω_m include



$\theta \simeq L/d$


anisotropy
power spec
l.

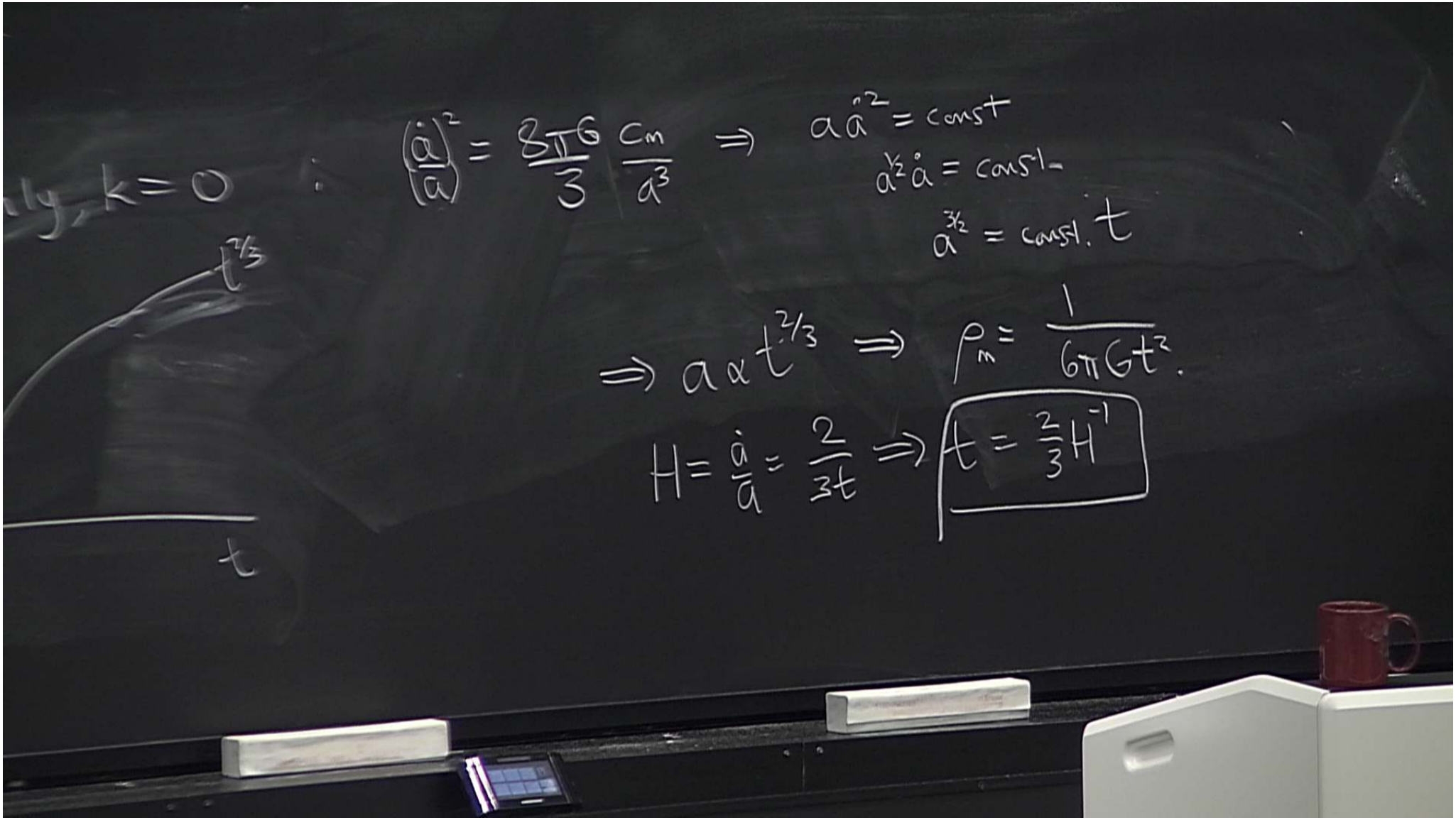

power spec
k.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{c_m}{a^3} \Rightarrow a\ddot{a}^2 = \text{const}$$

$$a^{1/2}\dot{a} = \text{const}$$

$$a^{3/2} = \text{const}\cdot t$$

$$\Rightarrow a \propto t^{2/3}$$


ly, k=0
$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{c_m}{a^3} \Rightarrow a\dot{a}^2 = \text{const}$
$a^{1/2}\dot{a} = \text{const}$
$a^{3/2} = \text{const}\cdot t$
$t^{2/3}$
t
$\Rightarrow a \propto t^{2/3} \Rightarrow \rho_m = \frac{1}{6\pi G t^2}$
$H = \frac{\dot{a}}{a} = \frac{2}{3t} \Rightarrow t = \frac{2}{3}H^{-1}$

$$\frac{C_m}{a^3} \Rightarrow a\dot{a}^2 = \text{const}$$

$$a^{1/2}\dot{a} = \text{const.}$$

$$a^{3/2} = \text{const.}\, t$$

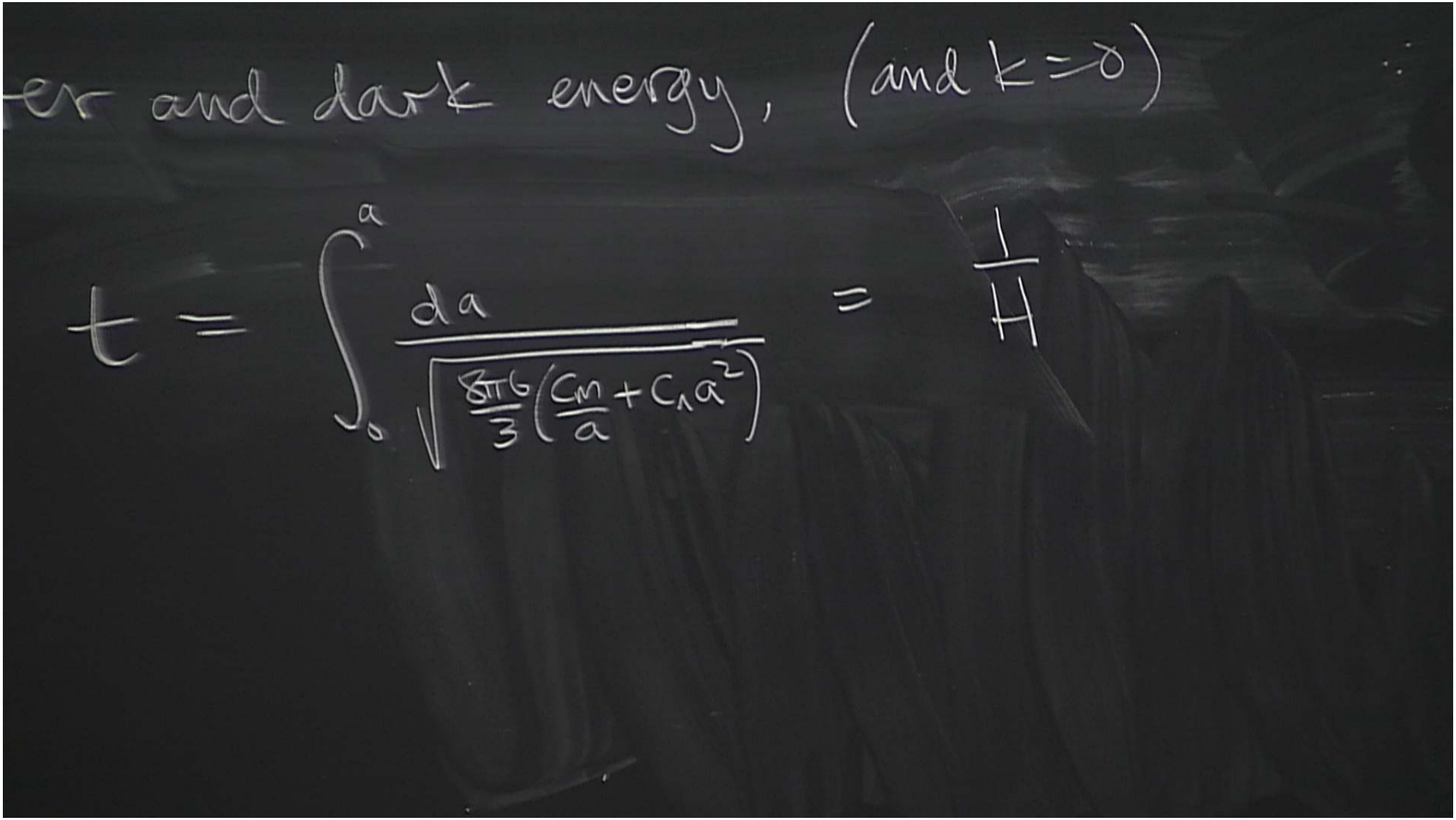
$$\Rightarrow a \propto t^{2/3} \Rightarrow \rho_m = \frac{1}{6\pi G t^2}.$$

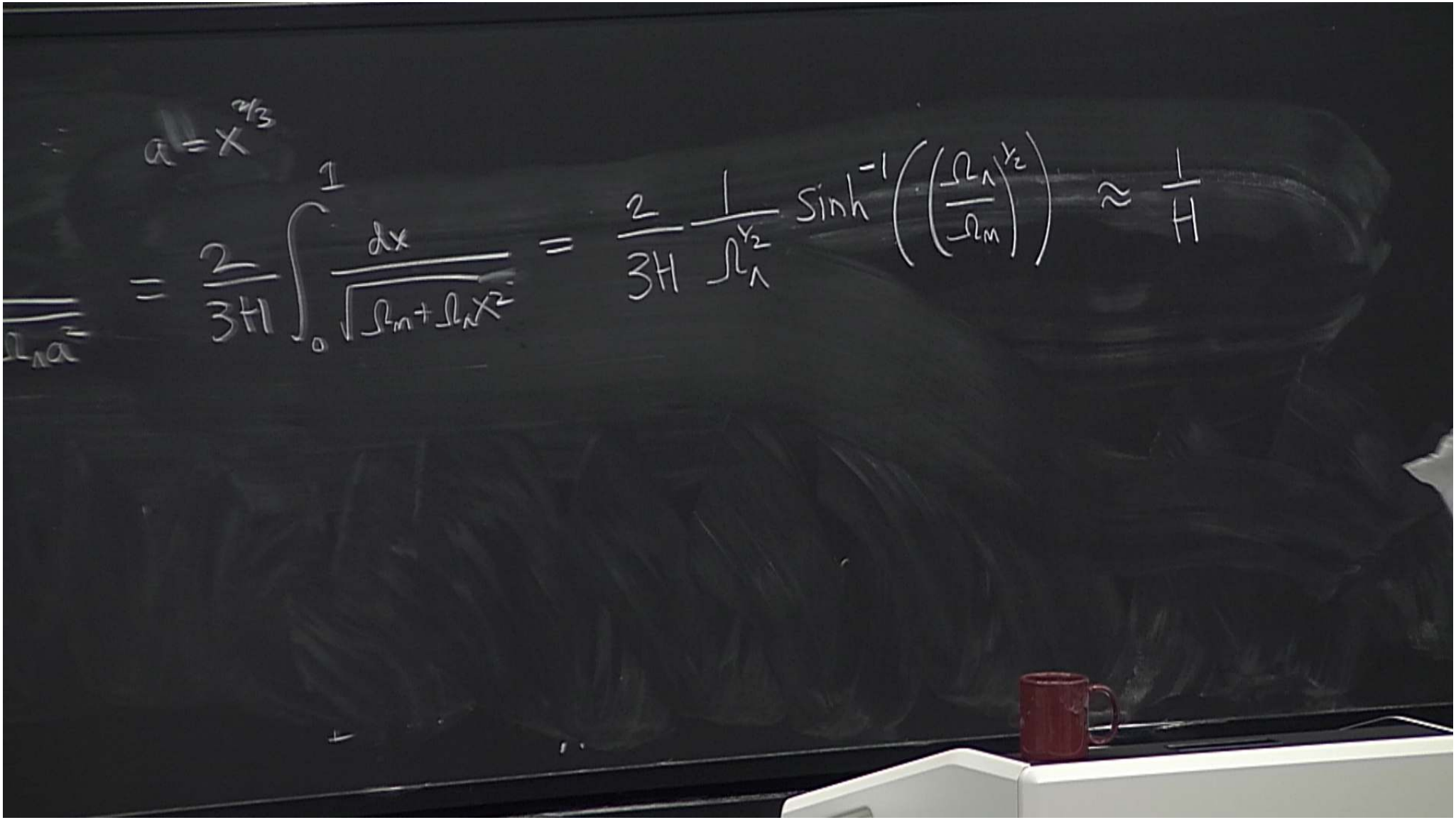
$$H = \frac{\dot{a}}{a} = \frac{2}{3t} \Rightarrow \boxed{t = \frac{2}{3}H^{-1}}$$

H^{-1} = time

$\simeq 15 \times 10^9$ yrs.

$\Rightarrow$ m.d.u. would be 10 billion years old.


er and dark energy, (and k=0)
$t = \int_0^a \frac{da}{\sqrt{\frac{8\pi G}{3}\left(\frac{C_M}{a} + C_\Lambda a^2\right)}} = \frac{1}{H}$


$a = x^{2/3}$
$= \frac{2}{3H} \int_0^1 \frac{dx}{\sqrt{\Omega_m + \Omega_\Lambda x^2}} = \frac{2}{3H} \frac{1}{\Omega_\Lambda^{1/2}} \sinh^{-1}\left(\left(\frac{\Omega_\Lambda}{\Omega_M}\right)^{1/2}\right) \approx \frac{1}{H}$
$\Omega_\Lambda a^2$

$t = x^{3/2}$

$$\frac{2}{3H}\int_0^1 \frac{dx}{\sqrt{\Omega_m + \Omega_\Lambda x^2}} = \frac{2}{3H}\frac{1}{\Omega_\Lambda^{1/2}} \sinh^{-1}\left(\left(\frac{\Omega_\Lambda}{\Omega_M}\right)^{1/2}\right) \approx \frac{1}{H}$$ for observed values