

Title: 13/14 PSI - Relativity - Lecture 11

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Abstract:

Sign error last time

$$g = \eta + h \quad R_{\mu\nu\rho\lambda} \sim \frac{1}{2}(\partial_\mu \partial_\nu g_{\rho\lambda} + \dots)$$

$$\begin{array}{c} \cancel{R_{\mu\nu\rho\lambda}} \\ \Downarrow \\ R_{\mu\nu\lambda\rho} \end{array}$$

$$= \frac{1}{2}(h_{\mu\rho,\nu\lambda} - \underline{h_{\nu\rho,\mu\lambda}} - h_{\mu\lambda,\nu\rho} + h_{\nu\lambda,\rho\mu}) + o(h^2)$$

$$\Rightarrow R_{00} = R^{\mu}_{\phantom{\mu}0\mu 0} = -\frac{1}{2}\eta^{\mu\rho} h_{00,\mu\rho} = -\frac{1}{4}\frac{\nabla^2 h_{00}}{\Box} = \kappa(T_{00} - \frac{1}{2}g_{00}T^{\lambda}_{\phantom{\lambda}\lambda})$$

$$= \eta^{\mu\rho} R_{\mu 0 \rho 0} + o(h^2)$$

$$h_{00} = -2\phi/c^2 \quad \nabla^2 \phi = 4\pi G\rho$$

$\phi = \text{Newtonian grav. pot.}$

$$\Rightarrow \boxed{G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}}$$



Sign error last time

$$g = \eta + h$$

$$R_{\mu\nu\rho\lambda} \sim \frac{1}{2}(\partial_\mu \partial_\nu h_{\rho\lambda} - \dots)$$

$$\begin{array}{c} \cancel{R_{\mu\nu\rho\lambda}} \\ \Downarrow \\ R_{\mu\nu\lambda\rho} \end{array}$$

$$= \frac{1}{2}(h_{\mu\rho,\nu\lambda} - \underline{h_{\nu\rho,\mu\lambda}} - h_{\mu\lambda,\nu\rho} + h_{\nu\lambda,\rho\mu}) + o(h^2)$$

$$\nabla_\alpha T^{\alpha\beta} = 0$$

$$\Rightarrow R_{00} = R^\mu{}_{0\mu 0} = -\frac{1}{2}\eta^{\mu\rho} h_{00,\mu\rho} = -\frac{1}{2}\nabla^2 h_{00} = \kappa(T_{00})$$

$$\Rightarrow \boxed{G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}}$$

$$+ c g_{\alpha\beta}$$

$$= \eta^{\mu\rho} R_{\mu 0 \rho 0} + o(h^2)$$

$$h_{00} = -2\phi/c^2 \quad ; \quad \nabla^2 \phi = 4\pi G \rho$$

$\phi = \text{Newtonian grav. pot.}$



# Cosmology

want to use GR to describe the whole (observable) universe!

→ study solutions to the field equations  $G_{\alpha\beta} = \frac{8\pi G}{c^2} T_{\alpha\beta}$  including matter

variable) universe!

$\frac{16\pi G}{3} T_{\mu\nu}$  including matter and time dependence.

Maximal Symmetry in Space : homogeneity and isotropy.

"cosmological principle"



# Cosmology

want to use GR to describe the whole (observable) universe!

→ study solutions to the field equations  $G_{\alpha\beta} = \frac{8\pi G}{c^2} T_{\alpha\beta}$  including matter

to simplify the solutions, impose symmetry: maximal symmetry in  $SA$

In  $d$  dimensions, greatest number of possible symmetries is  $\frac{d(d+1)}{2}$



to describe the whole (observable) universe!

to the field equations  $G_{\alpha\beta} = \frac{8\pi G}{c^2} T_{\alpha\beta}$  including matter and time dependence.

solutions, impose symmetry: maximal symmetry in space : homogeneity and isotropy

, greatest number of possible symmetries is  $\frac{d(d+1)}{2}$ . In  $d=3 \Rightarrow 6$  symmetries.



In  $d=3$ ,  $S^3$  : 6 rotations  $x\partial_y - y\partial_x$   
 $x^2 + y^2 + z^2 + u^2 = 1$  in  $\mathbb{E}^4$   $\underline{x}' = \underline{O}\underline{x}$

$\mathbb{E}^3$  : 3 rotations, 3 translations



In  $d=3$ ,

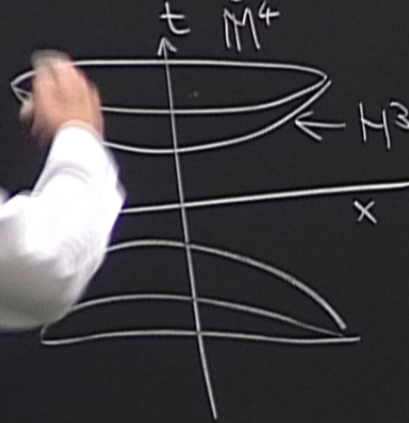
$S^3$  : 6 rotations

$$x^2 + y^2 + z^2 + u^2 = 1 \quad \text{in } \mathbb{E}^4$$

$$x\partial_y - y\partial_x$$

$$\underline{x}' = \underline{O}\underline{x}$$

$$t^2 - x^2 - y^2 - z^2 = 1$$



$\mathbb{E}^3$  : 3 rotations, 3 translations

$\leftarrow H^3$  : 3 boosts, 3 rotations.



Metri:

$$S^3: x^2 + y^2 + z^2 + u^2 = 1 \text{ in } \mathbb{E}^4$$

$$x = \sin \chi \sin \theta \cos \varphi$$

$$y = \sin \chi \sin \theta \sin \varphi$$

$$z = \sin \chi \cos \theta$$

$$u = \cos \chi$$

$$dx^2 + dy^2 + dz^2 + du^2 = d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2)$$

Also, commonly write this by defining  $r = \sin \chi$

(Here, I assumed the  $S^3$  had unit radius : if the radius was  $r_0$ , the



I assumed the  $S^3$  had unit radius. If the radius was  $r_0$ , then

$$ds^2 = \frac{dr^2}{1 - r^2/r_0^2} + r^2 d\Omega_2^2$$

obvious that for  $r \ll r_0$ , looks flat.

for

$$t = \cosh X$$

$$x = \sinh X \sin \theta \cos \phi$$

$$y = \sinh X \sin \theta \sin \phi$$

$$z = \sinh X \cos \theta$$

$$\Rightarrow ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = dX^2 + \sinh^2 X d\Omega_2^2$$

again, defining  $r = \sinh X$ , get  $\frac{dr^2}{1 + r^2/r_0^2} + r^2 d\Omega_2^2$   $\Rightarrow \frac{dr^2}{1 + (r/r_0)^2} + r^2 d\Omega_2^2$

$\frac{1}{r_0}$  of rad  $r_0$

$$z = \sinh X \cos \theta$$

$$u = \cosh X$$

Also, commonly write this by defining  $r = \sinh X$  (radius of  $S^3$  const.  $X$  surface)

$$\Rightarrow ds^2 = \boxed{\frac{dr^2}{1 + r^2/r_0^2} + r^2 d\Omega_2^2}$$

Spacetime metric :

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00} c^2 dt^2$$



Spacetime metric :

choose units  
where  $c=1$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \underbrace{g_{00}}_{g_{00}(t)} dt^2 + 2g_{0i} dt dx^i + g_{ij} dx^i dx^j$$

Homog + Iso in space  $\Rightarrow$

$$-g_{00} dt^2 \equiv dt'^2$$

$$\text{i.e. } t' \equiv \int \sqrt{-g_{00}(t)} dt$$

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$$= -dt'^2 + 2g'_{0i} dt' dx^i + g_{ij} dx^i dx^j$$

drop primes  $= -dt^2 + 2g_{0i} dt dx^i + g_{ij} dx^i dx^j$



Homog + Iso in space  $\Rightarrow$

$$\underbrace{g_{00}(t)}_{-g_{00} dt^2} = dt'^2$$

i.e.  $t' \equiv \int^t \sqrt{-g_{00}(t)} dt$

$$= -dt'^2 + 2g'_{0i} dt' dx^i + g_{ij} dx^i dx^j$$

drop  
primos

$$= -dt^2 + \underbrace{2g_{0i} dt dx^i}_{0 \text{ by isotropy}} + g_{ij} dx^i dx^j$$

$$= -dt^2 + 2g_{0i} dt dx^i + g_{ij} dx^i dx^j$$

drop  
primos

$$= -dt^2 + 2g_{0i} dt dx^i + g_{ij}^{(t,x)} dx^i dx^j$$

0 by isotropy

x-dependence has to  
be one of

$S^3$   
 $E^3$   
 $H^3$

$g_{\mu\nu}$



$$= -dt'^2 + 2g_{0i}' dt' dx^i + g_{ij}' dx^i dx^j$$

drop  
primos

$$= -dt^2 + 2\underbrace{g_{0i}}_{0 \text{ by isotropy}} dt dx^i + g_{ij}^{(t,x)} dx^i dx^j$$

x-dependence has to  
be one of

$$\Rightarrow g_{ij}(t,x) = a^2(t) \gamma_{ij}$$

$\Rightarrow$  can always  
choose coordinates  
such that

$$\underline{ds^2 = -dt^2 + a^2(t) \gamma_{ij} dx^i dx^j}$$

Friedmann- $\rho$



$$= -dt^2 + 2g_{0i}dt dx^i + g_{ij}dx^i dx^j$$

drop  
primos

$$= -dt^2 + 2g_{0i}dt dx^i + g_{ij}^{(t,x)} dx^i dx^j$$

0 by isotropy

x-dependence has to  
be one of

$S^3$   
 $E^3$   
 $H^3$

$g_{\mu\nu}$

$$\Rightarrow g_{ij}(t, \underline{x}) = a^2(t) \gamma_{ij}(\underline{x}), \text{ where } \gamma_{ij}(\underline{x}) \text{ is}$$

$$ds^2 = -dt^2 + a^2(t) \gamma_{ij}(\underline{x}) dx^i dx^j$$

Friedmann-Robertson-Walker line element.



Matter: assume perfect fluid

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}$$

Isotropy  $\Rightarrow u^\mu = (u^0, \underline{0})$  :  $g_{\mu\nu} u^\mu u^\nu = -1 \Rightarrow \underline{u^0 = 1}$  (future-pointing timelike)

+ fluid

$$p) u^\mu u^\nu + p g^{\mu\nu}$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & \\ & a^2 \gamma_{ij} \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & a^{-2} \gamma^{ij} \end{pmatrix}$$

$$u^0, \underline{0}) : g_{\mu\nu} u^\mu u^\nu = -1 \Rightarrow \underline{u^0 = 1} \quad (\text{future-pointing timelike vector})$$

$$\begin{pmatrix} 0 & 0 & 0 \\ \frac{p}{a^2} \gamma^{ij} \end{pmatrix}$$

[Actually, homogeneity force  $T^{\mu\nu}$  to take the form of a perfect fluid]



Einstein equations  $G_{\mu\nu} = 8\pi G T_{\mu\nu}$

$$g_{\mu\nu} \rightarrow \Gamma_{\mu\nu}^{\lambda} \rightarrow R_{\mu\nu\rho\lambda} \rightarrow R_{\mu\nu}$$



Einstein equations  $G_{\mu\nu} = 8\pi G T_{\mu\nu}$

$$g_{\mu\nu} \rightarrow \Gamma_{\mu\nu}^{\lambda} \rightarrow R_{\mu\nu\rho\lambda} \rightarrow R_{\mu\nu}$$

$$R_{00} = -3\frac{\ddot{a}}{a}$$

$$R_{0i} = 0$$

$$R_{ij} = R_{ij}^{(3)}(\gamma) + (a\ddot{a} + 2\dot{a}^2)\gamma_{ij}, \quad R = 6\dot{a}^2$$

for space of maximal symmetry

$$R_{ij}^{(3)}(\gamma) = 2k\gamma_{ij}, \text{ with } k \text{ a constant}$$

$$\underline{\underline{R^{(3)} = 6k}}$$



equations  $G_{\mu\nu} = 8\pi G T_{\mu\nu}$

$$R_{\mu\nu\rho\lambda} \rightarrow R_{\mu\nu}$$

$$R_{00} = -3\frac{\ddot{a}}{a}$$

$$R_{0i} = 0$$

$$R_{ij} = R_{ij}^{(3)}(\gamma) + (a\ddot{a} + 2\dot{a}^2)\gamma_{ij}, \quad R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right)$$

for a space of maximal symmetry

$$R_{ij}^{(3)}(\gamma) = 2k\gamma_{ij}, \text{ with } k \text{ a constant}$$

$$\underline{\underline{R^{(3)} = 6k}} \text{ in 3 dimensions}$$



maximal symmetry

$= 2k \delta_{ij}$ , with  $k$  a constant

$R = 6k$  in 3 dimensions

$$6 \left( \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right)$$

[ in fact, for  $S^3$  of radius  $r_0$ ,  $k = \frac{1}{r_0^2}$   
 $\mathbb{H}^3$  " "  $r_0$ ,  $k = -\frac{1}{r_0^2}$  ]



"scale factor" of the universe

$$= 3 \left( \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right) = 8\pi G\rho$$

$$-dt^2 + a^2(t) \left( \frac{dr^2}{1 - r^2/r_0^2} + r^2 d\Omega^2 \right)$$

FRW metric

"scale factor" of the universe

$$= 3 \left( \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right) = 8\pi G \rho$$

$$-dt^2 + \bar{a}^2(t) \left( \frac{dr^2}{1 - r^2/r_0^2} + r^2 d\Omega^2 \right)$$

FRW metric

imagine redefining  $a = \tilde{a}A$

$$\tilde{r} = Ar$$

$$\tilde{a}^2(t) \left( \frac{d\tilde{r}^2}{1 - \tilde{r}^2/\tilde{r}_0^2} + \tilde{r}^2 d\Omega^2 \right)$$

$$\text{choose } A = \frac{1}{r_0} \\ = \tilde{a}^2 \left( \frac{d\tilde{r}^2}{1 - \tilde{r}^2} + \tilde{r}^2 d\Omega^2 \right)$$

(can set  $\frac{1}{r_0^2} \rightarrow 1$ ,  
just by redefining  
 $a$  and  $r$ )



✓ "scale factor" of the universe

$$G_{00} = R_{00} - \frac{1}{2} g_{00} R = R_{00} + \frac{1}{2} R = 3 \left( \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right) = 8\pi G \rho$$

$$G_{ij} = R_{ij} - \frac{1}{2} a^2 \gamma_{ij} R =$$

(can see  
just



at this level of the discussion,  $p(t)$  is a completely

In fact, (3) is implied by (1) and (2) - Bianchi identity.



Finally, can use

$$\nabla_\mu T^{\mu\nu} = 0 = \partial_\mu T^{\mu\nu} + \Gamma_{\alpha\mu}^\mu T^{\alpha\nu} + \Gamma_{\alpha\mu}^\nu T^{\mu\alpha} \Rightarrow \dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) \quad (3)$$

(can set  $\frac{1}{f_0} \rightarrow$   
just by redefining  
 $a$  and

at this level of the discussion,  $p(t)$  is a completely

In fact, (3) is implied by (1) and (2) - Bianchi identity



a discussion,  $\rho(t)$  is a completely arbitrary function

and (2) - Bianchi identity.

e.g.  $p = 0$  "dust" : (3)  $\Rightarrow \dot{\rho}(t) = -3\frac{\dot{a}}{a}\rho \rightarrow \frac{d\rho}{\rho} = -3\frac{da}{a}$   
 $\Rightarrow \underline{\rho = \frac{c_m}{a^3}}$   
with  $c_m$  a constant.

$$u^\mu u^\nu + p g^{\mu\nu}$$

$$g_{\mu\nu} u^\mu u^\nu = -1 \Rightarrow \underline{u^0 = 1} \quad (\text{future-pointing timelike vector})$$

(Actually, homogeneity forces  $T^{\mu\nu}$  to take the form)



$$G_{00} = R_{00} - \frac{1}{2} g_{00} R = R_{00} + \frac{1}{2} R = \underline{3\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) = 8\pi G \rho}$$

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R = -(\ddot{a}^2 + 2a\ddot{a} + k)\gamma_{ij} = 8\pi G T_{ij} = 8\pi G p a^2 \gamma_{ij}$$

$$+2a\ddot{a} + 1 = -8\pi G p a^2 \quad (2)$$

Finally, can use

$$\nabla_\mu T^{\mu\nu} = 0 =$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) \quad (3)$$

(can see just)



$$G_{00} = R_{00} - \frac{1}{2} g_{00} R = R_{00} + \frac{1}{2} R = \underline{3\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) = 8\pi G \rho} \quad (1)$$

$$G_{ij} = R_{ij} - \frac{1}{2} a^2 \gamma_{ij} R \Rightarrow -(\ddot{a}^2 + 2a\ddot{a} + k)\gamma_{ij} = 8\pi G T_{ij} = 8\pi G p a^2 \gamma_{ij}$$

$$\Rightarrow \boxed{p = -\frac{1}{3}\rho}$$

$$\Rightarrow \underline{\ddot{a}^2 + 2a\ddot{a} + k = -8\pi G p a^2} \quad (2)$$

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$$\nabla_\mu T^{\mu\nu} = 0 = \partial_\mu T^{\mu\nu} + \Gamma_{\alpha\mu}^\mu T^{\alpha\nu} + \Gamma_{\alpha\mu}^\nu T^{\mu\alpha} \Rightarrow \underline{\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)} \quad (3)$$

(can set  $\frac{1}{a^2} \rightarrow 1$ ,  
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 $a$  and  $r$ )



Cosm const

$$P = -\rho$$

dust

$$P = 0$$

$$G_{00} = R_{00} - \frac{1}{2} g_{00} R = R_{00} + \frac{1}{2} R = 3 \left( \frac{\ddot{a}}{a} \right)$$

$$G_{ij} = R_{ij} - \frac{1}{2} a^2 \gamma_{ij} R \Rightarrow -(\ddot{a}^2 + 2a\ddot{a} + k)\gamma_{ij} = 8\pi G \rho a^2 \gamma_{ij}$$

$$\Rightarrow \boxed{P = -\frac{1}{3}\rho}$$

$$\Rightarrow \underline{\ddot{a}^2 + 2a\ddot{a} + k = -8\pi G \rho a^2}$$

Finally, can use

$$\nabla_\mu T^{\mu\nu} = 0 = \partial_\mu T^{\mu\nu} + \Gamma_{\alpha\mu}^\mu T^{\alpha\nu} + \Gamma_{\alpha\mu}^\nu T^{\mu\alpha} \Rightarrow \dot{\rho} = -3\frac{\dot{a}}{a} \rho$$