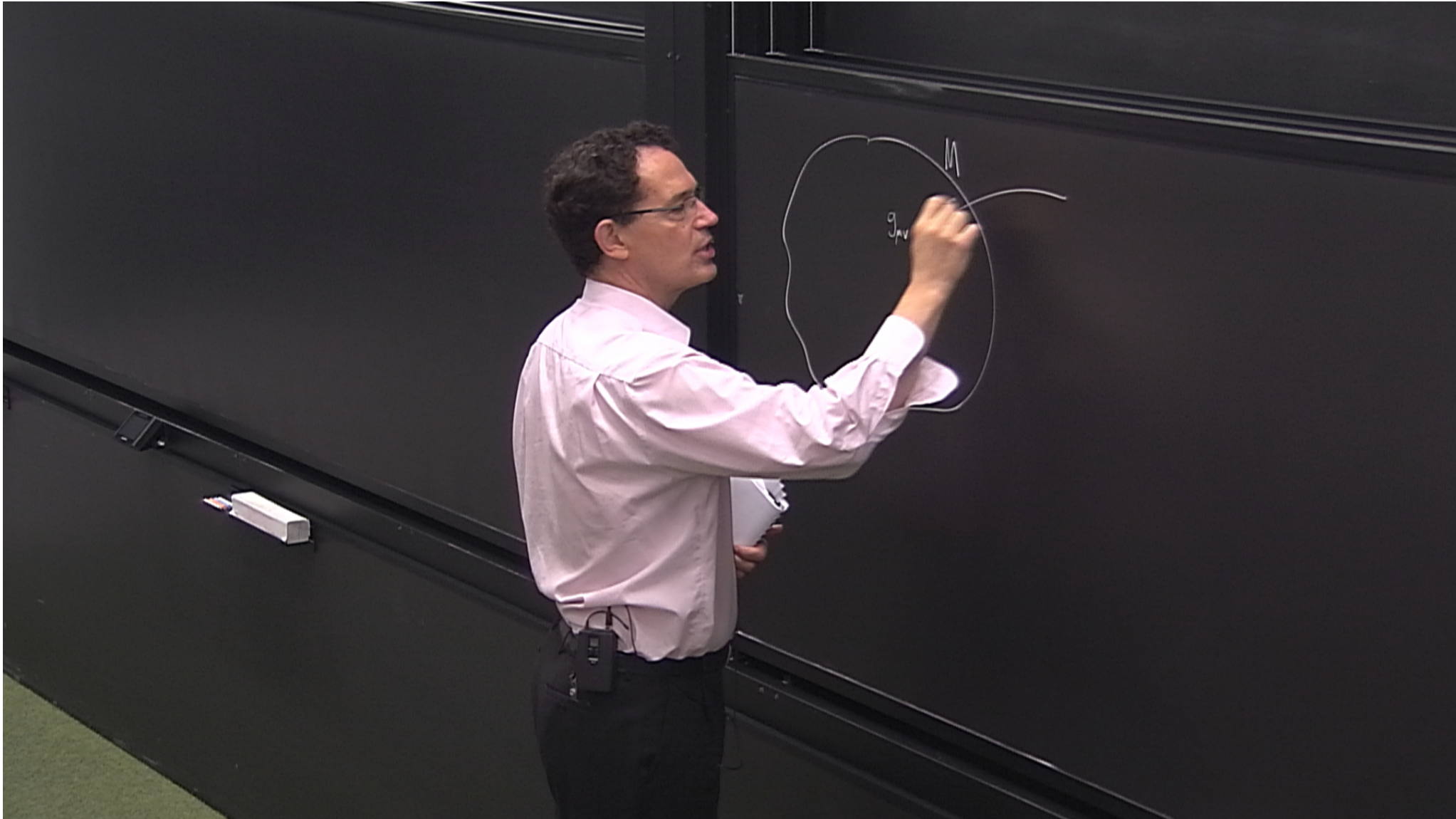


Title: 13/14 PSI - Relativity - Lecture 8

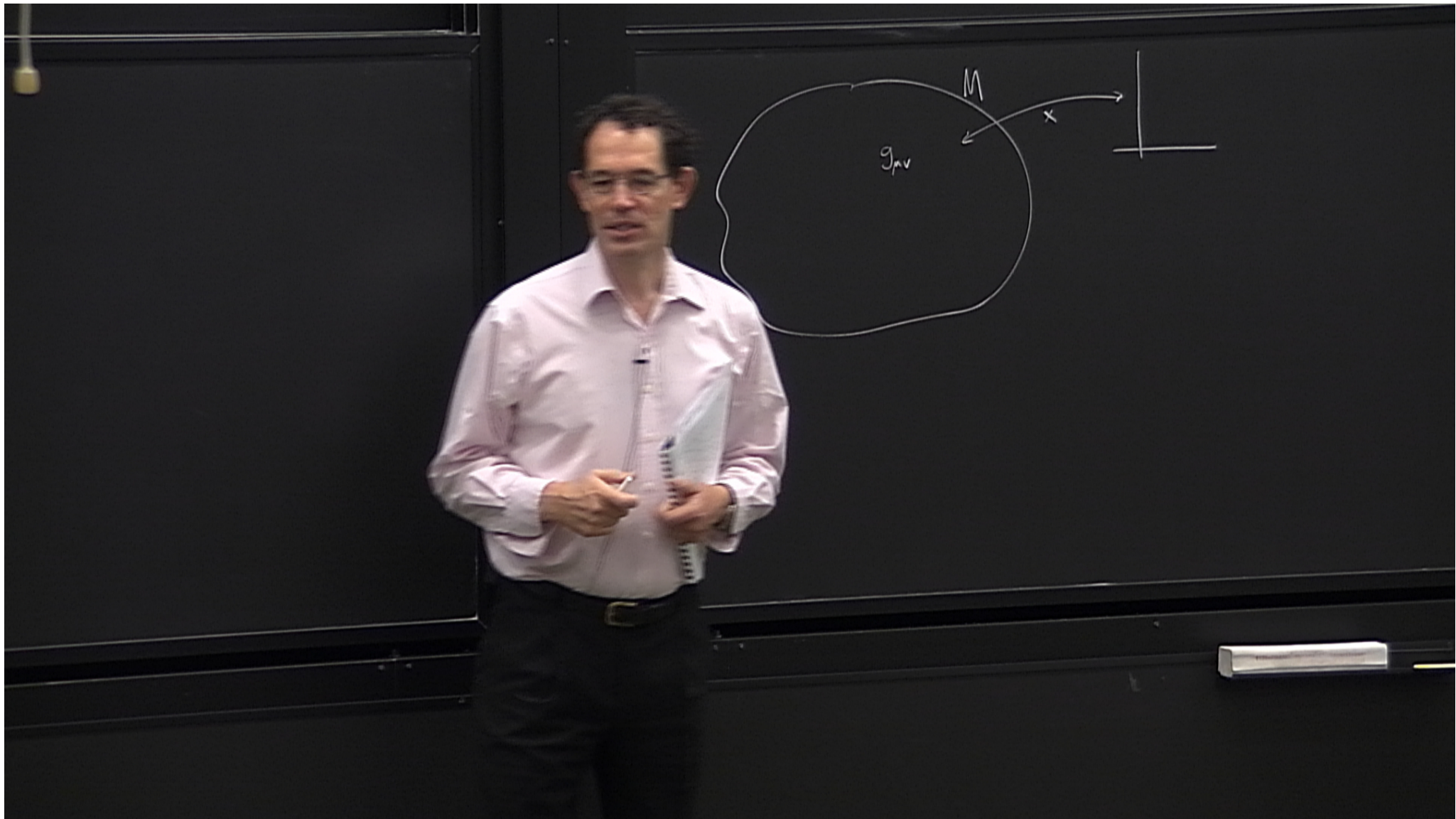
Date: Sep 18, 2013 09:00 AM

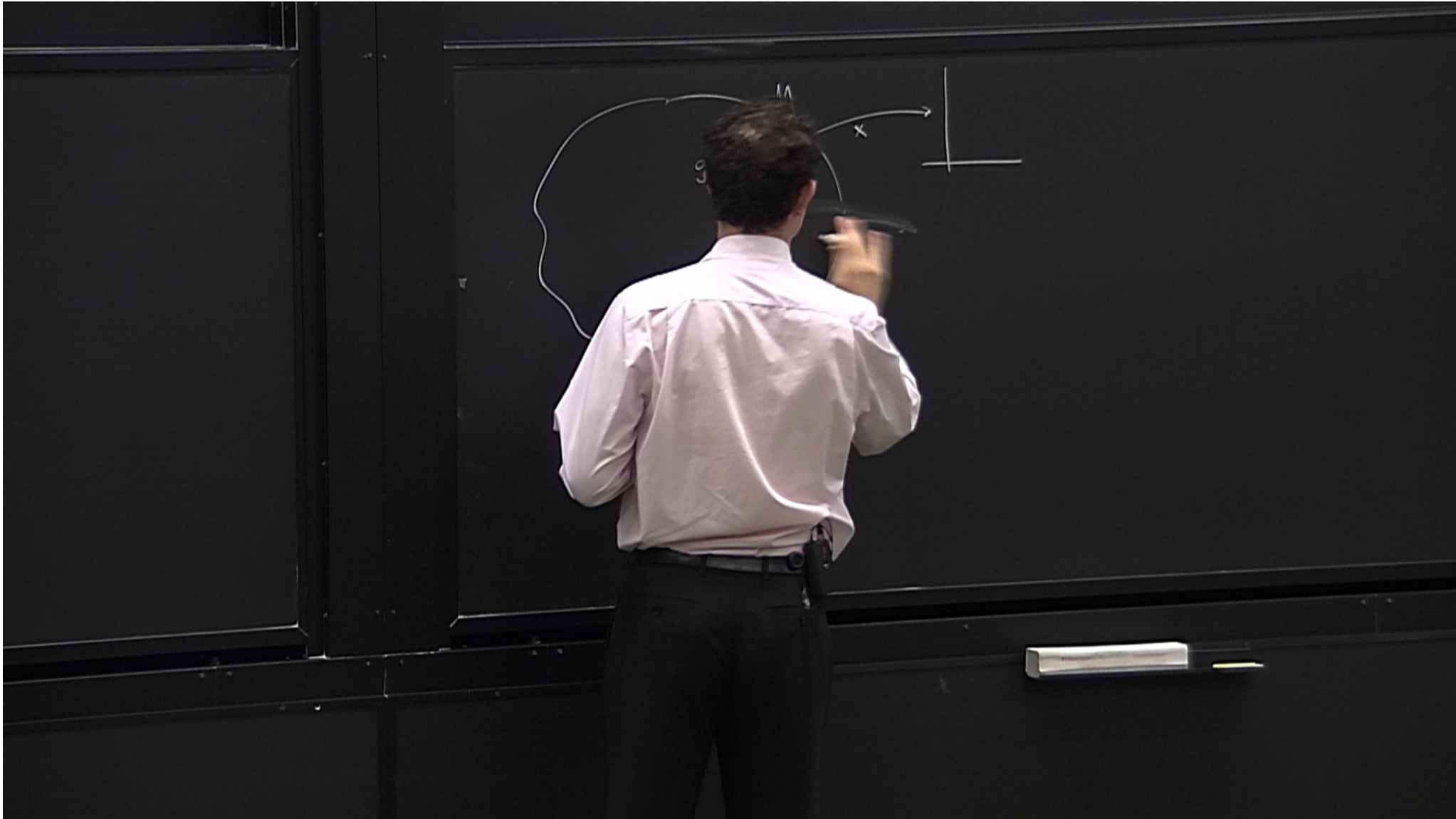
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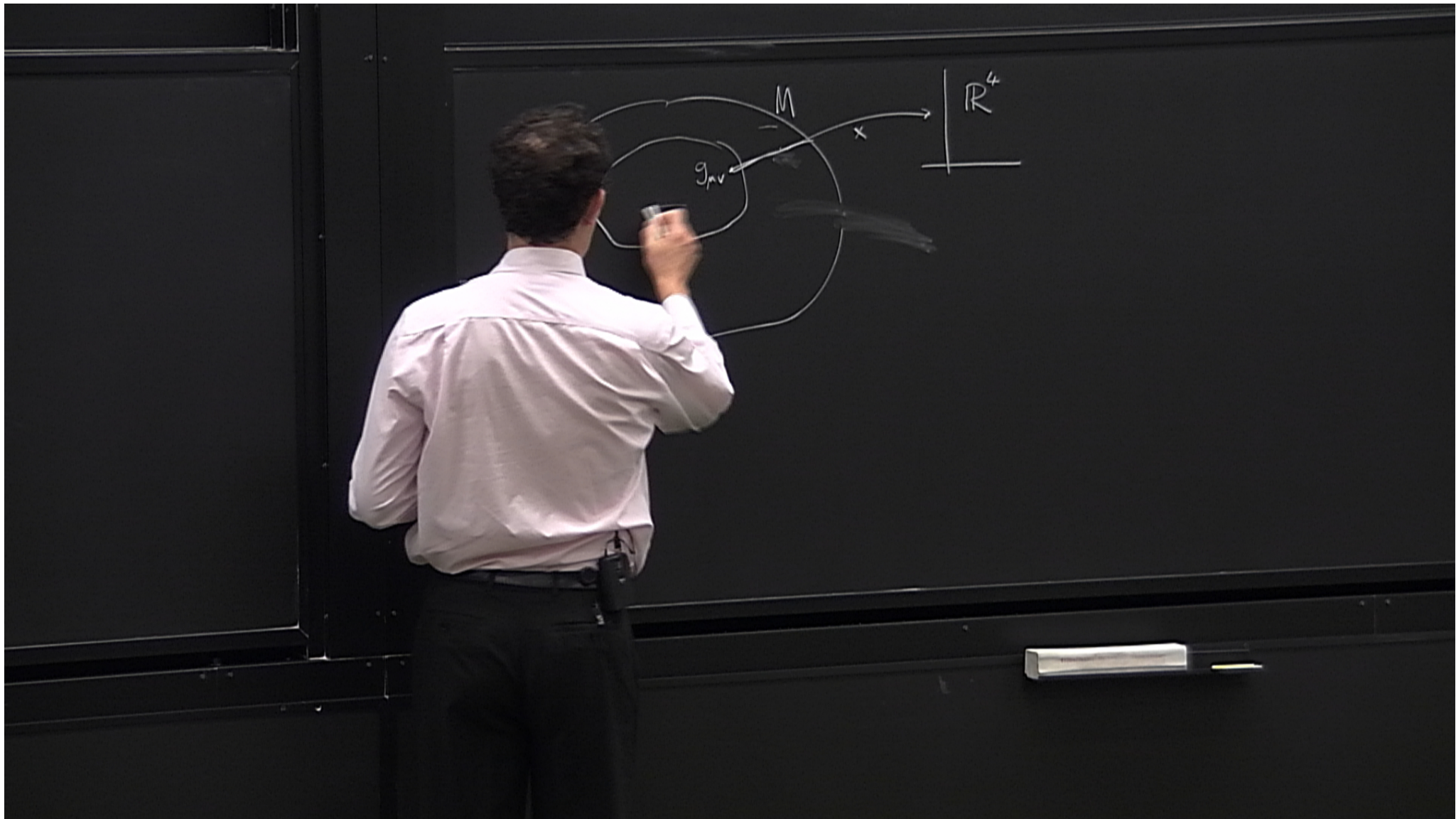
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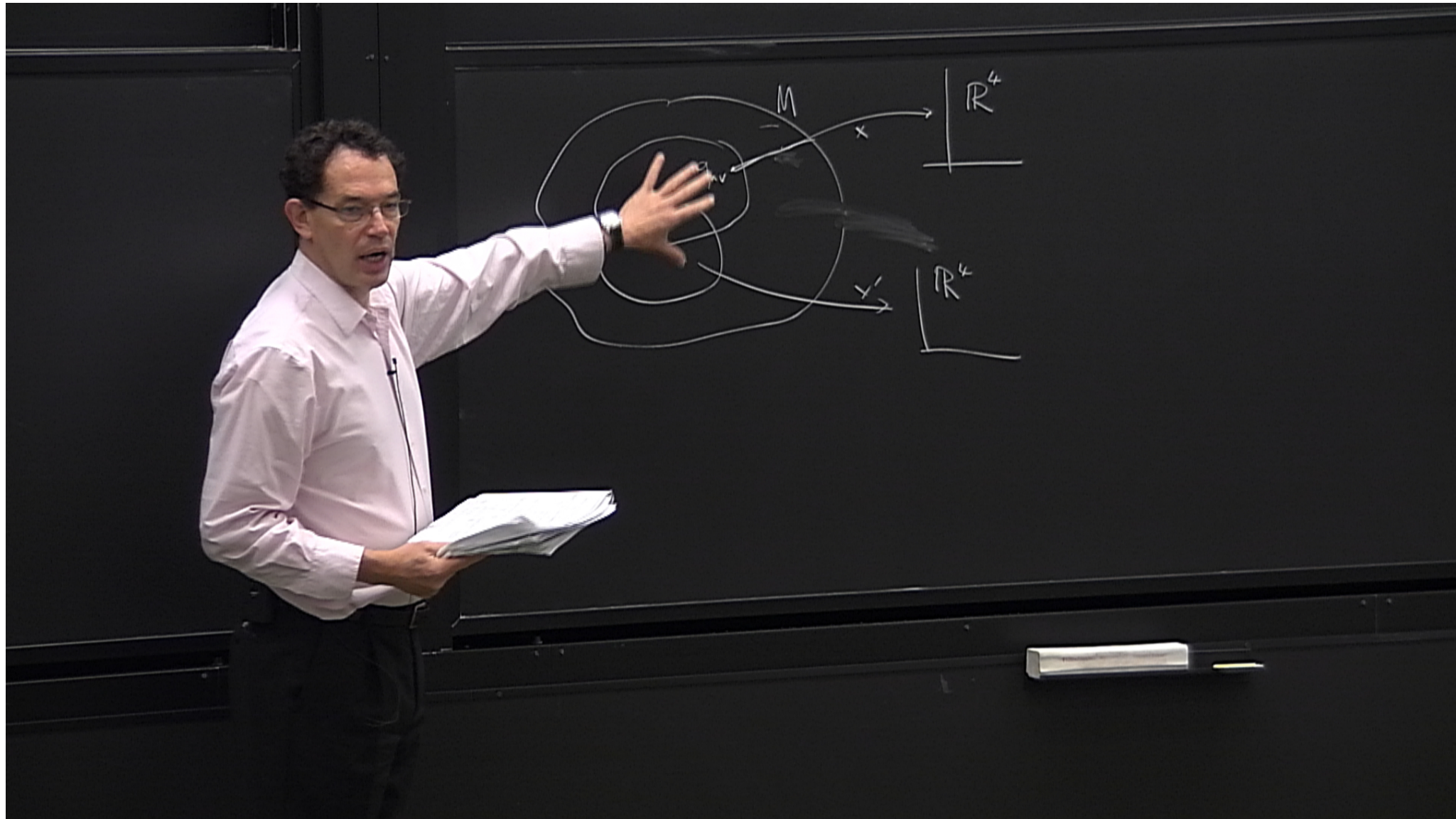


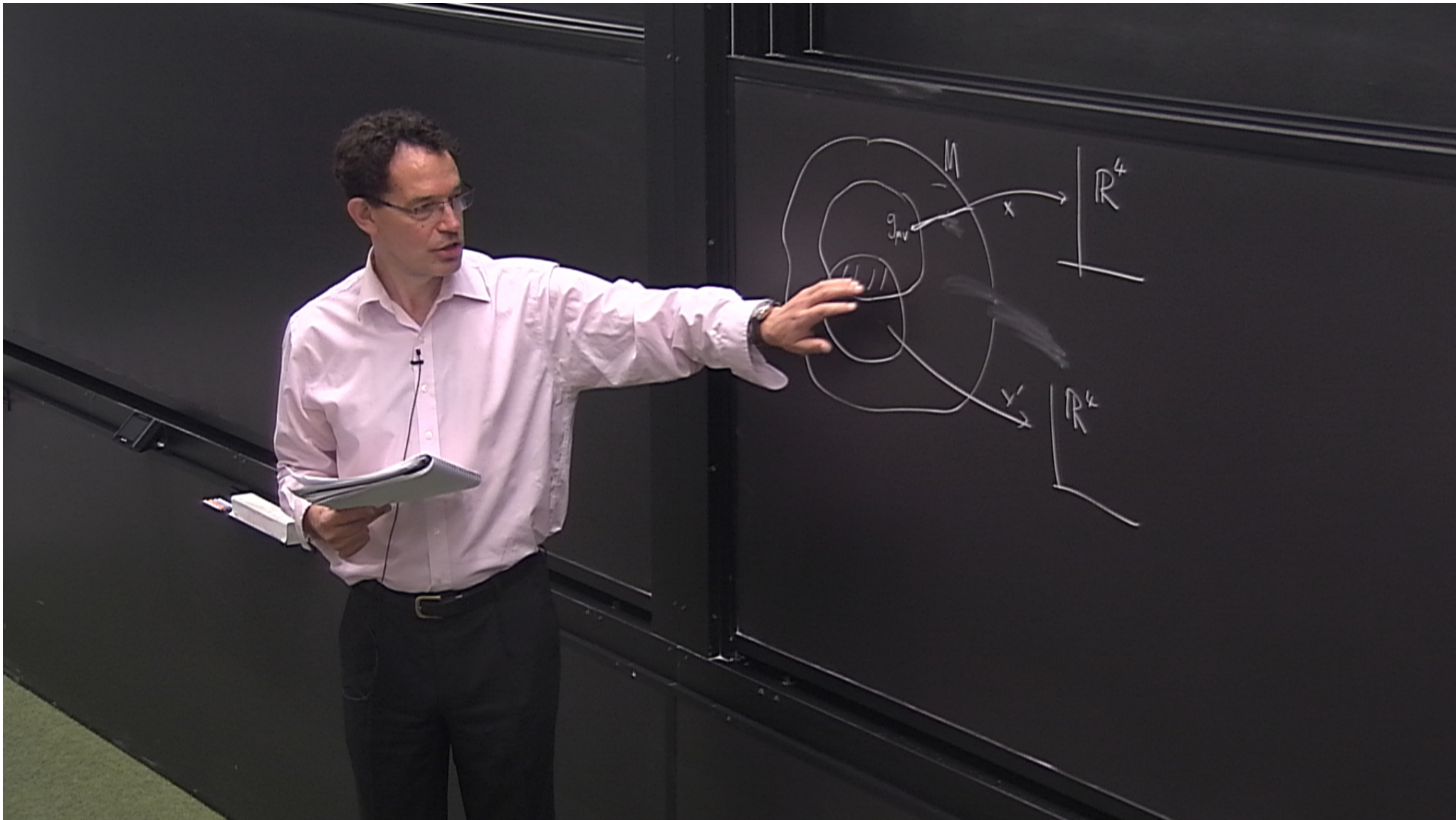










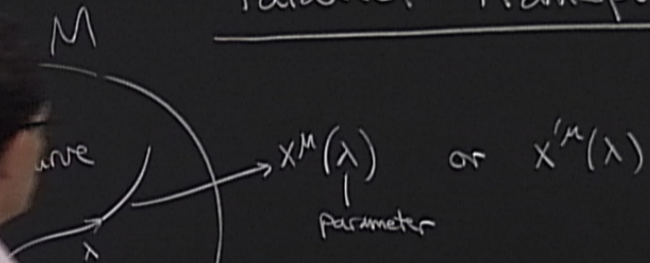


Parallel Trans and G

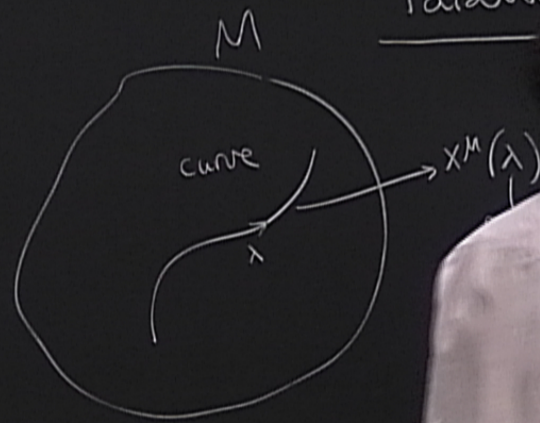




Parallel Transport and Geodesics



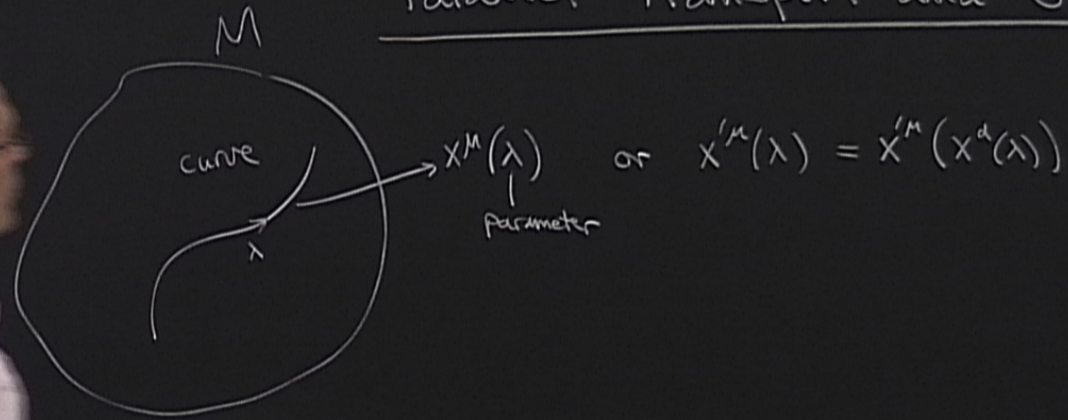
Parallel Transport and Geodesics

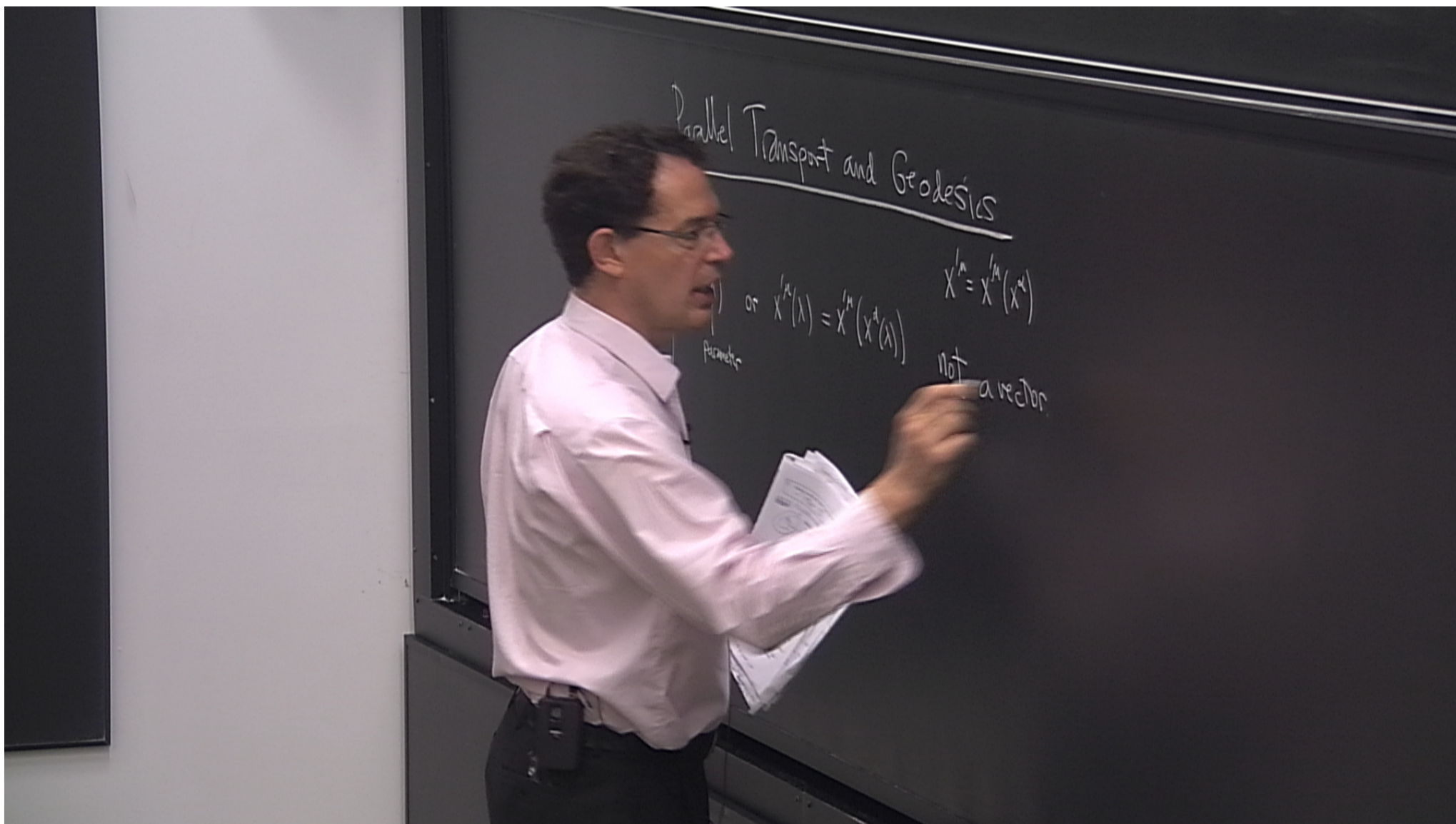


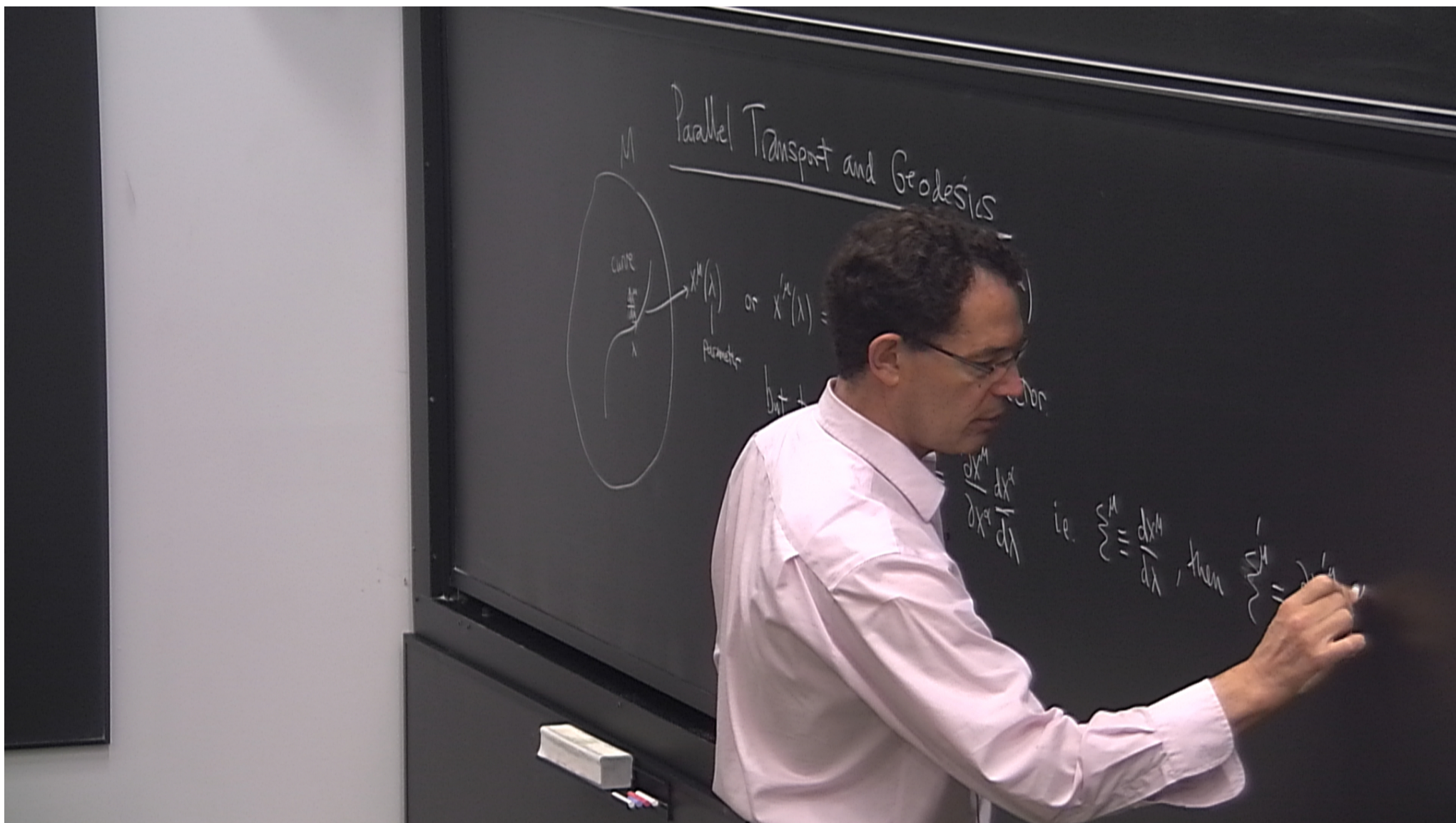
$$\dot{x}^\mu(\lambda) = \dot{x}^\mu(x^a(\lambda))$$

Parallel Transport and Geodesics

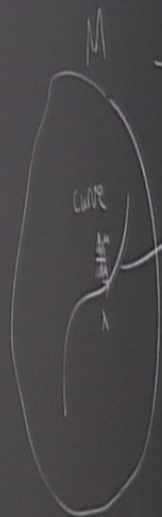
$$x'^{\mu} = x'^{\mu}(x^{\alpha})$$







Parallel Transport and Geodesics



$$x'^{\mu}(\lambda) \text{ or } x'^{\mu}(\lambda) = x'^{\mu}(x^{\alpha}(\lambda))$$

$$x'^{\mu} = x'^{\mu}(x^{\alpha})$$

not a vector

but tangent vector

$$\frac{dx'^{\mu}}{d\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{dx^{\alpha}}{d\lambda}$$

ie. $\xi^{\mu} = \frac{dx^{\mu}}{d\lambda}$, then $\xi'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \xi^{\alpha}$

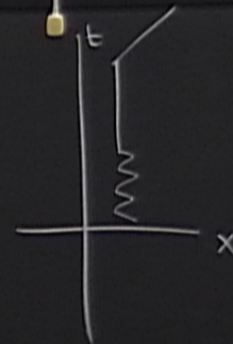
Parallel Transport and Geodesics

$x'^{\mu} = x'^{\mu}(x^{\alpha})$
 $x^{\mu}(\lambda)$ or $x'^{\mu}(\lambda) = x'^{\mu}(x^{\alpha}(\lambda))$ not a vector
 parameter but tangent vector $\frac{dx^{\mu}}{d\lambda} = \frac{\partial x^{\mu}}{\partial x^{\alpha}} \frac{dx^{\alpha}}{d\lambda}$

$\tilde{u}^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\alpha}} \tilde{u}^{\alpha}$

Parallel Transport and Geodesics

$x'^{\mu} = x'^{\mu}(x^{\alpha})$
 $x^{\mu}(\lambda)$ or $x'^{\mu}(\lambda) = x'^{\mu}(x^{\alpha}(\lambda))$ not a vector.
 parameter but tangent vector $\frac{dx'^{\mu}}{d\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{dx^{\alpha}}{d\lambda}$ i.e. $\sum^{\alpha} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \dot{x}^{\alpha}$



Parallel Transport and Geodesics

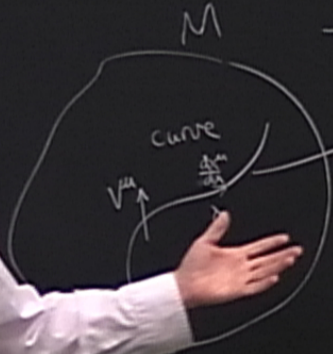
$$x'^{\mu} = x'^{\mu}(x^{\alpha})$$

$x^{\mu}(\lambda)$ or $x'^{\mu}(\lambda) = x'^{\mu}(x^{\alpha}(\lambda))$ not a vector
parameter

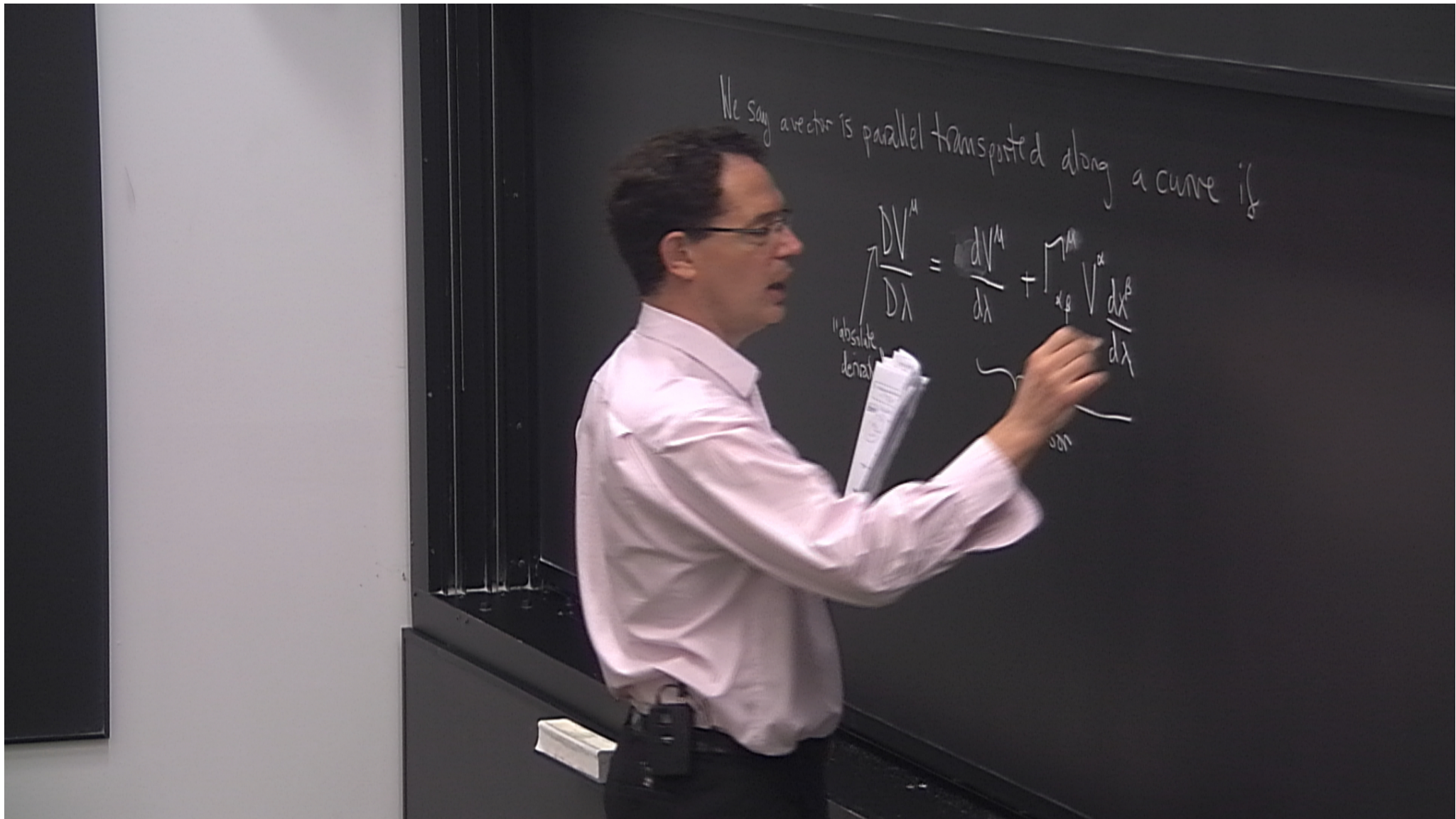
but tangent vector $\frac{dx'^{\mu}}{d\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{dx^{\alpha}}{d\lambda}$ ie $\xi^{\mu} = \frac{dx^{\mu}}{d\lambda}$, then $\xi'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \xi^{\alpha}$

vector $V^{\mu}(x)$ in spacetime

$V^{\mu}(x(\lambda))$ on curve; or can consider $V^{\mu}(\lambda)$ defined everywhere on the curve.



We say a vector is parallel transported along a curve if



We say a vector is parallel transported along a curve if

$$\frac{DV^M}{D\lambda} \equiv \underbrace{\frac{dV^M}{d\lambda} + \Gamma^M_{\alpha\beta} V^\alpha \frac{dx^\beta}{d\lambda}}_{\text{a 4-vector}} = 0$$

"absolute derivative"

We say a vector is parallel transported along a curve if

$$\frac{DV^\mu}{D\lambda} \equiv \frac{dV^\mu}{d\lambda} + \underbrace{\Gamma_{\alpha\beta}^\mu(x(\lambda)) V^\alpha(\lambda) \frac{dx^\beta}{d\lambda}}_{\text{a 4-vector}}$$

"absolute derivative"

We say a vector is parallel transported along a curve if

$$V^{\mu}(\lambda) \quad \nearrow \quad \frac{DV^{\mu}}{D\lambda} \equiv \underbrace{\frac{dV^{\mu}}{d\lambda} + \Gamma^{\mu}_{\alpha\beta}(x^{\nu}(\lambda)) V^{\alpha}(\lambda) \frac{dx^{\beta}}{d\lambda}}_{\text{a 4-vector}} = 0.$$

"absolute derivative"

vector $V^\mu(x)$ in spacetime

$V^\mu(x(\lambda))$ on curve, or can consider $V^\mu(\lambda)$ defined everywhere in the c
e.g spin

The simplest curves are geodesic curves

x^μ

vector $V^\mu(x)$ in spacetime

$V^\mu(x(\lambda))$ on curve, or can consider $V^\mu(\lambda)$ defined everywhere in the c
e.g spin

The simplest curves are geodesic curves

$x^\mu(\lambda)$, $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$ tangent vector

A curve is geodesic if $\frac{D\dot{x}^\mu}{d\lambda} =$

vector $V^\mu(x)$ in spacetime

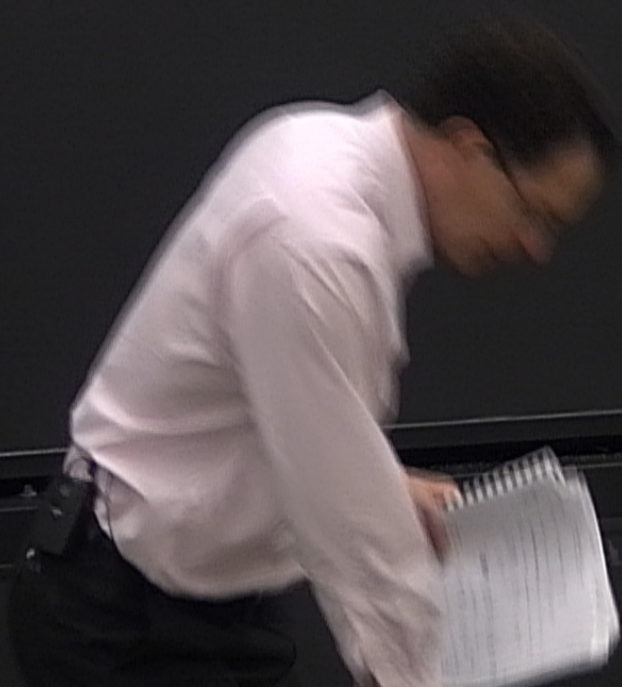
$V^\mu(x(\lambda))$ on curve, or can consider $V^\mu(\lambda)$ defined everywhere in the c
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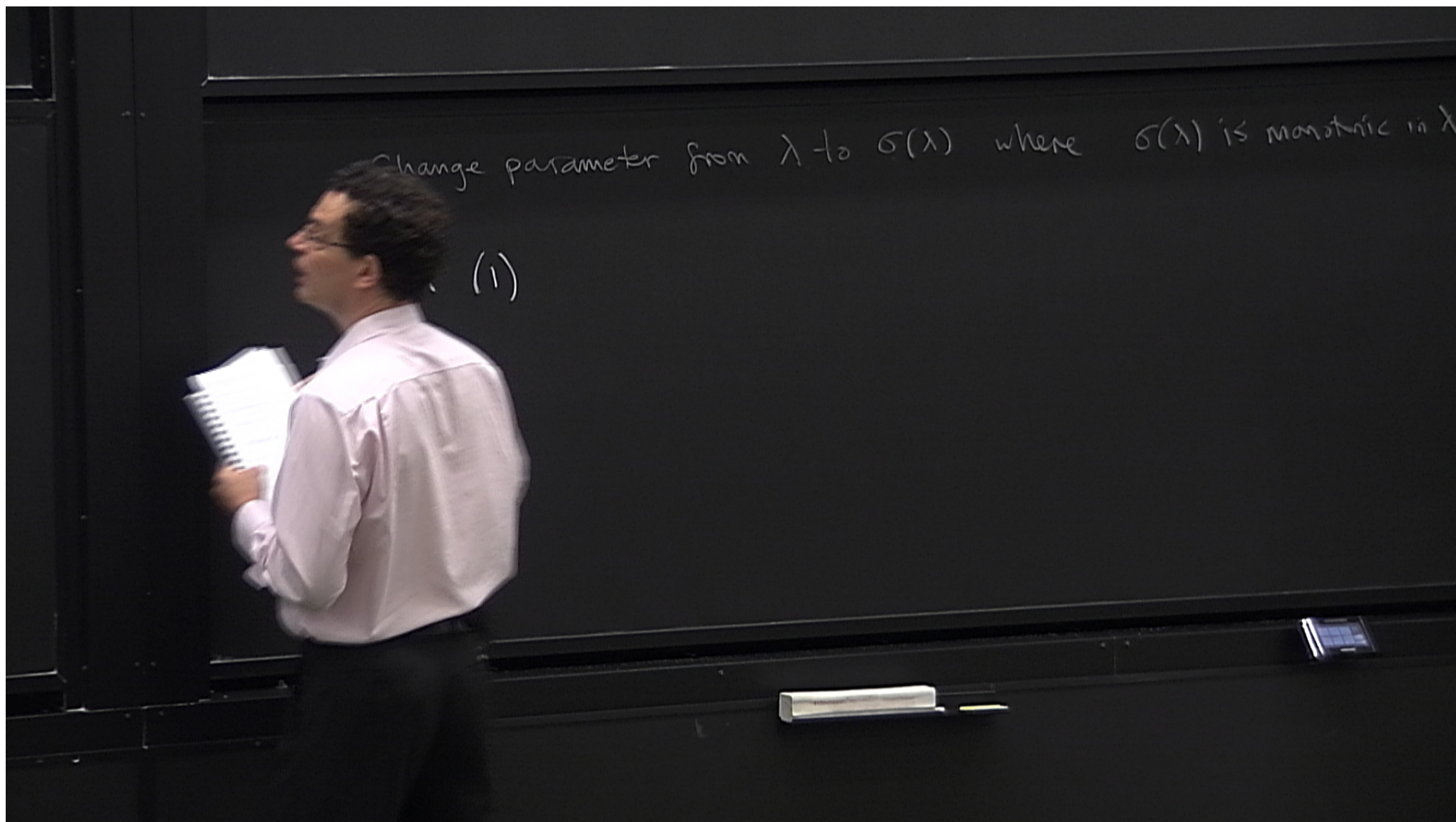
The simplest curves are geodesic curves

$x^\mu(\lambda)$, $\xi^\mu = \frac{dx^\mu}{d\lambda}$ tangent vector

A curve is geodesic if $\frac{D\xi^\mu}{d\lambda} = f(\lambda)\xi^\mu$

Change parameter from λ to $\sigma(\lambda)$





Change parameter from λ to $\sigma(\lambda)$ where $\sigma(\lambda)$ is monotonic in λ
then (i) come

Change parameter from λ to $\sigma(\lambda)$ where $\sigma(\lambda)$ is monotonic in λ

then (1) becomes $\frac{d}{dx} \left(\frac{d\sigma}{d\lambda} \frac{dx^\mu}{d\sigma} \right)$

$$\begin{aligned} x^\mu(\lambda) &\rightarrow x^\mu(\lambda(\sigma)) \\ &= x^\mu(\sigma) \end{aligned}$$

Change parameter from λ to $\sigma(\lambda)$ where $\sigma(\lambda)$ is monotonic in λ .

i) becomes
$$\frac{d}{d\lambda} \left(\frac{d\sigma}{d\lambda} \frac{dx^M}{d\sigma} \right) + \Gamma^M_{\beta\alpha} \frac{dx^\beta}{d\sigma} \frac{dx^\alpha}{d\sigma} \left(\frac{d\sigma}{d\lambda} \right)^2 = f(\lambda) \frac{dx^M}{d\sigma} \frac{d\sigma}{d\lambda}$$

$\rightarrow x^M(\lambda(\sigma))$
 $= x^M(\sigma)$

Change parameter from λ to $\sigma(\lambda)$ where $\sigma(\lambda)$ is monotonic in λ .

then (1) becomes $\frac{d}{d\lambda} \left(\frac{d\sigma}{d\lambda} \frac{dx^\mu}{d\sigma} \right) + \Gamma_{\beta\gamma}^\mu \frac{dx^\beta}{d\sigma} \frac{dx^\gamma}{d\sigma} \left(\frac{d\sigma}{d\lambda} \right)^2 = f(\lambda) \frac{dx^\mu}{d\sigma}$

$$\begin{aligned} x^\mu(\lambda) &\rightarrow x^\mu(\lambda(\sigma)) \\ &= x^\mu(\sigma) \end{aligned}$$

$$\Downarrow$$

$$\frac{d^2\sigma}{d\lambda^2} \frac{dx^\mu}{d\sigma} + \left(\frac{d\sigma}{d\lambda} \right)^2 \left(\frac{d^2x^\mu}{d\sigma^2} + \Gamma_{\beta\gamma}^\mu \frac{dx^\beta}{d\sigma} \frac{dx^\gamma}{d\sigma} \right) = f(\lambda) \frac{d\sigma}{d\lambda} \frac{dx^\mu}{d\sigma}$$

Change parameter from λ to $\sigma(\lambda)$ where $\sigma(\lambda)$ is monotonic in λ .

then (1) becomes $\frac{d\sigma}{d\lambda} \frac{dx^M}{d\sigma} + \Gamma_{\beta\alpha}^M \frac{dx^\beta}{d\sigma} \frac{dx^\alpha}{d\sigma} \left(\frac{d\sigma}{d\lambda}\right)^2 = f(\lambda) \frac{dx^M}{d\sigma} \frac{d\sigma}{d\lambda}$

$$\begin{aligned} x^M(\lambda) &\rightarrow x^M(\sigma) \\ &= x^M + \left(\frac{d\sigma}{d\lambda}\right)^2 \left(\frac{d^2 x^M}{d\sigma^2} + \Gamma_{\beta\alpha}^M \frac{dx^\beta}{d\sigma} \frac{dx^\alpha}{d\sigma} \right) = f(\lambda) \frac{d\sigma}{d\lambda} \frac{dx^M}{d\sigma} \end{aligned}$$

Change parameter from λ to $\sigma(\lambda)$ where $\sigma(\lambda)$ is monotonic in λ .

then (1) becomes
$$\frac{d}{d\lambda} \left(\frac{d\sigma}{d\lambda} \frac{dx^\mu}{d\sigma} \right) + \Gamma_{\beta\gamma}^\mu \frac{dx^\beta}{d\sigma} \frac{dx^\gamma}{d\sigma} \left(\frac{d\sigma}{d\lambda} \right)^2 = f(\lambda) \frac{dx^\mu}{d\sigma} \frac{d\sigma}{d\lambda}$$

$$x^\mu(\lambda) \rightarrow x^\mu(\lambda(\sigma)) \\ = x^\mu(\sigma)$$

$$\Downarrow \\ \frac{d^2\sigma}{d\lambda^2} \frac{dx^\mu}{d\sigma} + \left(\frac{d\sigma}{d\lambda} \right)^2 \left(\frac{d^2x^\mu}{d\sigma^2} + \Gamma_{\beta\gamma}^\mu \frac{dx^\beta}{d\sigma} \frac{dx^\gamma}{d\sigma} \right) = f(\lambda) \frac{d\sigma}{d\lambda} \frac{dx^\mu}{d\sigma}$$

cancel if $\frac{d^2\sigma}{d\lambda^2} = f(\lambda) \frac{d\sigma}{d\lambda}$

$$f \frac{d\sigma}{d\lambda^2} = f(\lambda) \frac{d\sigma}{d\lambda}$$

$$\Rightarrow \frac{d}{d\lambda} \left(\frac{d\sigma}{d\lambda} \right) = f(\lambda) \frac{d\sigma}{d\lambda} \Rightarrow \frac{d\sigma}{d\lambda} = A e^{\int f(\lambda) d\lambda}$$

$$\frac{d\sigma}{d\lambda} = f(\lambda) \frac{d\sigma}{d\lambda}$$

$$\Rightarrow \frac{d}{d\lambda} \left(\frac{d\sigma}{d\lambda} \right) = f(\lambda) \frac{d\sigma}{d\lambda} \Rightarrow \frac{d\sigma}{d\lambda} = A e^{\int f(\lambda) d\lambda} \text{ with } A \text{ an arbitrary constant.}$$

$$\frac{d\sigma}{d\lambda} = f(\lambda) \frac{d\sigma}{d\lambda}$$

$$\Rightarrow \frac{d}{d\lambda} \left(\frac{d\sigma}{d\lambda} \right) = f(\lambda) \frac{d\sigma}{d\lambda} \Rightarrow \frac{d\sigma}{d\lambda} = A e^{\int f(\lambda) d\lambda} \text{ with } A \text{ an arbit. constant.}$$

then geodesic eqⁿ. is just $\frac{dx^M}{d\sigma^2} + \Gamma_{\alpha\beta}^M \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} = 0$

ge

$$\frac{d\sigma}{d\lambda} = f(\lambda) \frac{d\sigma}{d\lambda}$$

$$\Rightarrow \frac{d}{d\lambda} \left(\frac{d\sigma}{d\lambda} \right) = f(\lambda) \frac{d\sigma}{d\lambda} \Rightarrow \frac{d\sigma}{d\lambda} = A e^{\int f(\lambda) d\lambda} \quad \text{with } A \text{ an arbitrary constant.}$$

then geodesic eqⁿ. is just $\frac{dx^M}{d\sigma^2} + \Gamma_{\alpha\beta}^M \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} = 0$ geodesic equation

and σ is called an affine parameter

canal if $\frac{d^2\sigma}{d\lambda^2} = f(\lambda) \frac{d\sigma}{d\lambda}$

$\Rightarrow \frac{d}{d\lambda} \left(\frac{d\sigma}{d\lambda} \right) = f(\lambda) \frac{d\sigma}{d\lambda} \Rightarrow d\sigma = A e^{\int f(\lambda) d\lambda}$ with A an arbitrary constant.

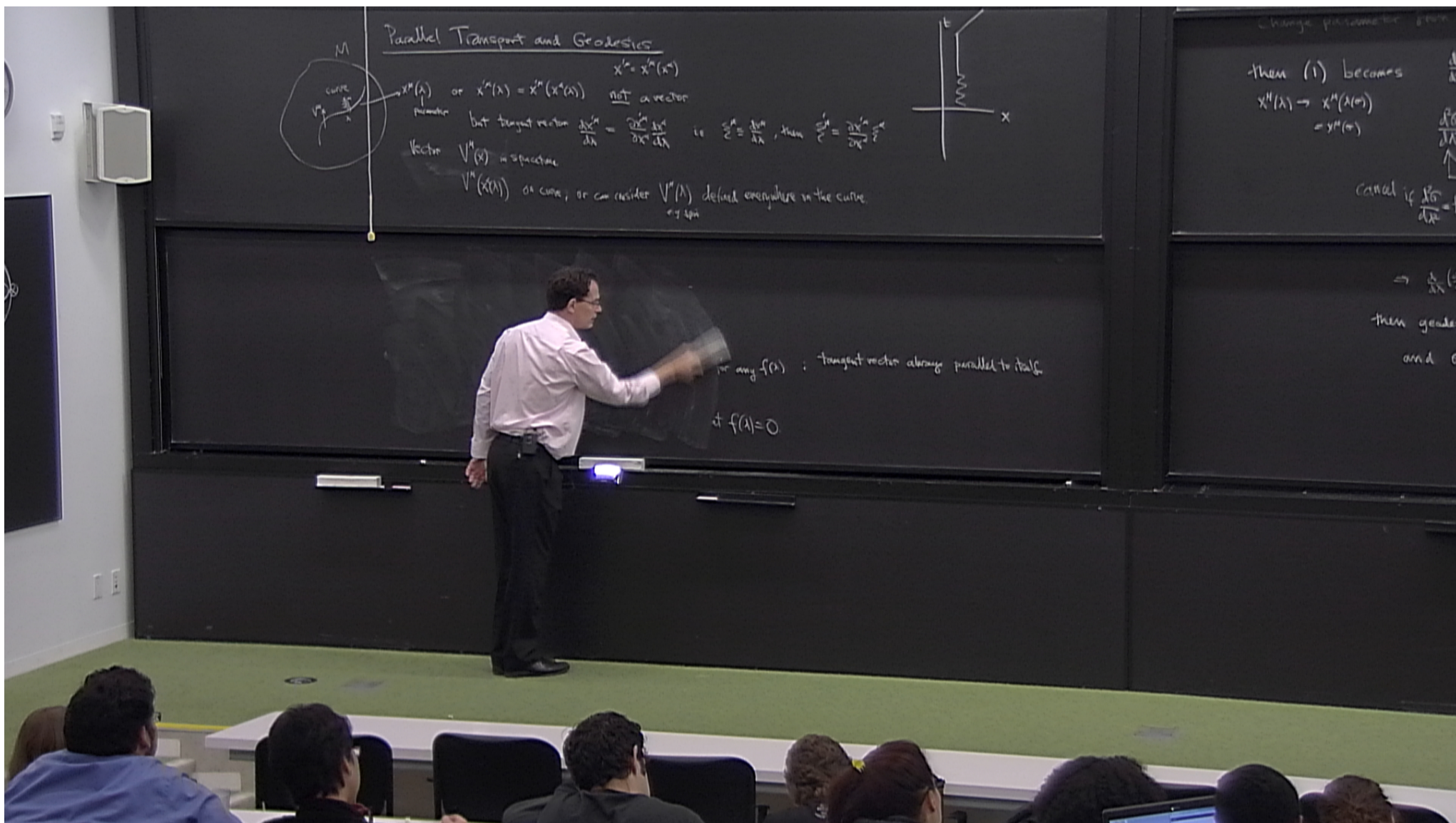
then geodesic eqⁿ is just

and σ is called

$\frac{d^2 x^\alpha}{d\sigma^2} = 0$

geodesic equation

(affine $\Rightarrow \sigma$ determined up to $\sigma' = A\sigma + B$, with A, B const)



Action for a particle moving in spacetime



Action for particle moving in spacetime

$$S = - \int d\tau$$

time
world

Action for a particle moving in spacetime

$$S = -mc^2 \int d\tau$$

proper time
along worldline

Action S for a particle moving in spacetime

recall

$$S = -mc^2 \int dt =$$

$t \rightarrow X^\mu(t)$ spacetime worldline

Action for a particle moving in spacetime
1 from SR

$$= -mc^2 \int d\tau$$

proper time
along worldline

$$= -mc^2 \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$$

Action for a particle moving in spacetime
recall from SR

$$S = -mc \int d\tau = -mc^2 \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$$

proper time
along worldline

$\tau \rightarrow X^\mu(\tau)$

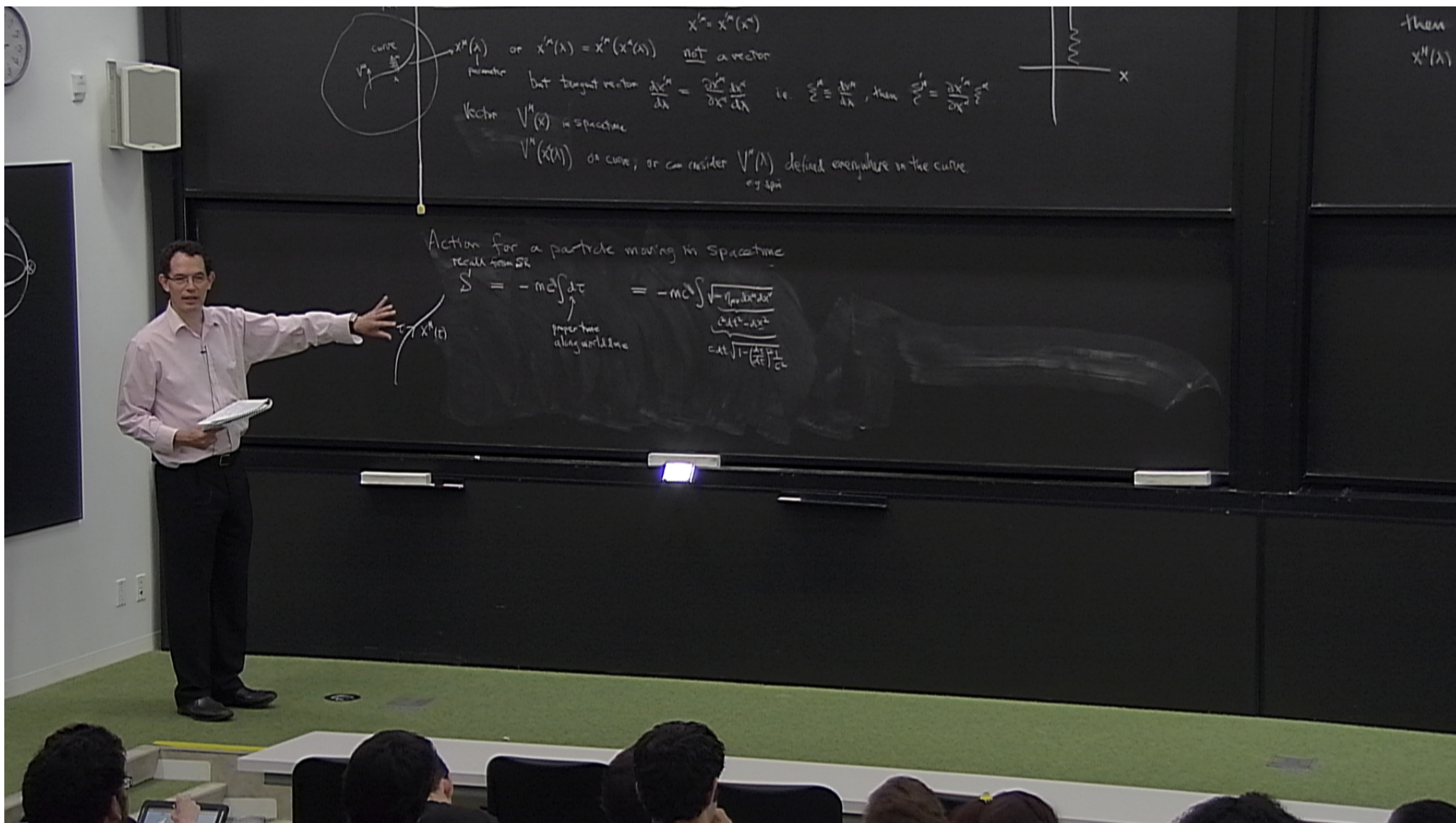
Action for a particle moving in spacetime
recall from SR

$$S = -mc \int d\tau = -mc \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$$

$$c^2 dt^2 - dx^2$$

proper time
along worldline

$\tau \rightarrow X^\mu(\tau)$



Action for a particle moving in spacetime
 recall from SR

$$S = -mc \int d\tau = -mc \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} = -mc \int \sqrt{c^2 dt^2 - d\mathbf{x}^2}$$

proper time
along worldline

$$= -mc \int dt \sqrt{1 - \left(\frac{d\mathbf{x}}{dt}\right)^2 \frac{1}{c^2}}$$

$V(X(\lambda))$ on curve, or can consider $V(\lambda)$ defined everywhere on the curve.
e.g spin

Action for a particle moving in spacetime

recall from SR

$$S = -mc \int d\tau$$

$\tau \nearrow X^\mu(\tau)$

proper time
along

$$-mc \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} \\ \underbrace{c^2 dt^2 - dx^2}_{c dt \sqrt{1 - \left(\frac{dx}{dt}\right)^2}} \frac{1}{c^2}$$

Equivalence
Principle



$V(\lambda)$ on curve, or can consider $V(\lambda)$ defined everywhere on the curve.
e.g. spin

for a particle moving in spacetime
from SR

$$= -mc \int dt$$

proper time
along worldline

$$= -mc \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$$

$$\underbrace{c^2 dt^2 - dx^2}_{c dt \sqrt{1 - \left(\frac{dx}{dt}\right)^2} \frac{1}{c^2}}$$

Equivalence
Principle

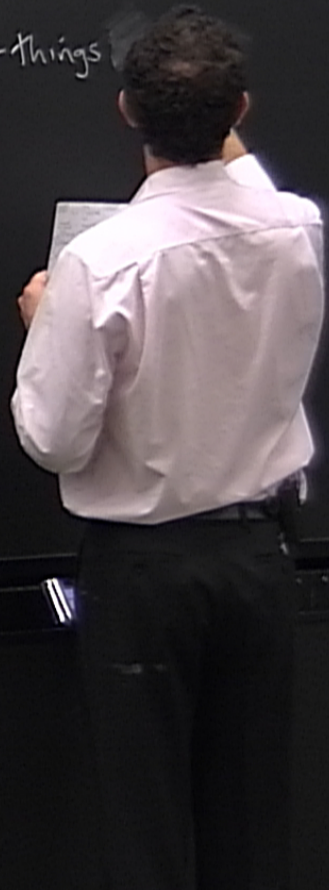
$$\Rightarrow -mc \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

$x^\mu(\sigma)$

$$\frac{1}{\lambda^2} \frac{dx^\mu}{d\sigma} + \left(\frac{d\sigma}{d\lambda} \right) \left(\frac{d\lambda}{d\sigma^2} + \frac{1}{\lambda^2} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} \right) = f(\lambda) \frac{d\sigma}{d\lambda} \frac{dx^\mu}{d\sigma}$$

cancel if $\frac{d\sigma}{d\lambda} = f(\lambda) \frac{d\sigma}{d\lambda}$

Aside: why $E=mc^2$, and why momentum + other things



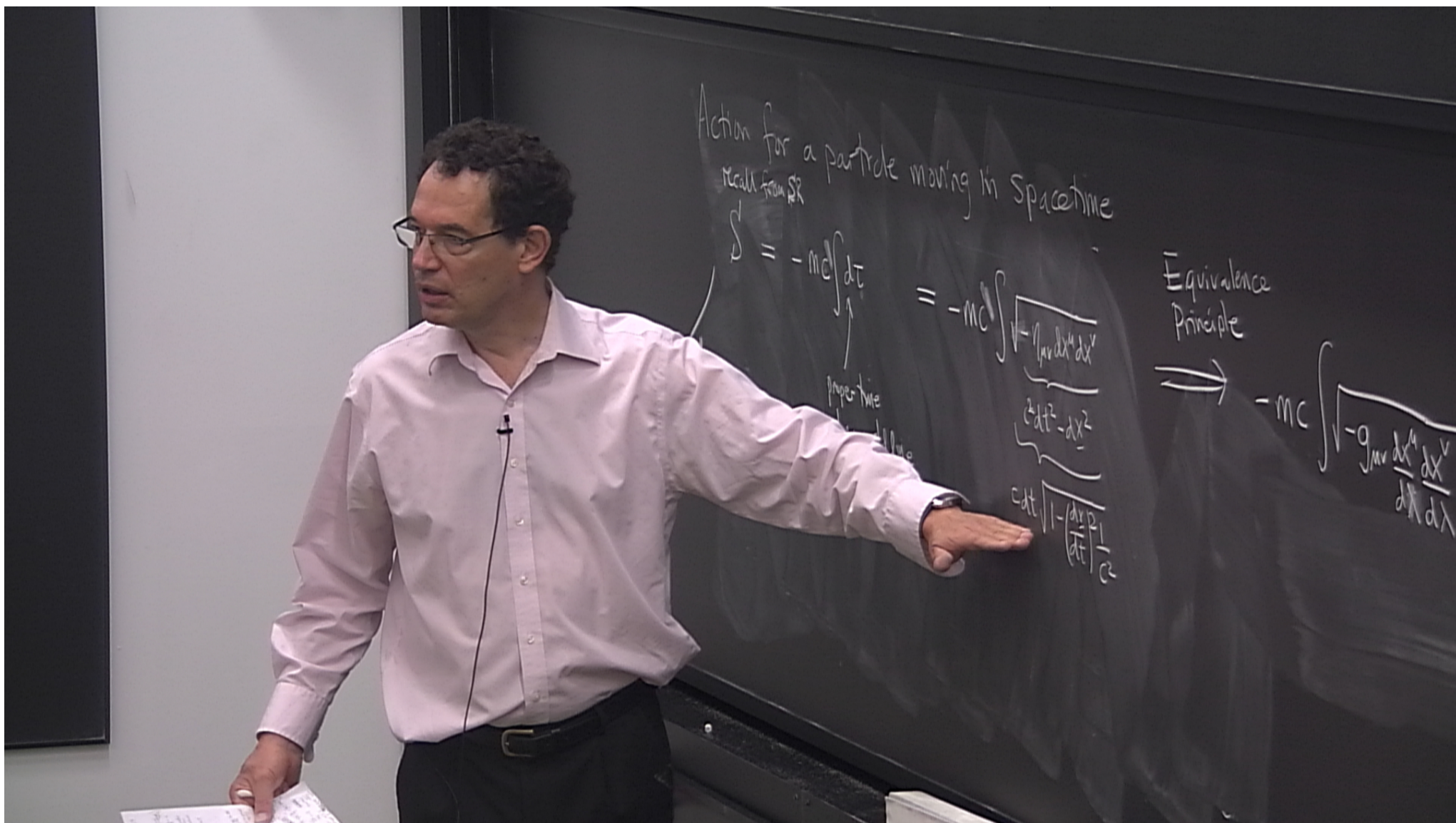
cancel if $\frac{d^2\sigma}{d\lambda^2} = f(\lambda) \frac{d\sigma}{d\lambda}$

Aside: why $E=mc^2$, and why momentum + other things are conserved

If $S' = -\int L dt$

$$\frac{d}{d\lambda} \left(\frac{dx^\mu}{d\sigma} \right)$$

$$= \frac{d\sigma}{d\lambda} \frac{d}{d\sigma}$$



Consider varying $x_c(t_i), x_c(t_f) \Rightarrow$ variations in classical path $x_c(t)$

Consider varying $x_i, t_i, x_f, t_f \Rightarrow$ variations in classical path $x_c(t)$

$$\delta S_{\text{classical}} = \delta t_f L(t_f) - \delta t_i L(t_i)$$

Consider varying $x_i, t_i, x_f, t_f \Rightarrow$ variation of classical path $x_c(t)$

$$\delta S'_{\text{classical}} = \delta t_f L(t_f) - \delta t_i L(t_i) + \int \frac{\partial L}{\partial x} \delta x$$

Consider varying $x_i, t_i, x_f, t_f \Rightarrow$ variations in classical path $x_c(t)$

$$\delta S'_{\text{classical}} = \delta t_f L(t_f) - \delta t_i L(t_i) + \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right) dt \quad (\text{to 1st order in } \delta x, \delta t)$$

Consider varying $x_i, t_i, x_f, t_f \Rightarrow$ variations in classical path $x_c(t)$

$$\delta S'_{\text{classical}} = \delta t_f L(t_f) - \delta t_i L(t_i) + \int_{t_i}^{t_f} \frac{\partial L}{\partial x} \delta x dt + \int_{t_i}^{t_f} \frac{\partial L}{\partial \dot{x}} \delta \dot{x} dt$$

der in $\delta x, \delta t$

$$= \left[\frac{\partial L}{\partial x} \delta x \right]_{t_i}^{t_f}$$

Consider varying $x_i, t_i, x_f, t_f \Rightarrow$ variations in classical path $\underline{x}_c(t)$

$$\delta S'_{\text{classical}} = \delta t_f L(t_f) - \delta t_i L(t_i) + \int_{t_i}^{t_f} \frac{\partial L}{\partial \underline{x}} \delta \underline{x} dt + \int_{t_i}^{t_f} \frac{\partial L}{\partial \dot{\underline{x}}} \delta \dot{\underline{x}} dt \quad \left(\text{to 1st order in } \delta x, \delta t \right)$$

$$= \left[\frac{\partial L}{\partial \dot{\underline{x}}} \delta \underline{x} \right]_{t_i}^{t_f} + \text{terms vanishing by } \dots$$

Consider varying $x_i, t_i, x_f, t_f \Rightarrow$ variations in classical path $x_c(t)$

$$\delta S'_{\text{classical}} = \delta t_f L(t_f) - \delta t_i L(t_i) + \int_{t_i}^{t_f} \frac{\partial L}{\partial x} \delta x \, dt + \int_{t_i}^{t_f} \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \, dt \quad (\text{to 1st order in } \delta x, \delta t)$$

$$= \left[\delta t L \right] + \text{terms vanishing by classical eq's}$$

Consider varying $x_i, t_i, x_f, t_f \Rightarrow$ variations in classical path $x_c(t)$

$$\delta S'_{\text{classical}} = \delta t_f L(t_f) - \delta t_i L(t_i) + \int_{t_i}^{t_f} \frac{\partial L}{\partial x} \delta x dt + \int_{t_i}^{t_f} \frac{\partial L}{\partial \dot{x}} \delta \dot{x} dt \quad (\text{to 1st order in } \delta x, \delta t)$$

$$= \left[\delta t L + \frac{\partial L}{\partial \dot{x}} \delta x \right]_{t_i}^{t_f} + \text{terms vanishing by classical eq's}$$

Consider varying $x_i, t_i, x_f, t_f \Rightarrow$ variations in classical path $x_c(t)$

$$\delta S'_{\text{classical}} = \delta t_f L(t_f) - \delta t_i L(t_i) + \int_{t_i}^{t_f} \frac{\partial L}{\partial \underline{x}} \delta \underline{x} dt + \int_{t_i}^{t_f} \frac{\partial L}{\partial t} \delta t dt \quad (\text{to 1st order})$$

$$= \left[\delta t L + \frac{\partial L}{\partial \underline{x}} \delta \underline{x} \right]_{t_i}^{t_f} + \text{terms vanishing by Euler-Lagrange}$$

subtlety: $\delta x_f = \delta x(t_f) + \dot{x}_f \delta t_f$
 $\uparrow$ var. in path

$$\frac{\partial L}{\partial \underline{x}} = \frac{1}{\dot{x}} \left(\frac{\partial L}{\partial \dot{x}} \right)$$

Consider varying $x_i, t_i, x_f, t_f \Rightarrow$ variations in classical path $x_c(t)$

$$\delta S'_{\text{classical}} = \delta t_f L(t_f) - \delta t_i L(t_i) + \int_{t_i}^{t_f} \frac{\partial L}{\partial \underline{x}} \delta \underline{x} dt \quad (\text{to 1st order})$$

$$= \left[\delta t L + \frac{\partial L}{\partial \underline{x}} \delta \underline{x} \right]_{t_i}^{t_f} + \text{terms vanishing by } \frac{\partial L}{\partial \dot{\underline{x}}} = 0$$

subtlety: $\delta x_f = \delta \underline{x}(t_f) + \dot{\underline{x}}_f \delta t_f$
 $\uparrow$ var. in path

Consider varying $x_i, t_i, x_f, t_f \Rightarrow$ variations in classical path $x_c(t)$

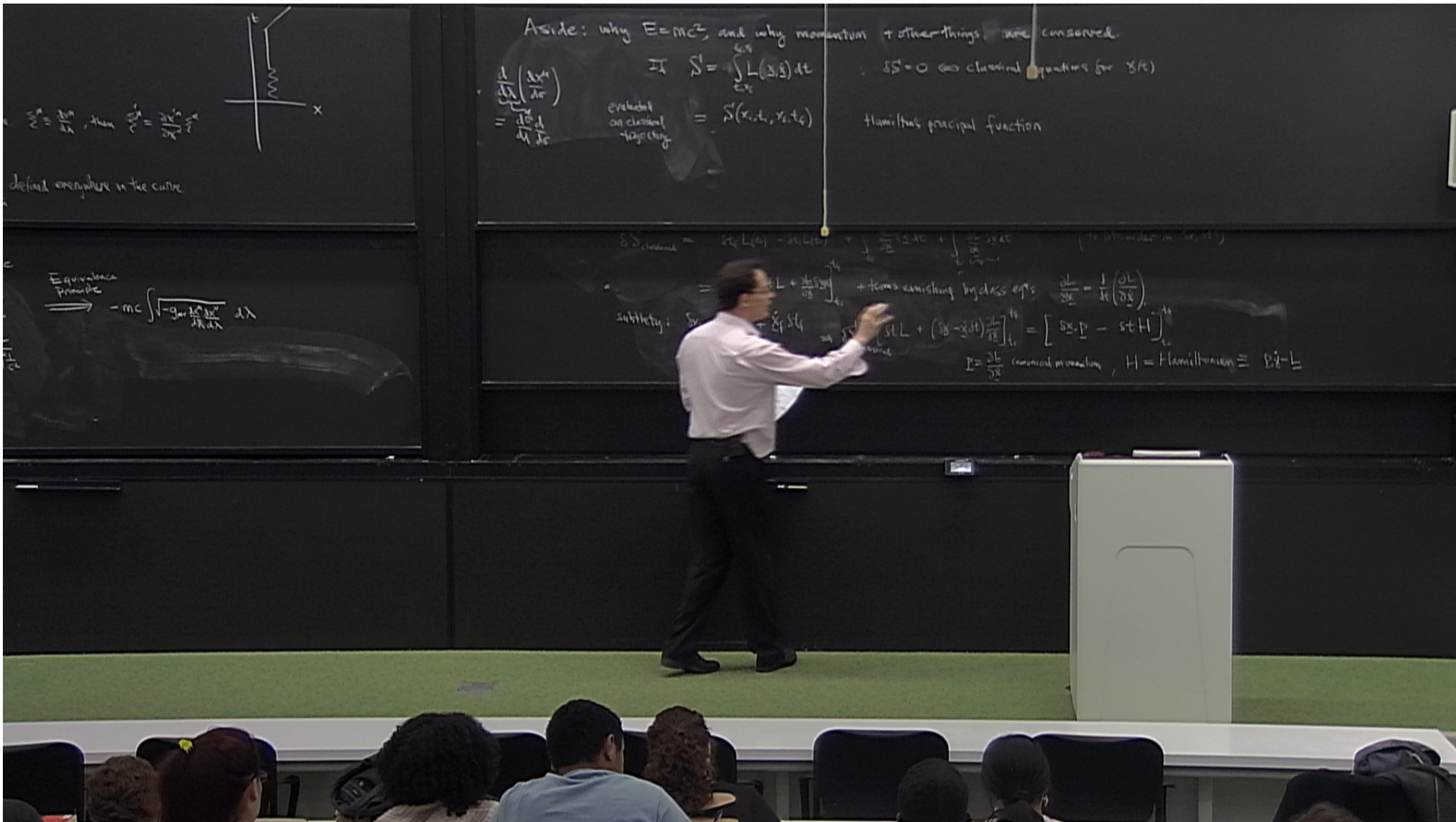
$$\delta S'_{\text{classical}} = \delta t_f L(t_f) - \delta t_i L(t_i) + \int_{t_i}^{t_f} \frac{\partial L}{\partial x} \delta x dt + \int_{t_i}^{t_f} \frac{\partial L}{\partial \dot{x}} \delta \dot{x} dt \quad (\text{to 1st order})$$

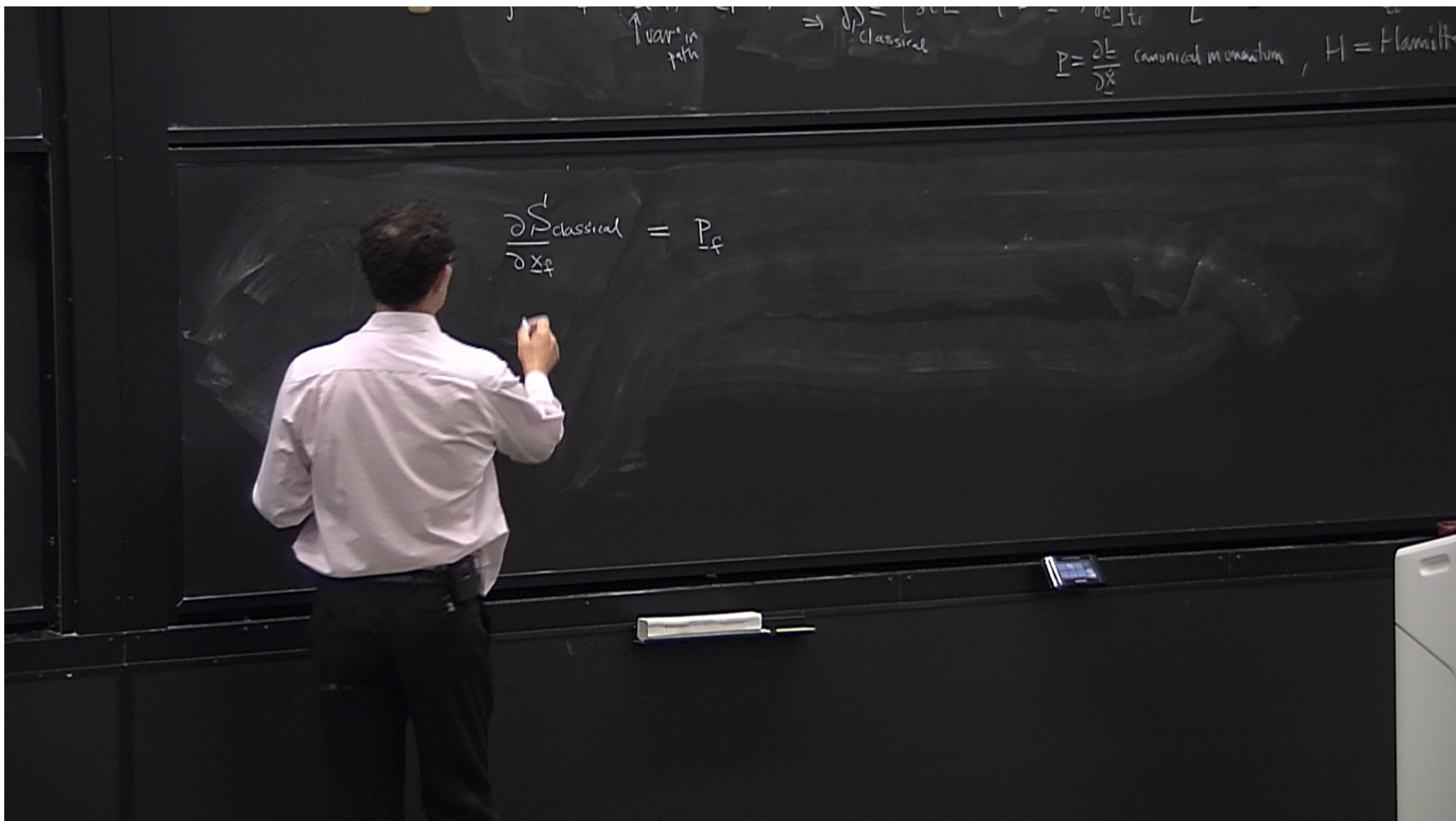
$$= \left[\delta t L + \frac{\partial L}{\partial \dot{x}} \delta x \right]_{t_i}^{t_f} + \text{terms vanishing by class eq.}$$

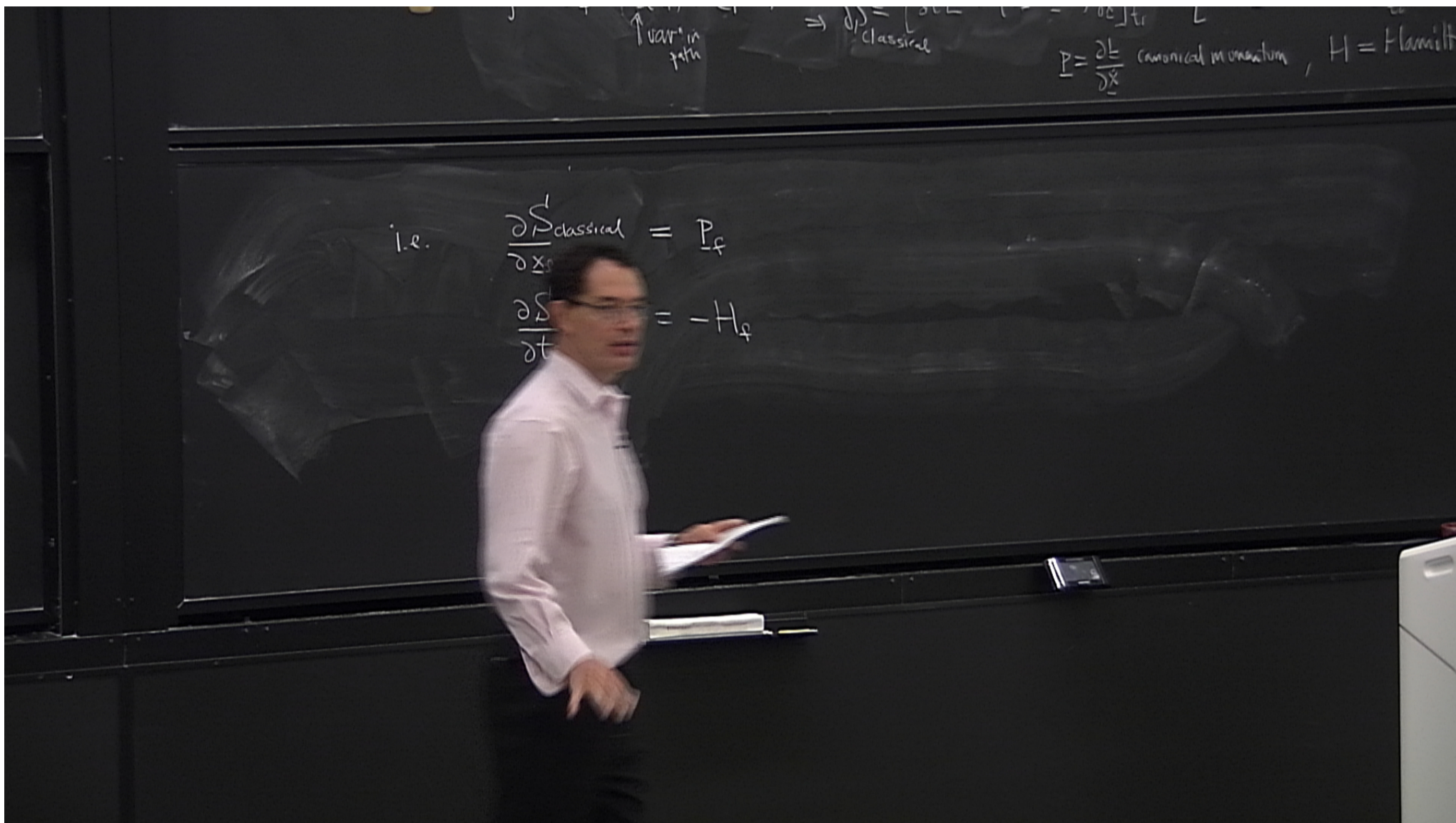
subtlety: $\delta x_f = \delta x(t_f) + \dot{x}_f \delta t_f$ $\Rightarrow \delta S' = \left[\delta t L + (\delta x - \dot{x} \delta t) \right]_{t_i}^{t_f}$

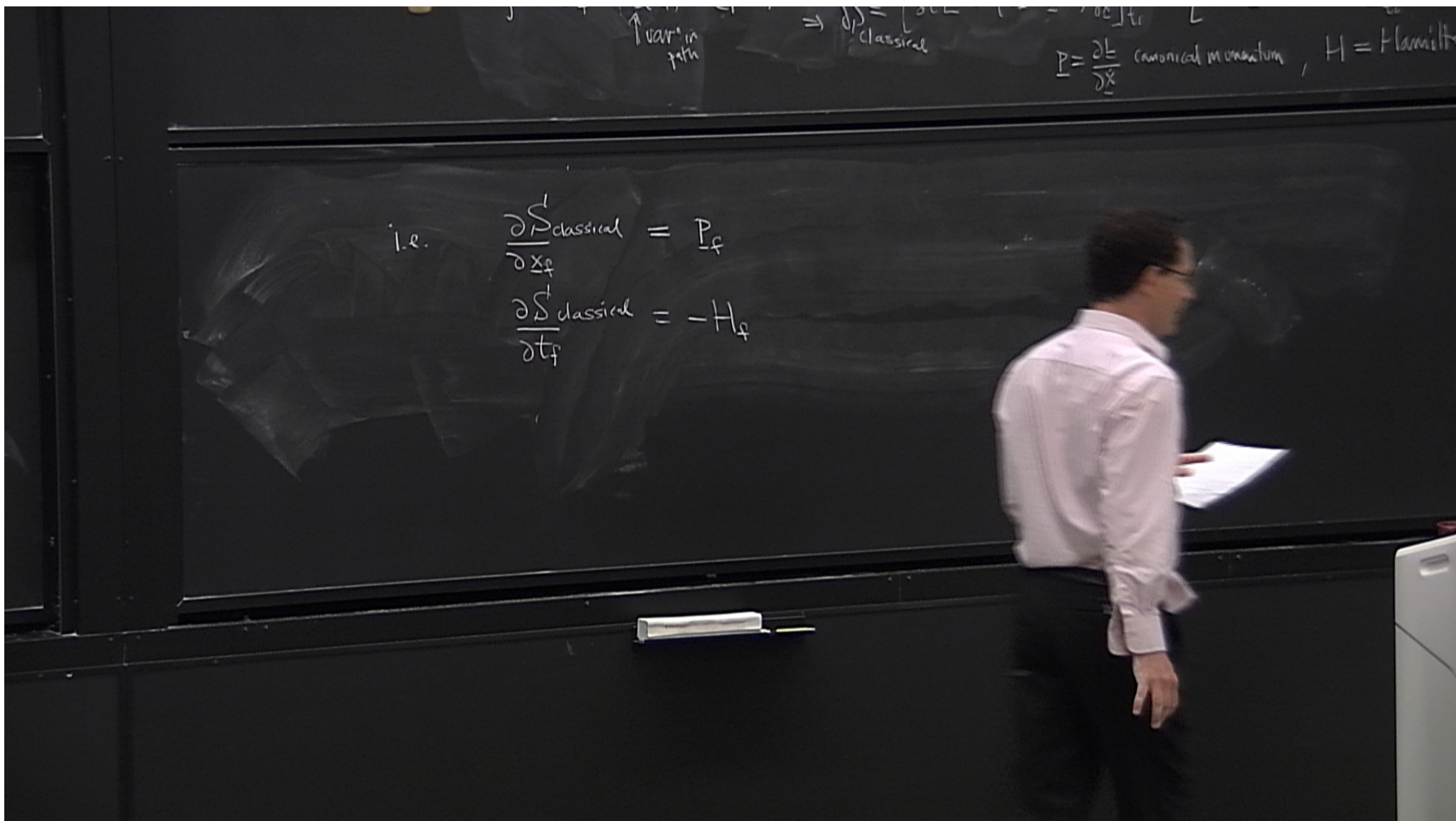
$\uparrow$ var. in path

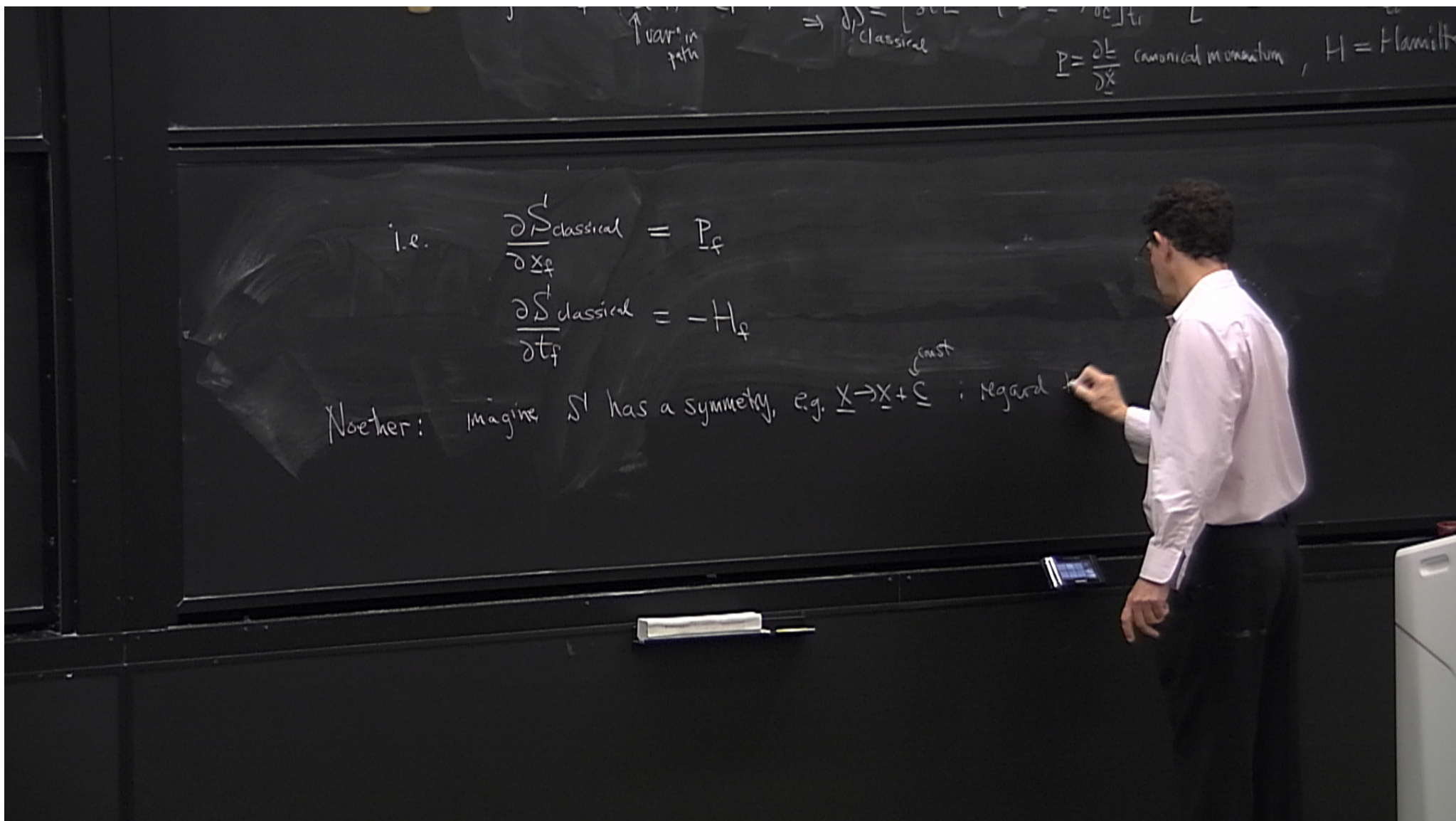
$$\begin{aligned}
 \delta S_{\text{classical}} &= \delta t_f L(t_f) - \delta t_i L(t_i) + \int_{t_i}^{t_f} \frac{\partial L}{\partial x} \delta x \, dt + \int_{t_i}^{t_f} \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \, dt \quad (\text{to 1st order in } \delta x, \delta \dot{x}) \\
 &= \left[\delta t L + \frac{\partial L}{\partial \dot{x}} \delta x \right]_{t_i}^{t_f} + \text{terms vanishing by class eq's} \quad \frac{\partial L}{\partial x} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) \\
 \text{subtlety: } \delta x_f &= \delta x(t_f) + \dot{x}_f \delta t_f \quad \Rightarrow \delta S = \left[\delta t L + \left(\delta x - \dot{x} \delta t \right) \frac{\partial L}{\partial \dot{x}} \right]_{t_i}^{t_f} \\
 &\quad \uparrow \text{var. in path}
 \end{aligned}$$











↑ var. in path

$\Rightarrow \delta S_{\text{classical}}$

$$\underline{p} = \frac{\partial L}{\partial \dot{x}} \text{ canonical momentum}$$

$$H = \text{Hamiltonian} \equiv \underline{p} \cdot \dot{x} - L$$

$$\frac{\partial S_{\text{classical}}}{\partial x_f} = p_f$$

$$\frac{\partial S_{\text{classical}}}{\partial t_f} = -H_f$$

Imagine S has a symmetry, e.g. $\underline{x} \rightarrow \underline{x} + \underline{c}$ (const); regard this as a

like wise if $t \rightarrow t + c$ is a symmetry $\Rightarrow H_f - H_i = 0 =$

$$\Rightarrow \delta S_{\text{class}} = 0 = \left[\underline{x} \cdot \underline{p} \right]_i^f \Rightarrow p_f = p_i$$

$V(X(\lambda))$ on curve, or can consider $V(\lambda)$ defined everywhere on the curve.
e.g spin

Action for a particle moving in spacetime

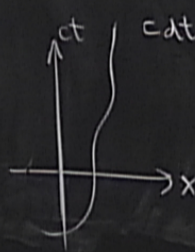
recall from SR

$$S = -mc \int dt$$

proper time
along worldline

$$= -mc \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$$

$$\underbrace{c^2 dt^2 - dx^2}_{c^2 dt^2 - dx^2}$$



$$c dt \sqrt{1 - \left(\frac{dx}{dt}\right)^2 \frac{1}{c^2}}$$

$$\Rightarrow \underline{p} - \frac{\partial L}{\partial \dot{x}} = + \dots^2$$

Equivalence
Principle

$$\Rightarrow -mc \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} d\lambda$$

$V(X(\lambda))$ on curve, or can consider $V(\lambda)$ defined everywhere on the curve.
e.g. spin

Action for a particle moving in spacetime

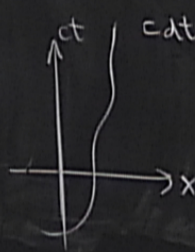
recall from SR

$$S = -mc \int dt$$

proper time
along worldline

$$= -mc \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$$

$$= \sqrt{c^2 dt^2 - dx^2}$$



$$= c dt \sqrt{1 - \left(\frac{dx}{dt}\right)^2 \frac{1}{c^2}}$$

$$\Rightarrow \underline{p} - \frac{\partial L}{\partial \dot{x}} = \underline{p} = \frac{mc \dot{x}}{\sqrt{1 - \dot{x}^2/c^2}}$$

Equivalence
Principle

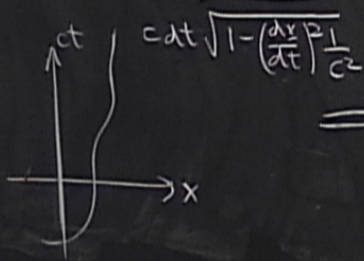
$$\Rightarrow -mc \int \sqrt{-g_{\mu\nu}} d\lambda$$

particle moving in spacetime

$$= -mc \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} \\ \underbrace{c^2 dt^2 - dx^2}_{c^2 dt^2 - dx^2}$$

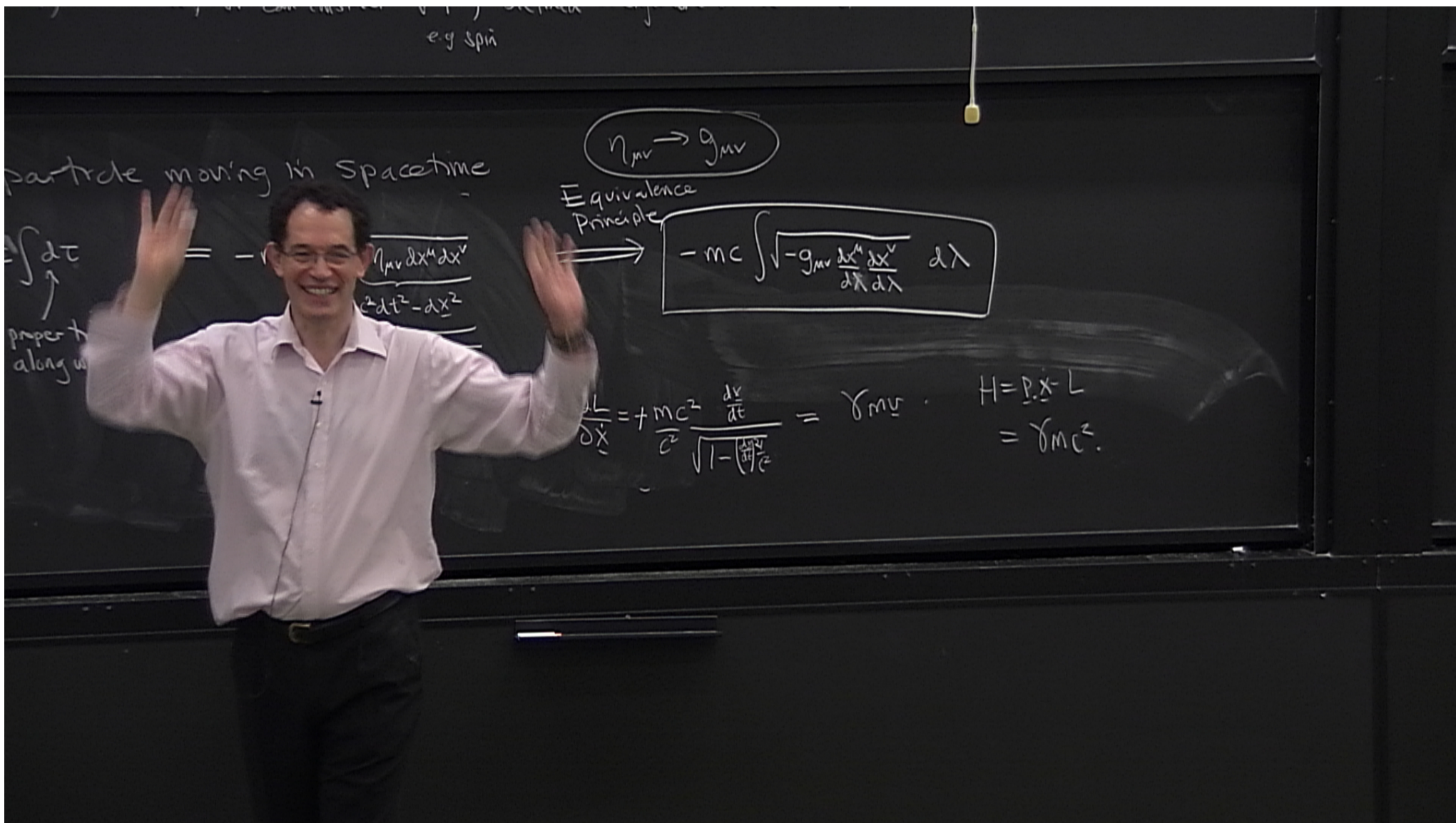
Equivalence
Principle

$$\Rightarrow -mc \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$



$$\Rightarrow \underline{p} - \frac{\partial L}{\partial \dot{x}} = +mc^2 \frac{\frac{dx}{dt}}{\sqrt{1 - (\frac{dx}{dt})^2}} = \gamma m \underline{v}$$

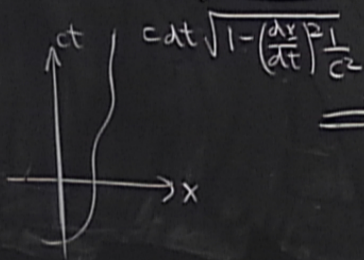
$$H = \underline{p} \cdot \underline{x} - L \\ = \gamma mc^2$$



particle moving in spacetime

$\int dt$
proper time
along worldline

$$= -mc \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$



$$\eta_{\mu\nu} \rightarrow g_{\mu\nu}$$

Equivalence
Principle

$$-mc \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

$$\Rightarrow \underline{p} - \frac{\partial L}{\partial \underline{x}} = +mc^2 \frac{\frac{d\underline{v}}{dt}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m \underline{v}$$

$$H = \underline{p} \cdot \underline{x} - L = \gamma mc^2$$

$$\partial_\mu F^{\mu\nu} = J^\nu (-\mu_0)$$

$$\partial_\alpha F_{\beta\gamma} + \dots = 0$$

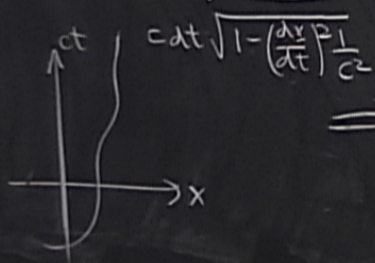
$$F^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} F_{\alpha\beta}$$

R

e.g spin

Moving in Spacetime

$$= -mc \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} dt} = -mc \int \sqrt{c^2 dt^2 - dx^2}$$



$$\eta_{\mu\nu} \rightarrow g_{\mu\nu}$$

Equivalence Principle

$$\Rightarrow -mc \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

$$\partial_\mu F^{\mu\nu} = J^\nu (-\mu_0)$$

$$\partial_\alpha F_{\beta\gamma} + \dots = 0$$

$$F^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} F_{\alpha\beta}$$

$$\Rightarrow \underline{p} - \frac{\partial L}{\partial \underline{\dot{x}}} = + \frac{mc^2}{c^2} \frac{\frac{d\underline{x}}{dt}}{\sqrt{1 - \left(\frac{d\underline{x}}{dt}\right)^2/c^2}} = \gamma m \underline{v}$$

$$H = \underline{p} \cdot \underline{\dot{x}} - L = \gamma mc^2$$



$$S = -mc \int \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\lambda$$

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

$$\delta f(x) \\ = \delta x \frac{\partial f}{\partial x}$$

$$\delta S' = -mc \int \frac{1}{2} \frac{d\lambda}{\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \left(-g_{\mu\nu, \lambda} \delta x^\lambda \dot{x}^\mu \dot{x}^\nu - 2g_{\mu\nu} \delta \dot{x}^\mu \dot{x}^\nu \right)$$

$$S = -mc \int \sqrt{-g_{\mu\nu}^{(s)} \dot{x}^\mu \dot{x}^\nu} d\lambda$$

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

$$\delta f(x) = \delta x \frac{\partial f}{\partial x}$$

$$\delta S = -mc \int \frac{1}{2} \frac{d\lambda}{\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} (-g_{\mu\nu, \lambda} \delta x^\lambda \dot{x}^\mu \dot{x}^\nu - 2g_{\mu\nu} \delta \dot{x}^\mu \dot{x}^\nu)$$

$$= -mc^2 \int \frac{1}{2} d\tau (-g_{\mu\nu, \lambda} \delta x^\lambda \dot{x}^\mu \dot{x}^\nu - 2g_{\mu\nu} \delta \dot{x}^\mu \dot{x}^\nu)$$

$$\dot{x}^{\mu} = \frac{dx^{\mu}}{d\lambda}$$

$$(-g_{\mu\nu,\lambda} \delta x^{\lambda} \dot{x}^{\mu} \dot{x}^{\nu} - 2g_{\mu\nu} \delta \dot{x}^{\mu} \dot{x}^{\nu})$$

notice. $\dot{x}^{\mu} = \frac{dx^{\mu}}{cd\tau} = \frac{dx^{\mu}}{d\lambda} \frac{1}{\sqrt{-g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}}}$

$$\propto \int (-g_{\mu\nu,\lambda} \delta x^{\lambda} \dot{x}^{\mu} \dot{x}^{\nu} - 2g_{\mu\nu} \delta \dot{x}^{\mu} \dot{x}^{\nu}) \propto \int (-g_{\mu\nu,\lambda} \dot{x}^{\mu} \dot{x}^{\nu} + 2(g_{\mu\nu} \ddot{x}^{\nu})^{\circ}) \delta x^{\lambda}$$

$$\frac{d\lambda}{d\tau} (-g_{\mu\nu,\lambda} \delta x^\lambda \dot{x}^\mu \dot{x}^\nu - 2g_{\mu\nu} \delta \dot{x}^\mu \dot{x}^\nu)$$

notice. $\dot{x}^\mu = \frac{dx^\mu}{cd\tau} = \frac{dx^\mu}{dx} \frac{1}{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$

$$\frac{1}{2} d\tau (-g_{\mu\nu,\lambda} \delta x^\lambda \dot{x}^\mu \dot{x}^\nu - 2g_{\mu\nu} \delta \dot{x}^\mu \dot{x}^\nu) \propto \int (-g_{\mu\nu,\lambda} \dot{x}^\mu \dot{x}^\nu + 2(g_{\mu\nu} \dot{x}^\nu)^\cdot) \delta x^\lambda = 0$$

$$-mc) \frac{1}{2} d\tau (-g_{\mu\nu,\lambda} \delta x^\lambda \dot{x}^\mu \dot{x}^\nu - 2g_{\mu\nu} \delta \dot{x}^\lambda \dot{x}^\nu) \propto \int (-g_{\mu\nu,\lambda} \dot{x}^\mu \dot{x}^\nu + 2(g_{\mu\nu} \dot{x}^\nu)') \delta x^\lambda$$

$$\Rightarrow -(g_{\mu\nu} \dot{x}^\nu)' = \frac{1}{2} g_{\mu\nu,\lambda} \dot{x}^\mu \dot{x}^\nu$$

$$\Rightarrow g_{\lambda\nu,\mu} \dot{x}^\mu \dot{x}^\nu + g_{\mu\nu} \ddot{x}^\nu = \frac{1}{2} g_{\mu\nu,\lambda} \dot{x}^\mu \dot{x}^\nu$$

$$\Rightarrow \ddot{x}^\mu = - \left\{ \begin{matrix} \mu \\ \nu\lambda \end{matrix} \right\} \dot{x}^\nu \dot{x}^\lambda$$

$$\left\{ \begin{matrix} \mu \\ \nu\lambda \end{matrix} \right\} = \frac{1}{2} g^{\mu\alpha} (g_{\nu\alpha,\lambda} + g_{\lambda\alpha,\nu} - g_{\alpha\lambda,\nu})$$

$$\Rightarrow \ddot{x}^\mu + \left\{ \begin{matrix} \mu \\ \nu\lambda \end{matrix} \right\} \dot{x}^\nu \dot{x}^\lambda = 0 \quad \text{geodesic eq.}$$