

Title: 13/14 PSI - Relativity - Lecture 7

Date: Sep 17, 2013 09:00 AM

URL: <http://pirsa.org/13090019>

Abstract:

orem (Nash)

Every Riemannian manifold can be isometrically embedded in a Euclidean space.

[any 4d manifold can be locally embedded in 10 Euclidean dimensions]

line, metric η_{AB}
" $\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

line element $\eta_{AB} dX^A dX^B = -dX^0^2 + dX^1^2 + \dots + dX_4^2$

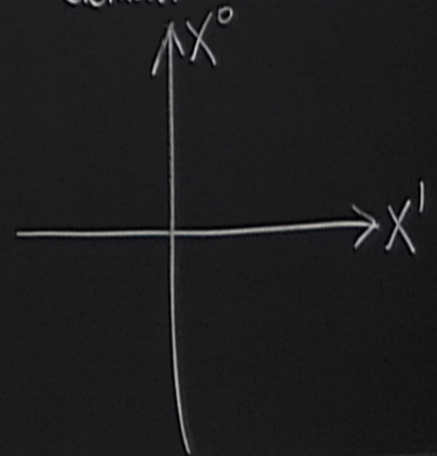
de Sitter spacetime

4-dimensional version of a sphere.

5-dimensional Minkowski spacetime, metric η_{AB} ^{0,1,2,3,4}
" $\begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$

Consider 4d hyperboloid $-X_0^2 + X_1^2 + X_2^2 + \dots + X_4^2 = L^2$

line element $\eta_{AB} dX^A dX^B = -dX^0^2 + dX^1^2 + dX^2^2 + dX^3^2 + dX^4^2$



spacetime, metric

$$\eta_{AB} = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}_{0,1,2,3,4}$$

$$-X_0^2 + X_1^2 + X_2^2 + \dots + X_4^2 = L^2$$

line element

$$\eta_{AB} dX^A dX^B = -dX_0^2 + dX_1^2 + \dots + dX_4^2$$

$$X_1^2 = X_0^2 + L^2$$

de Sitter spacetime

Solve constraint:

$$X^0 = L \sinh t$$

$$X^I = L \cosh t \, \Omega^I \quad I=1 \dots 4$$

where $\Omega^I \Omega^I = 1$

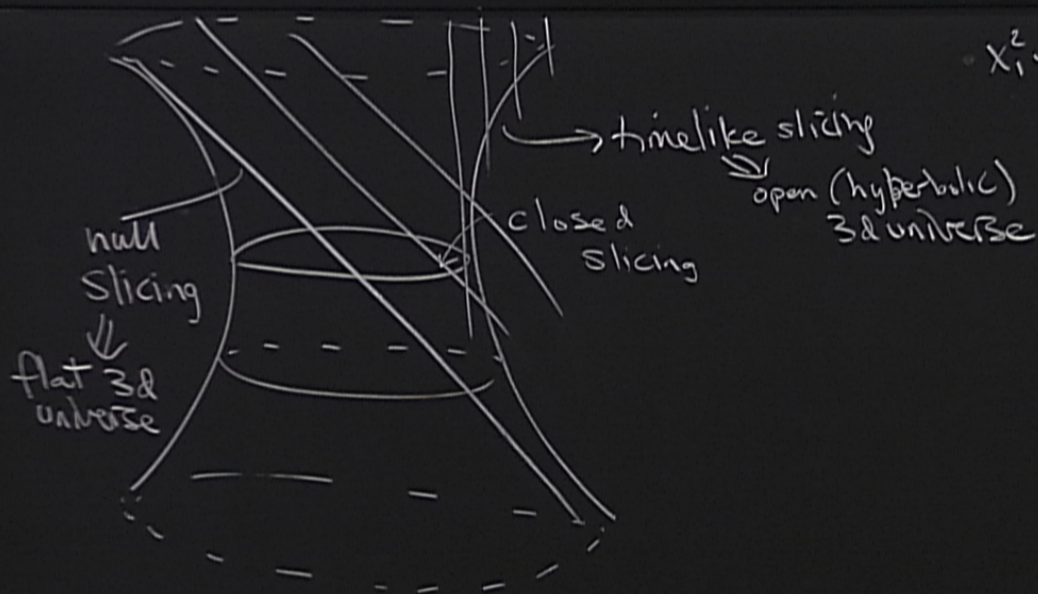
$\Omega^I = (\sin \chi \sin \theta \cos \varphi, \sin \chi \sin \theta \sin \varphi, \sin \chi \cos \theta, \cos \chi)$ parametrises S^3

$\Rightarrow$ line element becomes $L^2 [-dt^2 + \cosh^2 t d\Omega_3^2]$ where $d\Omega_3^2 = d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2)$

$$I=1\dots 4$$

$(\chi, \cos\chi)$ parametrises S^3

$\left[\begin{smallmatrix} \Omega^2 \\ 3 \end{smallmatrix} \right]$ where $d\Omega^2_3 = d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\varphi^2)$ is metric on S^3 .



$$x_1^2 + \dots + x_4^2 = 1 \quad S^3$$

flat slicing

$$X^0 = L(\sinh t + \frac{1}{2} r^2 e^t)$$

$$X^i = L e^t x^i \quad i=1,2,3$$

$$X^4 = L(\cosh t - \frac{1}{2} r^2 e^t)$$

$$r^2 = x^i x^i$$

$$\text{Ex: } -dX^0^2 + dX^1^2 + \dots + dX^3^2$$

$$x_4^2 = 1 \quad S^3$$

flat slicing

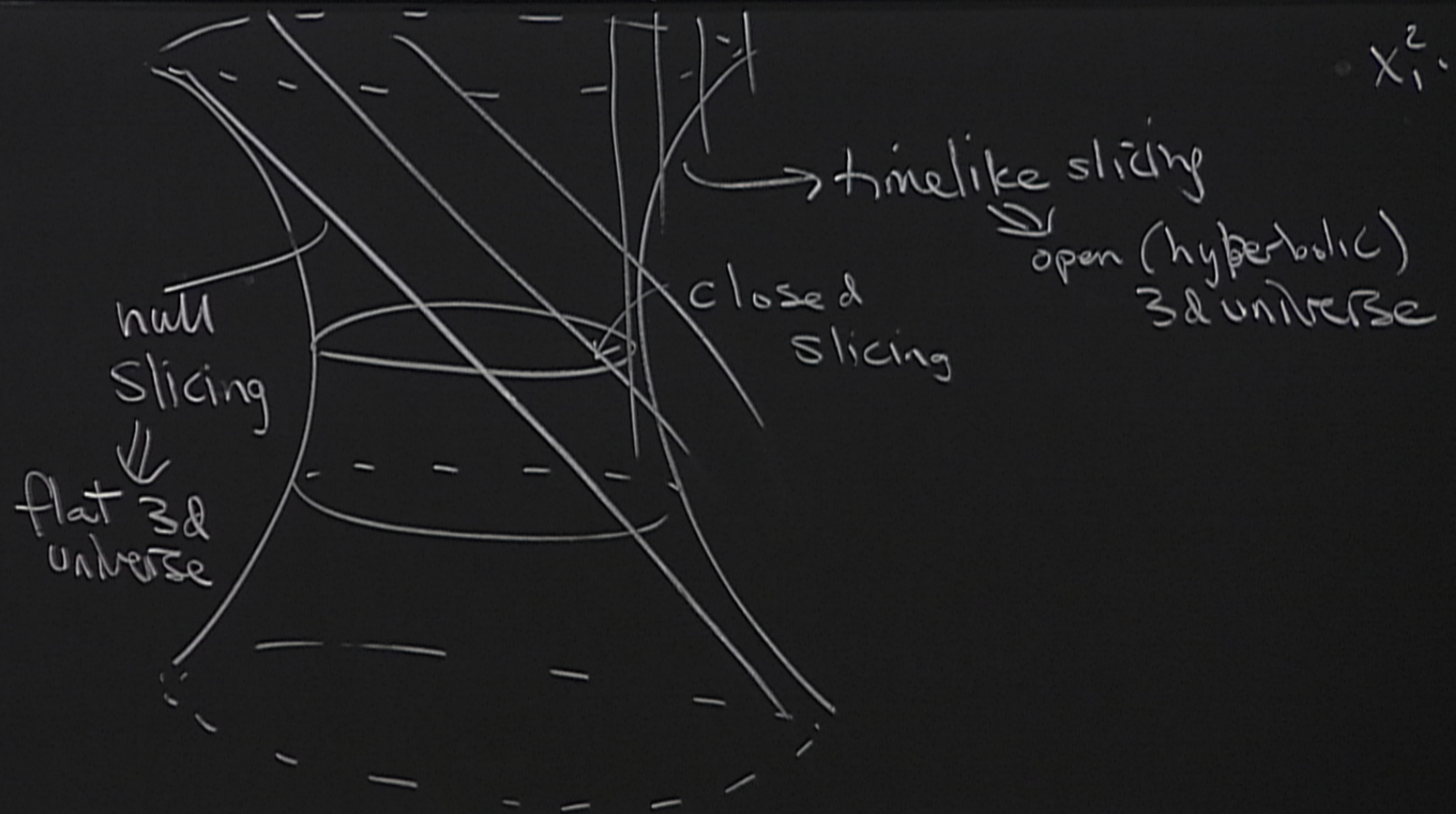
$$X^0 = L(\sinh t + \frac{1}{2} r^2 e^t)$$

$$X^i = L e^t x^i \quad i=1,2,3$$

$$X^4 = L(\cosh t - \frac{1}{2} r^2 e^t)$$

$$r^2 = x^i x^i$$

$$\text{Ex: } -dX^0^2 + dX^1^2 + \dots + dX^4^2 = L^2(-dt^2 + e^{2t} dx^i dx^i)$$



$$x_1^2 + \dots + x_4^2 = 1 \quad S^3$$

flat slicing

$$X^0 = L(\sinh t + \frac{1}{2} r^2 e^t)$$

$$X^i = L e^t x^i \quad i=1,2,3$$

$$X^4 = L(\cosh t - \frac{1}{2} r^2 e^t)$$

$$r^2 = x^i x^i$$

$$\text{Ex: } -dX^0^2 + dX^1^2 + \dots + dX^4^2 = L^2 \left(-dt^2 + e^{2t} dx^i dx^i \right)$$

$\mathbb{H}_3$

$$t + \frac{1}{2} r^2 e^t)$$

$$i=1,2,3$$

$$t - \frac{1}{2} r^2 e^t)$$

$$+ \dots + dx^2 = L^2 \left(-dt^2 + e^{2t} dx^i dx^i \right)$$

 $\mathbb{H}^3$

Note $-\infty < t < \infty$

metric is singular
at $t = -\infty$.

$$x_1^2 + \dots + x_4^2 = 1 \quad S^3$$

flat slicing

$$X^0 = L(\sinh t + \frac{1}{2} r^2 e^t)$$

$$X^i = L e^t x^i \quad i=1,2,3$$

$$X^4 = L(\cosh t - \frac{1}{2} r^2 e^t)$$

$$r^2 = x^i x^i$$

$$\text{Ex: } -dX^0^2 + dX^1^2 + \dots + dX^4^2 = L^2(-dt^2 + e^{2t} dx^i dx^i) \quad \mathbb{H}^3$$

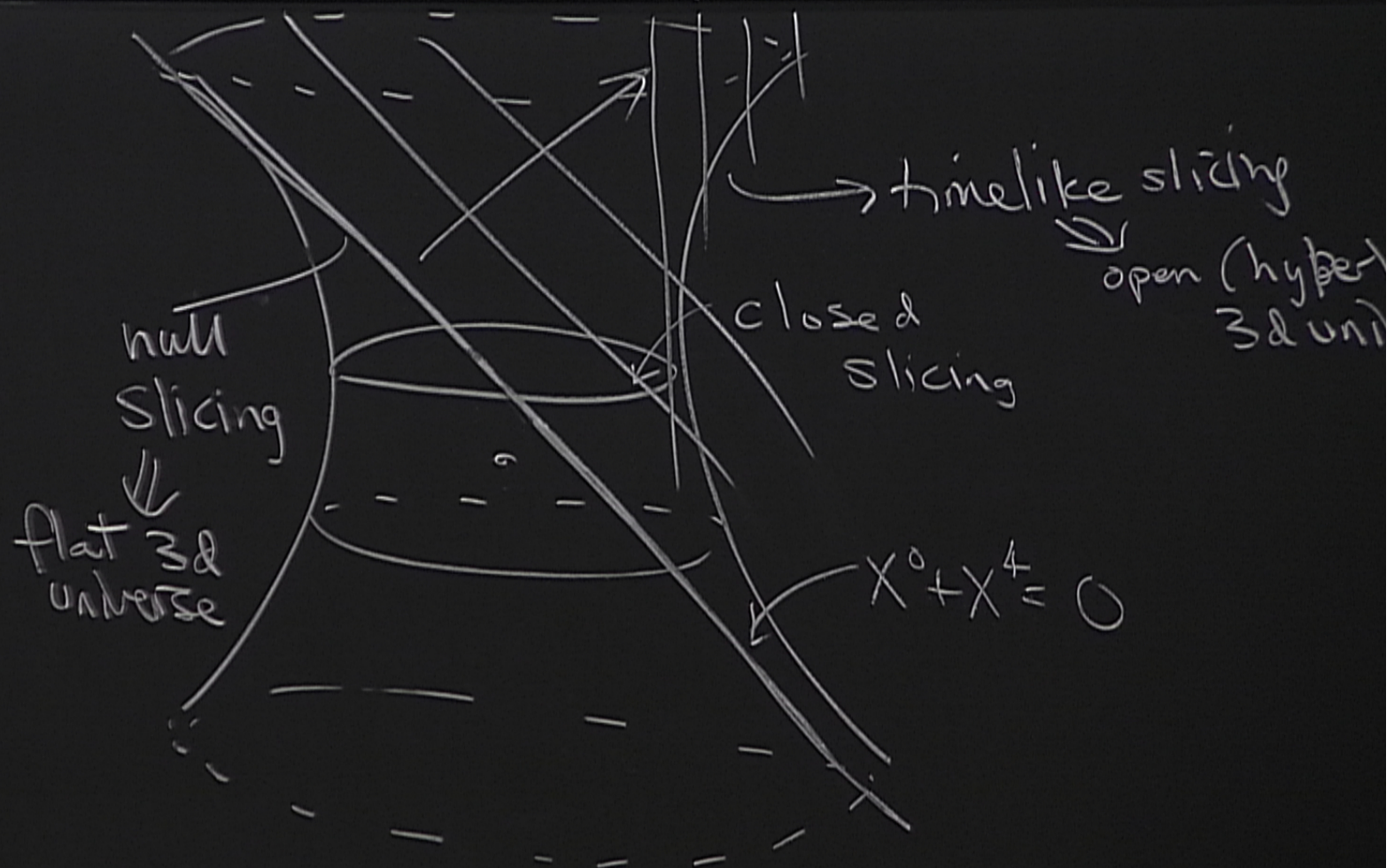
Note $-\infty < t < \infty$

metric is singular
at $t = -\infty$.

$$X^0 + X^4 = 0$$

$$X^i = 0$$

$\mathbb{H}^3$
↓
 $+ e^{2t} dx^i dx^i$



flat slicing

$$X^0 = L(\sinh t + \frac{1}{2} r^2 e^t)$$

$$X^i = L e^t x^i \quad i=1,2,3$$

$$X^4 = L(\cosh t - \frac{1}{2} r^2 e^t)$$

$$\text{Ex: } -dX^0^2 + dX^1^2 + \dots + dX^4^2 = L^2(-dt^2 + e^{2t} dx^i dx^i)$$

- only covers half of the spacetime.

Note

metric
at $t=0$

$X^0 + X^4$

X^0

flat slicing

$$X^0 = L(\sinh t + \frac{1}{2} r^2 e^t)$$

$$X^i = L e^t x^i \quad i=1,2,3$$

$$X^4 = L(\cosh t - \frac{1}{2} r^2 e^t)$$

$$\text{Ex: } -dX^0^2 + dX^1^2 + \dots + dX^4^2 = L^2(-dt^2 + e^{2t} dx^i dx^i)$$

- only covers half of the spacetime.

Note

metric at $t=0$

$X^0 + X^4$

X^4

- Only covers half of the spacetime.

The very early universe was a version of de Sitter spacetime
(L very small)

rimelike slicing
 open (hyperbolic)
 3d universe

flat slicing

$$X^0 = L(\sinh t + \frac{1}{2} r^2 e^t)$$

$$X^i = L e^t x^i \quad i=1,2,3$$

$$X^4 = L(\cosh t - \frac{1}{2} r^2 e^t)$$

$$r^2 = x^i x^i$$

$$\text{Ex: } -dX^0{}^2 + dX^1{}^2 + \dots + dX^4{}^2 = L^2(-dt^2 + e^{2t} dx^i dx^i)$$

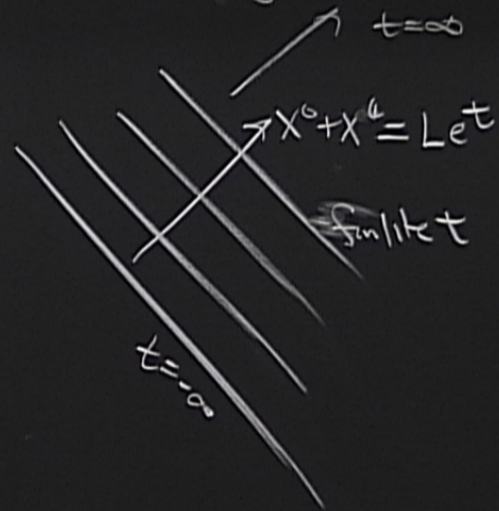
$$X^0 + X^4 = L e^t$$

- only covers half of the spacetime

flat slicing cosmology, the very early universe was a version of de Sitter space and the very late universe is too (L very large).

$$X^0 + X^6 = L e^t$$

According to inflationary cosmology, the



and the very late v

Solve constraint : $X^0 = L \sinh t$
 $X^I = L \cosh t \Omega^I \quad I=1 \dots 4$

where $\Omega^I \Omega^I = 1$

$\Omega^I = (\sin \chi \sin \theta \cos \varphi, \sin \chi \sin \theta \sin \varphi, \sin \chi \cos \theta, \cos \chi)$ parametrises S^3

$\Rightarrow$ line element becomes $L^2 [-dt^2 + \cosh^2 t d\Omega_3^2]$ where $d\Omega_3^2 = d\chi^2 + \sin^2 \chi$
 $t_p = Lt \quad \underbrace{-dt_p^2 + \cosh^2 \frac{t_p}{L} d\Omega_3^2}$

$(\sin\chi \sin\theta \cos\varphi, \sin\chi \sin\theta \sin\varphi, \sin\chi \cos\theta, \cos\chi)$ param

ent becomes $L^2[-dt^2 + \cosh^2 t d\Omega_3^2]$ where $d\Omega_3^2 =$

$$t_p = Lt \quad \underbrace{-dt_p^2 + L^2 \cosh^2 \frac{t_p}{L} d\Omega_3^2}_{R_3^2}$$

$$I=1\dots 4$$

$(\sin\chi \cos\theta, \cos\chi)$ parametrises S^3

where $d\Omega_3^2 = d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\varphi^2)$ is metric

trick $t_{\hat{t}} = i\tau$

$$\Rightarrow d\tau^2 + L^2 \cos^2\left(\frac{\tau}{L}\right) d\Omega_3^2$$

$$\left[\cosh^2 t d\Omega_3^2 \right]$$

$$\cosh^2 \frac{\tau}{L} d\Omega_3^2$$

$$R_3$$

$$\Omega^I \quad I=1\dots 4$$

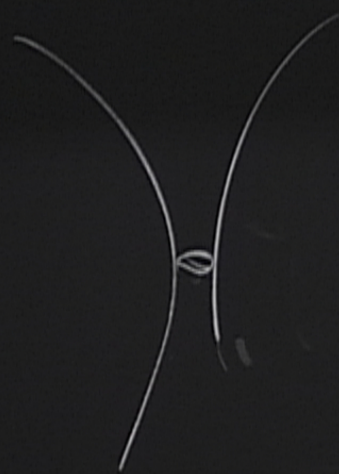
$(\sin\chi \cos\theta, \cos\chi)$ parametrises S^3

where $d\Omega_3^2 = d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\varphi^2)$ is metric

trick $t_{\pm} = i(\tau \mp \tau_0/2)$ $\rightarrow$ periodic

$$\Rightarrow d\tau^2 + L^2 \sin^2\left(\frac{\tau}{L}\right) d\Omega_3^2$$

all)



Solve constraint : $X^0 =$
 $X^I =$

where $\Omega^I \Omega^I = 1$

$\Omega^I = (\sin\chi \sin\theta \cos\phi, \sin\chi \sin\theta \sin\phi, \sin\chi \cos\theta, \cos\chi)$

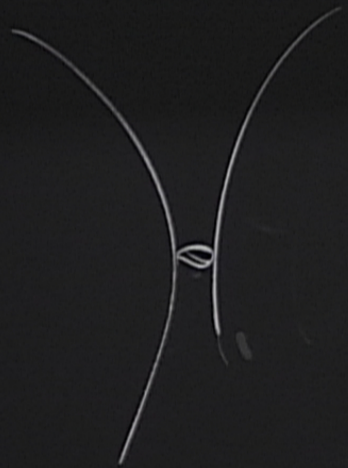
$\Rightarrow$ line element becomes

$$\text{period} = \frac{L}{2\pi}$$

$$e^{-\frac{H}{T}} = e^{-iHt} \quad t = -i\frac{1}{T}$$

$$\text{Tr}(e^{-\frac{H}{T}}) \Rightarrow \text{periodic, period } \Delta\tau = \frac{1}{T}$$

$$t_F = L t$$



Solve constraint : $X^0 = L \sinh$
 $X^I = L \cosh$

where $\Omega^I \Omega^I = 1$

$\Omega^I = (\sin \chi \sin \theta \cos \phi, \sin \chi \sin \theta \sin \phi, \sin \chi \cos \theta, \cos \chi)$

$\Rightarrow$ line element becomes $L^2 [-dt^2 + d\chi^2 + \sin^2 \chi d\theta^2 + \sin^2 \chi \sin^2 \theta d\phi^2]$

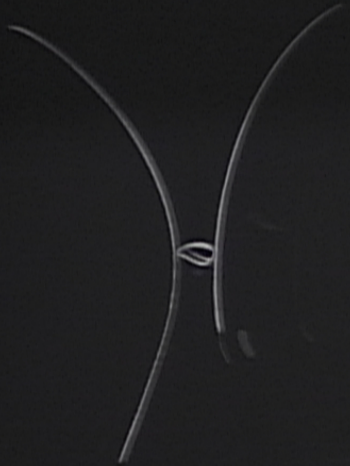
$T = \frac{2\pi}{L} = 2$

$t_p = Lt$

period = $\frac{L}{2\pi}$

$e^{-\frac{H}{T}} = e^{-iHt} \quad t = -i\frac{1}{T}$

$\text{Tr}(e^{-\frac{H}{T}}) \Rightarrow$ periodic, period $\Delta\tau = \frac{1}{T} = \frac{L}{2\pi}$



Solve constraint : $X^0 = L \sinh t$
 $X^I = L \cosh t \Omega^I$

where $\Omega^I \Omega^I = 1$

$\Omega^I = (\sin \chi \sin \theta \cos \varphi, \sin \chi \sin \theta \sin \varphi, \sin \chi$

period = $2\pi L$

$e^{-\frac{H}{T}} = e^{-iHt} \quad t = -i\frac{1}{T}$
 $\text{Tr}(e^{-\frac{H}{T}}) \Rightarrow \text{periodic, period } \Delta t = \frac{1}{T} = \frac{2\pi L}{\dots}$

$\Rightarrow$ line element becomes $L^2[-dt^2 + \cosh^2 t \underbrace{d\Omega^2}_{R_3^2}]$
 $T = \frac{1}{2\pi L} = \frac{H}{2\pi} \quad t_p = Lt$

The connection

Laws of physics expressed using derivatives, but ordinary partial derivatives are not tensors $\Rightarrow$ need to correct them using a "connection".

$$\dot{V}^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \dot{V}^{\alpha}(x)$$

Connection

Physics expressed using derivatives, but ordinary partial derivative tensors $\Rightarrow$ need to correct them using a "connection".

$$V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} V^{\alpha}(x) \quad ; \quad \frac{\partial V'^{\mu}(x')}{\partial x'^{\alpha}} = \frac{\partial x^{\lambda}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\lambda}}$$

derivatives, but ordinary partial derivatives of vectors (or more generally tensors) must be corrected using a "connection".

$$\frac{\partial V'^{\mu}}{\partial x'^{\alpha}}(x') = \frac{\partial x^{\lambda}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\lambda}} \left(\frac{\partial x'^{\mu}}{\partial x^{\beta}} V^{\beta} \right) = \frac{\partial x^{\lambda}}{\partial x'^{\alpha}} \frac{\partial V^{\beta}}{\partial x^{\lambda}}$$

derivatives, but ordinary partial derivatives of vectors (or more generally tensors) correct them using a "connection".

$$\frac{\partial V^{\mu}}{\partial x^{\alpha}} = \frac{\partial x^{\lambda}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\lambda}} \left(\frac{\partial x'^{\mu}}{\partial x^{\beta}} V^{\beta} \right) = \frac{\partial x^{\lambda}}{\partial x'^{\alpha}} \frac{\partial V^{\beta}}{\partial x^{\lambda}} \frac{\partial x'^{\mu}}{\partial x^{\beta}}$$

derivatives, but ordinary partial derivatives of vectors (or more generally tensors) must be corrected using a "connection".

$$\frac{\partial V'^{\mu}}{\partial x'^{\alpha}} = \frac{\partial x^{\lambda}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\lambda}} \left(\frac{\partial x'^{\mu}}{\partial x^{\beta}} V^{\beta} \right) = \frac{\partial x^{\lambda}}{\partial x'^{\alpha}} \frac{\partial V^{\beta}}{\partial x^{\lambda}} \frac{\partial x'^{\mu}}{\partial x^{\beta}}$$

of partial derivatives of vectors (or more generally, tensors)
 "connection".
 for a tensor $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
 ✓

$$\frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial}{\partial x^\alpha} \left(\frac{\partial x'^\mu}{\partial x^\beta} V^\beta \right) = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial V^\beta}{\partial x^\alpha} \frac{\partial x'^\mu}{\partial x^\beta} + \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\beta} V^\beta$$

derivatives of vectors (or more generally, tensors)

for a tensor $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

✓

WRONG for $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor.

$$\left(\frac{\partial x'^{\mu}}{\partial x^{\beta}} V^{\beta} \right) = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial V^{\beta}}{\partial x^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\beta}} + \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial^2 x'^{\alpha}}{\partial x^{\alpha} \partial x^{\beta}} V^{\beta}$$

Riemann: define $\Gamma_{\nu\alpha}^{\mu}(x)$ st.

$$\Gamma_{\nu\alpha}^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \Gamma_{\beta\gamma}^{\rho} + \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial^2 x^{\lambda}}{\partial x'^{\nu} \partial x'^{\alpha}}$$

Riemann: define $\Gamma^M_{\nu\alpha}(x)$ NOT A TENSOR! st.

$$\Gamma'^M_{\nu\alpha}(x') = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\gamma}{\partial x'^\alpha} \Gamma^{\rho}_{\beta\gamma} + \frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial x'^\nu \partial x'^\alpha}$$

Riemann: define $\Gamma^M_{\nu\alpha}(x)$ NOT A TENSOR! s.t.

$$\Gamma'^M_{\nu\alpha}(x') = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\gamma}{\partial x'^\alpha} \Gamma^{\rho}_{\beta\gamma} + \frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial x'^\nu \partial x'^\alpha}$$

then $\underbrace{\frac{\partial}{\partial x^\mu}, \partial_\mu}_{\text{COVARIANT DERIVATIVE}} \nabla_\mu V^\nu \equiv \frac{\partial V^\nu}{\partial x^\mu} + \Gamma^{\nu}_{\alpha\mu} V^\alpha$ is a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor.

$$\frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\mu}{\partial x^\beta} = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta$$

$$\frac{\partial}{\partial x}$$

$$\frac{\partial^2 x}{\partial x' \partial x'} \frac{\partial x'}{\partial x} \cdot \frac{\partial x'}{\partial x} + \frac{\partial x}{\partial x'} \frac{\partial^2 x'}{\partial x \partial x} = 0$$

Ex: show, using

$T = (2\pi/\omega) \Rightarrow$ periodic, period $\Delta t = L = 2\pi L$

Similaty $\nabla_\mu V_\beta \equiv \partial_\mu V_\beta - \Gamma_{\beta\mu}^\alpha V_\alpha$

$\Rightarrow$ periodic, period $\Delta\tau = L = 2\pi L$

Similarity $\nabla_\mu V_\beta \equiv \partial_\mu V_\beta - \Gamma_{\beta\mu}^\alpha V_\alpha$

derivative index last

$\Rightarrow$ periodic, period $\Delta\tau = L = 2\pi L$

Similaty $\nabla_\mu V_\beta \equiv \partial_\mu V_\beta - \Gamma_{\beta\mu}^\alpha V_\alpha$

derivative index last

- derivative index last

$$\partial_\lambda T^{\mu\nu\dots} + \overbrace{(\Gamma^\lambda{}_\mu{}^\nu \Gamma^\mu{}_\lambda{}^\nu \dots)}^m - \overbrace{(\Gamma^\lambda{}_\mu{}^\nu + \Gamma^\nu{}_\mu{}^\lambda \dots)}^n$$

ing two simplifying assumptions.

$$\Gamma_{[\mu\alpha]}^{\nu} = \frac{1}{2}(\Gamma_{\mu\alpha}^{\nu} - \Gamma_{\alpha\mu}^{\nu}) \text{ is a tensor, called the torsion } T_{\mu\alpha}^{\nu}.$$

We can fix $\Gamma_{\mu\nu}^{\lambda}$ by making two simpli

Note: Γ isn't a tensor, but $\Gamma_{[\mu\alpha]}^{\nu} = \frac{1}{2}(\Gamma_{\mu}^{\nu} - \Gamma_{\alpha}^{\nu})$

$\Rightarrow$ can impose $T_{\mu\alpha}^{\nu} = 0$.

We can fix $\Gamma_{\mu\nu}^\lambda$ by making two simplifying assumptions

Note: Γ isn't a tensor, but $\Gamma_{[\mu\alpha]}^\nu = \frac{1}{2}(\Gamma_{\mu\alpha}^\nu - \Gamma_{\alpha\mu}^\nu)$ is a tensor.
 $\Rightarrow$ can impose $T_{\mu\alpha}^\nu = 0$. i.e. torsion is zero.

ing two simplifying assumptions.

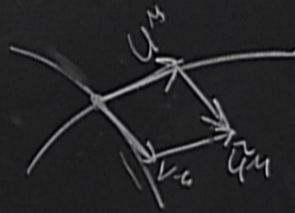
$$\Gamma_{[\mu\alpha]}^{\nu} = \frac{1}{2}(\Gamma_{\mu\alpha}^{\nu} - \Gamma_{\alpha\mu}^{\nu}) \text{ is a tensor, called the torsion } T_{\mu\alpha}^{\nu}.$$

○. i.e. torsion is zero. $\Leftrightarrow \Gamma_{\mu\alpha}^{\nu} = \Gamma_{\alpha\mu}^{\nu}$

simplifying assumptions.

$2(\Gamma_{\mu\alpha}^{\nu} - \Gamma_{\alpha\mu}^{\nu})$ is a tensor, called the torsion $T_{\mu\alpha}^{\nu}$.

if torsion is zero, $\Leftrightarrow \Gamma_{\mu\alpha}^{\nu} = \Gamma_{\alpha\mu}^{\nu}$

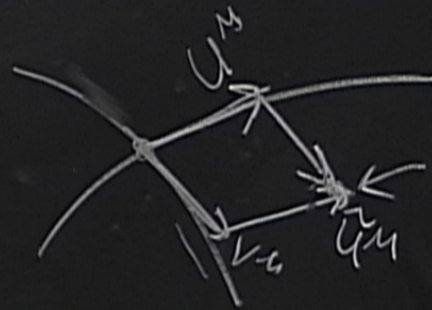


$$\frac{dX^{\mu}(\lambda)}{d\lambda} = U^{\mu}$$

Assumptions.

Γ is a tensor, called the torsion $T_{\mu\alpha}^{\nu}$.

$$\Leftrightarrow \Gamma_{\mu\alpha}^{\nu} = \Gamma_{\alpha\mu}^{\nu}$$



$$\frac{dX^M(\lambda)}{d\lambda} = U^M$$

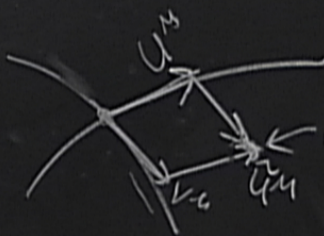
agree if $T_{\mu\nu}^{\lambda} = 0$.

assumptions.

is a tensor

the torsion $T_{\mu\nu}^{\lambda}$.

$\Leftrightarrow$



$$\frac{dX^M(\lambda)}{d\lambda} = U^M$$

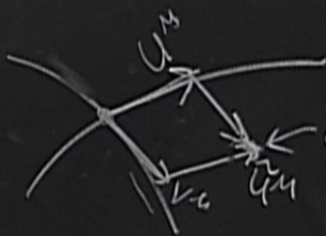
agree if $T_{\mu\nu}^{\lambda} = 0$.

$$\nabla_{\nu}\nabla_{\mu}\phi = \nabla_{\mu}\nabla_{\nu}\phi \cdot \forall \phi \Rightarrow T_{\mu\nu}^{\lambda} = 0.$$

assumptions.

is a tensor, called the torsion $T_{\mu\alpha}^{\nu}$.

$$\Leftrightarrow \Gamma_{\mu\alpha}^{\nu} = \Gamma_{\alpha\mu}^{\nu}$$



$$\frac{dX^M(\lambda)}{d\lambda} = U^M$$

agree if $T_{\mu\nu}^{\lambda} = 0$.

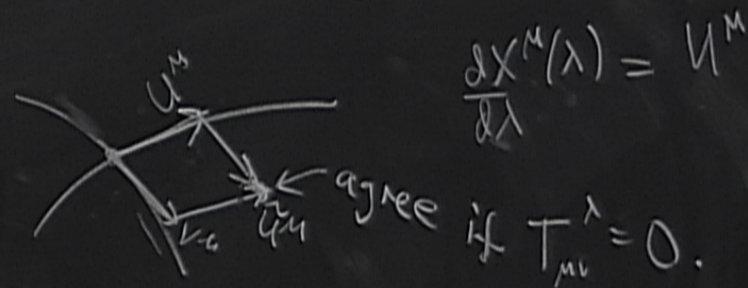
$$\nabla_{\nu}\phi = \partial_{\nu}\phi$$

$$\nabla_{\mu}\nabla_{\nu}\phi = \nabla_{\nu}\nabla_{\mu}\phi \cdot \forall \phi \Rightarrow T_{\mu\nu}^{\lambda} = 0.$$

two simplifying assumptions.

$$\Gamma_{[\mu\alpha]}^{\nu} = \frac{1}{2}(\Gamma_{\mu\alpha}^{\nu} - \Gamma_{\alpha\mu}^{\nu}) \text{ is a tensor, called the torsion } T_{\mu\alpha}^{\nu}.$$

i.e. torsion is zero. $\Leftrightarrow \Gamma_{\mu\alpha}^{\nu} = \Gamma_{\alpha\mu}^{\nu}$



$$\nabla_{\mu} \nabla_{\nu} \phi = \nabla_{\mu} \partial_{\nu} \phi$$

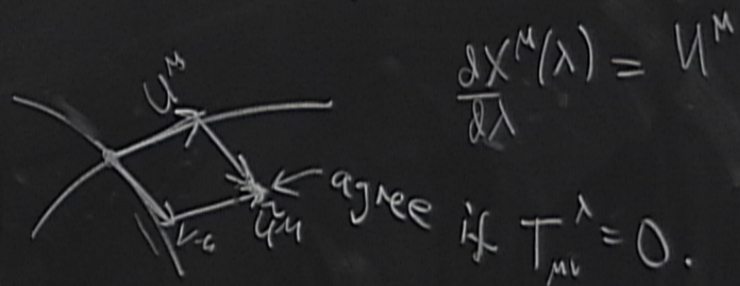
$$= \partial_{\mu} \partial_{\nu} \phi - \Gamma_{\nu\mu}^{\alpha} \partial_{\alpha} \phi$$

$$\nabla_{\mu} \nabla_{\nu} \phi = \nabla_{\nu} \nabla_{\mu} \phi \quad \forall \phi \Rightarrow T_{\mu\nu}^{\alpha} = 0$$

simplifying assumptions.

$\frac{1}{2}(\Gamma_{\mu\alpha}^{\nu} - \Gamma_{\alpha\mu}^{\nu})$ is a tensor, called the torsion $T_{\mu\alpha}^{\nu}$.

torsion is zero. $\Leftrightarrow \Gamma_{\mu\alpha}^{\nu} = \Gamma_{\alpha\mu}^{\nu}$



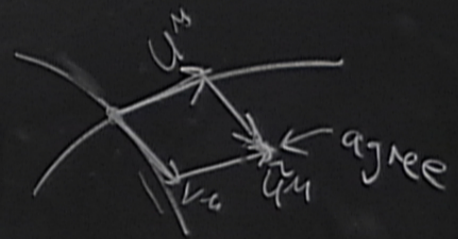
$$\boxed{\nabla_{\mu} \nabla_{\nu} \phi = \nabla_{\mu} \partial_{\nu} \phi}$$

$$= \partial_{\mu} \partial_{\nu} \phi - \Gamma_{\nu\mu}^{\alpha} \partial_{\alpha} \phi$$

$$\nabla_{\mu} \nabla_{\nu} \phi = \nabla_{\nu} \nabla_{\mu} \phi \quad \forall \phi \Rightarrow T^{\alpha}_{\mu\nu} = 0.$$

making two simplifying assumptions.

but $\Gamma_{[\mu\alpha]}^{\nu} = \frac{1}{2}(\Gamma_{\mu\alpha}^{\nu} - \Gamma_{\alpha\mu}^{\nu})$ is a tensor, called the torsion $T_{\mu\alpha}^{\nu}$.
 $T_{\mu\alpha}^{\nu} = 0$. i.e. torsion is zero. $\Leftrightarrow \Gamma_{\mu\alpha}^{\nu} = \Gamma_{\alpha\mu}^{\nu}$



$$\boxed{\nabla_{\mu} \nabla_{\nu} \phi = \nabla_{\nu} \nabla_{\mu} \phi}$$

$$= \partial_{\mu} \partial_{\nu} \phi - \Gamma_{\nu\mu}^{\alpha} \partial_{\alpha} \phi$$

$$\nabla_{\mu} \nabla_{\nu} \phi = \nabla_{\nu} \nabla_{\mu} \phi$$

We can fix $\Gamma_{\mu\nu}^{\lambda}$ by making two simplifying ass

Note: Γ isn't a tensor, but $\Gamma_{[\mu\alpha]}^{\nu} = \frac{1}{2}(\Gamma_{\mu\alpha}^{\nu} - \Gamma_{\alpha\mu}^{\nu})$ is

$\Rightarrow$ can impose $T_{\mu\alpha}^{\nu} = 0$. i.e. torsion is zero.

Consider $\nabla_{\alpha} g_{\mu\nu}$, a $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ tensor.

$\Rightarrow$ Set it zero ($\nabla_{\alpha} g_{\mu\nu}$ is called metricity)

then $\Gamma_{\alpha\beta}^{\mu} = \frac{1}{2} g^{\mu\lambda} (g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda}) \equiv \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\}$

Christoffel symbol.

If $\nabla_\alpha g_{\mu\nu} = 0$, and $T_{\mu\alpha}^\nu = 0$, then $\Gamma_{\alpha\beta}^\mu = \frac{1}{2} g^{\mu\lambda} (g_{\alpha\lambda,\beta} +$

proof: $\partial_\alpha g_{\beta\gamma} - \Gamma_{\beta\alpha}^\delta g_{\delta\gamma} - \Gamma_{\gamma\alpha}^\delta g_{\beta\delta}$

If $\nabla_\alpha g_{\mu\nu} = 0$, and $T^\nu_{\mu\alpha} = 0$, then $\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} (g_{\lambda\alpha} \partial_\beta + g_{\lambda\beta} \partial_\alpha - g_{\alpha\beta} \partial_\lambda)$

proof: $\partial_\alpha g_{\beta\gamma} - \Gamma^\delta_{\beta\alpha} g_{\delta\gamma} - \Gamma^\delta_{\gamma\alpha} g_{\beta\delta} = 0$
 + cycles $\partial_\gamma g_{\alpha\beta} - \partial_\beta g_{\gamma\alpha} - \Gamma^\delta_{\beta\gamma} g_{\delta\alpha} - \Gamma^\delta_{\alpha\gamma} g_{\beta\delta} = 0$
 $\partial_\beta g_{\gamma\alpha} - \Gamma^\delta_{\gamma\beta} g_{\delta\alpha} - \Gamma^\delta_{\alpha\beta} g_{\gamma\delta} = 0$

If $\nabla_\alpha g_{\mu\nu} = 0$, and $T^\nu_{\mu\alpha} = 0$, then $\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda}$

proof:

$$\partial_\alpha g_{\beta\gamma} - \Gamma^\delta_{\beta\alpha} g_{\delta\gamma} - \Gamma^\delta_{\gamma\alpha} g_{\beta\delta} = 0$$

+ cycles

$$\partial_\gamma g_{\alpha\beta} - \Gamma^\delta_{\alpha\gamma} g_{\delta\beta} - \Gamma^\delta_{\beta\gamma} g_{\alpha\delta}$$

$$\partial_\beta g_{\gamma\alpha} - \Gamma^\delta_{\gamma\beta} g_{\delta\alpha} - \Gamma^\delta_{\alpha\beta} g_{\gamma\delta}$$

If $\nabla_\alpha g_{\mu\nu} = 0$, and $T_{\mu\alpha}^\nu = 0$, then $\Gamma_{\alpha\beta}^\mu = \frac{1}{2} g^{\mu\lambda}$

proof:

$$\partial_\alpha g_{\beta\gamma} - \Gamma_{\beta\alpha}^\delta g_{\delta\gamma} - \Gamma_{\gamma\alpha}^\delta g_{\beta\delta} = 0$$

+ cycles.

$$\partial_\gamma g_{\alpha\beta} - \Gamma_{\alpha\gamma}^\delta g_{\delta\beta} - \Gamma_{\beta\gamma}^\delta g_{\alpha\delta} = 0$$

$$\partial_\beta g_{\gamma\alpha} - \Gamma_{\gamma\beta}^\delta g_{\delta\alpha} - \Gamma_{\alpha\beta}^\delta g_{\gamma\delta} = 0$$

If $\nabla_\alpha g_{\mu\nu} = 0$, and $T_{\mu\alpha}^\nu = 0$, then $\Gamma_{\alpha\beta}^\mu = \frac{1}{2} g^{\mu\lambda}$

proof:

$$\partial_\alpha g_{\beta\gamma} - \Gamma_{\beta\alpha}^\delta g_{\delta\gamma} - \Gamma_{\gamma\alpha}^\delta g_{\beta\delta} = 0$$

+ cycles

$$\partial_\gamma g_{\alpha\beta} - \Gamma_{\alpha\gamma}^\delta g_{\delta\beta} - \Gamma_{\beta\gamma}^\delta g_{\alpha\delta} = 0$$

$$\partial_\beta g_{\gamma\alpha} - \Gamma_{\gamma\beta}^\delta g_{\delta\alpha} - \Gamma_{\alpha\beta}^\delta g_{\gamma\delta} = 0$$

Christoffel symbol.

$$\frac{1}{2} g^{\mu\lambda} (g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda}) \equiv \left\{ \begin{matrix} \mu \\ \alpha \beta \end{matrix} \right\}$$

}

$$\partial_{\alpha} g_{\beta\gamma} + \partial_{\beta} g_{\gamma\alpha} - \partial_{\gamma} g_{\alpha\beta} - \Gamma^{\delta}_{\gamma\alpha} g_{\beta\delta}$$

Christoffel symbol.

$$\frac{1}{2} g^{\mu\lambda} (g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda}) \equiv \sum_{\alpha\beta}^{\mu}$$

using symmetry of Γ .

$$\partial_{\alpha} g_{\beta\gamma} + \partial_{\beta} g_{\gamma\alpha} - \partial_{\gamma} g_{\alpha\beta} - 2 \Gamma_{\gamma\alpha}^{\delta} g_{\beta\delta}$$

Christoffel symbol.

$$\frac{1}{2} g^{\mu\lambda} (g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda}) \equiv \sum_{\alpha\beta}^{\mu}$$

using symmetry of Γ

$$\partial_{\alpha} g_{\beta\gamma} + \partial_{\beta} g_{\gamma\alpha} - \partial_{\gamma} g_{\alpha\beta} - 2 \Gamma_{\gamma\alpha}^{\delta} g_{\beta\delta} = 0 \quad \times g^{\beta\gamma}$$

$$\Rightarrow \Gamma_{\gamma\alpha}^{\gamma} = \frac{1}{2} g^{\gamma\beta} (\partial_{\alpha} g_{\beta\gamma} + \partial_{\beta} g_{\gamma\alpha} - \partial_{\gamma} g_{\alpha\beta})$$

$$(g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda}) \equiv \sum_{\alpha\beta} \{$$

using symmetry of Γ

$$\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta} - 2 \Gamma_{\gamma\alpha}^\epsilon g_{\beta\epsilon} = 0$$

$$\Rightarrow \Gamma_{\gamma\alpha}^\nu = \frac{1}{2} g^{\nu\beta} (\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta})$$

$\Rightarrow$ can set $\Gamma_{\gamma\alpha}^\nu = 0$ at any spacetime point, through a coord transtm.

Christoffel symbol.

$$\frac{1}{2} g^{\mu\lambda} (g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda}) \equiv \sum_{\alpha\beta}^{\mu} \delta_s^v$$

using symmetry of Γ

$$\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta} - 2 \Gamma_{\beta\alpha}^\gamma g_{\gamma\delta} = 0$$

$$\Rightarrow \Gamma_{\alpha\beta}^\gamma = \frac{1}{2} g^{\gamma\delta} (\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta})$$

$\Rightarrow$ can set $\Gamma_{\gamma\alpha}^\nu = 0$ at any spacetime point, through a coord transfmⁿ

derivative index last

$$\nabla_\lambda T^{\mu\nu\cdots} = \partial_\lambda T^{\mu\nu\cdots} + \overbrace{(\Gamma^\mu{}_\lambda{}^\alpha T^{\alpha\nu\cdots})}^m - \overbrace{(\Gamma^\nu{}_\lambda{}^\alpha T^{\mu\alpha\cdots})}^n$$

then

length squared

$$\phi = g_{\mu\nu} V^\mu V^\nu$$

$$\nabla_\alpha \phi = \frac{\partial}{\partial x^\alpha} (g_{\mu\nu} V^\mu V^\nu) = \nabla_\alpha (g_{\mu\nu} V^\mu V^\nu)$$

derivative index last

$$\partial_\lambda T^{\mu\nu} + \overbrace{(\Gamma^\mu_\lambda \Gamma^\nu_{\dots})}^m - \overbrace{(\Gamma^\mu_\lambda + \Gamma^\nu_{\dots})}^n$$

$$\begin{aligned} \nabla_\alpha \phi &= \frac{\partial}{\partial x^\alpha} (g_{\mu\nu} V^\mu V^\nu) = \nabla_\alpha (g_{\mu\nu} V^\mu V^\nu) \quad \text{Leibniz} \\ &= (\nabla_\alpha g_{\mu\nu}) V^\mu V^\nu + g_{\mu\nu} \nabla_\alpha V^\mu V^\nu + g_{\mu\nu} V^\mu \nabla_\alpha V^\nu \end{aligned}$$

and for general tensor $\nabla_\lambda T^{\mu\nu \dots}_{\alpha\beta\dots}$ $\underbrace{\alpha\beta\dots}_n = \partial_\lambda T^{\mu\nu \dots}_{\alpha\beta\dots} + (\Gamma T)_{\dots}$

If metricity is not zero, then

$$\text{if } \nabla_\alpha V^\nu = 0$$

" V^ν is const"

length squared

$$\phi = g_{\mu\nu} V^\mu V^\nu$$

$$\nabla_\alpha \phi = \frac{\partial}{\partial x^\alpha} (g_{\mu\nu} V^\mu V^\nu)$$

is not constant, if metricity is not zero.