

Title: 13/14 PSI - Relativity - Lecture 6

Date: Sep 16, 2013 09:00 AM

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Abstract:

Minkowski spacetime, Lorentz-invariant distance is given by

$$-c^2 dt^2 + dx^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \text{ invariant under } x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$$

given by

$$\Lambda^\mu_\nu \Lambda^\alpha_\beta \eta_{\mu\alpha} = \eta_{\nu\beta}.$$

under $x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$

given by

$$\Lambda^\mu{}_\nu \Lambda^\alpha{}_\beta \eta_{\mu\alpha} = \eta_{\nu\beta}.$$

under $x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$\Rightarrow$ 6 indept parameters
- 3 boosts
- 3 rotations.

sk1 = spacetime, Lorentz-invariant distance is given by

$$-c^2 dt^2 + dx^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

$\Rightarrow$ all laws of physics written in tensor form (+ spinors), invariant under $x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$, invariant under L.Ts.

III, in Minkowski-spacetime, Lorentz-invariant distance is given by

$$-c^2 dt^2 + dx^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \text{ invariant under } x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$$

$\Rightarrow$ all laws of physics written in tensor form, invariant under L.Ts.
(+ spinors)

Recall, in Minkowski spacetime, Lorentz

$$x' = x + \frac{1}{2}at^2$$

$$\frac{d^2 x_p}{dt^2} = 0$$

$$-c^2 dt^2 + dx^2$$

$\Rightarrow$ all laws of physics

Recall, in Minkowski spacetime, Lorentz

$$x' = x + \frac{1}{2}at^2$$

$$-c^2dt^2 + dx^2$$

$$\text{If } \frac{d^2x_p}{dt^2} = 0$$

$$\text{then } m \frac{d^2x'}{dt^2} = ma$$

$\Rightarrow$ all laws of physics

Recall, in Minkowski spacetime, Lorentz

$$x' = x + \frac{1}{2}at^2$$

$$-c^2dt^2 + dx^2$$

$$\begin{aligned} \text{If } \frac{d^2x_p}{dt^2} &= 0 \\ \text{then } m \frac{d^2x'}{dt^2} &= ma = \text{"F"} \end{aligned}$$

$\Rightarrow$ all laws of physics

Recall, in Minkowski spacetime

eg.
 $x' = x + \frac{1}{2}at^2$

$$-c^2dt^2 +$$

If $\frac{d^2x_p}{dt^2} = 0$
then $m \frac{d^2x'_p}{dt^2} = \underbrace{ma}_{= "F"} = "F"$

$\Rightarrow$ all laws of

Einstein: $F_{\text{grav}}?$

in Minkowski^(FLAT) spacetime, Lorentz-invariant distance is given by

$$-c^2 dt^2 + dx^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

$\begin{matrix} \nwarrow & \nearrow \\ (-1, +1, +1, +1) & \text{Minkowski metric} \end{matrix}$

$\Rightarrow$ all laws of physics written in tensor form (+ spinors), invariant under $x^\mu \rightarrow$

$g_{\mu\nu}(x) dx^\mu dx^\nu$, where $g_{\mu\nu}(x)$ is an arbitrary 4×4 symmetric matrix.
(10 indept components)

using metric $g_{\mu\nu}(x) dx^\mu dx^\nu$, where $g_{\mu\nu}(x)$ is an arbitrary 4×4

Condⁿs 1) $g_{\mu\nu}$ invertible ($\Leftrightarrow \det g \neq 0$)

2) 3 +ve and 1 -ve eigenvalues

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(
Condⁿs 1) $g_{\mu\nu}$ invertible ($\Leftrightarrow \det g \neq 0$)
2) 3 +ve and 1 -ve eigenvalues ("pseudo-Riemannian")

say that a space possessing such a metric is locally Minkowski

because we can always choose coordinates so that the metric at one point is equal to the Minkowski metric.

We say that a space possessing such a metric is locally Minkowski

because we can always choose coordinates so that the metric is the Minkowski metric.

g symmetric
 $\Rightarrow \exists O_{\alpha}^{\mu}$ s.t. $O_{\alpha}^{\mu} O_{\beta}^{\nu} g_{\mu\nu}(x_p) = D_{\alpha\beta}$, diagonal
($OgO^T = D$)

one particular point

... that a space possessing such a metric is locally Minkowski

because we can always choose coordinates so that the metric at a particular point is the Minkowski metric.

g symmetric

$$\Rightarrow \exists O_\alpha^\mu \text{ s.t. } O_\alpha^\mu O_\beta^\nu g_{\mu\nu}(x_p) = D_{\alpha\beta}, \text{ diagonal} = \begin{pmatrix} -l_0^2 & & & \\ & l_1^2 & & \\ & & l_2^2 & \\ & & & l_3^2 \end{pmatrix}$$

(orthogonal) $(OgO^T = D)$ $\Rightarrow g = O^T D O$
 $O^T O = \mathbb{I}$

locally Minkowski

so that the metric at one point is equal to

$$\text{diagonal} = \begin{pmatrix} -l_0^2 & & & \\ & l_1^2 & & \\ & & l_2^2 & \\ & & & l_3^2 \end{pmatrix} \quad ; \quad \text{defining } L = \begin{pmatrix} l_0 & & & \\ & l_1 & & \\ & & l_2 & \\ & & & l_3 \end{pmatrix}$$

$$\Rightarrow g = O^T \eta O$$

$$\Rightarrow g = O L \eta L^T O^T$$

$$\text{or } dx^\mu dx^\nu g_{\mu\nu} = dy^\mu dy^\nu \eta_{\mu\nu},$$

$$y = O L x$$

Minkowski metric.

or particular point

$$\text{s.t. } O_{\alpha}^{\mu} O_{\beta}^{\nu} g_{\mu\nu}(x_p) = D_{\alpha\beta}, \text{ diagonal} = \begin{pmatrix} -l_0^2 & & & \\ & l_1^2 & & \\ & & l_2^2 & \\ & & & l_3^2 \end{pmatrix}$$

$$(OgO^T = D) \\ O^T O = \mathbb{1}$$

$$\Rightarrow g = O^T D O$$

defining $L = \begin{pmatrix} l_0 & & & \\ & l_1 & & \\ & & l_2 & \\ & & & l_3 \end{pmatrix}$

$$\Rightarrow g = O^T L \eta L^T O$$

$$dx^{\mu} dx^{\nu} g_{\mu\nu} = dy^{\alpha} dy^{\beta} \eta_{\alpha\beta},$$

$$y = O L x$$

$$dx^T g dx = \underbrace{dx^T O^T L \eta L O}_{dy^T \eta dy} dx$$

Minkowski metric.

or particular point

s.t. $O_\alpha^\mu O_\beta^\nu g_{\mu\nu}(x_p) = D_{\alpha\beta}$, diagonal = $\begin{pmatrix} -l_0^2 & & & \\ & l_1^2 & & \\ & & l_2^2 & \\ & & & l_3^2 \end{pmatrix}$

(diagonal)
 $(OgO^T = D)$
 $O^T O = \mathbb{1}$

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$\Rightarrow g = O^T L \eta L^T O$

$dx^\mu dx^\nu g_{\mu\nu} = dy^\alpha dy^\beta \eta_{\alpha\beta}$

$y = L O x$

$dx^T g dx = \underbrace{dx^T O^T L \eta L O dx}_{dy^T \eta dy}$

Minkowski metric.

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$dx = \underbrace{dx^T O^T L \eta L^T O}_{dy^T \eta dy} dx$

Minkowski metric.

or particular point

s.t. $O_\alpha^\mu O_\beta^\nu g_{\mu\nu}(x_p) = D_{\alpha\beta}$, diagonal = $\begin{pmatrix} -l_0^2 & & & \\ & l_1^2 & & \\ & & l_2^2 & \\ & & & l_3^2 \end{pmatrix}$

(diagonal)
 $(OgO^T = D)$
 $O^T O = \mathbb{1}$

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defining $L = \begin{pmatrix} l_0 & & & \\ & l_1 & & \\ & & l_2 & \\ & & & l_3 \end{pmatrix}$

$\Rightarrow g = O^T L \eta L^T O$

$dx^\mu dx^\nu g_{\mu\nu} = dy^\alpha dy^\beta \eta_{\alpha\beta}$

$y = L O x$

$dx^T g dx = \underbrace{dx^T O^T L \eta L O dx}_{dy^T \eta dy}$

$$dx^\mu g_{\mu\nu} = dy^\mu dy^\nu \eta_{\mu\nu},$$

$$\underline{y = L O x}$$

$$dx^\mu g_{\mu\nu} dx^\nu = \underbrace{dx^\mu O^\mu{}_\alpha L^\alpha{}_\beta \eta_{\alpha\beta} L^\beta{}_\gamma O^\gamma{}_\nu}_{dy^\alpha \eta_{\alpha\beta} dy^\beta} dx^\nu$$

$\Rightarrow$ we can transform $g_{\mu\nu}$ into $\eta_{\mu\nu}$ using O and L
 $\uparrow$ $\uparrow$
 6 parameters 4 parameters

2

$$g_{\mu\nu}(x) dx^\mu dx^\nu$$

$$= g'_{\mu\nu}(x') dx'^\mu dx'^\nu$$

$$\Rightarrow g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$$

$$= \eta_{\mu\nu} \quad \text{at one point.}$$

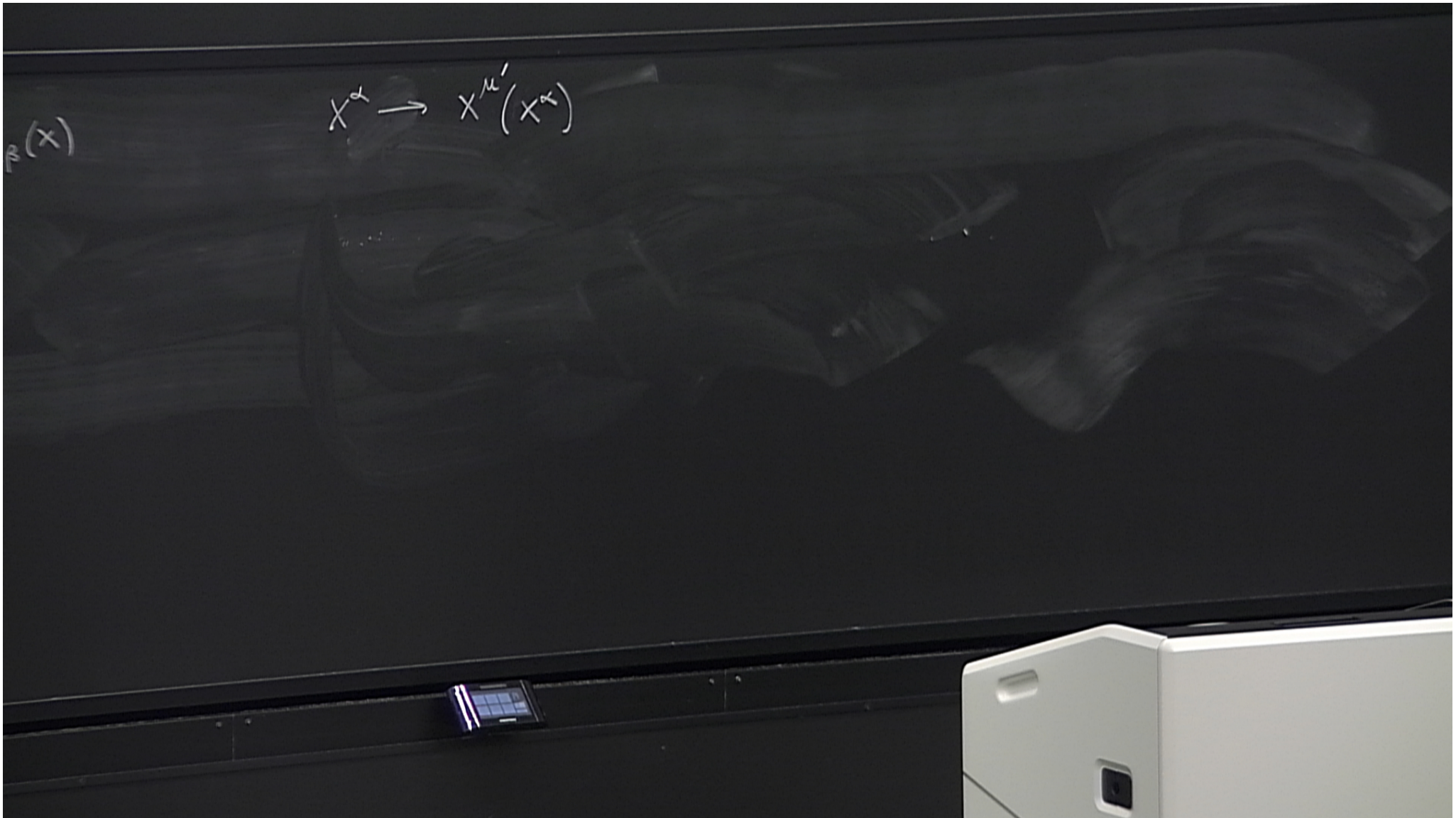
at every spacetime point it is possible to choose a "free-fall" or "locally inertial" coordinate system within a sufficiently small neighbourhood, $g_{\mu\nu} = \eta_{\mu\nu}$ and $\frac{\partial}{\partial x^\alpha} g_{\mu\nu} \approx 0$,

choose a "free-fall" or "locally inertial" coordinate system such that

$$g_{\mu\nu} = \eta_{\mu\nu} \quad \text{and} \quad \frac{\partial}{\partial x^\alpha} g_{\mu\nu} \approx 0, \quad \text{then the laws of physics}$$

Proof:

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$$



$x^\alpha \rightarrow x'^\mu(x)$: locally 1-1 if $\frac{\partial x^\alpha}{\partial x'^\mu}$ is invertible

(Jacobian
matrix)

$$\frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\mu} = \frac{\partial x^\beta}{\partial x^\alpha} = \delta_\alpha^\beta$$

Proof:

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$$

Assume
coord transfn.
has a Taylor
series about
the point a^α .
 $x^\mu(a^\alpha) = a^\mu$

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$$

$x^\alpha \rightarrow x'^\mu(x)$: locally

$$x'^\mu(x') = a^\mu + \left. \frac{\partial x'^\mu}{\partial x^\nu} \right|_a (x^\nu - a^\nu) + \frac{1}{2} \frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\beta} (x^\alpha - a^\alpha) (x^\beta - a^\beta) + \dots$$

from
on
about
 a^μ

$$= a^\mu$$

$$\begin{aligned}
 &= \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \\
 &= a^\mu + \frac{\partial x^\mu}{\partial x'^\nu} (x'^\nu - a'^\nu) + \frac{\partial^2 x^\mu}{\partial x'^\alpha \partial x'^\beta} (x'^\alpha - a'^\alpha) (x'^\beta - a'^\beta) + \dots \\
 g'_{\mu\nu}(x') &= \underbrace{\left(\frac{\partial x}{\partial x'} \right)^2 g}_{\downarrow} + \left(\frac{\partial x}{\partial x'} \frac{\partial^2 x}{\partial x' \partial x'} g + \frac{\partial x}{\partial x'} \frac{\partial x}{\partial x'} \frac{\partial g}{\partial x} \right) (x' - a') + \dots
 \end{aligned}$$

$x^\alpha \rightarrow x'^\mu(x)$: locally 1-1 if
 (Jacobian matrix)

① choose $\frac{\partial x}{\partial x'}|_a$ to set $g'(a') = \eta$

② $\frac{\partial^2 x^\alpha}{\partial x'^\beta \partial x'^\gamma} \Big|_a$

① choose $\frac{\partial x}{\partial x'}|_a$ to set $g'(a') = \eta$

② $\frac{\partial^2 x^\alpha}{\partial x'^\beta \partial x'^\gamma}|_a \rightarrow 4 \cdot 10$ numbers
↑
indep components of 4×4 Symmetric

to adjust $g'_{\mu\nu,\beta}|_{a'} = \frac{\partial}{\partial X^\beta} g'_{\mu\nu}|_{a'}$
- 40 quantities

① choose $\frac{\partial x}{\partial x'}|_a$ to set $g'(a') = \eta$

② $\frac{\partial^2 x^a}{\partial x'^b \partial x'^c} \Big|_a \rightarrow 4 \cdot 10 \uparrow$ num
indep components of

$\Rightarrow$ can set $g'_{\alpha\beta} = 0$.

③

$x^\alpha \rightarrow x'^\mu(x)$: locally 1-1 if $\frac{\partial x'^\mu}{\partial x^\alpha}$ is invertible
 (Jacobian matrix)

$$\frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} g_{\mu\beta}(x)$$

$$= a^\mu + \frac{\partial x'^\mu}{\partial x'^\nu} \Big|_a (x'^\nu - a'^\nu) + \frac{\partial^2 x'^\mu}{\partial x'^\nu \partial x'^\rho} \Big|_a (x'^\nu - a'^\nu)(x'^\rho - a'^\rho) + \dots$$

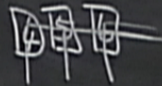
$$g_{\mu\alpha}(x') = \underbrace{\left(\frac{\partial x}{\partial x'} \right)^2 g}_{\downarrow} \Big|_a + \underbrace{\left(\frac{\partial x}{\partial x'} \frac{\partial^2 x}{\partial x' \partial x'} g + \frac{\partial x}{\partial x'} \frac{\partial x}{\partial x'} \frac{\partial g}{\partial x} \right)}_{\mu\nu, \alpha} (x' - a') + \dots \left(\frac{\partial^3 x}{\partial x' \partial x' \partial x'} + \dots \right) (x')$$

③ Can I choose

$$\left. \frac{\partial x^m}{\partial x^{a'} \partial x^{b'} \partial x^{c'}} \right|_{a'}$$

T_{112}

Young Tableaux



$$\frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3} = 20$$

Symm 3-index tensor

$$\left. \begin{array}{l} \text{3 indices same} \Rightarrow 4 \\ \text{2 indices same} \Rightarrow 12 \\ \text{No indices same} \Rightarrow 4 \end{array} \right\} 20$$

$$\left. \frac{\partial x^\mu}{\partial x^{\alpha'} \partial x^{\beta'} \partial x^{\gamma'}} \right|_{a'}$$

so that

$$\underbrace{g_{\mu\nu, \alpha\beta}}_{100} = 0?$$

tensor

$$\left. \begin{array}{l} \Rightarrow 4 \\ \Rightarrow 12 \\ \Rightarrow 4 \end{array} \right\} 20 \times 4 = 80$$

μ index

$\Rightarrow$ can set 80 of the $g_{\mu\nu, \alpha\beta}$ to zero, leaving 20 nonzero.

$\Rightarrow$ Riemann tensor.

choose $\left. \frac{\partial x^\mu}{\partial x'^{\alpha'} \partial x'^{\beta'} \partial x'^{\gamma'}} \right|_{a'}$ so that $\underbrace{g_{\mu\nu, \alpha\beta}}_{100} = 0?$

Symm 3-index tensor

3 indices same $\Rightarrow$	4	} 20×4 $= 80$	$\swarrow$ μ index
2 indices same $\Rightarrow$	12		
No indices same $\Rightarrow$	4		

$\Rightarrow$ can set 80 of the $g_{\mu\nu}$ leaving 20 nonzero.

$\Rightarrow$ Riemann tensor

(Ex: compute in D dims)

Metric can be used to measure distances and volumes.

metric transforms

Metric can be used to measure distances and volumes

metric transforms $g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$

$\Rightarrow$ determinant $g' = \det g'_{\mu\nu} = \left| \frac{\partial x}{\partial x'} \right|^2 g$

metric can be used to measure distances and volumes

metric transforms $g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$

$\Rightarrow$ determinant $g' = \det g'_{\mu\nu} = \left| \frac{\partial x}{\partial x'} \right|^2 g \Rightarrow \sqrt{-g'} = \left| \frac{\partial x}{\partial x'} \right| \sqrt{-g}$

c.f. $d^4 x' = \left| \frac{\partial x'}{\partial x} \right| d^4 x$

$\Rightarrow \boxed{d^4 x' \sqrt{-g'} = d^4 x \sqrt{-g}}$

inv

Inverse metric $g^{\mu\nu}$, defined to be inverse of $g_{\mu\nu}$, so

$$g^{\mu\nu} g_{\nu\alpha} = \delta^\mu_\alpha$$

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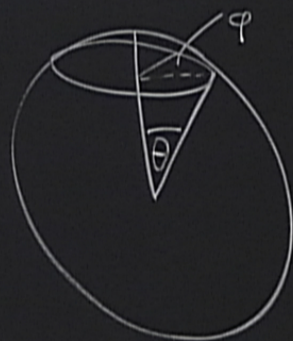
$$g^{\mu\nu} g_{\nu\alpha} = \delta^\mu_\alpha$$

transforms
by
 $\frac{\partial x'}{\partial x}$

$$g'_{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} g_{\alpha\beta}$$

$\Rightarrow$ use $g_{\mu\nu}, g^{\mu\nu}$ to raise and lower indices, just as in special relativity. =

Examples:



$$S^2 \subset \mathbb{R}^3$$

$$x^2 + y^2 + z^2 = 1$$

$$x = \sin \theta \cos \varphi$$

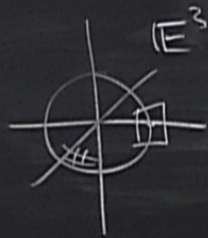
$$y = \sin \theta \sin \varphi$$

$$z = \cos \theta$$

} embedding coordinates
 θ, φ intrinsic coords.

Metric : inherited from $\mathbb{R}^3$: $dx^2 + dy^2 + dz^2 = d\theta^2 + \sin^2 \theta d\varphi^2$

Examples:



$$S^2 \subset \mathbb{E}^3$$

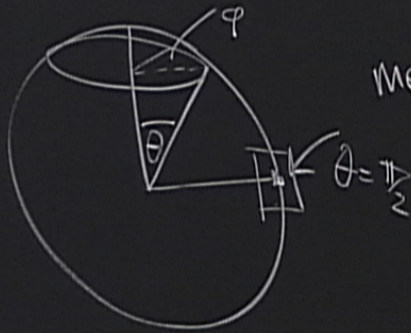
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Metric: inherited from $\mathbb{E}^3$: $dx^2 + dy^2 + dz^2 = d\theta^2 + \sin^2 \theta d\varphi^2$

$\approx d\theta^2 + d\varphi^2$ when $\theta \approx \frac{\pi}{2}$

2d : only 1 component of Riemann.

embedding coordinates

θ, φ intrinsic coords.

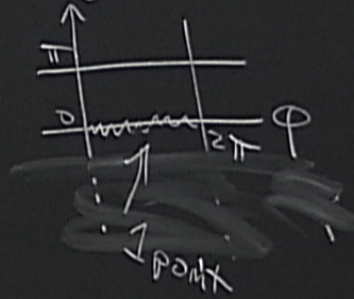
$$\text{from } \mathbb{H}^3 : dx^2 + dy^2 + dz^2 = d\theta^2 + \sin^2\theta d\varphi^2$$

$$\approx d\theta^2 + d\varphi^2 \quad \text{when } \theta \approx \frac{\pi}{2} \\ + \left(\theta - \frac{\pi}{2}\right)^2 d\varphi^2$$

$$\begin{aligned}x &= \theta \cos \varphi \\y &= \theta \sin \varphi\end{aligned}$$

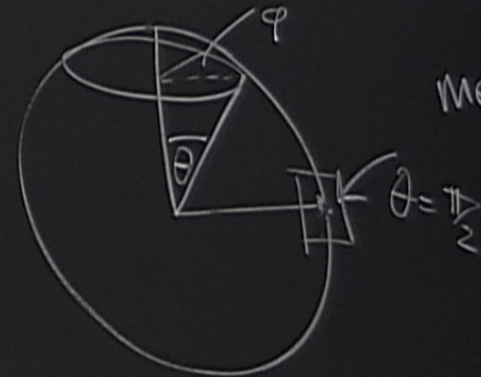
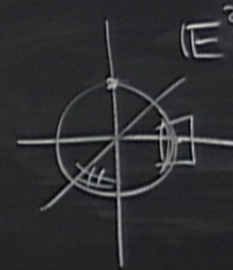
$$\underbrace{d\theta^2 + \theta^2 d\varphi^2}_{dx^2 + dy^2}$$

$$d\theta^2 + d\varphi^2$$



Examples:

$$\begin{aligned}0 &\leq \theta \leq \pi \\0 &< \varphi \leq 2\pi\end{aligned}$$



$$S^2 \subset \mathbb{E}^3$$

$$x^2 + y^2 + z^2 =$$

$$x = \sin \theta \cos \varphi$$

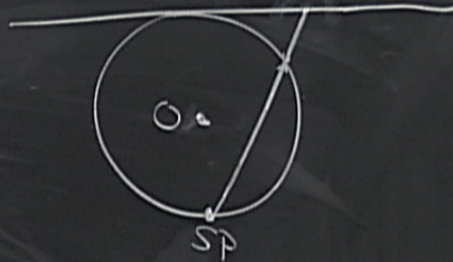
$$y = \sin \theta \sin \varphi$$

$$z =$$

Me:

Stereographic coords in S^2

$\mathbb{R}^3$
 (x, y, z)



ds in S^2

$$\begin{aligned} z_1 &= R \cos \theta \\ z_2 &= R \sin \theta \end{aligned}$$

$$\frac{z_1}{2} = \frac{x_1}{1+x_3}$$

Ex. $dx_1^2 + dx_2^2 + dx_3^2 =$

$$\frac{4(dz_1^2 + dz_2^2)}{(1+z_1^2+z_2^2)^2} = \frac{4(R^2 + R^2 d\theta^2)}{(1+R^2)^2}$$

and $\frac{z_2}{2} = \frac{x_2}{1+x_3}$