

Title: 13/14 PSI - Relativity - Lecture 5

Date: Sep 13, 2013 09:00 AM

URL: <http://pirsa.org/13090015>

Abstract:

WHERE WE ARE?

PE $\rightarrow$ GRAVITY IS GEOMETRY $\rightarrow g_{\mu\nu}$. (FIE)

3 PARTS

FIELD $g_{\mu\nu}$

EQ. OF MOTION; ... $\frac{d^2 x^\mu}{d\tau^2} + \dots$

FIELD EQS. ... NEED TO UNDER

? PE $\rightarrow$ GRAVITY IS GEOMETRY $\rightarrow g_{\mu\nu}$ (FIELD)

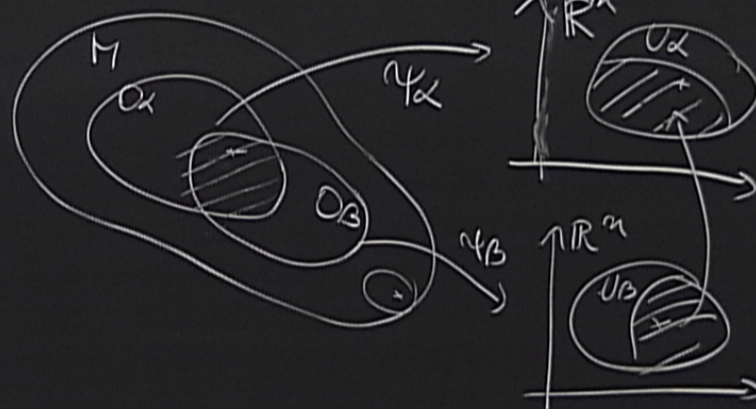
3 PARTS $\swarrow$ FIELD $g_{\mu\nu}$

EQ. OF MOTION; ... $\frac{d^2 x^\mu}{d\tau^2} + \dots$ $\mu, \nu = 0$

FIELD EQS. ... NEED TO UNDERSTAND THE GEOMETRY.

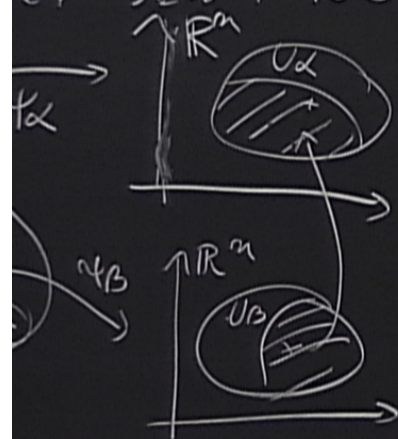
a) MANIFOLDS. "MANIFOLD MADE OF PIECES THAT LOOK LIKE OPEN
SMOOTH

a) MANIFOLDS. "MANIFOLD MADE OF PIECES THAT LOOK LIKE OPEN SUBSETS
SMOOTHLY SEWN TOGETHER."



$$\phi_\alpha \circ \phi_\beta^{-1} \in C^\infty$$

OLD MADE OF PIECES THAT LOOK LIKE OPEN SUBSETS OF $\mathbb{R}^n$, WHICH ARE
 LY SEWN TOGETHER."



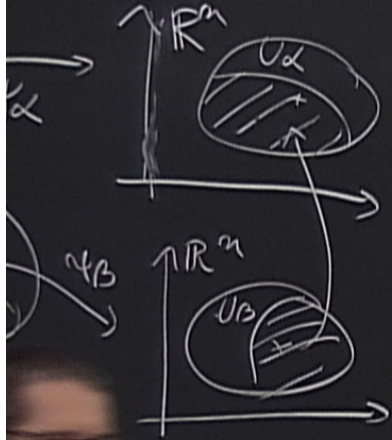
$$\phi_\alpha \circ \phi_\beta^{-1} \dots (\infty)$$

• ϕ_α CHART (MATH) .. COORDINATE SYSTEM (PHYS)

• COLLECTION OF MAPS $\{$

2D MADE OF PIECES THAT LOOK LIKE OPEN SUBSETS OF $\mathbb{R}^n$ WHICH ARE

GLUED TOGETHER."



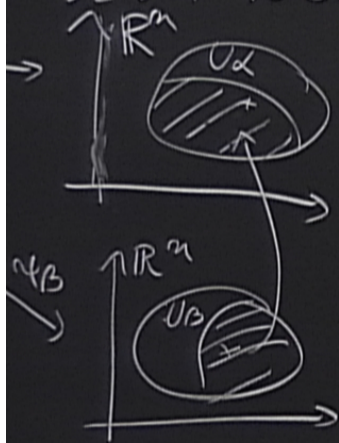
ψ_α CHART (MATH) ... COORDINATE SYSTEM (PHYS)

• COLLECTION OF MAPS $\{O_\alpha, \psi_\alpha\}$ ATLAS

EXAMPLES:

$$\psi_\alpha \circ \psi_\beta^{-1} \dots \infty$$

MADE OF PIECES THAT LOOK LIKE OPEN SUBSETS OF $\mathbb{R}^n$, WHICH ARE
SEWN TOGETHER."



$$\varphi_\alpha \circ \varphi_\beta^{-1} \dots \infty$$

EXAMPLES:

$$\bullet \mathbb{R}^n$$

$$\bullet \underline{S^n}:$$

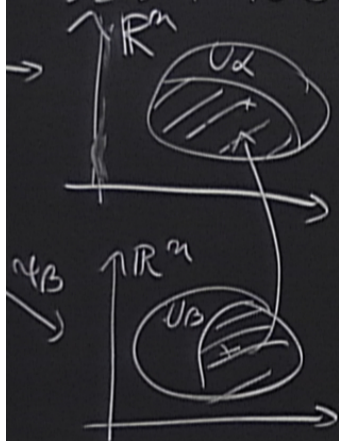
$$x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1$$



φ_α CHART (MATH) COORDINATE SYSTEM (PHYS)

• COLLECTION OF MAPS $\{\varphi_\alpha, \varphi_\alpha^{-1}\}$ ATLAS

MADE OF PIECES THAT LOOK LIKE OPEN SUBSETS OF $\mathbb{R}^n$, WHICH ARE
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$$\varphi_\alpha \circ \varphi_\beta^{-1} \dots \infty$$

φ_α CHART (MATH) .. COORDINATE SYSTEM (PHYS)

• COLLECTION OF MAPS $\{\varphi_\alpha, \varphi_\beta\}$ ATLAS

EXAMPLES:

• $\mathbb{R}^n$

• S^n :

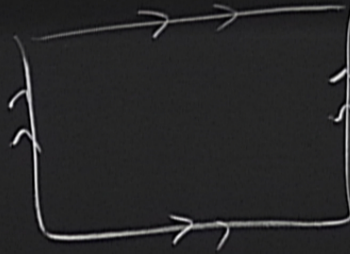
EX S^2

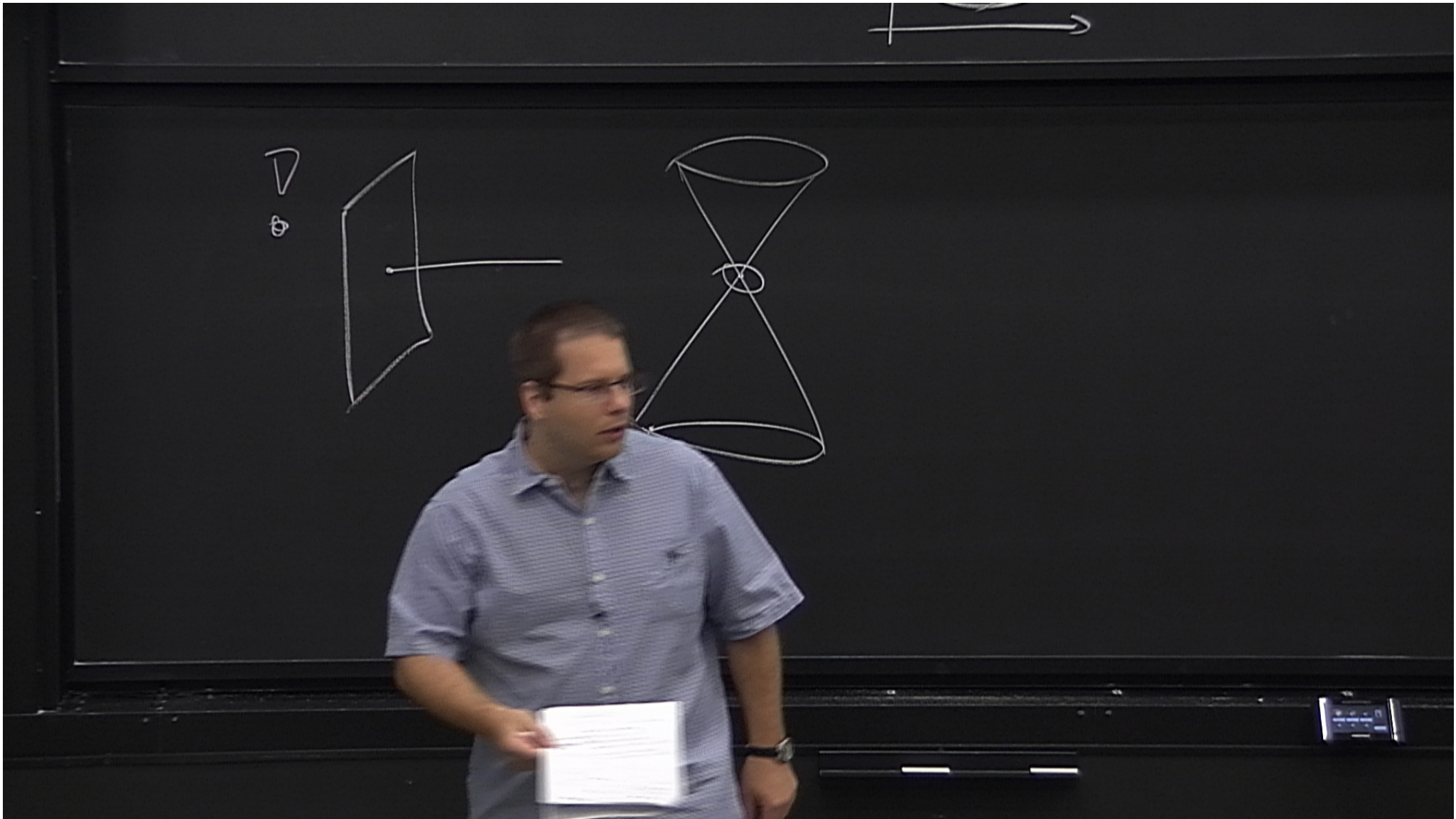
$$x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1$$



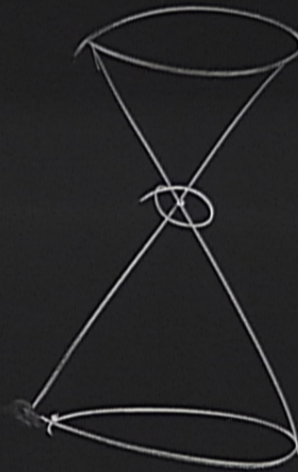
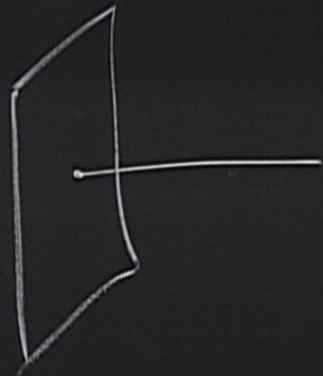
$\underline{EX}$ $\underline{S}$

$\cdot T^2$



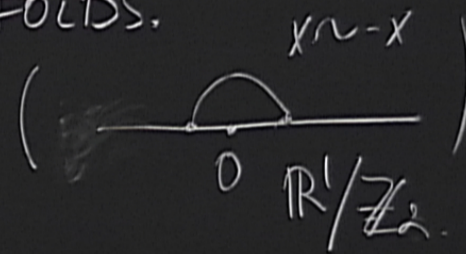


$\mathbb{D}$
 $\odot$

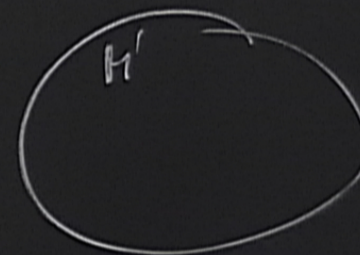
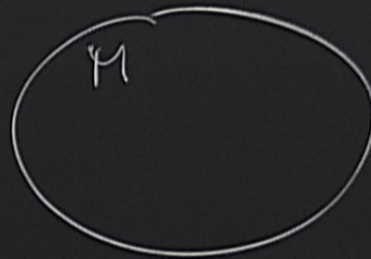


NOTE : GR (PARTICLES) ... MANIFOLDS.

• ST ... ORBI FOLDS

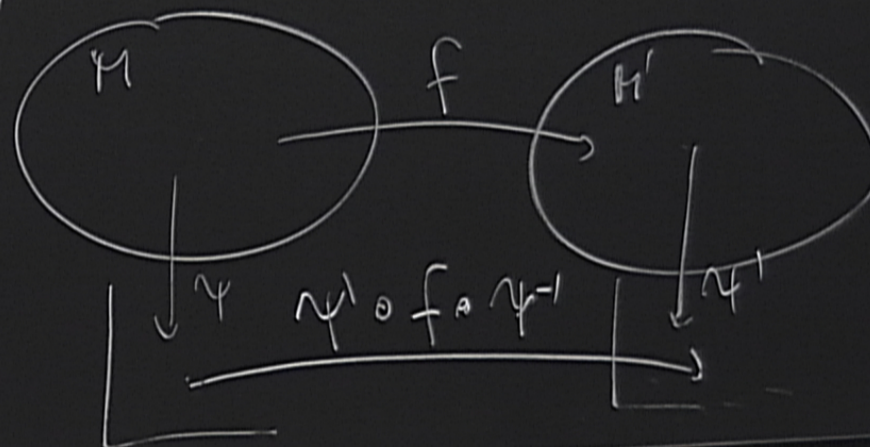


• A MAP $f: M \rightarrow M'$

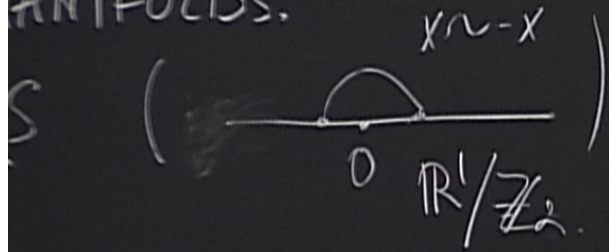


$\mathbb{R}'/\mathbb{Z}_2$

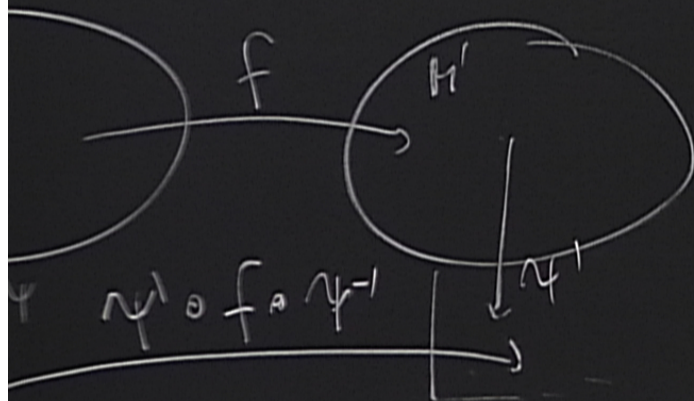
• A MAP $f: M \rightarrow M'$



MANIFOLDS.



IF $f: M \rightarrow M'$ IS C^0 , $1-1$, ONTO



IF $f: M \rightarrow M$ IS C^1 , f^{-1} , ONTO, HAS C^1 INVERSE,

$\Rightarrow f$.. DIFFEOMORPHISM

• DIFFEOMORPHIC MANIFOLDS... HAVE THE SAME n

IF $f: M \rightarrow M'$ IS C^∞ , 1-1, ONTO, HAS C^∞ INVERSE,

$\Rightarrow f$ IS DIFFEOMORPHISM

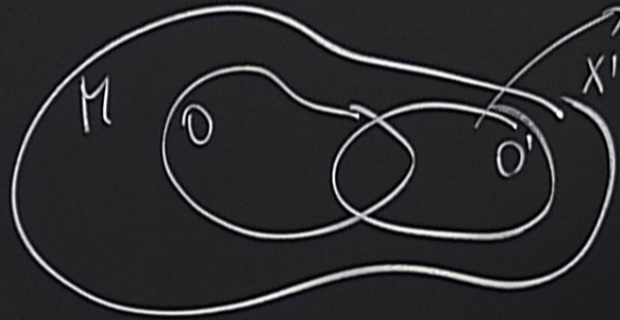
• DIFFEOMORPHIC MANIFOLDS... HAVE THE SAME MANIFOLD STRUCTURE.

b) TENSORS (INVARIANT OBJECTS . ACTORS ON OUR STAGE)

• SCALAR FUNCTION f IS A MAP $f: M \rightarrow \mathbb{R}'$

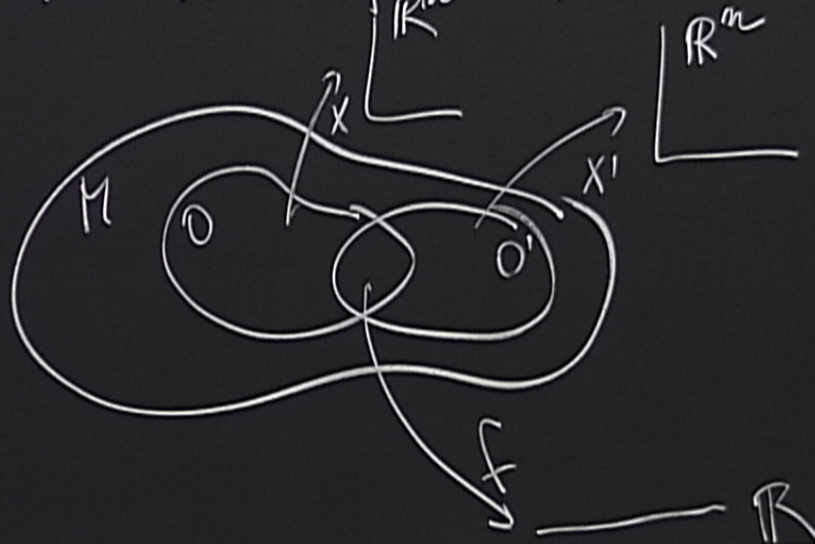
b) TENSORS (INVARIANT OBJECTS: ACTORS ON OUR SPACE)

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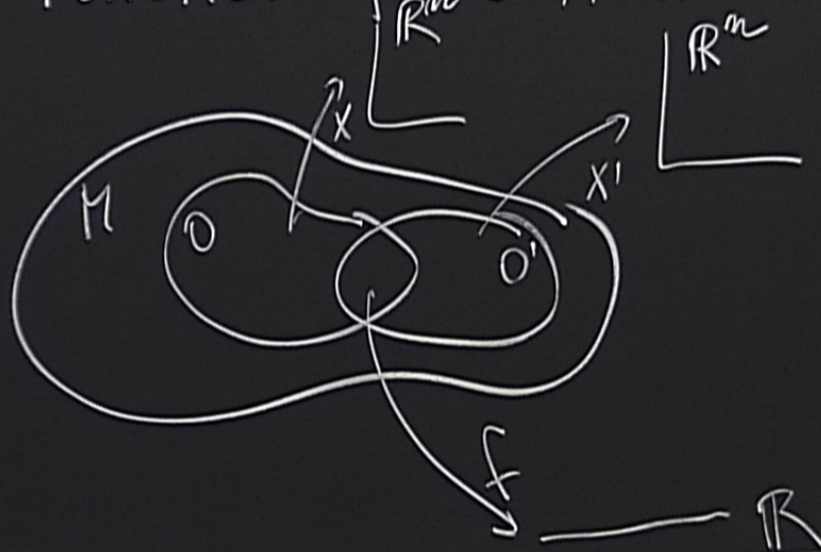
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TENSORS (INVARIANT OBJECTS . ACTORS ON OUR STAGE)

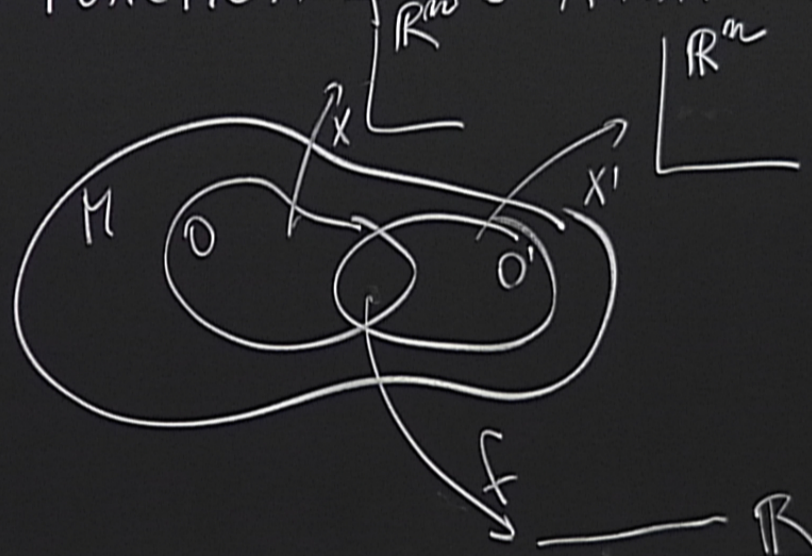
• SCALAR FUNCTION f IS A MAP $f: M \rightarrow \mathbb{R}$



$$f(x(p)) = f'(x'(p))$$
$$f'(x') = f(x)$$

6) TENSORS (INVARIANT OBJECTS . ACTORS ON OUR STAGE)

• SCALAR FUNCTION $f: M \rightarrow \mathbb{R}$ IS A MAP $f: M \rightarrow \mathbb{R}$



$$f(x(p)) = f'(x'(p))$$

$$f'(x') = f(x)$$

• TANGENT VECTOR "DIRECTION OF A DERIVATIVE AT A POINT"

• TANGENT VECTOR

"DIRECTION OF A DERIVATIVE AT A POINT"

IN $\mathbb{R}^n$: $\nabla f = (v_1^1, v_1^2, \dots, v_1^n) \iff$ DIRECTIONAL DERIVATIVE OP.

"DIRECTION OF A DERIVATIVE AT A POINT"

$$x^1, \dots, x^n$$



DIRECTIONAL
DERIVATIVE OP.

$$\hat{\nabla} = \nabla^\mu \frac{\partial}{\partial x^\mu}$$

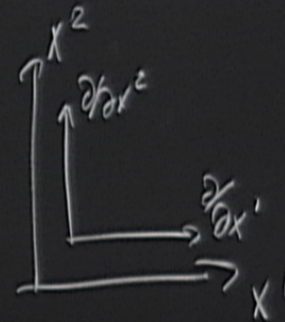
A DERIVATIVE AT A POINT"

DIRECTIONAL
DERIVATIVE OP.

$$\hat{N} = N^{\mu} \frac{\partial}{\partial x^{\mu}}$$

$$\hat{N} f = N^{\mu} \frac{\partial f}{\partial x^{\mu}} = \#$$

LEIBNITZ RULE
LINEAR.



• TANGENT VECTOR "DIRECTION OF A DERIVATIVE AT A POINT"

IN $\mathbb{R}^n$

$$v^M = (v^1, v^2, \dots, v^n) \iff$$

DIRECTIONAL
DERIVATIVE OP.

$$\hat{v} = v^M \frac{\partial}{\partial x^M}$$

DEF: LET $\mathcal{F}$ BE THE COLLECTION OF C^∞ SCALAR FUNCTIONS ON M .
A TANGENT VECTOR V AT POINT $p \in M$ IS A MAP $v: \mathcal{F}$

• TANGENT VECTOR "DIRECTION OF A DERIVATIVE AT A POINT"

IN $\mathbb{R}^n$: $v^M = (v^1, v^2, \dots, v^n) \iff$ DIRECTIONAL DERIVATIVE OP. $\hat{v} = v^M \frac{\partial}{\partial x^M}$

DEF: LET $\mathcal{F}$ BE THE COLLECTION OF C^∞ SCALAR FUNCTIONS ON M .
A TANGENT VECTOR V AT POINT $p \in M$ IS A MAP $v: \mathcal{F} \rightarrow \mathbb{R}$

TANGENT VECTOR

"DIRECTION OF A DERIVATIVE AT A POINT"

IN $\mathbb{R}^n$

$$v^M = (v^1, v^2, \dots, v^n)$$



DIRECTIONAL
DERIVATIVE OP.

$$\hat{v} = v^M \frac{\partial}{\partial x^M}$$

DEF: LET $\mathcal{F}$ BE THE COLLECTION OF C^∞ SCALAR FUNCTIONS ON M .

A TANGENT VECTOR

V AT POINT $p \in M$ IS A MAP $v: \mathcal{F} \rightarrow \mathbb{R}$

i) LINEAR: $V(af + bg) =$

LEIBNITZ RULE
LINEAR.

Let $\mathcal{F}$ be the collection of C^∞ scalar functions on M .

TANGENT VECTOR V AT POINT $p \in M$ IS A MAP $\mathcal{F} \rightarrow \mathbb{R}$:

i) LINEAR: $V(af + bg) = aV(f) + bV(g)$ $a, b \in \mathbb{R}$

ii) LEIBNITZ: $V(fg) = V(f)g(p) + V(g)f(p)$

PROPOSITION: $\nabla(f+g) = \nabla f + \nabla g$

THEOREM: THE SET OF TANGENT VECTORS AT p FORMS A TANGENT

LEIBNIZ: $V(fg) = V(f)g(p) + V(g)f(p)$

THE SET OF ALL VECTORS AT p FORMS A TANGENT VECTOR SPACE $T_p(M)$ WHICH HAS

LINEAR: $V(af + bg) = aV(f) + bV(g)$

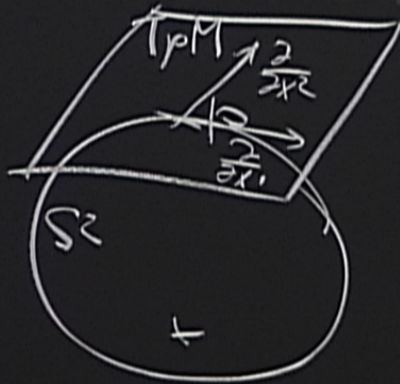
LEIBNIZ: $V(fg) = V(f)g(p) + V(g)f(p)$

VECTORS AT p FORMS A TANGENT VECTOR SPACE $T_p(M)$ WHICH HAS
AS M , WITH THE (COORDINATE) BASIS $\frac{\partial}{\partial x^i}$. ANY VECTOR $\underline{V}$

ii) LEIBNITZ: $V(fg) = V(f)g(p) + V(g)f(p)$

THEM: THE SET OF TANGENT VECTORS AT p FORMS A TANGENT VECTOR SPACE OF THE SAME DIMENSIONALITY AS M , WITH THE (COORDINATE) BASIS CAN BE EXPRESSED AS $V = V^M \frac{\partial}{\partial x^M}$ (WALD 1)

THEOREM: THE SET OF TANGENT VECTORS AT p FORMS A TANGENT SPACE
 THE SAME DIMENSIONALITY AS M , WITH THE (COORDINATE)
 CAN BE EXPRESSED AS $V = \sum V^{\mu} \left(\frac{\partial}{\partial x^{\mu}} \right)$ (WALD p. 122)
 COMPONENTS BASIS.



$$\bullet V(f) = V^M \frac{\partial}{\partial x^M} (f \circ \gamma^{-1})|_{\gamma(p)} \in \mathbb{R}$$

TRANSFORMATION OF COMPONENTS:

$$V^{\nu'}(x') = \frac{\partial x^{\nu'}}{\partial x^{\mu}} V^{\mu}(x)$$



END THE CONSTRUCTION TO THE WHOLE MANIFOLD.

NT VECTOR FIELD $\equiv \{ V/p \in T_p M, \forall p \in M, V(f) \text{ SMOOTH} \}$
 $\forall f \in \mathcal{F}$

$$V^{\mu}(x) \frac{\partial x}{\partial x^{\mu}} \left(\frac{\partial}{\partial x^{\mu}} \right)$$

$$V^{\mu}(x') = \frac{\partial x^{\mu}}{\partial x^{\nu}} V^{\nu}(x)$$

$V^{\mu}(x)$ $(V_p(f) \in \mathbb{R})$ • EXTEND THE CONSTRUCTION TO THE WHOLE MANIFOLD.

TANGENT VECTOR FIELD $\equiv \{V/p \in T_p M, \forall p \in M\}$

• THE SET OF ALL $T_p M$... TANGENT BUNDLE $T(M)$

$$\frac{\partial}{\partial x^i} \left(\frac{\partial}{\partial x^j} \right)$$

$$V^i(x') = \frac{\partial x^i}{\partial x'^j} V^j(x)$$

ω AT A POINT IS A MAP $\omega: T_p(M) \rightarrow \mathbb{R}$

$$(x) \frac{\partial x^{\mu}}{\partial x^{\nu}} \left(\frac{\partial}{\partial x^{\nu}} \right)$$

$$\underline{V^{\mu}(x') = \frac{\partial x^{\mu}}{\partial x^{\nu}} V^{\nu}(x)}$$

ONE-FORM) ω AT A POINT IS A MAP $\omega: T_p(M) \rightarrow \mathbb{R}$

TANGENT VECTOR SPACE T_p^*M WITH THE BASIS dx^{μ}

$$V = V^M(x) \left(\frac{\partial}{\partial x^M} \right) = V^M(x) \frac{\partial x^{\nu}}{\partial x^M} \left(\frac{\partial}{\partial x^{\nu}} \right)$$

$$V^{\nu}(x') =$$

COTANGENT VECTOR (ONE-FORM) ω AT A POINT, IS A MAP

• FORM A COTANGENT VECTOR SPACE T_p^*M WITH

DEFINED

$$dx^M$$

$$V^M(x) \left(\frac{\partial}{\partial x^M} \right) = V^M(x) \frac{\partial x^{\nu}}{\partial x^M} \left(\frac{\partial}{\partial x^{\nu}} \right) \quad \left| \quad V^{\nu}(x') = \frac{\partial x^{\nu}}{\partial x^M} V^M(x) \right|$$

COTANGENT VECTOR (ONE-FORM) ω AT A POINT, IS A MAP ω :

• FORM A COTANGENT VECTOR SPACE T_p^*M WITH THE B

DEFINED BY $dx^M \left(\frac{\partial}{\partial x^{\nu}} \right) = \delta^M_{\nu}$. CAN WRITE ω

FORM A COTANGENT VECTOR

DEFINED BY $dx^\mu \left(\frac{\partial}{\partial x^\nu} \right) = \delta^\mu_\nu$ CAN

• CHANGING COORDINATES:

$$\omega = \omega_\mu dx^\mu =$$

FORM A COTANGENT VECTOR SPACE $|p|$ WITH THE BASIS

DEFINED BY $dx^\mu \left(\frac{\partial}{\partial x^\nu} \right) = \delta^\mu_\nu$ CAN WRITE $\omega = \omega_\mu$

COORDINATES:

ω_ν

$$\omega dx^\mu = \omega_\mu \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$$

$$\omega_\nu(x') = \frac{\partial x^\mu}{\partial x'^\nu} \omega_\mu(x)$$

FORM A COTANGENT VECTOR SPACE $|p|$ WITH THE BASIS

DEFINED BY $dx^\mu \left(\frac{\partial}{\partial x^\nu} \right) = \delta^\mu_\nu$ CAN WRITE $\omega = \omega_\mu dx^\mu$

COORDINATES:

$$\omega dx^\mu = \omega_\mu \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$$

INVERSE

$$\omega_{\nu'}(x') = \frac{\partial x^\mu}{\partial x'^{\nu'}} \omega_\mu(x)$$

THE SET OF ALL TPT

• TENSOR OF TYPE (r, l) IS A MULTILINEAR MAP

$$T: \underbrace{T_p^* \times T_p^* \dots \times T_p^*}_r \times \underbrace{T_p \times \dots \times T_p}_l \rightarrow \mathbb{R}$$

$$T' \alpha \beta \gamma \dots \underbrace{\mu \nu \dots}_l (x') =$$

THE SET OF ALL TENSOR LANG

• TENSOR OF TYPE (k, l) . IS A MULTILINEAR MAP

$$T: \underbrace{T_p^* \times T_p^* \dots \times T_p^*}_k \times \underbrace{T_p \times T_p \dots \times T_p}_l \rightarrow \mathbb{R}$$

$$T(\underbrace{\alpha \beta \gamma \dots}_k)$$

$$\underbrace{\mu \nu \dots}_l$$

$$(x')$$

$$\frac{\partial x^{i_1}}{\partial x^{\sigma_1}} \dots$$

$$\frac{\partial x^0}{\partial x^{i_1}} \dots$$

$$\underbrace{\dots}_l$$

THE SET OF ALL $T_p M$... TANGENT BUNDLE

PE (k, l) . IS A MULTILINEAR MAP

$$\underbrace{p^* \times T_p^* \dots \times T_p^*}_k \times \underbrace{T_p \times \dots \times T_p}_l \rightarrow \mathbb{R}$$

$$\underbrace{\mu \nu \dots}_{l} (x') = \frac{\partial x^{i\alpha}}{\partial x^{j\sigma}} \dots \underbrace{\frac{\partial x^0}{\partial x^{i\alpha}}}_{l} \dots \underbrace{\tau^{\delta \dots}}_{l} \dots (x)$$

COOK BOOK (TENSOR ALGEBRA)

LET T, S TENSORS $\Rightarrow$

i) $T + S$ IS A TENSOR (SAME VALENCE)

ii) $T^\alpha \dots S^\beta \dots$

TENSOR PRODUCT

iii) CONTRACTION

✓
TENSOR ALGEBRA

LET T/S TENSORS $\Rightarrow$

ATENSOR (SAME VALENCE)

$S^{\beta} \dots$

TENSOR PRODUCT... TENSOR $(k+k', l+l')$

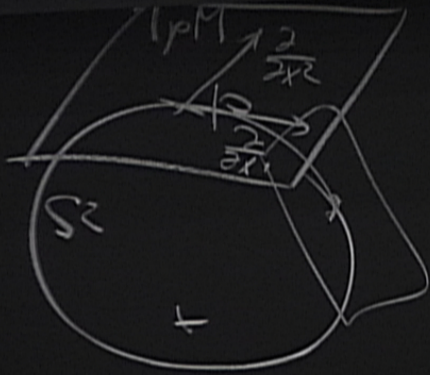
ION

$T^{\alpha\beta} \dots$

$S_d g_i$

RANK $(k+k'-1, l+l'-1)$

(iii) CONTRACTION $T^{\alpha\beta} S_{\alpha\beta}$ RANK $(l+l'-1, l+l'-1)$
 IV) IF TENSOR ZERO IN ONE COORD. SYSTEM $\Rightarrow$ ZERO IN ALL

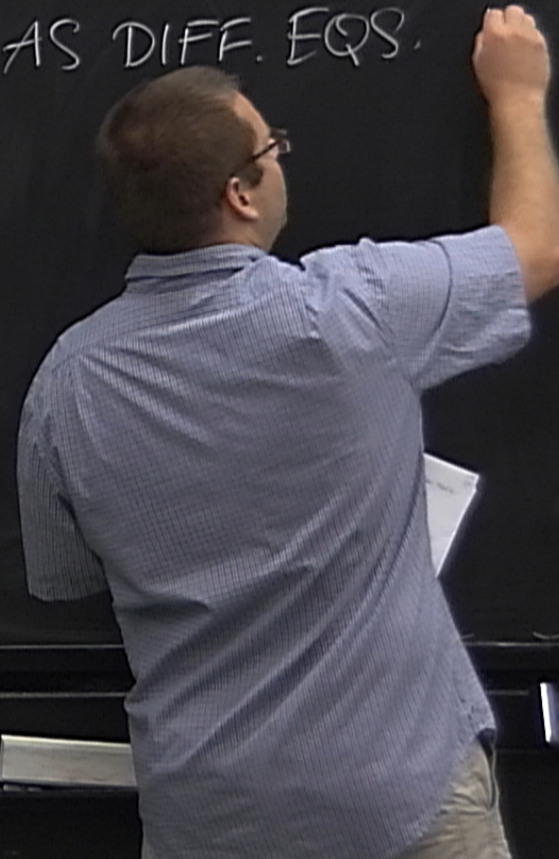


$\downarrow$ $\nabla \cdot \mathbf{M}$ — BASIS COMPONENTS

THE SET OF ALL T_{pM} ... MAN

C) CONNECTION

WE EXPRESS LAWS OF PHYSICS AS DIFF. EQS.



- $\partial_\mu \phi(x)$... TENSOR (0,1)
- $\frac{\partial}{\partial x^{\mu'}} V^\alpha(x') = \partial_{x^{\mu'}} \left(\frac{\partial x'^\alpha}{\partial x^\nu} V^\nu \right) = \frac{\partial x'^\alpha}{\partial x^\nu} \frac{\partial x'^\beta}{\partial x^{\mu'}} V^\nu$

$$T^{\alpha\beta\gamma\dots}_{\mu\nu\dots}(x') = \frac{\partial}{\partial x^{\mu'}} \dots$$

$$\frac{\partial}{\partial x^\nu} \left(\frac{\partial x^\alpha}{\partial x^\nu} V^\nu \right) = \underbrace{\frac{\partial x^\alpha}{\partial x^\nu} \frac{\partial x^\beta}{\partial x^\mu} V^\beta}_{\text{OK FOR } (1,1)} + \underbrace{\frac{\partial x^\beta}{\partial x^\mu} \frac{\partial^2 x^\alpha}{\partial x^\beta \partial x^\nu} V^\nu}_{\text{PROBLEM}}$$

$$\frac{\partial}{\partial x^\sigma} \left(\frac{\partial x^\alpha}{\partial x^\sigma} \right) = \frac{\partial^2 x^\alpha}{\partial x^\sigma \partial x^\sigma} \quad \dots \quad \frac{\partial}{\partial x^\sigma} \left(\frac{\partial x^\alpha}{\partial x^\sigma} \right) = \frac{\partial^2 x^\alpha}{\partial x^\sigma \partial x^\sigma}$$

• RIEMANN ... 1830 ... PH.D.

$$\nabla_\mu V^\nu = \partial_\mu V^\nu$$