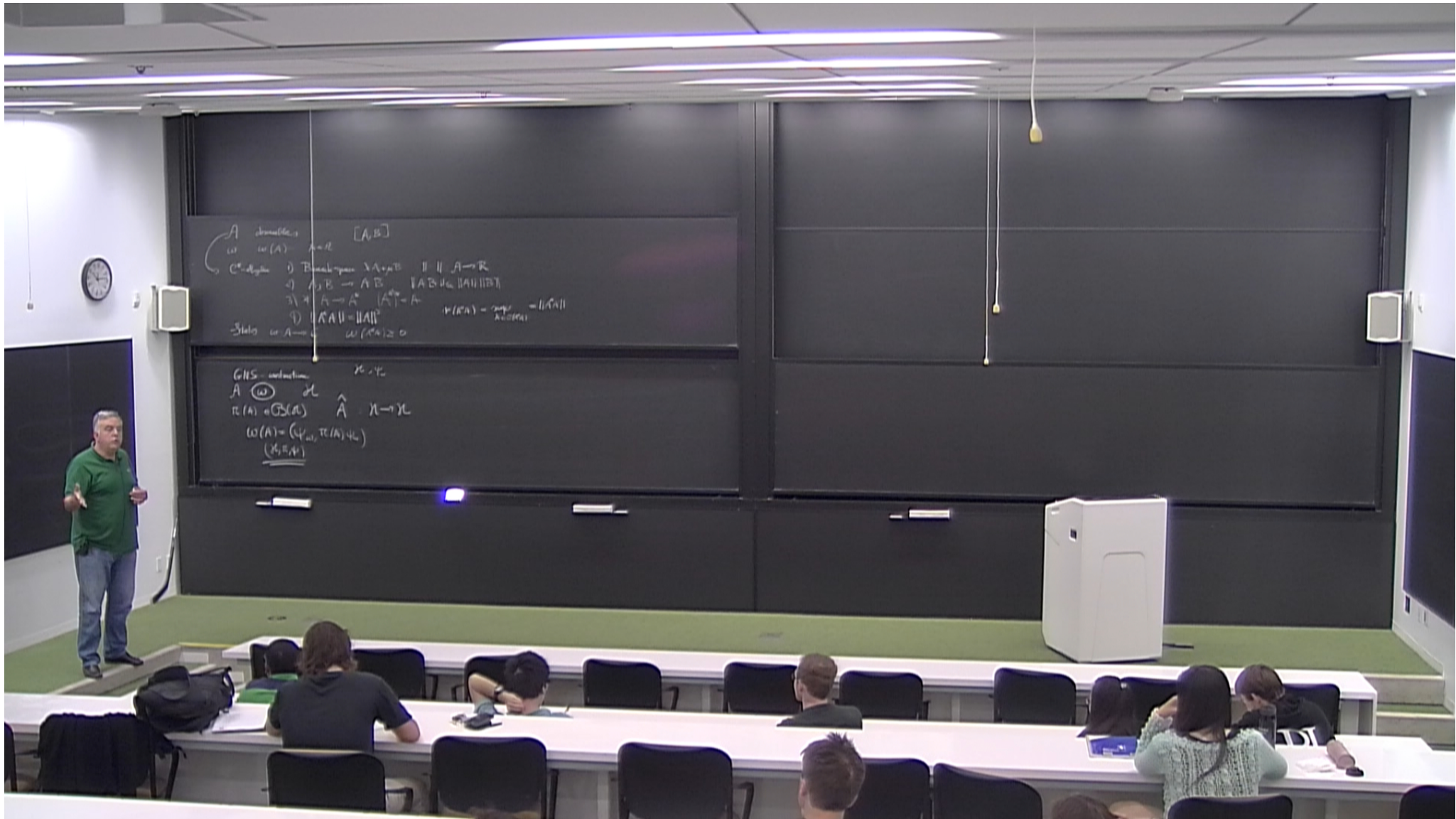


Title: Topics in QFT on Flat and Curved Spacetimes - Lecture 3

Date: Sep 30, 2013 10:00 AM

URL: <http://pirsa.org/13090008>

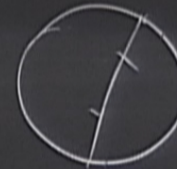
Abstract:



A observable, $[A, B]$
 $\omega(A) = \lambda \in \mathbb{R}$
 C^* -algebra 1) Banach space $\lambda A + \mu B$ $\|\cdot\|: A \rightarrow \mathbb{R}$
 2) $A, B \rightarrow AB$ $\|AB\| \leq \|A\| \|B\|$
 3) $*$ $A \rightarrow A^*$ $(A^*)^* = A$ $\|A^*A\| = \sup_{\lambda \in \sigma(A^*A)} \lambda = \|A\|^2$
 States $\omega: A \rightarrow \mathbb{C}$ $\omega(A^*A) \geq 0$

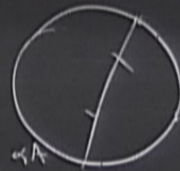
GNS-construction \mathcal{H}, Ψ_ω
 $A \in \mathcal{A}$ $\hat{A}: \mathcal{H} \rightarrow \mathcal{H}$
 $\pi(A) \in \mathcal{B}(\mathcal{H})$
 $\omega(A) = (\Psi_\omega, \pi(A)\Psi_\omega)$
 (\mathcal{H}, π, Ψ)

$\lambda \omega_1 + (1-\lambda) \omega_2 = \omega$
 λ



A observable, $[A, B]$
 $\omega(A) = \lambda \in \mathbb{R}$
 C^* -algebra: 1) Banach space $\lambda A + \mu B$ $\|\cdot\|: A \rightarrow \mathbb{R}$
 2) $A, B \rightarrow AB$ $\|AB\| \leq \|A\| \|B\|$
 3) $*$ $A \rightarrow A^*$ $(A^*)^* = A$
 4) $\|A^*A\| = \|A\|^2$ $\nu(A^*A) = \sup_{\lambda \in \sigma(A^*A)} \lambda = \|A^*A\|$
 States $\omega: A \rightarrow \mathbb{C}$ $\omega(A^*A) \geq 0$

GNS-construction \mathcal{H}, Ψ_ω $F(t) = F(t+1)$ $\lambda \omega_1 + (1-\lambda) \omega_2 = \omega$
 $A \in \mathcal{B}(\mathcal{H})$ $\hat{A}: \mathcal{H} \rightarrow \mathcal{H}$ $\frac{dF(t)}{dt} = e^{-tA} B e^{tA} e^{-\frac{t}{2}C} = 0$ $[A, e^{tA}]$
 $\pi(A) \in \mathcal{B}(\mathcal{H})$ $A, B \in \mathcal{B}(\mathcal{H})$
 $\omega(A) = (\Psi_\omega, \pi(A) \Psi_\omega)$ $\frac{dG(t)}{dt} = e^{-tA} B e^{tA} e^{-tC} = -e^{-tA} [A, B] e^{tA}$
 (\mathcal{H}, π, Ψ) $G(0) = B$ $-e^{-tA} C e^{tA} = -C$
 $G(t) = B - tC$

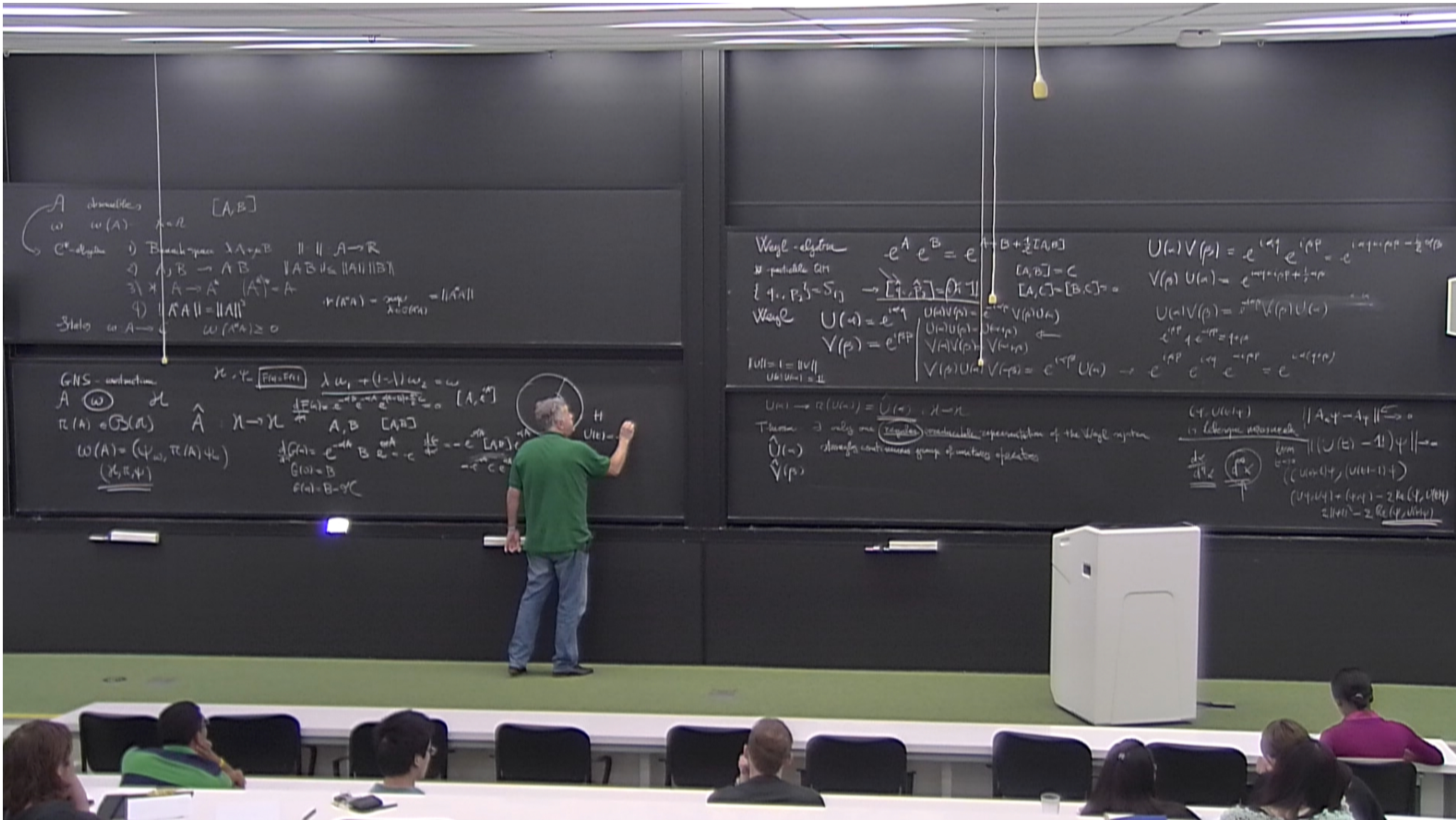


A hermitisch $[A, B]$
 $\omega(A) = \dots$
 C^* -Algebra $\lambda A + \mu B \implies \| \lambda A + \mu B \| \leq \| \lambda A \| + \| \mu B \|\|$
 $\lambda A, B \implies AB = BA \implies \| AB \| \leq \| A \| \| B \|$
 $\lambda A \implies A^* \implies (A^*)^* = A$
 $\| \lambda A \| = \| \lambda \| \| A \|$
 $\| A^* A \| = \| A \|^2$
 $\| A^* A \| = \| A \|^2 \implies \| A \| \geq 0$

GNS - construction $\mathcal{H}, \psi = \langle \cdot, \cdot \rangle$
 $A \in \mathcal{A} \implies \pi(A) \in \mathcal{B}(\mathcal{H})$
 $\omega(A) = \langle \psi, \pi(A) \psi \rangle$
 $\hat{A} : \lambda \rightarrow \pi(A)$
 $\frac{d}{dt} \pi(e^{-itA}) = -i \pi(A) \pi(e^{-itA})$
 $\pi(B) = B$
 $\pi(A) = B - iC$

Weyl - algebra $e^A e^B = e^{A+B + \frac{1}{2}[A, B]}$
 $[A, B] = C$
 $[A, C] = [B, C] = 0$
 $U(\alpha) = e^{i\alpha A}$
 $V(\beta) = e^{i\beta B}$
 $U(\alpha)V(\beta) = e^{i\alpha A} e^{i\beta B} = e^{i(\alpha A + \beta B + \frac{1}{2}\alpha\beta C)}$
 $V(\beta)U(\alpha) = e^{i\beta B} e^{i\alpha A} = e^{i(\alpha A + \beta B - \frac{1}{2}\alpha\beta C)}$
 $U(\alpha)V(\beta) = e^{i\alpha\beta C} V(\beta)U(\alpha)$
 $V(\beta)U(\alpha) = e^{-i\alpha\beta C} U(\alpha)V(\beta)$





A abelian \mathcal{A} $[A, B]$
 $\omega(A) = \lambda \in \mathbb{R}$
 C^* -algebra \mathcal{A} Banach space $\lambda A + \mu B$ $\| \cdot \|$ $A \rightarrow \mathbb{R}$
 $\lambda A, B \rightarrow \lambda A B$ $\| \lambda A \| \leq \| \lambda \| \| A \|$
 $\lambda A \rightarrow A^*$ $(A^*)^* = A$
 $\| \lambda A \| = \| \lambda \| \| A \|$ $\| A^* \| = \| A \|$
 $\omega(A^*) = \overline{\omega(A)}$ $\| A^* \| = \| A \|$
 $\omega(A^* A) \geq 0$

GNS-construction $\mathcal{H} = \mathcal{L}(\mathcal{A}) / \mathcal{I}$
 $A \in \mathcal{A}$ \mathcal{H} $\hat{A} : \mathcal{H} \rightarrow \mathcal{H}$
 $\pi(A) = \mathcal{G}(\mathcal{A})$ $\hat{A} : \mathcal{H} \rightarrow \mathcal{H}$
 $\omega(A) = \langle \psi_\omega, \pi(A) \psi_\omega \rangle$
 (\mathcal{H}, π, ψ)
 $\lambda \omega_1 + (1-\lambda) \omega_2 = \omega$
 $\frac{d}{dt} \pi(tA) = \pi(tA) \hat{A} - \hat{A} \pi(tA)$
 $\hat{A} = \frac{d}{dt} \pi(tA) \Big|_{t=0}$
 $\hat{A} = \frac{d}{dt} \pi(tA) \Big|_{t=0} = \frac{d}{dt} \pi(tA) \Big|_{t=0}$
 $\hat{A} = \frac{d}{dt} \pi(tA) \Big|_{t=0}$

Weyl-algebra $e^A e^B = e^{A+B + \frac{1}{2}[A, B]}$
 \mathcal{H} particle \mathcal{A} \mathcal{H} \mathcal{H} \mathcal{H}
 $\{ \psi, \phi \} = \langle \psi, \hat{A} \phi \rangle - \langle \hat{A} \psi, \phi \rangle$
 $\hat{A} = \frac{d}{dt} \pi(tA) \Big|_{t=0}$
 $\hat{A} = \frac{d}{dt} \pi(tA) \Big|_{t=0}$
 $\hat{A} = \frac{d}{dt} \pi(tA) \Big|_{t=0}$

$U(t) V(p) = e^{i(tq - pt + \frac{1}{2}tp^2)}$
 $V(p) U(t) = e^{i(tq - pt + \frac{1}{2}tp^2)}$
 $U(t) V(p) = e^{i(tq - pt + \frac{1}{2}tp^2)}$
 $V(p) U(t) = e^{i(tq - pt + \frac{1}{2}tp^2)}$
 $U(t) V(p) = e^{i(tq - pt + \frac{1}{2}tp^2)}$
 $V(p) U(t) = e^{i(tq - pt + \frac{1}{2}tp^2)}$

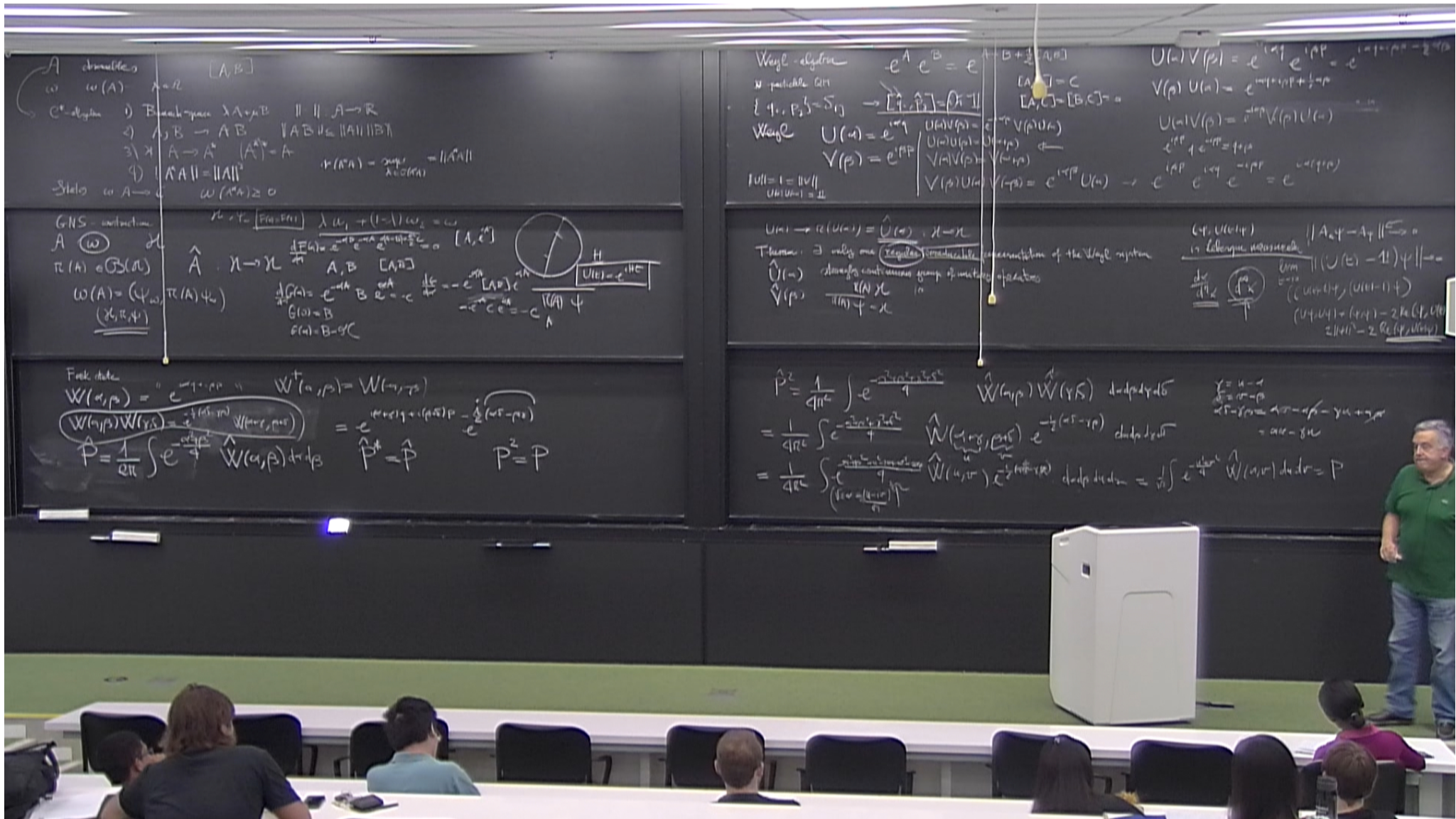
$U(t) \rightarrow \pi(U(t)) = \hat{U}(t) : \mathcal{H} \rightarrow \mathcal{H}$
 Theorem: \exists only one (regular) irreducible representation of the Weyl algebra
 $\hat{U}(t)$ strongly continuous group of unitary operators
 $\hat{V}(p)$

$\langle \psi, U(t) \psi \rangle = \langle \psi, \psi \rangle = 1$
 $\langle \psi, U(t) \psi \rangle = \langle \psi, \psi \rangle = 1$
 $\langle \psi, U(t) \psi \rangle = \langle \psi, \psi \rangle = 1$
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 $\langle \psi, U(t) \psi \rangle = \langle \psi, \psi \rangle = 1$

A Hermitian $[A, B]$
 $\omega(A) = \lambda \in \mathbb{R}$
 C^* -algebra \mathcal{A} Banach space $\lambda A + \mu B \in \mathcal{A} \implies \| \cdot \| : \mathcal{A} \rightarrow \mathbb{R}$
 $\langle A, B \rangle \rightarrow AB \quad \|AB\| \leq \|A\| \|B\|$
 $\exists \lambda A \rightarrow A^* \quad (A^*)^* = A \quad r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| = \|A\|$
 $\| \lambda A \| = |\lambda| \|A\|$
 $\sigma(A^*) = \overline{\sigma(A)}$
 $\omega(A^*) \geq 0$
 GNS construction $\lambda \omega_1 + (1-\lambda)\omega_2 = \omega$
 $A \in \mathcal{A} \subset \mathcal{H}$
 $\pi(A) \in \mathcal{B}(\mathcal{H}) \quad \hat{A} : \mathcal{H} \rightarrow \mathcal{H} \quad \frac{dF(t)}{dt} = e^{-itA} A e^{itA} = A \quad [A, e^{itA}] = 0$
 $\omega(A) = \langle \psi, \pi(A)\psi \rangle \quad \frac{dG(t)}{dt} = e^{-itA} B e^{itA} = B \quad [A, B] = C$
 $\frac{dG(t)}{dt} = e^{-itA} B e^{itA} = B \quad [A, B] = C$
 $G(t) = B \quad G'(t) = B - G(t)C$
 $\frac{dG(t)}{dt} = e^{-itA} B e^{itA} = B \quad [A, B] = C$
 $G(t) = B - G(t)C$

Weyl algebra $e^A e^B = e^{A+B + \frac{1}{2}[A, B]}$
 \mathbb{R} -partielle GNT $[A, B] = C$
 $\{ \psi, \psi^\dagger \} = \delta_{ij} \rightarrow [\hat{p}_i, \hat{q}_j] = \delta_{ij} \hbar$
 $[A, C] = [B, C] = 0$
 Weyl $U(\alpha) = e^{i\alpha A} \quad U(\alpha) V(\beta) = e^{-i\alpha\beta} V(\beta) U(\alpha)$
 $V(\beta) = e^{i\beta B} \quad U(\alpha) V(\beta) = e^{-i\alpha\beta} V(\beta) U(\alpha)$
 $U(1) = 1 = \|U\| \quad U(0) = 1$
 $V(\beta) U(\alpha) = e^{i\alpha\beta} U(\alpha) V(\beta)$
 $U(\alpha) \rightarrow \pi(U(\alpha)) = \hat{U}(\alpha) : \mathcal{H} \rightarrow \mathcal{H}$
 Theorem: \exists only one (regular) irreducible representation of the Weyl algebra
 $\hat{U}(\alpha)$ strongly continuous group of unitary operators
 $\hat{V}(\beta) : \frac{d}{dt} \hat{U}(t) = i \hat{A} \hat{U}(t)$
 $\langle \psi, U(\alpha)\psi \rangle = \langle U(\alpha)\psi, \psi \rangle = \langle U(\alpha)\psi, U(\alpha)\psi \rangle = \|U(\alpha)\psi\|^2 = \|\psi\|^2 = 1$
 $\langle \psi, U(\alpha)\psi \rangle = \langle U(\alpha)\psi, \psi \rangle = \langle U(\alpha)\psi, U(\alpha)\psi \rangle = \|U(\alpha)\psi\|^2 = \|\psi\|^2 = 1$
 $\langle \psi, U(\alpha)\psi \rangle = \langle U(\alpha)\psi, \psi \rangle = \langle U(\alpha)\psi, U(\alpha)\psi \rangle = \|U(\alpha)\psi\|^2 = \|\psi\|^2 = 1$
 $\langle \psi, U(\alpha)\psi \rangle = \langle U(\alpha)\psi, \psi \rangle = \langle U(\alpha)\psi, U(\alpha)\psi \rangle = \|U(\alpha)\psi\|^2 = \|\psi\|^2 = 1$





C^* -algebra A normed space $X \rightarrow B$ $\| \cdot \| : A \rightarrow \mathbb{R}$
 $\| \cdot \|$ norm $\| \cdot \|$ norm $\| \cdot \|$ norm
 $\exists \lambda A \rightarrow A^*$ $(A^*)^* = A$
 $\| \lambda A \| = \| \lambda \| \| A \|$
 $\| A^* A \| = \| A \|^2$
 $\| A^* A \| \geq 0$

GNS-construction
 $A \subset B(H)$
 $\pi(A) \subset B(H)$
 $\omega(A) = \langle \psi_\omega, \pi(A) \psi_\omega \rangle$
 $\hat{A} : \mathcal{H} \rightarrow \mathcal{H}$
 $\frac{d}{dt} \omega(A) = \langle \psi_\omega, \pi(A) \psi_\omega \rangle$
 $\frac{d}{dt} \omega(A) = \langle \psi_\omega, \pi(A) \psi_\omega \rangle$
 $\frac{d}{dt} \omega(A) = \langle \psi_\omega, \pi(A) \psi_\omega \rangle$

Fock state
 $W(\alpha, \beta) = e^{-\frac{1}{2}(\alpha^2 + \beta^2) - i\alpha\beta}$
 $W(\alpha, \beta) W(\gamma, \delta) = e^{-\frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) - i(\alpha\beta + \gamma\delta) - i(\alpha\delta - \beta\gamma)}$
 $\hat{P} = \frac{1}{2\pi} \int e^{-\frac{1}{2}(\alpha^2 + \beta^2) - i\alpha\beta} W(\alpha, \beta) d\alpha d\beta$
 $\hat{P}^2 = \hat{P}$
 $P^2 = P$

Weyl-algebra $e^A e^B = e^{A+B + \frac{1}{2}[A,B]}$
 $[A, C] = C$
 $[A, C] = [B, C] = 0$
 $U(\alpha) = e^{i\alpha\hat{p}}$
 $V(\beta) = e^{i\beta\hat{q}}$
 $U(\alpha)V(\beta) = e^{i\alpha\hat{p}} e^{i\beta\hat{q}} = e^{i\alpha\hat{p} + i\beta\hat{q} + \frac{1}{2}[\alpha\hat{p}, \beta\hat{q}]}$
 $U(\alpha)V(\beta) = e^{i\alpha\hat{p} + i\beta\hat{q} + \frac{1}{2}[\alpha\hat{p}, \beta\hat{q}]}$
 $U(\alpha)V(\beta) = e^{i\alpha\hat{p} + i\beta\hat{q} + \frac{1}{2}[\alpha\hat{p}, \beta\hat{q}]}$

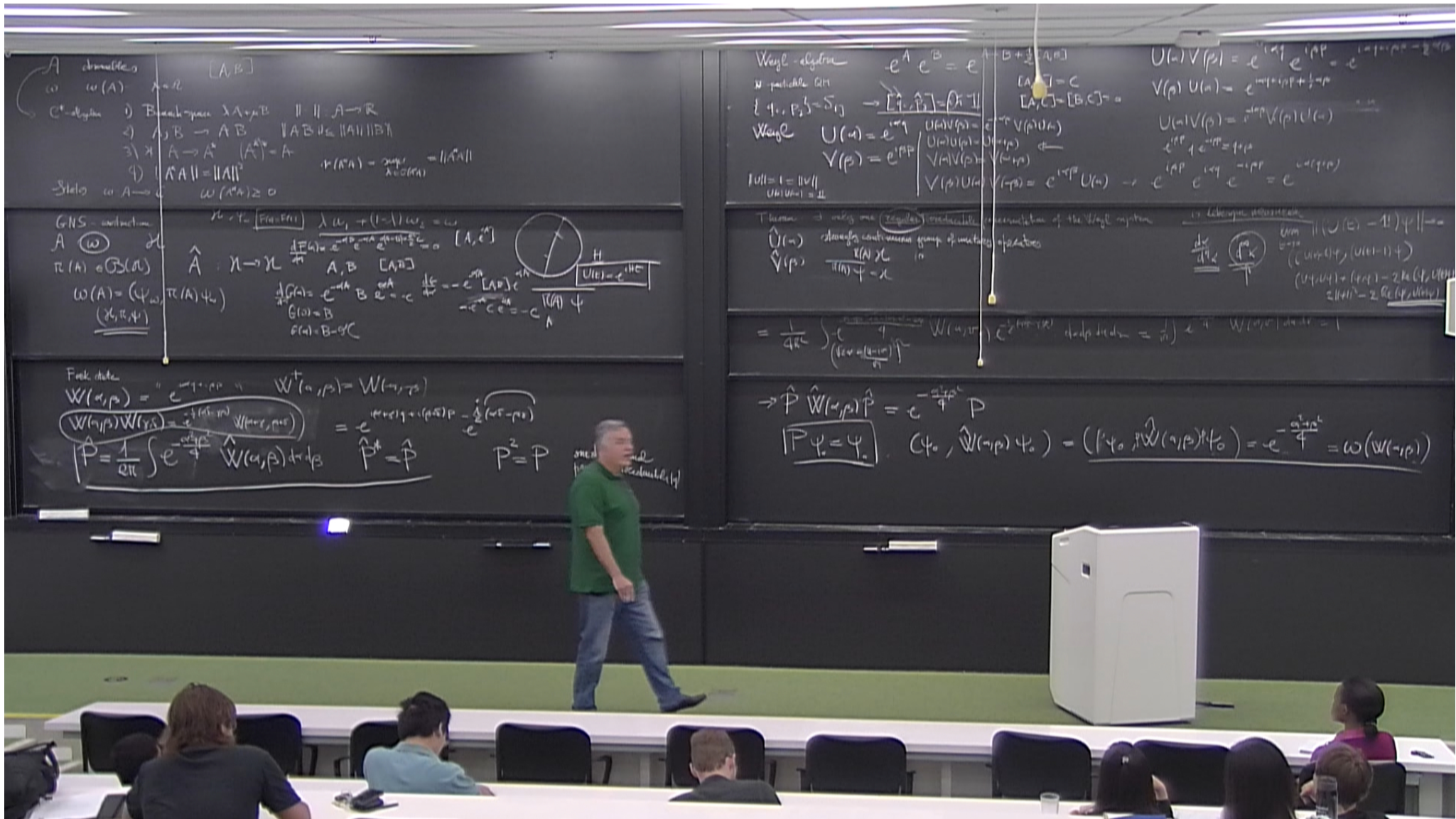
$U(\alpha) \rightarrow \pi(U(\alpha)) = \hat{U}(\alpha) : \mathcal{H} \rightarrow \mathcal{H}$
 Theorem: \exists only one (regular) irreducible representation of the Weyl system
 $\hat{U}(\alpha)$ strongly continuous group of unitary operators
 $\hat{V}(\beta)$

$\hat{P}^2 = \frac{1}{4\pi^2} \int e^{-\frac{1}{2}(\alpha^2 + \beta^2) - i\alpha\beta} \hat{W}(\alpha, \beta) \hat{W}(\gamma, \delta) d\alpha d\beta d\gamma d\delta$
 $= \frac{1}{4\pi^2} \int e^{-\frac{1}{2}(\alpha^2 + \beta^2) - i\alpha\beta} \hat{W}(\alpha, \beta) e^{-\frac{1}{2}(\gamma^2 + \delta^2) - i\gamma\delta} d\alpha d\beta d\gamma d\delta$
 $= \frac{1}{4\pi^2} \int e^{-\frac{1}{2}(\alpha^2 + \beta^2) - i\alpha\beta} \hat{W}(\alpha, \beta) e^{-\frac{1}{2}(\gamma^2 + \delta^2) - i\gamma\delta} d\alpha d\beta d\gamma d\delta = \frac{1}{4\pi^2} \int e^{-\frac{1}{2}(\alpha^2 + \beta^2) - i\alpha\beta} \hat{W}(\alpha, \beta) d\alpha d\beta = \hat{P}$

$U(\alpha)V(\beta) = e^{i\alpha\hat{p}} e^{i\beta\hat{q}} = e^{i\alpha\hat{p} + i\beta\hat{q} + \frac{1}{2}[\alpha\hat{p}, \beta\hat{q}]}$
 $V(\beta)U(\alpha) = e^{i\beta\hat{q}} e^{i\alpha\hat{p}} = e^{i\beta\hat{q} + i\alpha\hat{p} + \frac{1}{2}[\beta\hat{q}, \alpha\hat{p}]}$
 $U(\alpha)V(\beta) = e^{i\alpha\hat{p} + i\beta\hat{q} + \frac{1}{2}[\alpha\hat{p}, \beta\hat{q}]}$
 $V(\beta)U(\alpha) = e^{i\beta\hat{q} + i\alpha\hat{p} + \frac{1}{2}[\beta\hat{q}, \alpha\hat{p}]}$
 $U(\alpha)V(\beta) = e^{i\alpha\hat{p} + i\beta\hat{q} + \frac{1}{2}[\alpha\hat{p}, \beta\hat{q}]}$
 $V(\beta)U(\alpha) = e^{i\beta\hat{q} + i\alpha\hat{p} + \frac{1}{2}[\beta\hat{q}, \alpha\hat{p}]}$

$\langle \psi, U(\alpha)\psi \rangle = \langle U(\alpha)\psi, \psi \rangle = \langle \psi, \psi \rangle = 1$
 $\| (U(\alpha) - 1)\psi \|^2 = \langle (U(\alpha) - 1)\psi, (U(\alpha) - 1)\psi \rangle$
 $= \langle U(\alpha)\psi, U(\alpha)\psi \rangle - \langle U(\alpha)\psi, \psi \rangle - \langle \psi, U(\alpha)\psi \rangle + \langle \psi, \psi \rangle$
 $= 1 - \langle U(\alpha)\psi, \psi \rangle - \langle \psi, U(\alpha)\psi \rangle + 1$
 $= 2 - \langle U(\alpha)\psi, \psi \rangle - \langle \psi, U(\alpha)\psi \rangle$

$\hat{P}^2 = \frac{1}{4\pi^2} \int e^{-\frac{1}{2}(\alpha^2 + \beta^2) - i\alpha\beta} \hat{W}(\alpha, \beta) \hat{W}(\gamma, \delta) d\alpha d\beta d\gamma d\delta$
 $= \frac{1}{4\pi^2} \int e^{-\frac{1}{2}(\alpha^2 + \beta^2) - i\alpha\beta} \hat{W}(\alpha, \beta) e^{-\frac{1}{2}(\gamma^2 + \delta^2) - i\gamma\delta} d\alpha d\beta d\gamma d\delta$
 $= \frac{1}{4\pi^2} \int e^{-\frac{1}{2}(\alpha^2 + \beta^2) - i\alpha\beta} \hat{W}(\alpha, \beta) e^{-\frac{1}{2}(\gamma^2 + \delta^2) - i\gamma\delta} d\alpha d\beta d\gamma d\delta = \frac{1}{4\pi^2} \int e^{-\frac{1}{2}(\alpha^2 + \beta^2) - i\alpha\beta} \hat{W}(\alpha, \beta) d\alpha d\beta = \hat{P}$



A C^* -algebra $[A, B]$
 $\omega(A) = \|A\|$
 C^* -algebra B Banach space $\lambda A + \mu B$ $\| \cdot \| : A \rightarrow \mathbb{R}$
 $\lambda A, B \rightarrow \lambda A + \mu B$ $\| \lambda A + \mu B \| \leq |\lambda| \|A\| + |\mu| \|B\|$
 $\exists \lambda A \rightarrow \lambda^* A$ $(A^*)^* = A$ $\|A^*\| = \|A\|$
 $\| \lambda A \| = |\lambda| \|A\|$ $\|A^* A\| = \|A\|^2$
 $\|A\| \geq 0$

GNS-construction $\lambda \omega_1 + (1-\lambda) \omega_2 = \omega$
 $A \subset \mathcal{K}$ $\hat{A} : \mathcal{X} \rightarrow \mathcal{X}$ $\frac{d}{dt} \hat{A} = e^{-tA} A e^{tA}$ $[A, e^{tA}] = 0$
 $\pi(A) \in \mathcal{B}(\mathcal{H})$ $\hat{A} : \mathcal{X} \rightarrow \mathcal{X}$ $A, B \in [A, B]$ \mathcal{H}
 $\omega(A) = \langle \psi_\omega, \pi(A) \psi_\omega \rangle$ $\frac{d}{dt} G(t) = e^{-tA} B e^{tA} = e^{-tA} B e^{tA}$ $\frac{d}{dt} e^{-tA} = -e^{-tA} A$
 $(\mathcal{X}, \mathcal{B}, \psi)$ $G(t) = B - tC$ $F(t) = B - tC$

Fock state $W(\alpha, \beta) = e^{-\frac{1}{2}(\alpha^2 + \beta^2)} e^{i\alpha \hat{p} - i\beta \hat{q}}$ $W^\dagger(\alpha, \beta) = W(-\alpha, -\beta)$
 $W(\alpha, \beta) W(\alpha', \beta') = e^{-\frac{i}{2}(\alpha \beta' - \alpha' \beta)} W(\alpha + \alpha', \beta + \beta')$
 $\hat{P} = \frac{1}{2\pi} \int e^{-\frac{i}{2}(\alpha \hat{p} - \beta \hat{q})} W(\alpha, \beta) d\alpha d\beta$ $\hat{P}^\dagger = \hat{P}$ $P^2 = P$

Weyl-algebra $e^A e^B = e^{A+B + \frac{1}{2}[A, B]}$
 \mathcal{W} particular GNS $[A, B] = C$
 $\{ \hat{q}, \hat{p} \} = \delta_{ij} \rightarrow [\hat{q}, \hat{p}] = i\mathbb{1}$ $[A, C] = [B, C] = 0$
 Weyl $U(\alpha) = e^{i\alpha \hat{q}}$ $V(\beta) = e^{i\beta \hat{p}}$
 $U(\alpha)V(\beta) = e^{i\alpha\beta} V(\beta)U(\alpha)$
 $U(\alpha)V(\beta) = e^{i\alpha\beta} V(\beta)U(\alpha)$
 $U(\alpha)V(\beta) = e^{i\alpha\beta} V(\beta)U(\alpha)$
 $U(\alpha)V(\beta) = e^{i\alpha\beta} V(\beta)U(\alpha)$

Theorem \rightarrow every (regular) irreducible representation of the Weyl algebra
 $\hat{U}(t)$ strongly continuous group of unitary operators
 $\hat{V}(p) = \frac{d}{dt} \hat{U}(t) \Big|_{t=0}$
 $\frac{d}{dt} \hat{U}(t) = i\hat{H} \hat{U}(t)$
 $\frac{d}{dt} \hat{U}(t) = i\hat{H} \hat{U}(t)$
 $\frac{d}{dt} \hat{U}(t) = i\hat{H} \hat{U}(t)$

$\rightarrow \hat{P} \hat{W}(\alpha, \beta) \hat{P} = e^{-\frac{i}{2}(\alpha \hat{p} - \beta \hat{q})} P$
 $\langle P \psi_0, \psi_0 \rangle = \langle \psi_0, \hat{W}(-\alpha, -\beta) \psi_0 \rangle = \langle \psi_0, W(\alpha, \beta) \psi_0 \rangle = e^{-\frac{\alpha^2 + \beta^2}{2}} = \omega(W(\alpha, \beta))$

A - C^* algebra
 $\omega(A) = \lambda \in \mathbb{R}$
 C^* algebra: 1) Banach space $\lambda A + \mu B$ $\| \cdot \| : A \rightarrow \mathbb{R}$
 2) $A, B \rightarrow AB$ $\|AB\| \leq \|A\| \|B\|$
 3) $\lambda A \rightarrow \lambda A^*$ $(A^*)^* = A$
 4) $\| \lambda A \| = |\lambda| \|A\|$
 States: $\omega : A \rightarrow \mathbb{C}$ $\omega(A^*A) \geq 0$

GNS construction
 $A \subset \mathcal{B}(H)$
 $\pi(A) \subset \mathcal{B}(H)$
 $\omega(A) = \langle \psi, \pi(A)\psi \rangle$
 $(\chi, \pi(A)\psi)$
 $\hat{A} : \chi \rightarrow \chi$
 $\frac{d}{dt} \hat{A} = -A \hat{A} + \hat{A} A$
 $\hat{G}(t) = e^{-tA} B e^{tA}$
 $\hat{G}(0) = B$
 $\hat{G}(t) = B - t[A, B]$

Weyl algebra: $e^A e^B = e^{A+B + \frac{1}{2}[A, B]}$
 $[A, B] = C$
 $[A, C] = [B, C] = 0$
 $U(t) = e^{itA}$
 $V(p) = e^{ipB}$
 $U(t)V(p) = e^{itA} e^{ipB} = e^{itA + ipB + \frac{1}{2}itip} = e^{itA + ipB + \frac{1}{2}itip}$
 $V(p)U(t) = e^{ipB} e^{itA} = e^{ipB + itA - \frac{1}{2}itip} = e^{ipB + itA - \frac{1}{2}itip}$
 $U(t)V(p) = V(p)U(t)$
 $U(t)V(p) = e^{itA} V(p) U(t)$
 $V(p)U(t) = e^{ipB} U(t) V(p)$

$\omega(\hat{W}(t, p)) = \exp\left(-\frac{\sigma^2 p^2}{4}\right)$
 $\hat{W}(t, p) = e^{-itA} e^{ipB} e^{itA}$
 $U(t)V(p) = e^{itA} e^{ipB} e^{-itA} = e^{ipB + itA - \frac{1}{2}itip}$
 $U(t)V(p) = \hat{W}(t, p) e^{ipB}$
 $\langle \psi_0, \hat{W}(t, p)\psi_0 \rangle = \langle \psi_0, e^{ipB} \psi_0 \rangle = e^{-\frac{\sigma^2 p^2}{4}}$
 $\langle \psi_0, \hat{W}(t, p)\psi_0 \rangle = \langle \psi_0, e^{ipB} \psi_0 \rangle = e^{-\frac{\sigma^2 p^2}{4}}$

$\hat{P} \hat{W}(t, p) \hat{P} = e^{-\frac{\sigma^2 p^2}{4}} \hat{P}$
 $\langle \hat{P}\psi_0, \hat{W}(t, p)\hat{P}\psi_0 \rangle = \langle \psi_0, \hat{W}(t, p)\psi_0 \rangle = e^{-\frac{\sigma^2 p^2}{4}} = \omega(\hat{W}(t, p))$



A derivables $[A, B]$
 $\omega(A) = A \in \mathcal{R}$
 C^* -algebra
 1) Banach space $\lambda A + \mu B$ $\| \cdot \| : A \rightarrow \mathcal{R}$
 2) $A, B \rightarrow AB$ $\|AB\| \leq \|A\| \|B\|$
 3) $\lambda A \rightarrow \lambda A^*$ $(A^*)^* = A$
 4) $\|A^*A\| = \|AA^*\|$ $\|A^*A\| = \sup_{\lambda \in \sigma(A^*A)} |\lambda| = \|A\|^2$
 State $\omega : A \rightarrow \mathbb{C}$ $\omega(A^*A) \geq 0$

GNS construction
 $A \curvearrowright \mathcal{H}$
 $\pi(A) \in \mathcal{B}(\mathcal{H})$ $\hat{A} : \mathcal{X} \rightarrow \mathcal{X}$
 $\omega(A) = (\psi_\omega, \pi(A)\psi_\omega)$
 $(\mathcal{X}, \mathcal{R}, \psi)$
 $\forall (\omega, \rho) \in \mathcal{P}(\mathcal{S})$

$\lambda \omega_1 + (1-\lambda)\omega_2 = \omega$
 $\frac{dF(t)}{dt} = e^{-tA} B e^{-tA} = e^{-2tA} B = e^{-2tA} B e^{-2tA} = e^{-2tA} B e^{-2tA}$
 $\frac{dG(t)}{dt} = e^{-tA} B e^{-tA} = e^{-2tA} B = e^{-2tA} B e^{-2tA}$
 $G(0) = B$
 $F(0) = B = G(0)$

$\frac{d}{dt} \left(\frac{1}{\| \psi \|} \frac{d}{dt} \int_{\mathcal{X}} \psi(x) dx \right) = \dots$

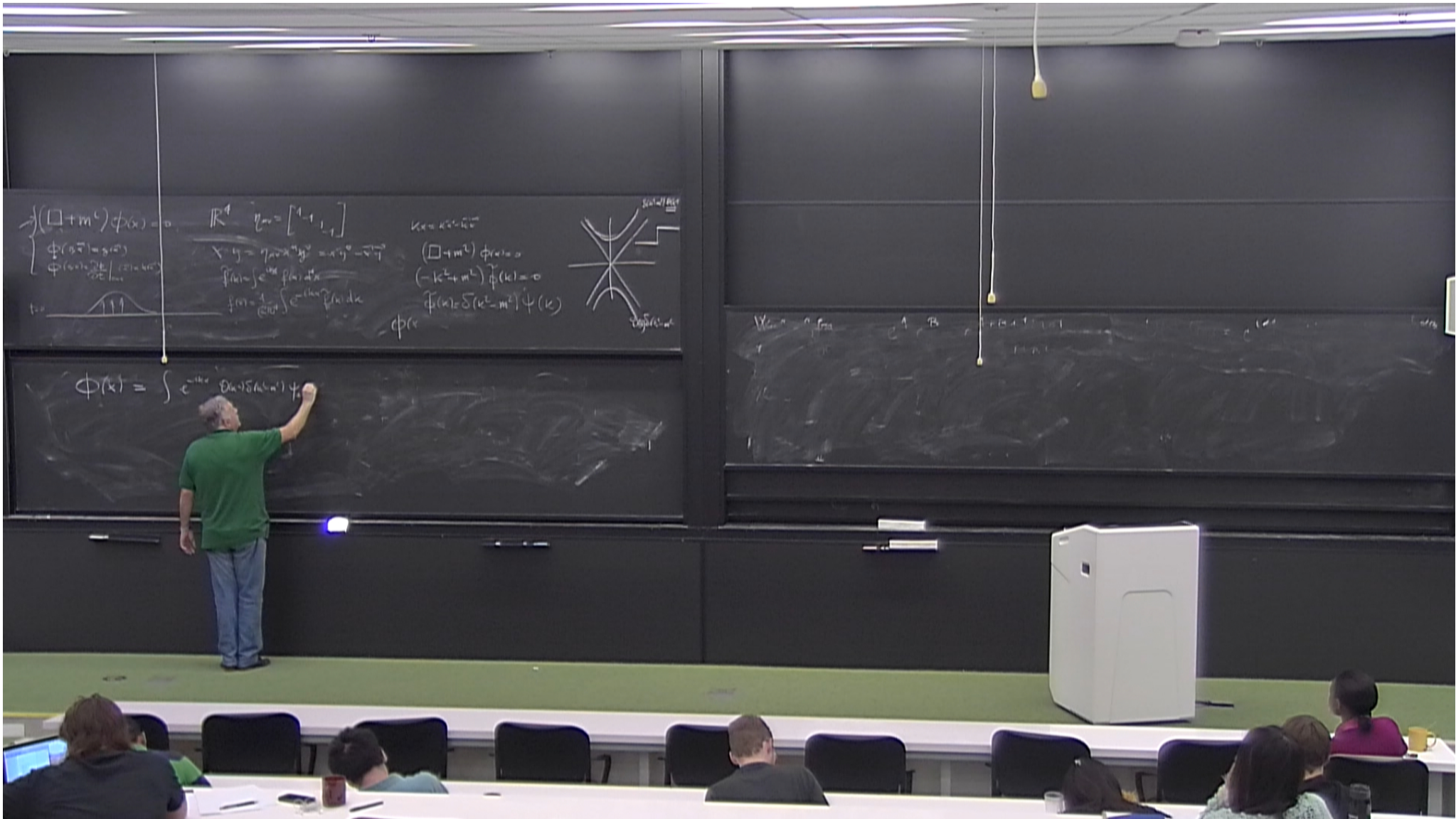
$U(t) \psi(x) = e^{itH} \psi(x)$ $L(\mathbb{R}, d\omega)$
 $V(p) \psi(x) = \psi(x + p)$
 Proof the unitarity

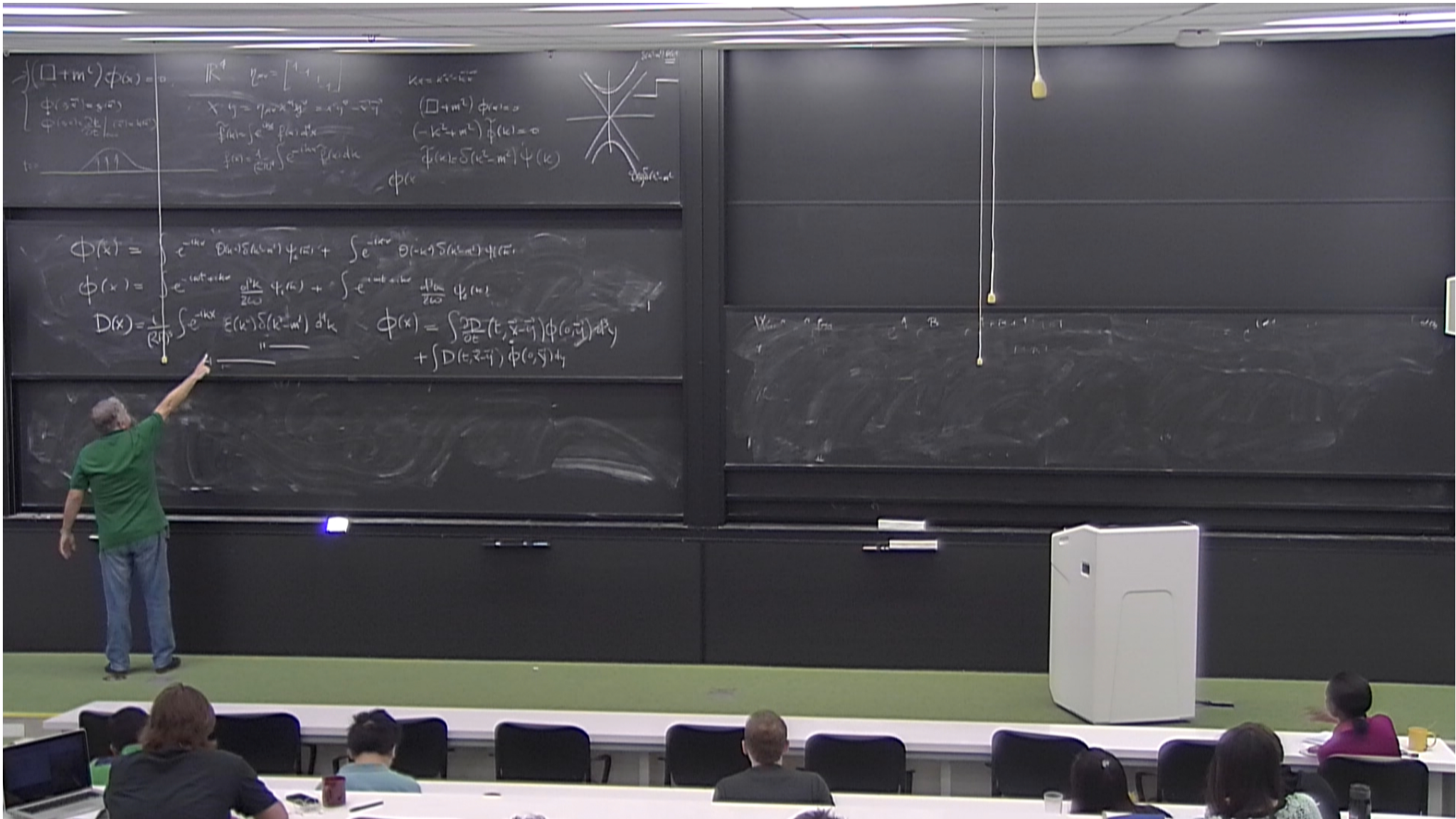
Weyl algebra $e^A e^B = e^{A+B + \frac{1}{2}[A, B]}$
 \mathcal{W} path-ordered GM
 $\{ \dot{q}_i, p_j \} = \delta_{ij} \rightarrow \{ \hat{q}_i, \hat{p}_j \} = \delta_{ij} \mathbb{1}$
 Weyl $U(\omega) = e^{i\omega \hat{q}}$ $V(p) = e^{ip \hat{p}}$
 $U(\omega)V(p) = e^{i\omega \hat{q} + ip \hat{p} + \frac{1}{2}[\omega \hat{q}, ip \hat{p}]} = e^{i\omega \hat{q} + ip \hat{p} + \frac{1}{2} \omega p}$
 $V(p)U(\omega) = e^{ip \hat{p} + i\omega \hat{q} + \frac{1}{2}[ip \hat{p}, \omega \hat{q}]} = e^{ip \hat{p} + i\omega \hat{q} - \frac{1}{2} \omega p}$
 $U(\omega)V(p) = e^{i\omega \hat{q}} e^{ip \hat{p}} e^{\frac{1}{2} \omega p} = e^{i\omega \hat{q} + ip \hat{p} + \frac{1}{2} \omega p}$
 $V(p)U(\omega) = e^{ip \hat{p}} e^{i\omega \hat{q}} e^{-\frac{1}{2} \omega p} = e^{ip \hat{p} + i\omega \hat{q} - \frac{1}{2} \omega p}$
 $U(\omega)V(p) = e^{i\omega \hat{q}} e^{ip \hat{p}} e^{\frac{1}{2} \omega p} = e^{i\omega \hat{q} + ip \hat{p} + \frac{1}{2} \omega p}$
 $V(p)U(\omega) = e^{ip \hat{p}} e^{i\omega \hat{q}} e^{-\frac{1}{2} \omega p} = e^{ip \hat{p} + i\omega \hat{q} - \frac{1}{2} \omega p}$

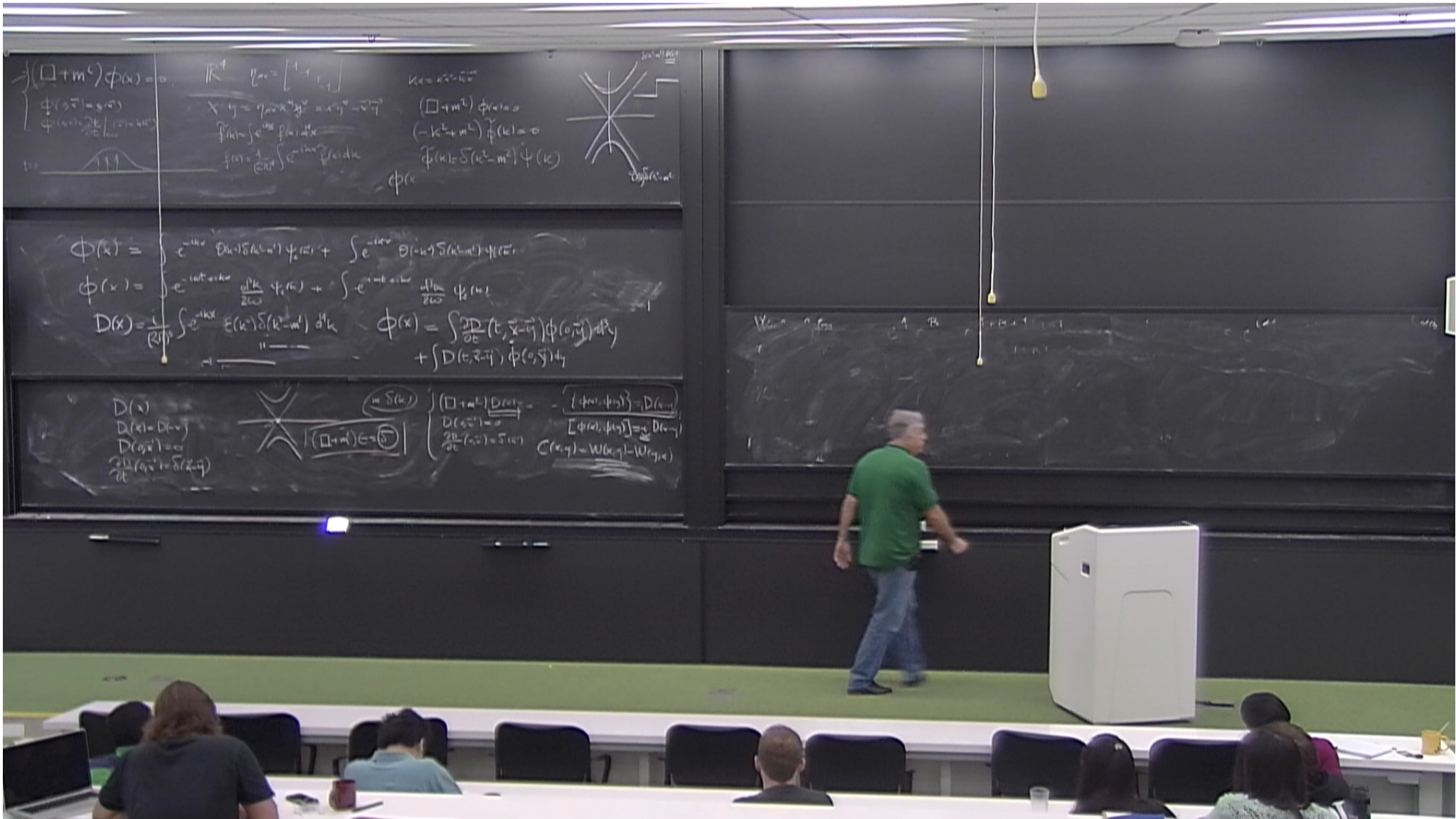
Theorem: \rightarrow every max. (regular) irreducible representation of the Weyl algebra
 $\hat{U}(t)$ strongly continuous group of unitary operators
 $\hat{V}(p)$ $\frac{d}{dt} \int_{\mathcal{X}} \psi(x) dx$
 $\lim_{t \rightarrow 0} \frac{1}{t} (\hat{U}(t) - \mathbb{1}) \psi = \frac{d}{dt} \int_{\mathcal{X}} \psi(x) dx$
 $(\hat{U}(t)\psi, \psi) = (\hat{U}(t) - \mathbb{1}) \psi, (\hat{U}(t) - \mathbb{1}) \psi = 2i \text{Re} \int_{\mathcal{X}} \psi(x) dx$
 $= \| \psi \|^2 = 2 \text{Re} \int_{\mathcal{X}} \psi(x) dx$

$$\rightarrow \hat{P} \hat{W}(t, p) \hat{P} = e^{-\frac{p^2}{2t}} P$$


$$\langle P \psi_0, \psi_0 \rangle = (\psi_0, \hat{W}(-t, p) \psi_0) = (\psi_0, \hat{W}(t, p) \psi_0) = e^{-\frac{p^2}{2t}} = \omega(\omega(-t, p))$$





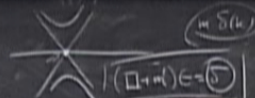


$(\square + m^2)\phi(x) = 0$ \mathbb{R}^4 $\eta_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$ $\kappa_\mu = \kappa^0 \epsilon_{\mu 0}$
 $\phi(\vec{x}, t) = \phi(\vec{x}, 0) + \frac{\partial \phi}{\partial t}(\vec{x}, 0) t$ $\square \phi = 0$ $(-\square + m^2)\phi(x) = 0$
 $\phi(x) = \int e^{i k x} \tilde{\phi}(k) d^4 k$ $(-k^2 + m^2)\tilde{\phi}(k) = 0$
 $\tilde{\phi}(k) = \delta(k^2 - m^2)\psi(k)$



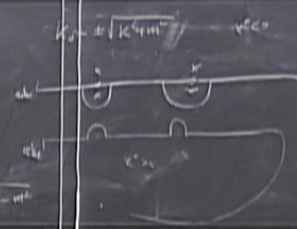
$\phi(x) = \int e^{-i k x} \theta(k^0) \delta(k^2 - m^2) \psi(k) d^4 k + \int e^{-i k x} \theta(-k^0) \delta(k^2 - m^2) \psi(k) d^4 k$
 $\phi(x) = \int e^{i k x} \frac{d^4 k}{2\omega} \psi(k) + \int e^{-i k x} \frac{d^4 k}{2\omega} \psi(k)$
 $D(x) = \frac{1}{(2\pi)^4} \int e^{-i k x} \delta(k^2 - m^2) d^4 k$ $\phi(x) = \int \frac{\partial D}{\partial t}(t, \vec{x} - \vec{y}) \phi(\vec{0}, \vec{y}) d^3 y + \int D(t, \vec{x} - \vec{y}) \dot{\phi}(\vec{0}, \vec{y}) d^3 y$

$D(x)$
 $D(x) = D(-x)$
 $D(0, \vec{x}) = 0$
 $\frac{\partial D}{\partial t}(0, \vec{x}) = \delta(\vec{x} - \vec{y})$



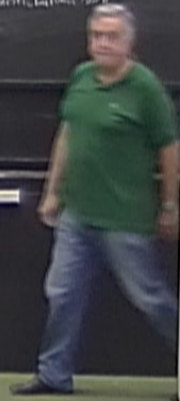
$(\square + m^2) D(x) = \delta(x)$
 $D(0, \vec{x}) = 0$
 $\frac{\partial D}{\partial t}(0, \vec{x}) = \delta(\vec{x})$
 $C(x, y) = W(x, y) - W(y, x)$

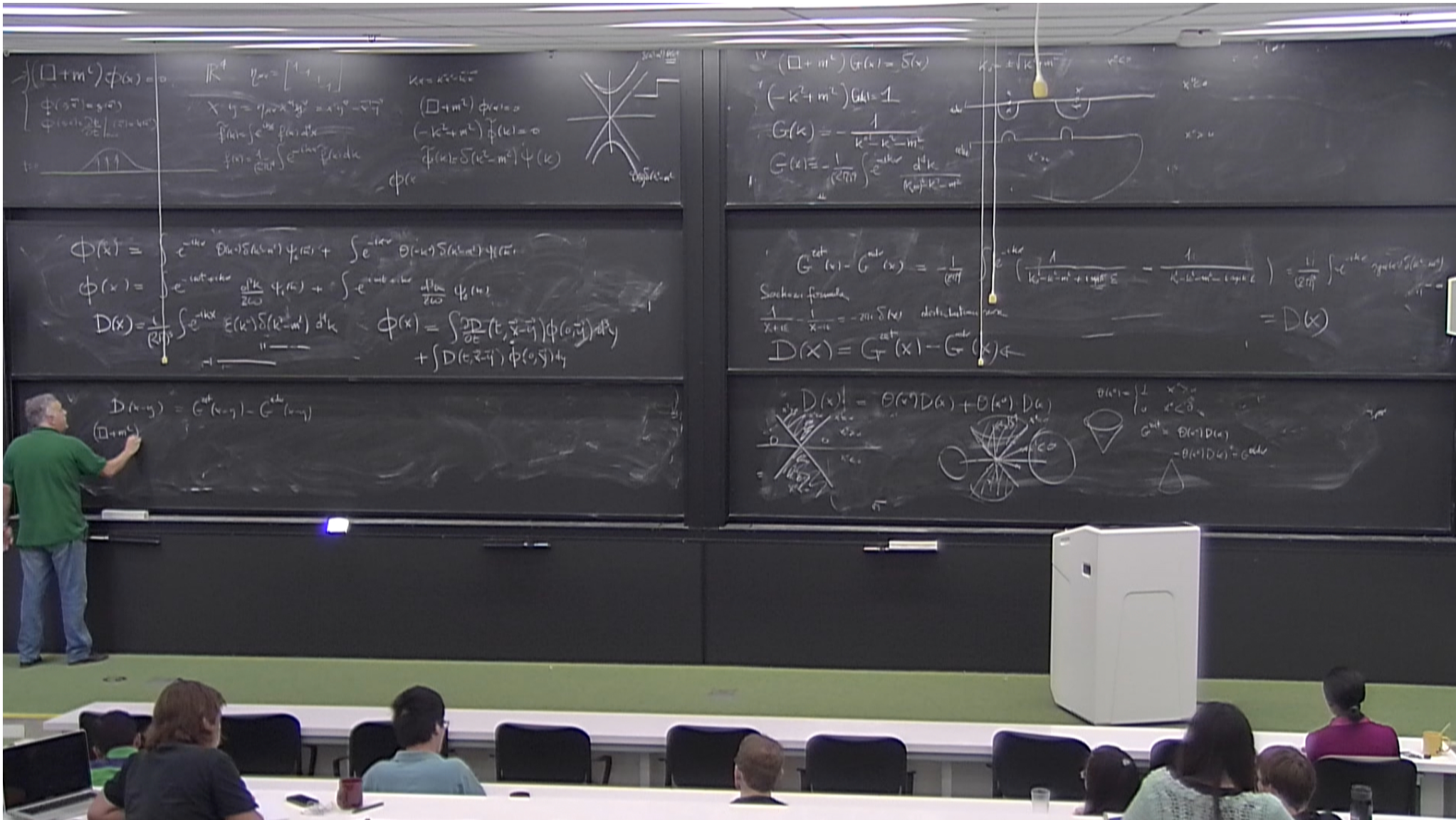
$(\square + m^2) G(x) = \delta(x)$
 $(-k^2 + m^2) G(k) = 1$
 $G(k) = -\frac{1}{k^2 - k^2 - m^2}$
 $G(x) = -\frac{1}{(2\pi)^4} \int e^{-i k x} \frac{d^4 k}{k^2 - k^2 - m^2}$



$G(x) = G^+(x) - G^-(x) = \frac{1}{(2\pi)^4} \int e^{-i k x} \left(\frac{1}{k^2 - k^2 - m^2 + i\epsilon} - \frac{1}{k^2 - k^2 - m^2 - i\epsilon} \right) d^4 k = \frac{1}{(2\pi)^4} \int e^{-i k x} \delta(k^2 - m^2) d^4 k = D(x)$

Suchman's formula:
 $\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon} = -2i\epsilon \delta(x)$ distribution sense

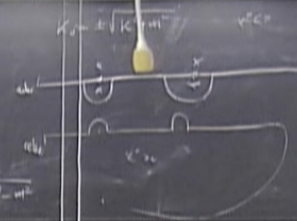




$$\begin{aligned}
 & (\square + m^2)\phi(x) = 0 \quad \mathbb{R}^4 \quad \eta_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \quad k_\mu = k^0, \vec{k} \\
 & \phi(\vec{x}, t) = \phi(\vec{x}, -t) \\
 & \phi(\vec{x}, t) = \frac{1}{(2\pi)^4} \int d^4k \delta(k^2 - m^2) \phi(k) \\
 & \phi(x) = \int d^4k \delta(k^2 - m^2) \phi(k) \\
 & \phi(x) = \int d^4k \delta(k^2 - m^2) \phi(k) \\
 & \phi(x) = \int d^4k \delta(k^2 - m^2) \phi(k)
 \end{aligned}$$



$$\begin{aligned}
 & (\square + m^2)G(x) = \delta(x) \\
 & (-k^2 + m^2)G(k) = 1 \\
 & G(k) = -\frac{1}{k^2 - m^2} \\
 & G(x) = -\frac{1}{(2\pi)^4} \int d^4k \frac{e^{-ikx}}{k^2 - m^2}
 \end{aligned}$$



$$\begin{aligned}
 \phi(x) &= \int d^4k \delta(k^2 - m^2) \phi(k) \\
 \phi(x) &= \int d^4k \delta(k^2 - m^2) \phi(k) \\
 D(x) &= \frac{1}{(2\pi)^4} \int d^4k \delta(k^2 - m^2) \phi(k) \\
 \phi(x) &= \int d^4k \delta(k^2 - m^2) \phi(k) \\
 \phi(x) &= \int d^4k \delta(k^2 - m^2) \phi(k)
 \end{aligned}$$

$$\begin{aligned}
 G^{\text{ret}}(x) - G^{\text{adv}}(x) &= \frac{1}{(2\pi)^4} \int d^4k \left(\frac{1}{k^2 - m^2 + i\epsilon} - \frac{1}{k^2 - m^2 - i\epsilon} \right) e^{-ikx} \\
 \frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon} &= -2i\pi \delta(x) \\
 D(x) &= G^{\text{ret}}(x) - G^{\text{adv}}(x) = D(x)
 \end{aligned}$$

$$\begin{aligned}
 D(x-y) &= G^{\text{ret}}(x-y) - G^{\text{adv}}(x-y) \\
 (\square + m^2)D(x) &= \delta(x)
 \end{aligned}$$

