Title: General Relativity for Cosmology - Lecture 2

Date: Sep 19, 2013 04:00 PM

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Abstract:

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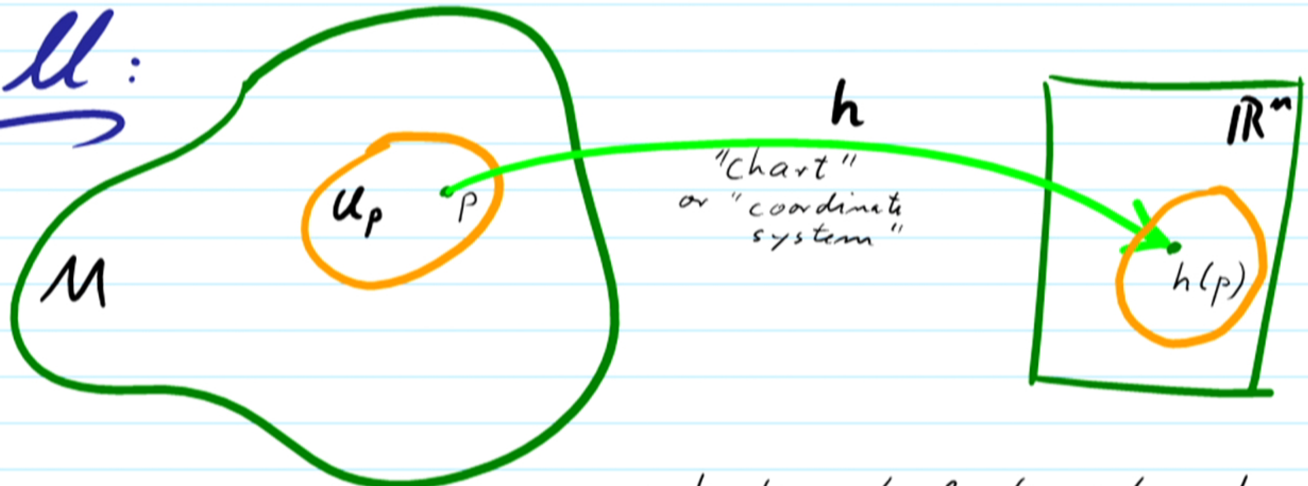
# GR for Cosmology, Fall 13, Achim Kempf, Lecture 2

Note Title

## Recall:




→ charts are tools to get a handle at the otherwise nameless abstract points of the manifold.

## Problem:

How to define the abstract "Tangent space, $T_p(M)$"

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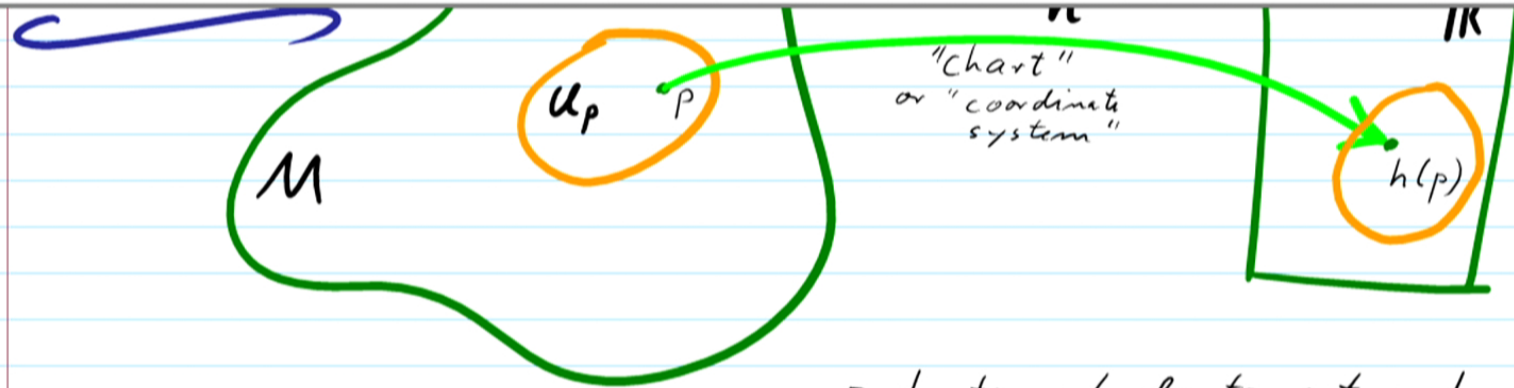


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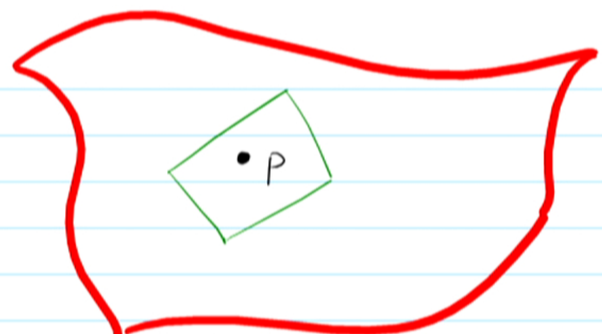
**Intuition:**




E.g. 2 dim manifold has 2 dim vector space of

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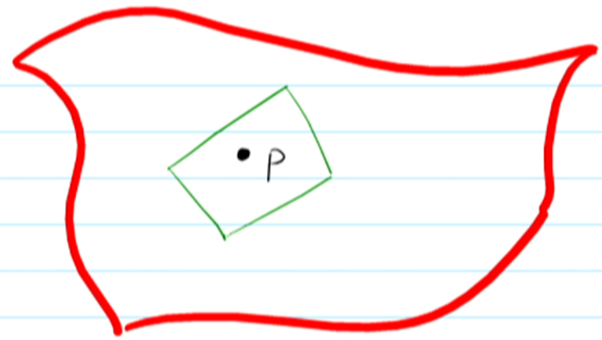


E.g. 2 dim manifold has 2 dim vector space of tangent vectors.

⇝ Proper definition should imply:

An $n$-dim mfld possesses for every point $p$ an $n$-dim vector space


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⇝ Proper definition should imply:

An n-dim mfld possesses for every point p an n-dim vector space of tangent vectors.

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# 3 equivalent definitions of $T_p(M)$ :

## 1. "Algebraic" definition of $T_p(M)$

lengthy and abstract
but modern and powerful!

**Idea:** A tangent vector can denote a directional derivative, which are recognizable by Leibniz rule of derivatives:

$$(fg)' = f'g + fg'$$

## 2. "Physicist" definition of $T_p(M)$

**Idea:** The elements of $T_p(M)$ are

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2. "Physicist" definition of $T_p(M)$

Idea: The elements of $T_p(M)$ are to be vectors $\Rightarrow$ recognizable by how their components change with charts.

3. "Geometric definition of $T_p(M)$"

Idea: The elements of $T_p(M)$ are

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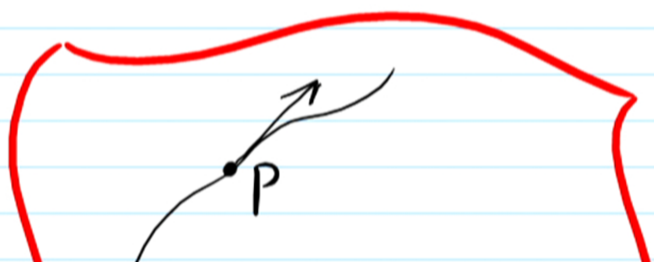
# 3. "Geometric definition of $T_p(M)$"

Idea: The elements of $T_p(M)$ are to be actual tangent vectors of one-dim. paths in the manifold, that pass through $p$.

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The 3 defs are equivalent, but:

One tends to need all 3 occasionally!

$\rightsquigarrow$ we will do all 3:

## 1. Algebraic definition of $T_p(M)$

Idea: Tangent vectors denote directional derivatives, and derivatives are recognizable through the Leibniz rule:

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# 1. Algebraic definition of $T_p(M)$

Idea: Tangent vectors denote directional derivatives, and derivatives are recognizable through the Leibniz rule:

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Example: In the case of $M = \mathbb{R}^n$, the tangent vectors $\xi$ at a point $p$ are in 1 to 1 correspondence to the directional 1st

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**Example:** In the case of $M = \mathbb{R}^n$, the tangent vectors $\xi$ at a point $p$ are in 1 to 1 correspondence to the directional 1st derivatives:

$$\xi = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x^i}\Big|_{x=p}$$

**Thus:** Each tangent vector maps functions into numbers:

$$\xi : f \to \xi(f) = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x^i} f(x)\Big|_{x=p}$$

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⇒ **Characteristic property "Leibniz rule":**

We'll try to identify tangent vectors by this property.

$$\boxed{\xi(fg) = \xi(f)\,g + f\,\xi(g)}$$

But how to define an 'algebra of functions' at a single point $p$?

**Def:**
- Assume $M, N$ are diffable mflds and $p \in M$.
- We say that two differentiable functions $\phi, \psi$ are equivalent about $p$ if in a neighborhood $\mathcal{U} \subset M$ of $p$ (i.e. an open set containing $p$):

$$\phi(q) = \psi(q) \quad \forall q \in \mathcal{U}$$

- Each such equivalence class of functions is called a germ at ...

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$$\phi(q) = \psi(q) \quad \forall q \in \mathcal{U}$$

- Each such equivalence class of functions is called a germ at $p$.
- Then, the "germ" of $\phi$ at $p$, denoted $\Phi_p$, is the equivalence class of all functions $\psi$ which are identical to $\phi$ in some neighborhood of $p$:

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class of all functions $\psi$ which are identical to $\phi$ in some neighborhood of $p$:

some open neighborhood of $p$ in $M$.

$$\psi \in \bar{\Phi}_p \text{ if } \exists\, U_p \;\; \forall q \in U_p : \;\; \phi(q) = \psi(q)$$

"there exists"

Notice: Each $\bar{\Phi}_p$ is an equivalence class of functions

$$\psi : M \to N$$

which possess the same 1st derivative at $p$.

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**For example:**

Consider germs of scalar functions $f$:


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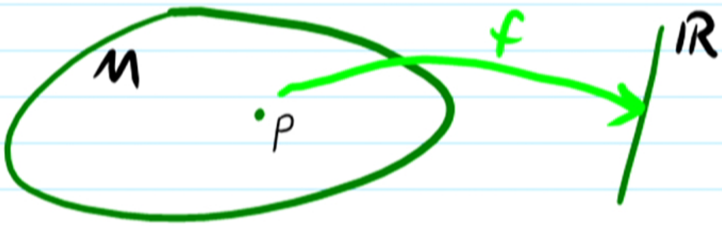


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Note:

- To specify a germ, it suffices to specify any arbitrary one of its functions.

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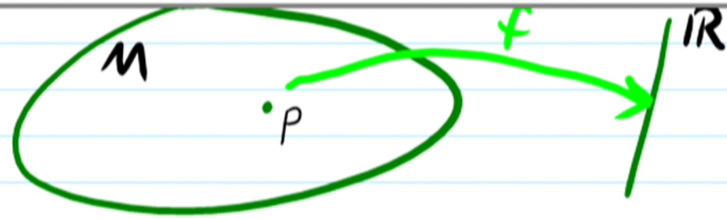


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- The set of all germs at $p$ is denoted $\mathcal{F}(p)$.

**Note:**

- One has for all $c \in \mathbb{R}$ and $f, g \in \mathcal{F}(p)$:

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$$\overline{c \cdot f} = c\,\overline{f}$$

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$\Rightarrow$ $\mathcal{F}(p)$ obeys the axioms of an associative algebra.

↑ it inherits the axioms from the full algebra of functions over $M$.

**Finally:** Algebraic definition of $T_p(M)$

Recall idea: The elements of $T(p)$ are to be 1st derivatives $\Rightarrow$ recognizable by Leibniz rule.

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# Finally: Algebraic definition of $T_p(M)$

**Recall idea:** The elements of $T(p)$ are to be 1st derivatives $\Rightarrow$ recognizable by Leibniz rule.

**Definition:** The tangent space $T_p(M)$ is the set of "derivations" of $\mathcal{F}(p)$, i.e. the set of linear maps $\xi: \mathcal{F}(p) \to \mathbb{R}$ which obey: (the Leibniz rule for differentiable functions $g, f$)

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$$\xi\left(\bar{f}_p \bar{g}_p\right) = \xi\left(\bar{f}_p\right) \cdot \underbrace{\bar{g}_p(p)}_{= g(p)} + \underbrace{\bar{f}_p(p)}_{= f(p)} \xi\left(\bar{g}_p\right)$$

(*) remember this: $\longrightarrow$

**Remark:**

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## Remark:

- this definition is abstract enough not only for arbitrary diff-able mflds!

- this definition (as derivations of the algebra of functions) is also suitable for "Noncommutative Geometry": There, (Quantum Gravity) the algebra of functions $F(p)$ is noncommutative.

- Note: Can't do Newton's derivatives then

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Simple example: a constant function $c$:


$c(p) := c$ and $c$ is a constant: $c \in \mathbb{R}$

Then: $\xi(\bar{c}) = 0$ for all $\xi \in T_p(M)$

Leibniz rule

Proof: $\xi(\bar{c}) = c\,\xi(1) = c\,\xi(1 \cdot 1) = c\big(\xi(1)\,1 + 1\,\xi(1)\big)$

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**Proof:** $\xi(\bar{c}) = c\,\xi(1) = c\,\xi(1\cdot 1) = c\big(\xi(1)\,1 + 1\,\xi(1)\big)$

$= 2c\,\xi(1) \Longrightarrow \xi(\bar{c}) = 0$ ✓

**Example:** The case $M = \mathbb{R}^n$

If our definition for $T_p(M)$ is good, we expect that

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**Example:** The case $M = \mathbb{R}^n$

If our definition for $T_p(M)$ is good, we expect that every $\xi \in T_p(M)$ is of the form:

$$\xi = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}\Big|_{x=p}$$

**Proof:** The proof is assigned reading:

- We choose $p$ to have coordinates $x = (0, 0, \ldots)$.

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**Proof:** The proof is assigned reading:

- We choose $p$ to have coordinates $x = (0, 0, \dots)$.
- Assume $\xi \in T_p(M)$ and $\bar{f} \in \mathcal{F}(p)$.

- Notation: $h_{,i}(a^1, \dots, a^n) := \frac{\partial}{\partial a^i} h(a^1, \dots, a^n)$

**Then:**

(Note: these are not 3 numbers! These are 3 function germs, i.e., 3 equivalence classes of functions.)

germ of a constant function

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$\Rightarrow$ Indeed, every $\xi \in T_p(M)$ is of the form

$$\xi = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial x^i}\Big|_{x=p} \qquad (\mathrm{I})$$

namely with

$$\xi^i = \xi(\bar{x}^i) \qquad (\mathrm{II})$$

$\square$

Notice: Knowing how $\xi$ acts on the coordinate functions $\bar{x}^i$ yields $\xi^i$ (from II) and thus

it means we know how $\xi$ acts on all

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**Notice:** Knowing how $\xi$ acts on the coordinate functions $\bar{x}^i$ yields $\xi^i$ (from II) and thus it means we know how $\xi$ acts on all functions $\bar{f} \in \mathcal{F}(p)$, namely through (I).

**But:**

- This was the simple example:

$$M = \mathbb{R}^n$$

- How does our definition of $T_p(M)$

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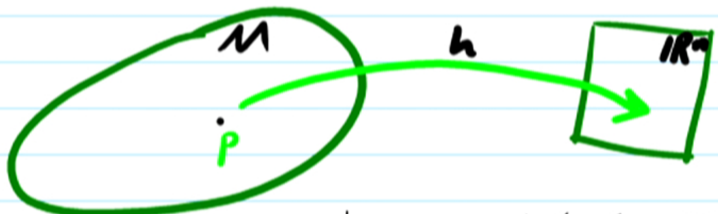
But:

- This was the simple example:

$$M = \mathbb{R}^n$$

- How does our definition of $T_p(M)$ work for $M \neq \mathbb{R}^n$, concretely?

- Recall:




h gives abstract points a name, i.e. makes them concrete.

- Problem: How to make abstract $\xi \in T_p(M)$ concrete?

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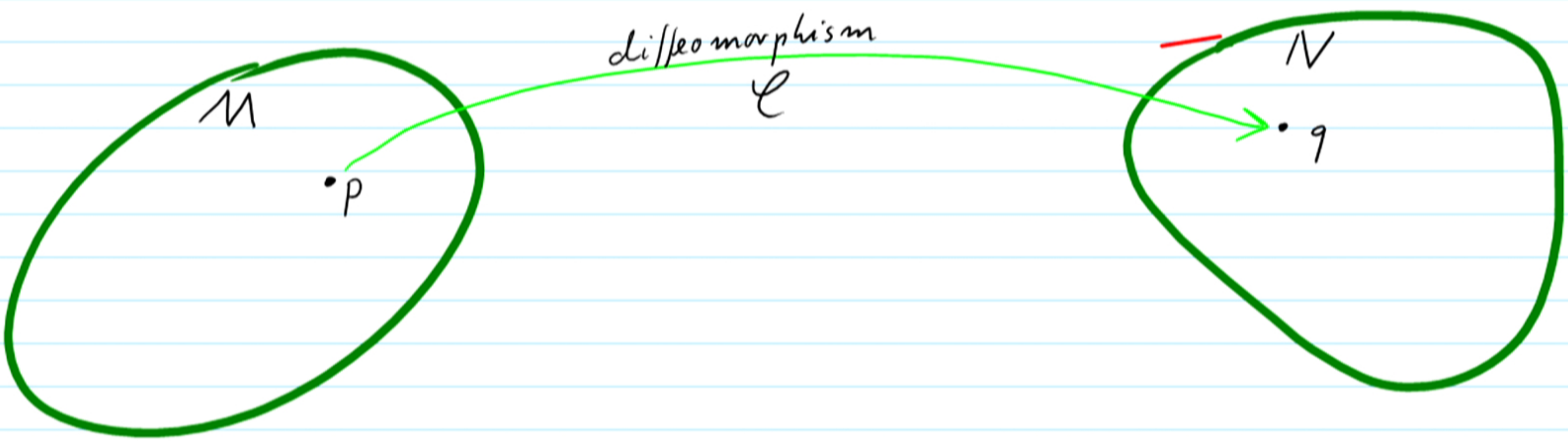
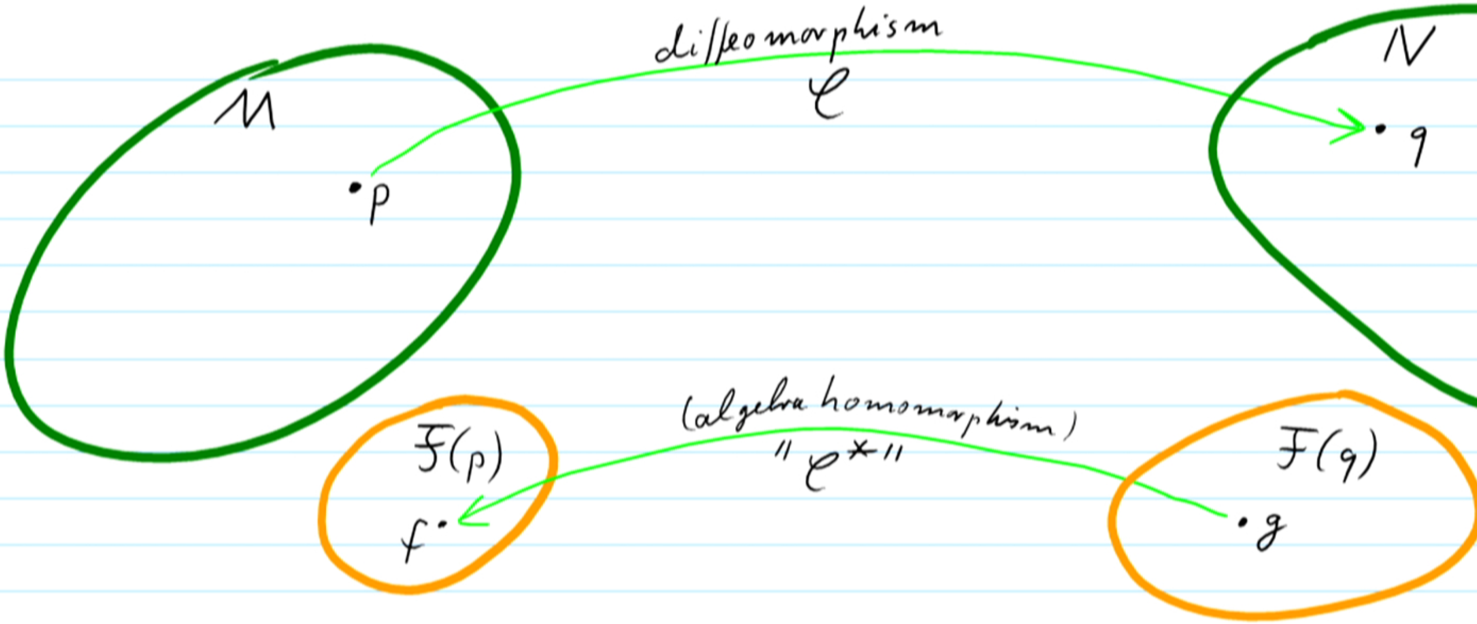
# Preparation: $T_p(M)$ and Diffeomorphisms.

Consider two diff'able manifolds, $M$ and $N$:

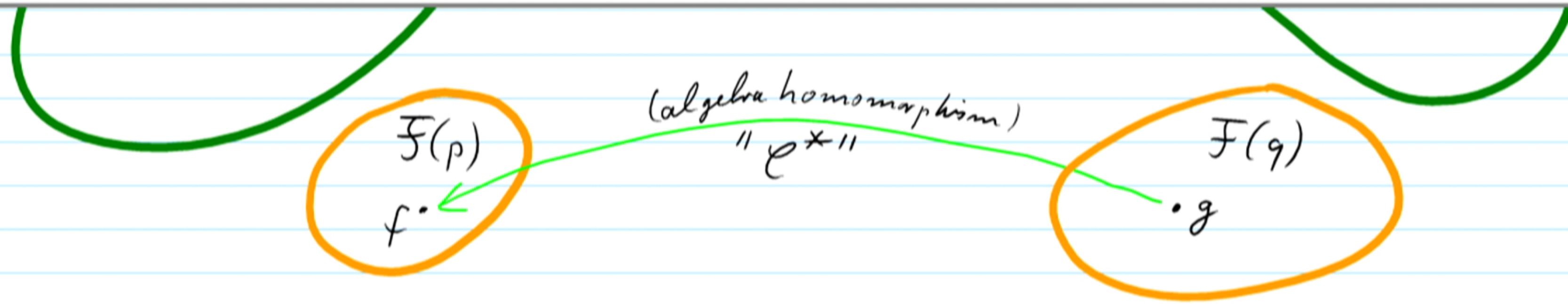



**Note:** If $N = \mathbb{R}^n$, then $\varphi$ is a chart.




Here: ▫ $\mathcal{F}(q)$ and $\mathcal{F}(p)$ are algebras of function (germs).

▫ Given $\varphi$ we obtain a map $\varphi^*: \mathcal{F}(q) \to \mathcal{F}(p)$




Here: $\mathcal{F}(q)$ and $\mathcal{F}(p)$ are algebras of function (germs).

Given $\varphi$ we obtain a map $\varphi^*: \mathcal{F}(q) \to \mathcal{F}(p)$

$$\varphi^*: g \to f = \varphi^*(g) \quad \text{with } f(x) = g(\varphi(x)) \ \forall x \in M$$

i.e.: $f = \varphi^*(g) = g \circ \varphi$ $\quad (+)$

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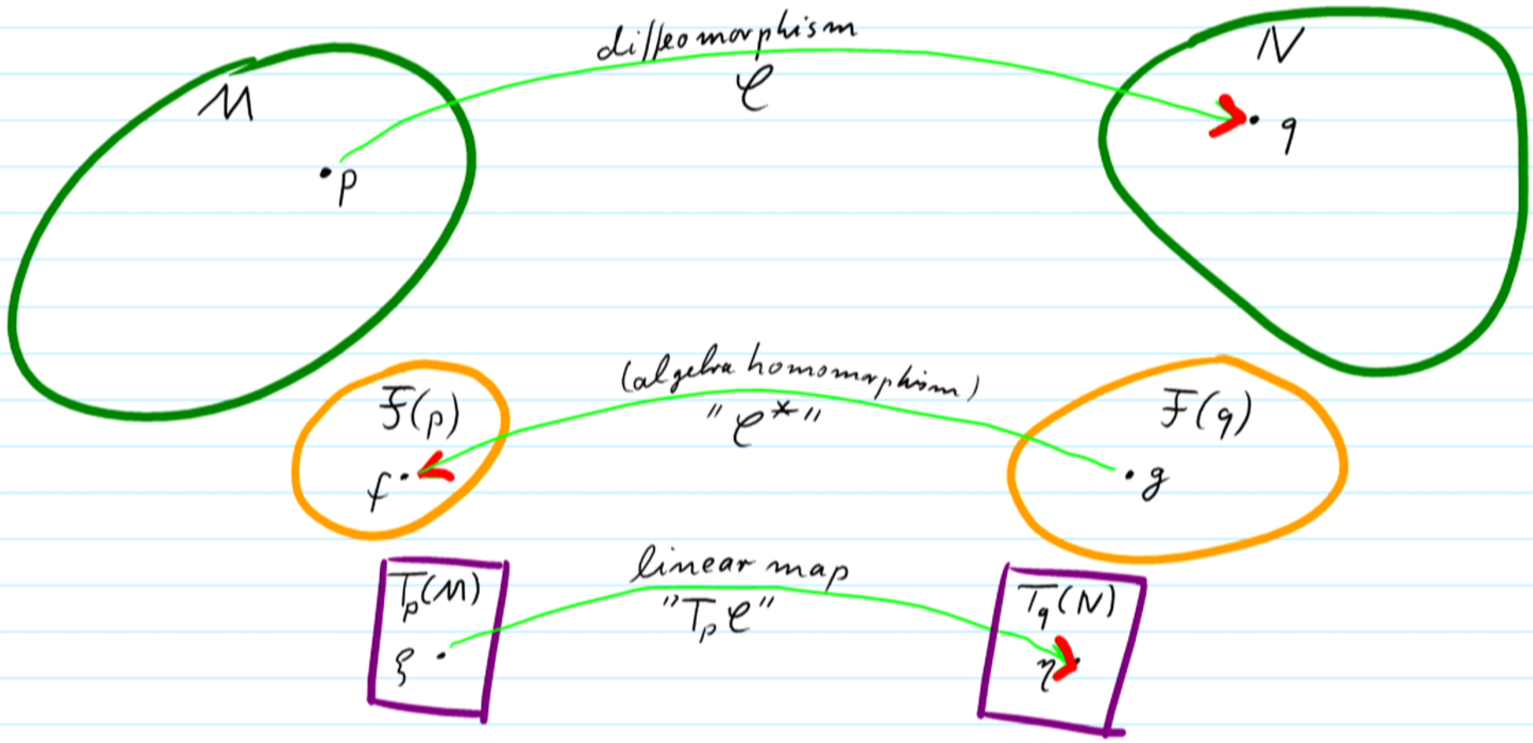


**Here:** □ Given $\varphi^*: \mathcal{F}(q) \to \mathcal{F}(p)$ we obtain the "tangent map":

$$T_p\varphi : T_p(M) \longrightarrow T_q(N)$$

(When choosing $M = \mathbb{R}$ ,

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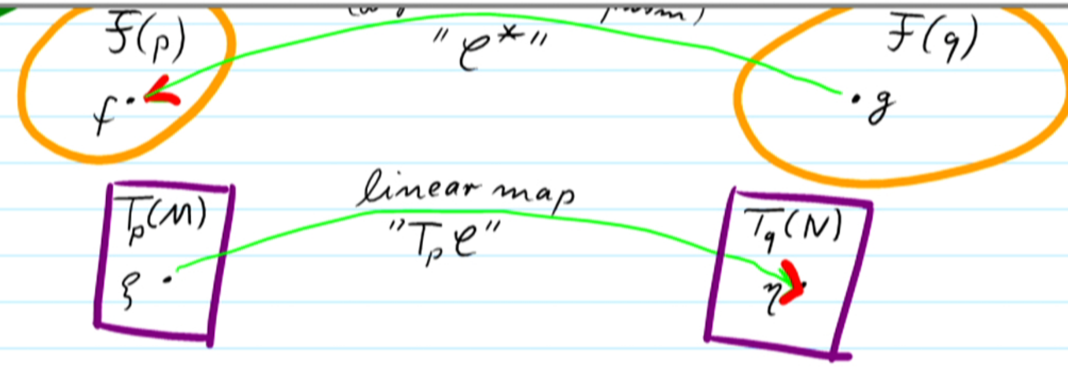


Here: □ Given $\mathcal{C}^*: \mathcal{F}(q) \to \mathcal{F}(p)$ we obtain the "tangent map":

$$T_p \mathcal{C}: T_p(M) \longrightarrow T_q(N)$$

$$T_p \mathcal{C}: \quad \xi \longrightarrow \eta$$

(When choosing $M = \mathbb{R}^n$, we obtain the desired concrete representation of $T_p(M)$ this way)

□ Namely: $\quad \eta = \xi \circ \mathcal{C}^*$

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- Namely: $\eta = \xi \circ \varphi^*$

  i.e.: $\eta(g) = \xi(\varphi^*(g))$

- From (+) $\Rightarrow$

  $$\eta(g) = \xi(g \circ \varphi)$$

## The crucial special case:

- $N = \mathbb{R}^n$ (with $n = \dim(N)$)
- $\varphi$ is invertible

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# The crucial special case:

- $N = \mathbb{R}^n$ (with $n = \dim(N)$)
- $\varphi$ is invertible
- ($\Rightarrow$ $\varphi^*$ is algebra isomorphism)
- $\Rightarrow$ $T_p\varphi$ is vector space isomorphism

$\Rightarrow$ We do obtain a concrete handle on the abstract tangent vectors $\xi \in T_p(M)$, given a chart $h$:

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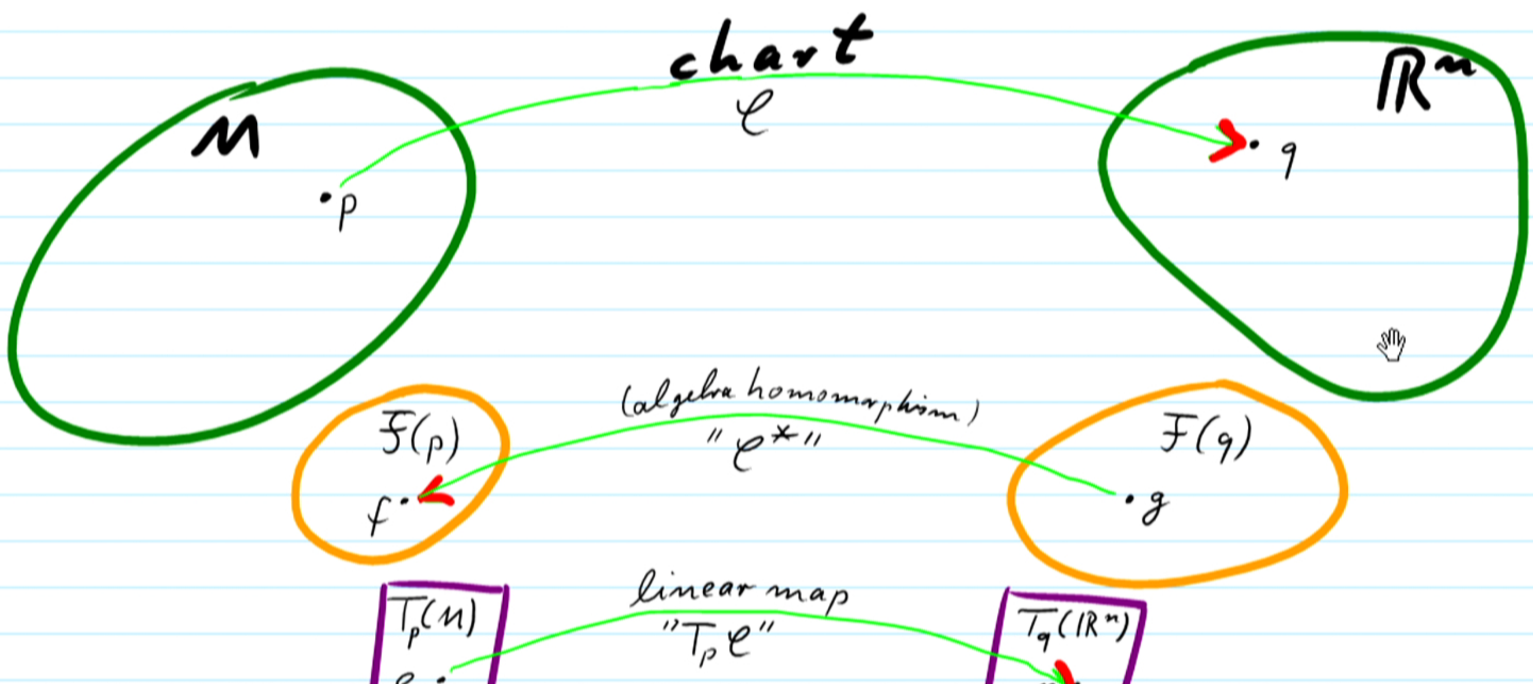
⇒ We do obtain a concrete handle on the abstract tangent vectors $\xi \in T_p(M)$, given a chart $h$:



- Given a chart $\mathcal{C}$, every abstract point $p \in M$ has a concrete image $\mathcal{C}(p) \in \mathbb{R}^n$, and:

- Every abstract vector $\xi \in T_p(M)$ has a concrete image

$$\underbrace{T_p\mathcal{C}(\xi)}_{=\eta} \in T_{\underbrace{\mathcal{C}(p)}_{=q}}(\mathbb{R}^n)$$

- The image $\eta$ is concrete because

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$\varphi(p) \in \mathbb{R}^n$, <u>and</u>:

- Every abstract vector $\xi \in T_p(M)$ has a concrete image

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- The image $\eta$ is concrete because $\eta$ is tangent vector to a point $q \in \mathbb{R}^n$, and it therefore must take the

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$$\underbrace{T_p \varphi(\xi)}_{=\eta} \in T_{\underbrace{\varphi(p)}_{=q}}(\mathbb{R}^n)$$

The image $\eta$ is concrete because $\eta$ is tangent vector to a point $q \in \mathbb{R}^n$, and it therefore must take the form (we showed this):

$$\eta = \sum_{i=1}^{n} \eta^i \frac{\partial}{\partial x^i}\Big|_{x=q}$$

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$$= \sum_{i=1}^{n} \xi(\bar{x}^i) \frac{\partial}{\partial x^i} f(x^1, \dots, x^n) \Big|_{x=p=0}$$

$\Rightarrow$ Indeed, every $\xi \in T_p(M)$ is of the form

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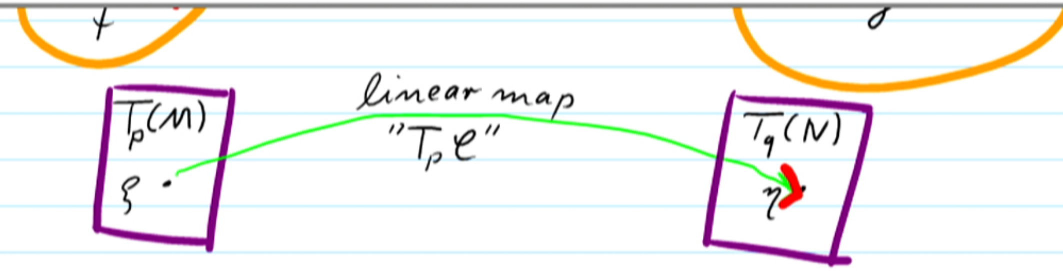


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$$T_p\varphi: \xi \longrightarrow \eta$$

(When choosing $M = \mathbb{R}^n$, we obtain the desired concrete representation of $T_p(M)$ this way)

□ Namely: $\eta = \xi \circ \varphi^*$

i.e.: $\eta(q) = \xi(\varphi^*(q))$

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$$\underbrace{T_p\varphi(\xi)}_{=\eta} \in T_{\underbrace{\varphi(p)}_{=q}}(\mathbb{R}^n)$$

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concrete numbers.

**Conversely:** (and very conveniently)

- Assuming a fixed $\varphi$, any choice of a $q = (x^1, \ldots, x^n)$ denotes a $p \in M$

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of a $q = (x^1, \dots, x^n)$ denotes a $p \in M$

and any choice of a $(\eta^1, \dots, \eta^n)$ denotes

a $\xi \in T_p(M)$.

some numbers

- E.g. $\eta = \frac{\partial}{\partial x^i}\Big|_{x=q}$ is the image

of some abstract $\xi \in T_p(M)$, for fixed $\varphi$.

Notation: $\xi = "\frac{\partial}{\partial x^i}\Big|_{x=p}"$

symbolic notation

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Notation: $\xi =$ "$\frac{\partial}{\partial x^i}\Big|_{x=p}$"

symbolic notation

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symbolic notation

**Next:**

If we hold $p$ and $\xi \in T_p(M)$ fixed,

how do the numbers $(x^1, \dots, x^n)$

and $(\eta^1, \dots, \eta^n)$ change when we

change the chart? $\rightarrow$ Physicists' def of $T_p(M)$

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