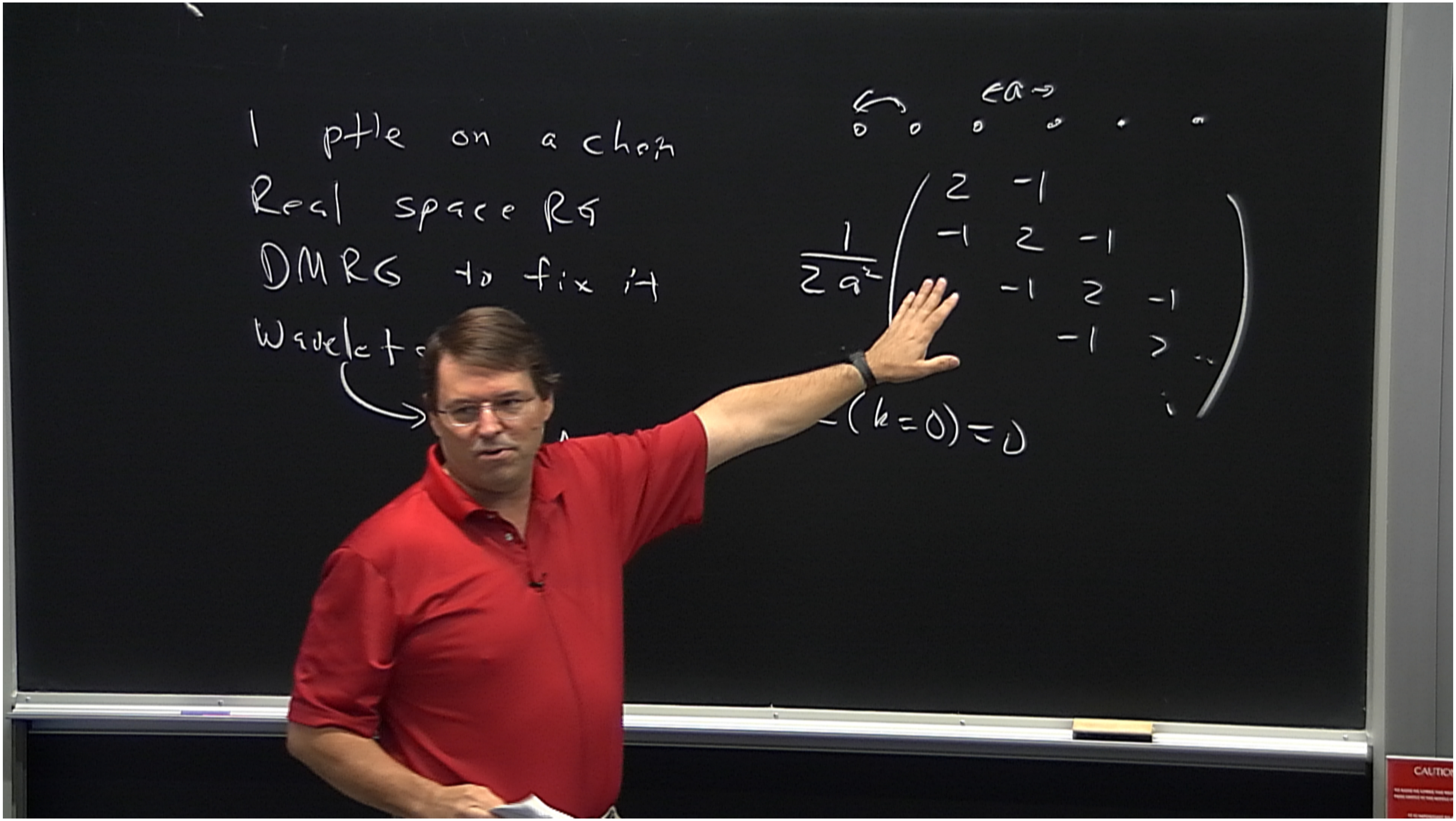
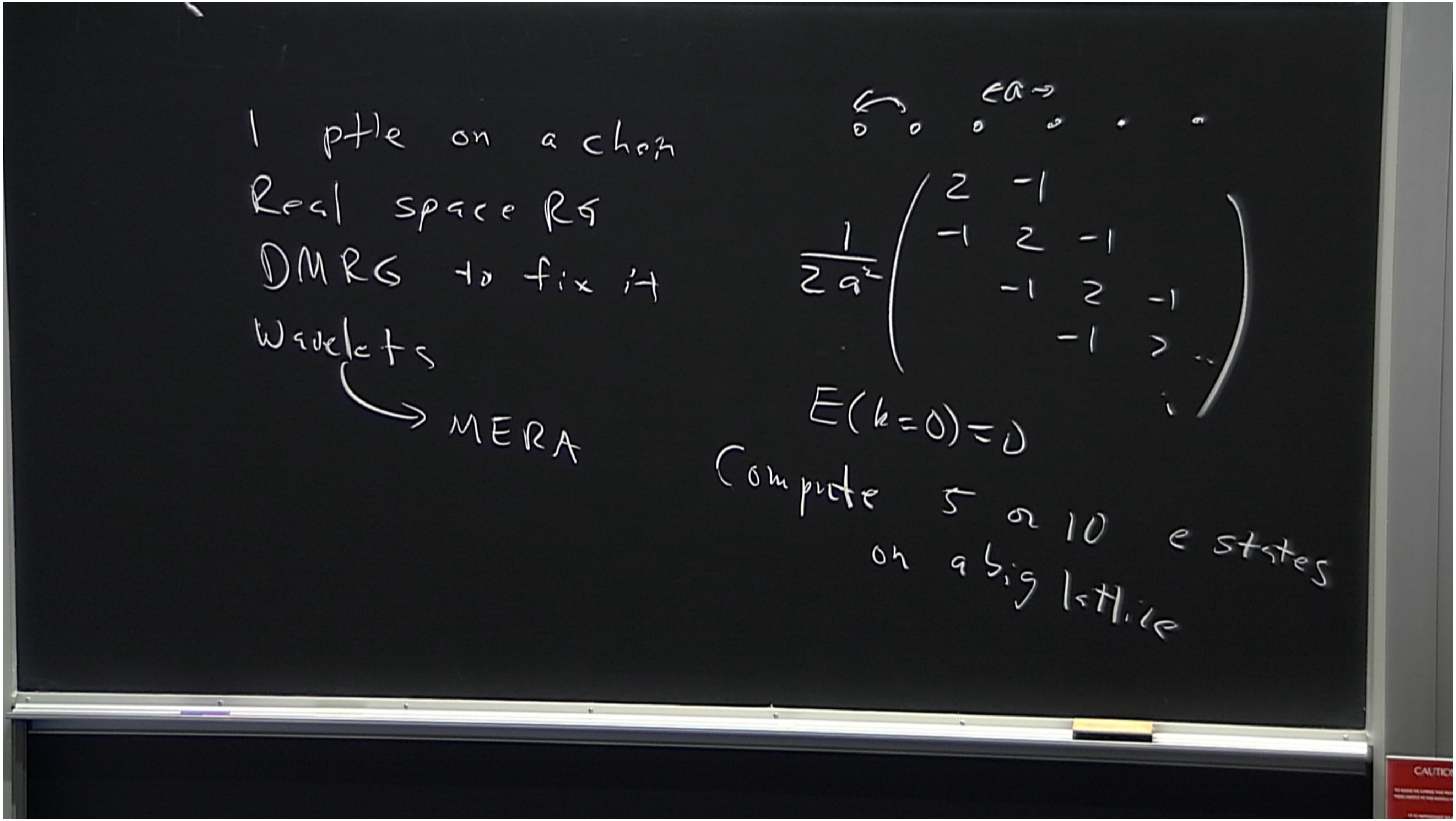
Title: Wavelets and MERA

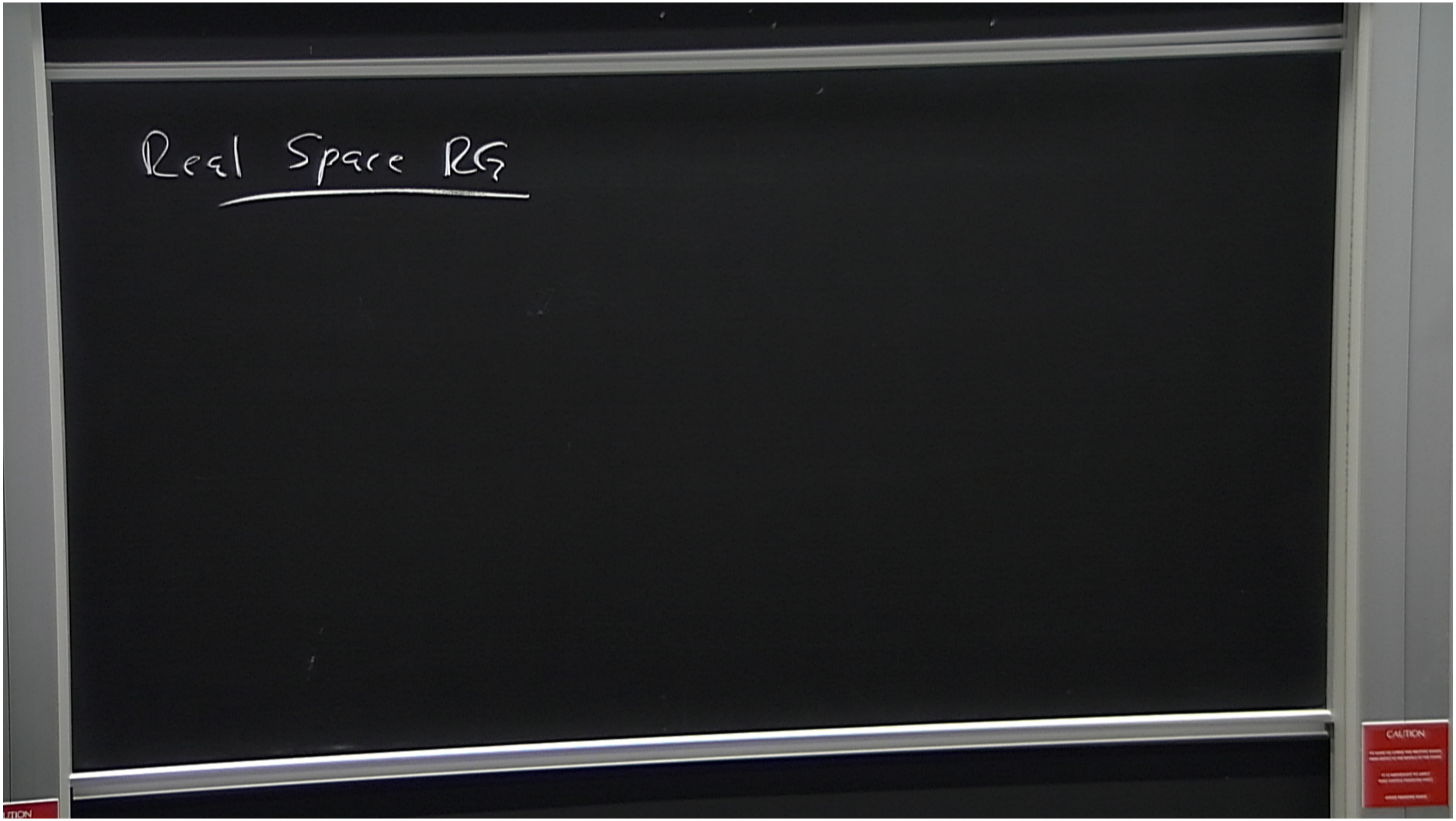
Date: Aug 30, 2013 10:30 AM

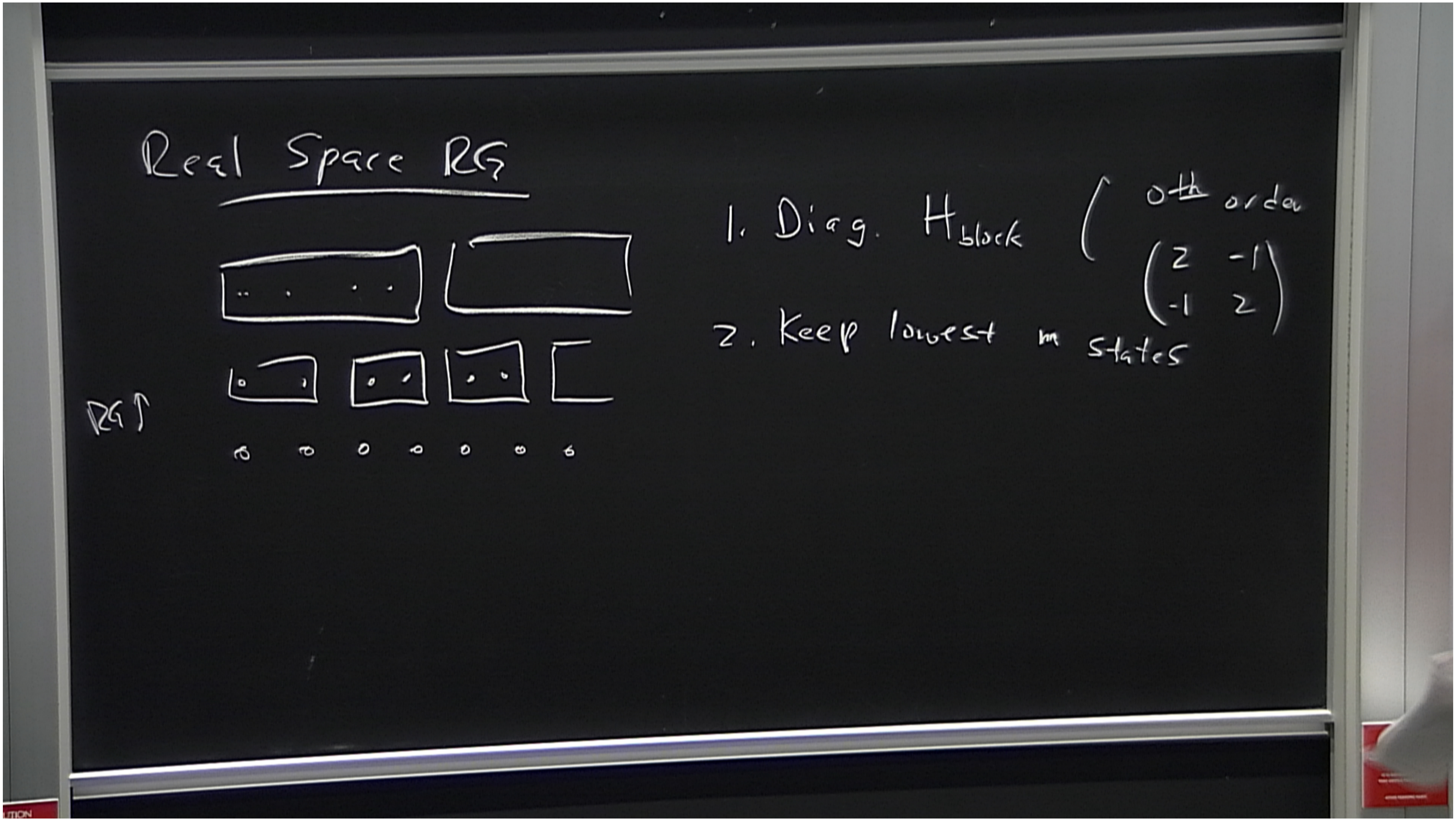
URL: http://pirsa.org/13080057

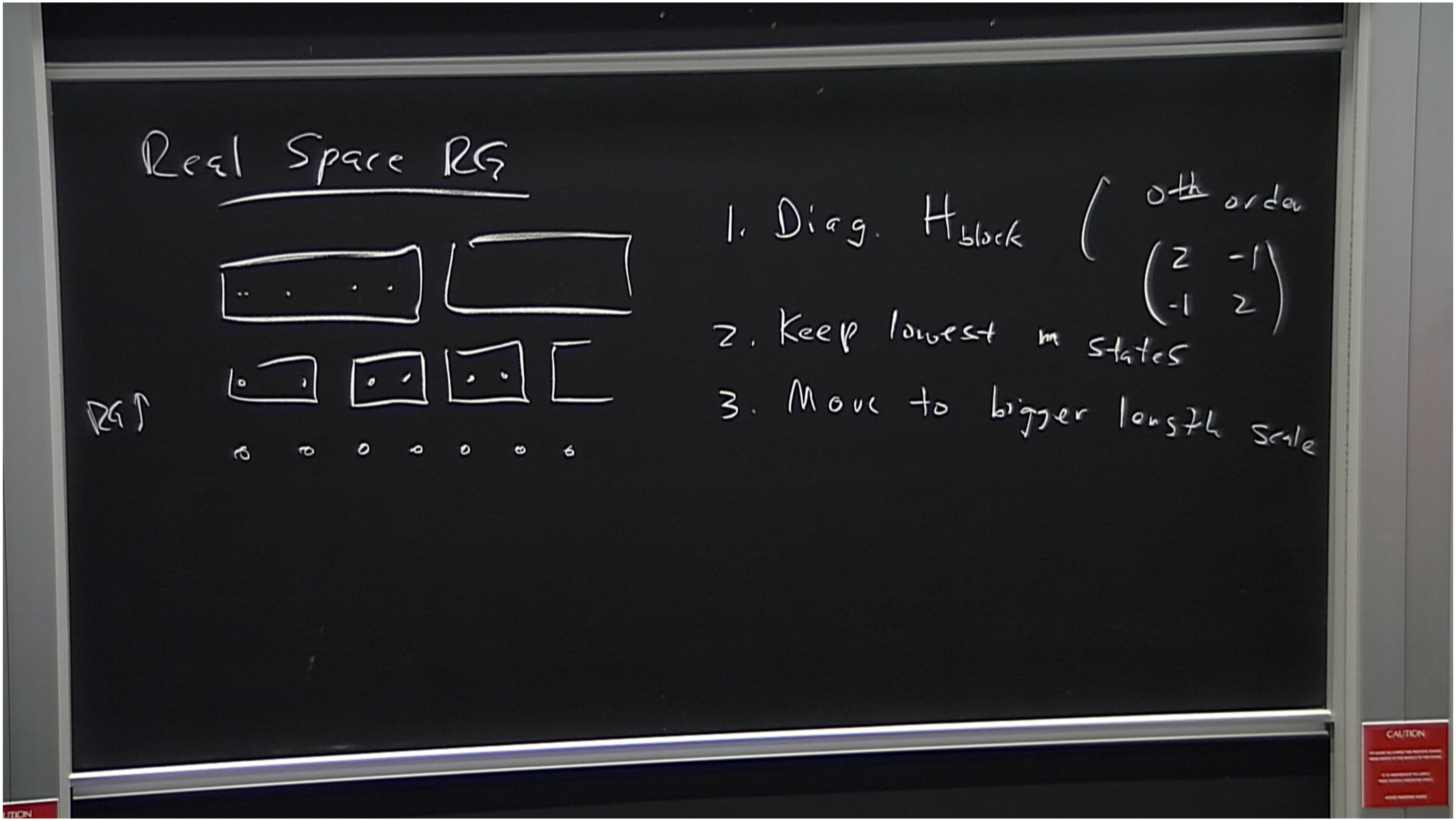
Abstract: Some of the key insights that led to the development of DMRG stemmed from studying the behavior of real space RG for single particle wavefunctions, a much simpler context than the many-particle case of main interest. Similarly, one can gain insight into MERA by studying wavelets. I will introduce basic wavelet theory and show how one of the most well-known wavelets, a low order orthogonal wavelet of Daubechies, can be realized as the fixed point of a specific MERA (in single-particle direct-sum space). Higher order wavelets and the conflict between compactness in real and Fourier space may provide insight into generalized MERAs for many particle systems.

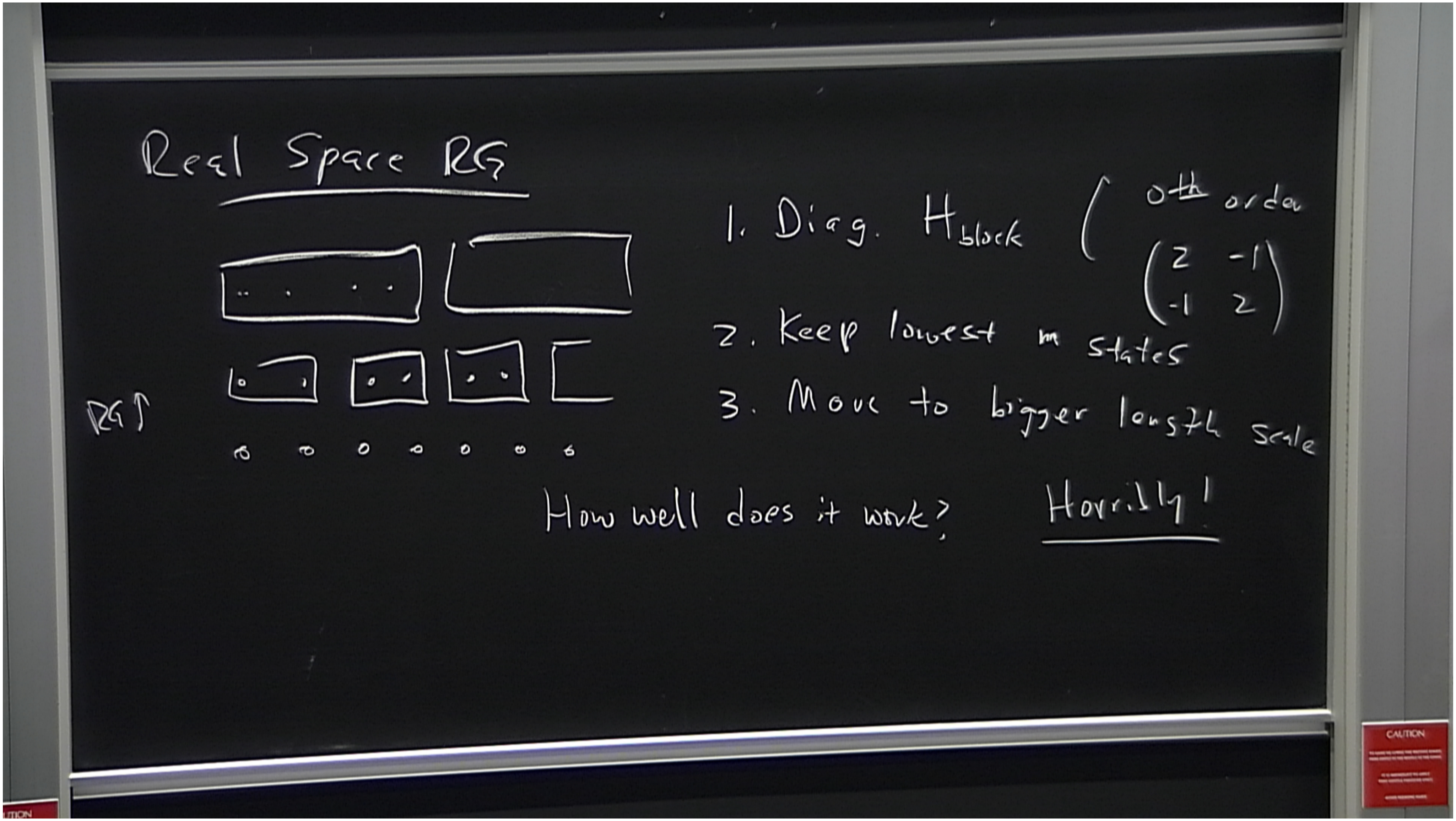

1 ptle on a chain
Real space RG
DMRG to fix it
Wavelets

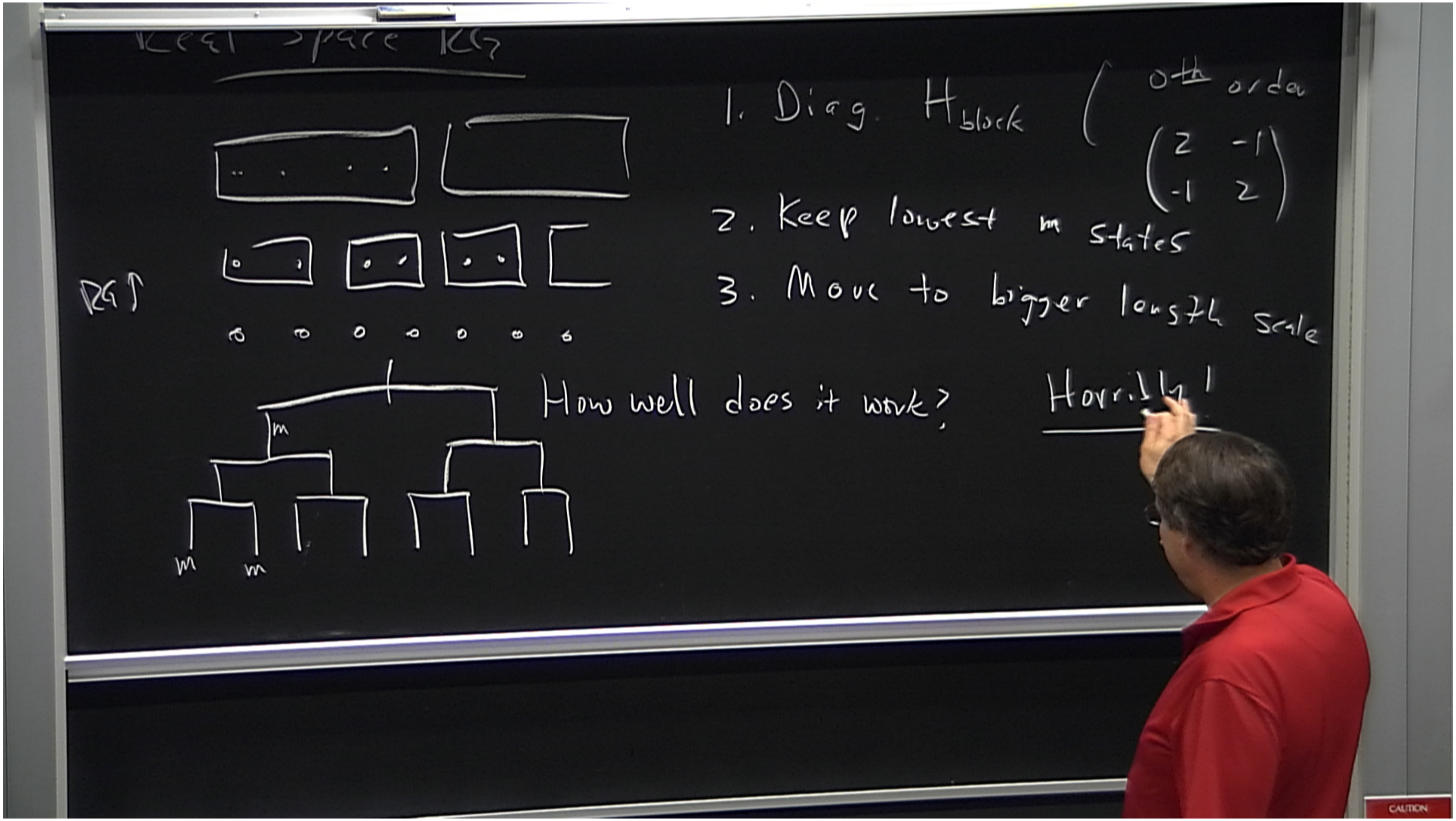

1 ptle on a chain
Real space RG
DMRG to fix it
Wavelets
MERA
E(k=0)=0
Compute 5 or 10 e states
on a big lattice

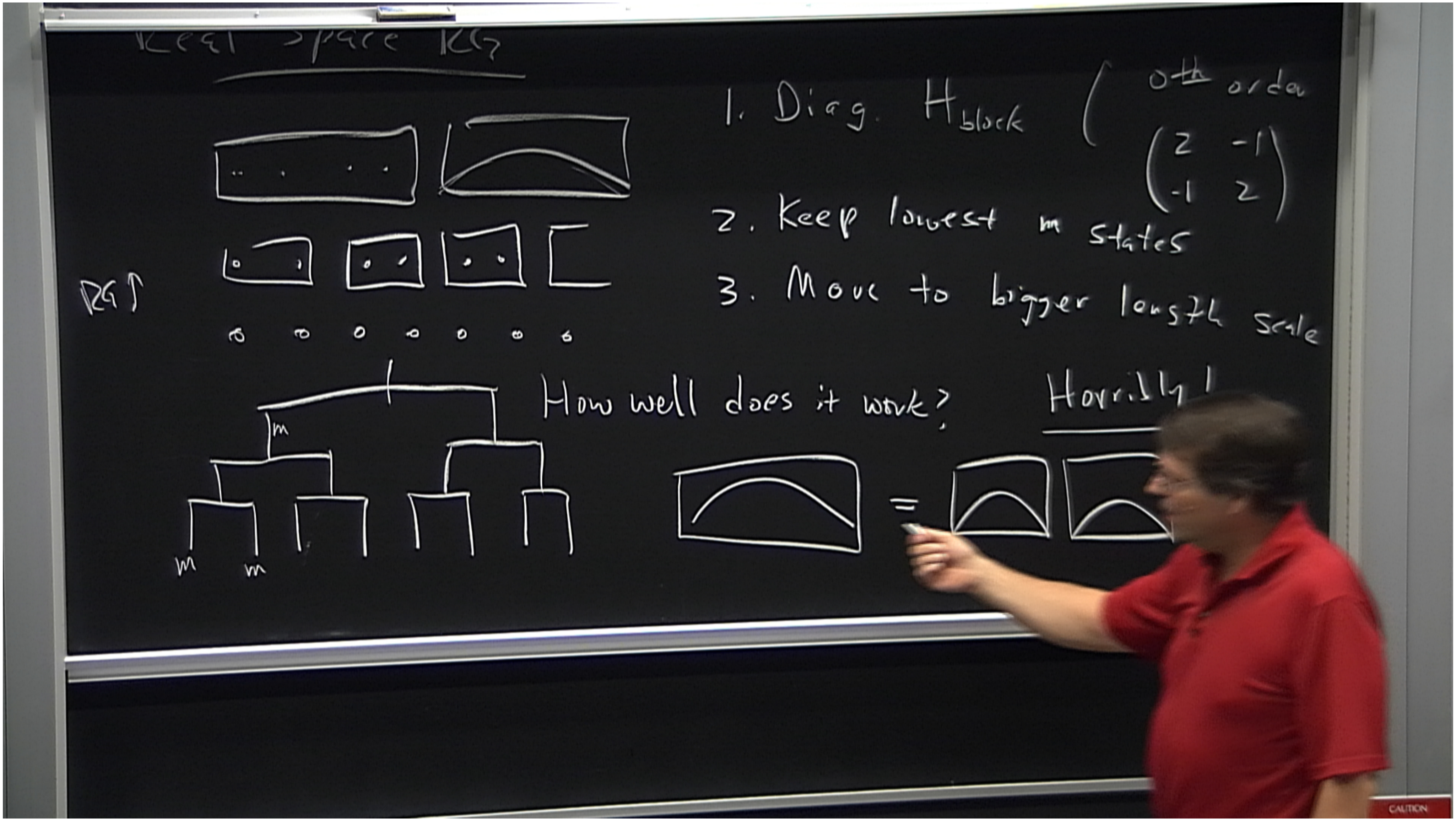

Real Space RG

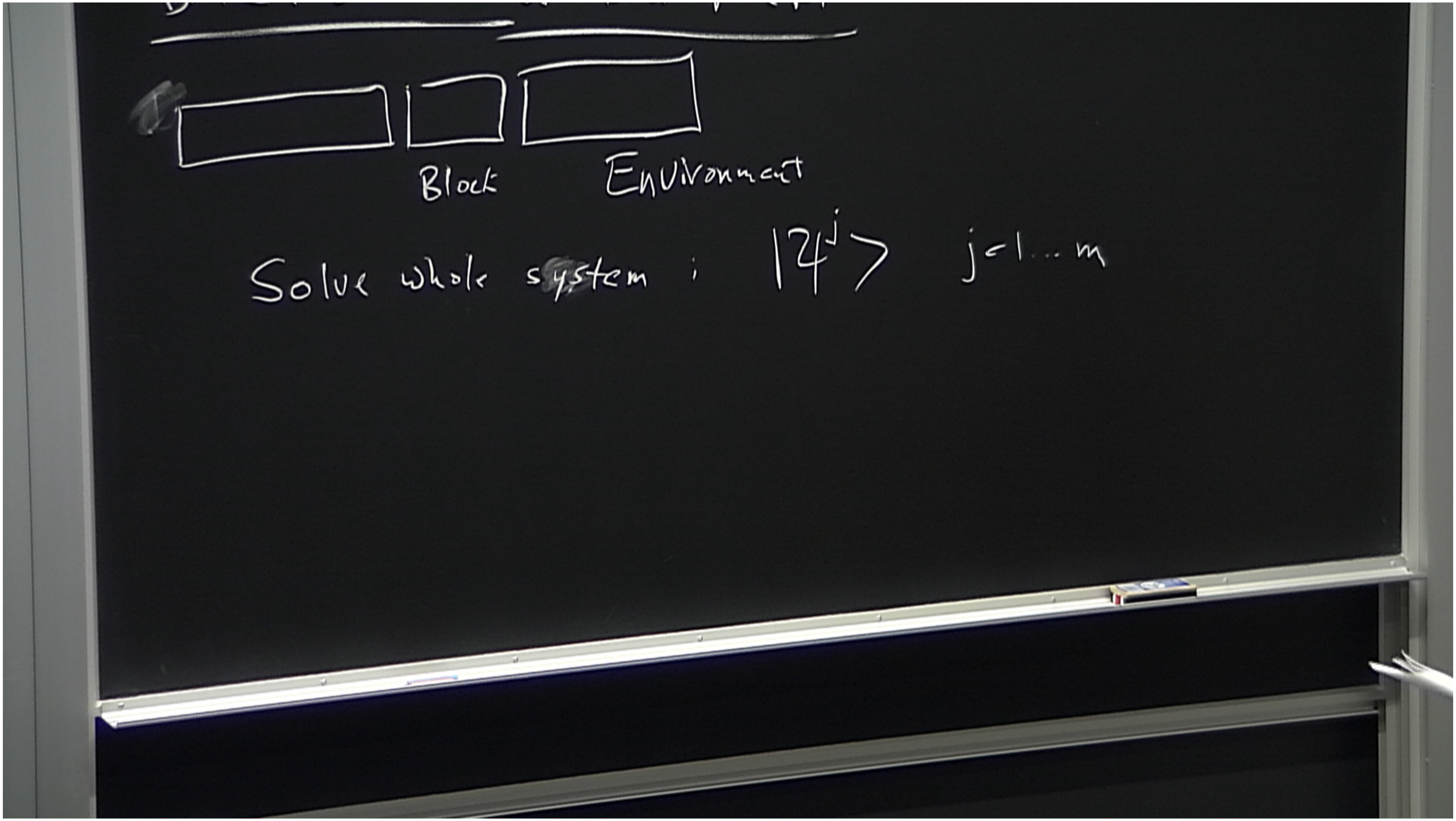

Real Space RG
RG ↑
1. Diag. H block (0th order
2 -1
-1 2
2. Keep lowest m states

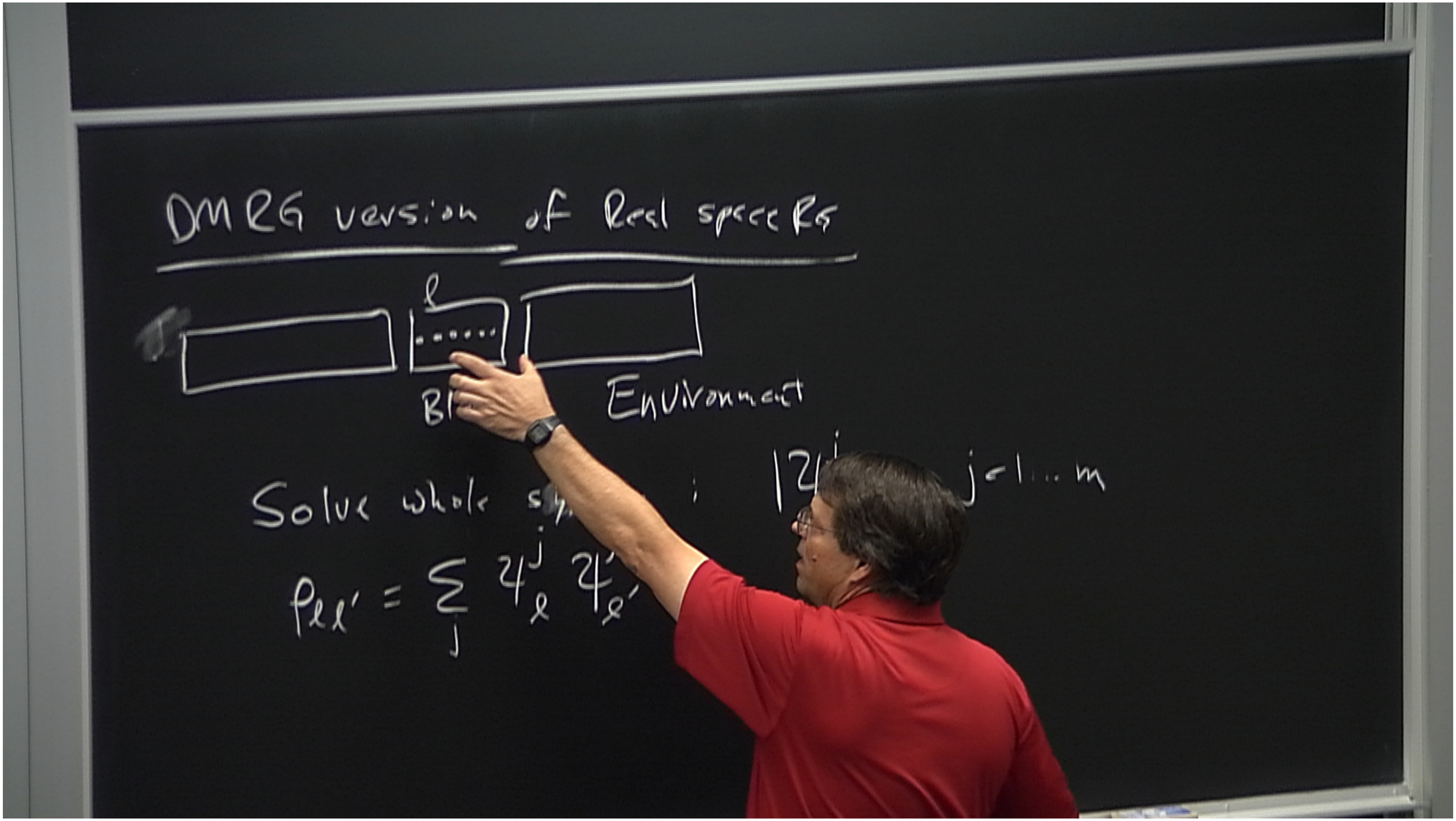

Real Space RG
1. Diag. H block (0th order
(2 -1; -1 2)
2. Keep lowest m states
3. Move to bigger length scale
RG ↑

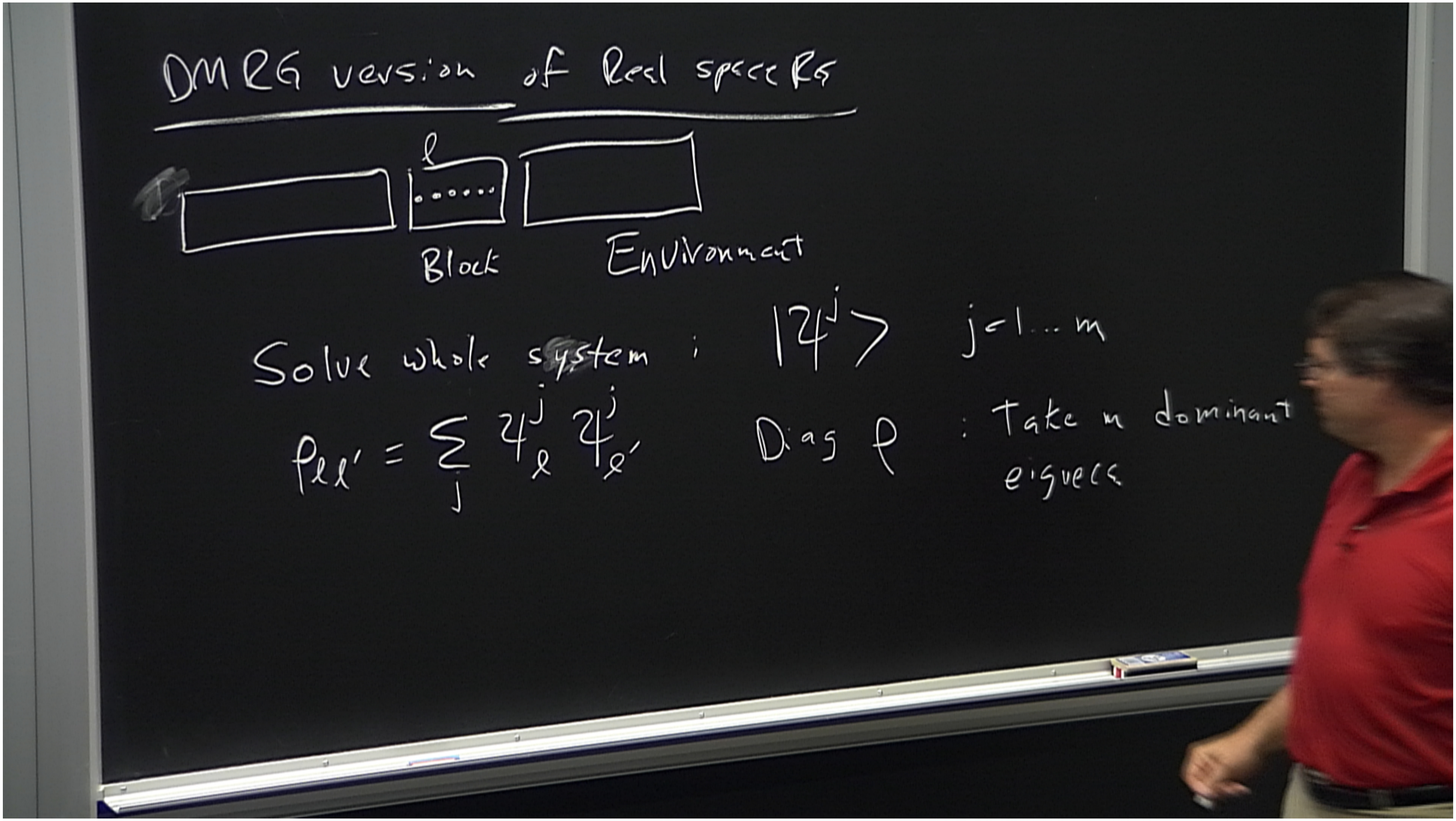

Real Space RG
1. Diag. H block (0th order
2 -1
-1 2
2. Keep lowest m states
3. Move to bigger length scale
RG
How well does it work? Horribly!
CAUTION

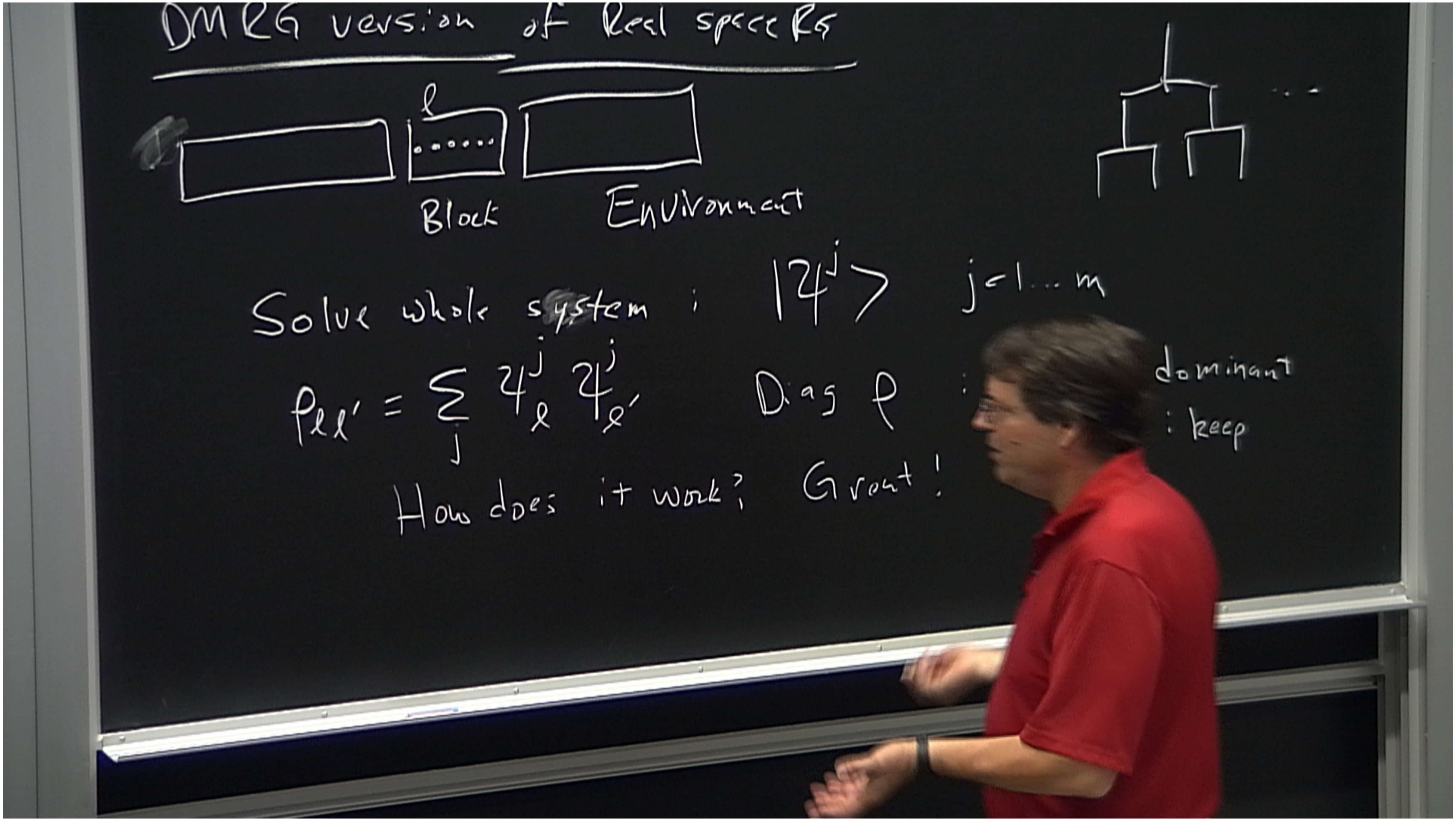

Real Space RG
1. Diag. H block
0th order
2 -1
-1 2
2. Keep lowest m states
3. Move to bigger length scale
RG
How well does it work?
Horribly!
m
m
m

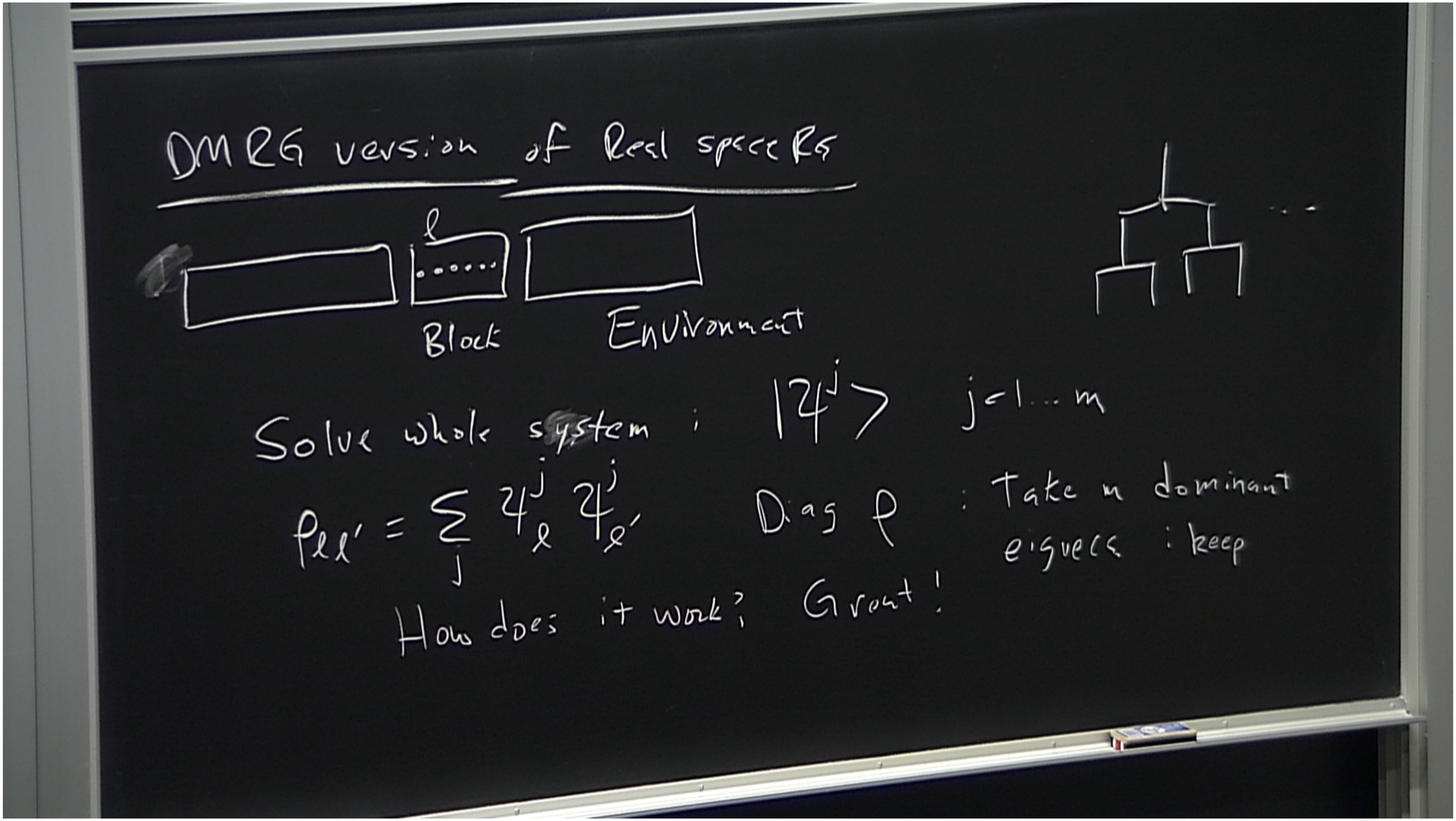

Real Space RG
1. Diag. H block (0th order
2 -1
-1 2
2. Keep lowest m states
3. Move to bigger length scale
RG
How well does it work? Horribly!
m
m m

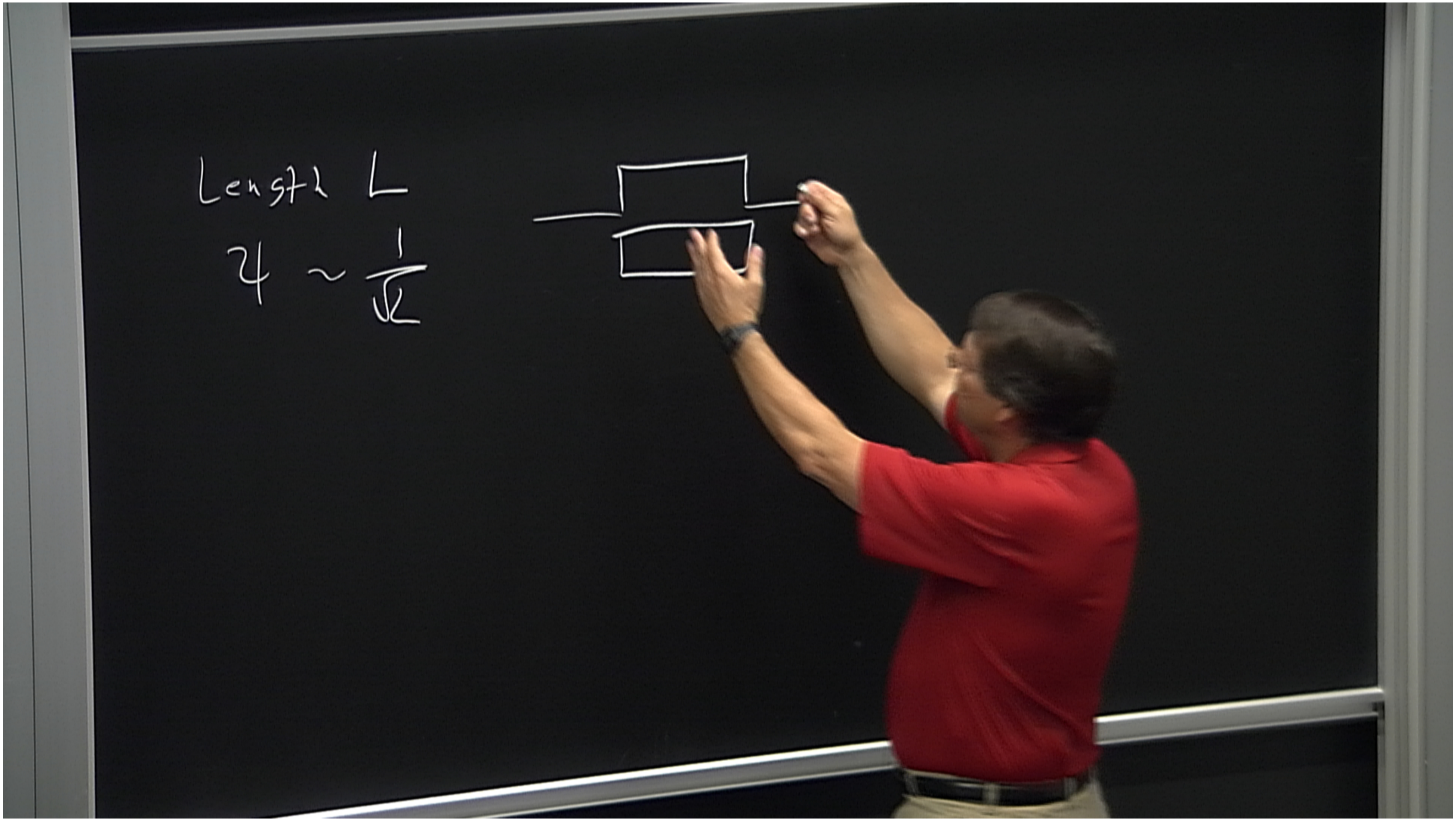

Block
Environment
Solve whole system : |ψ^j⟩ j=1...m

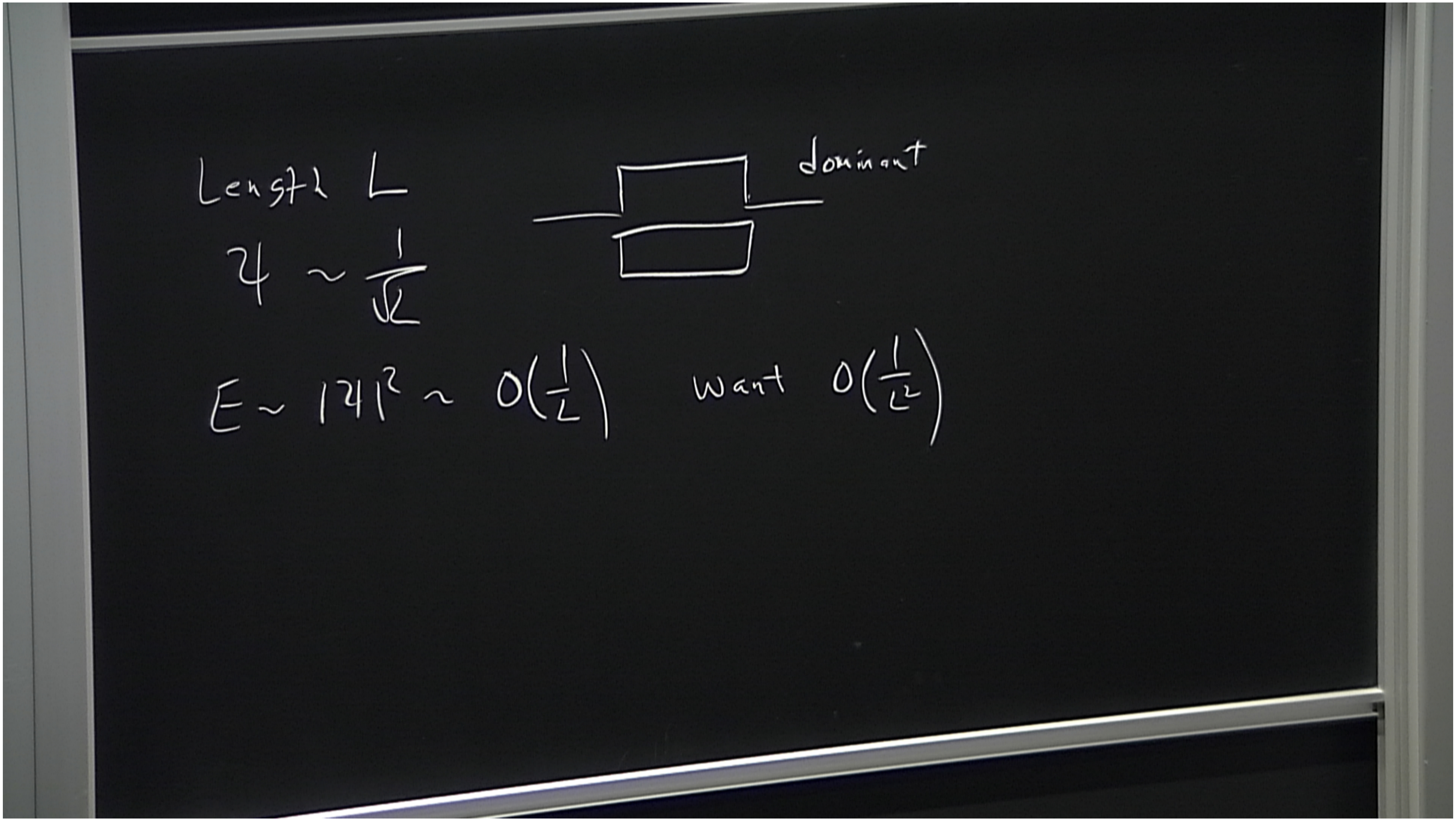

DMRG version of Real space RG
Environment
Solve whole
j=1...m

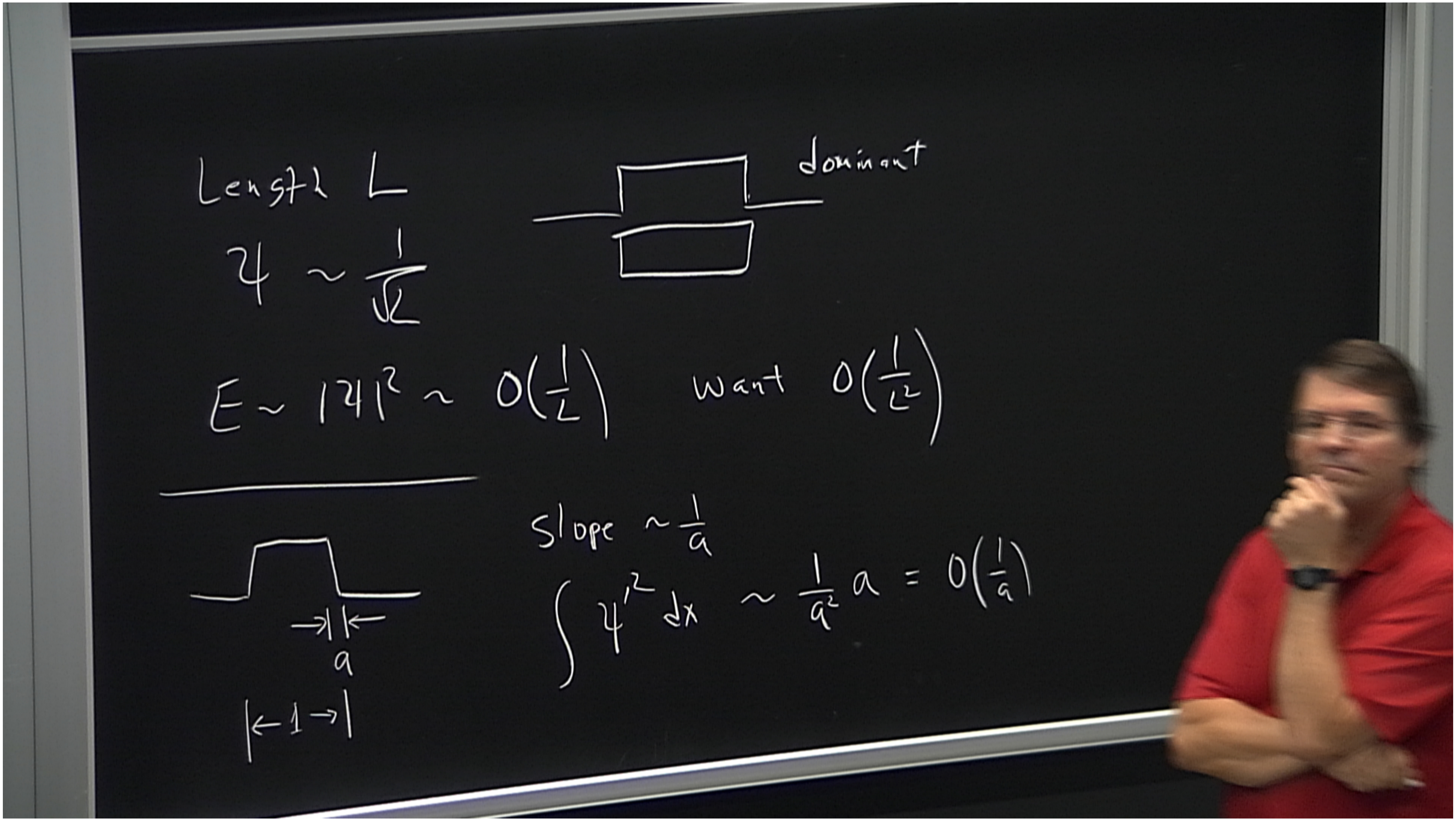

DMRG version of Real space RG
ℓ
Block
Environment
Solve whole system : |ψ^j⟩ j=1...m
ρ_ℓℓ' = Σ_j ψ^j_ℓ ψ^j_ℓ'
Diag ρ : Take m dominant
eigvecs

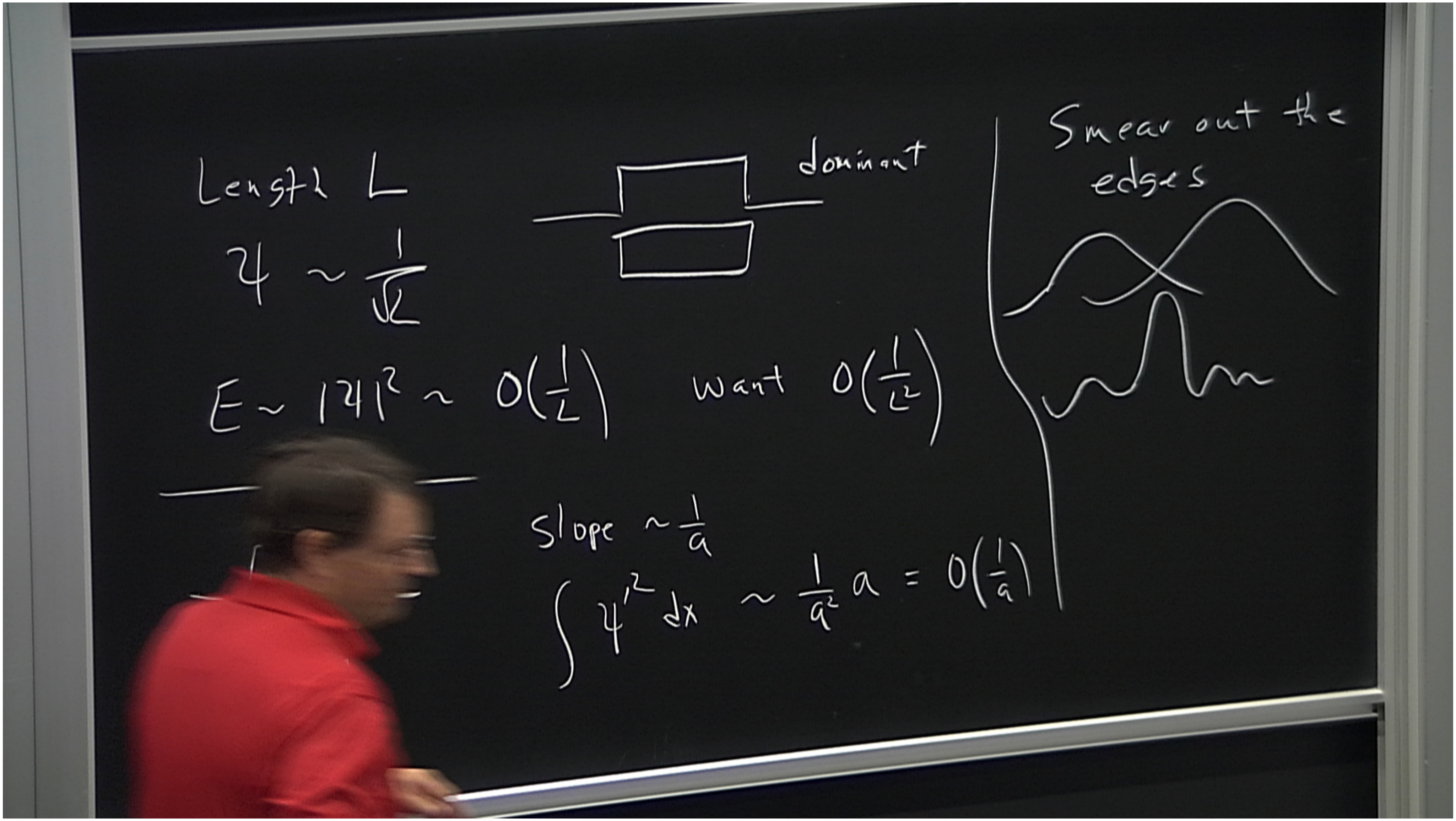

DMRG version of Real space RG
Block
Environment
Solve whole system
Diag
dominant
: keep
How does it work? Great!

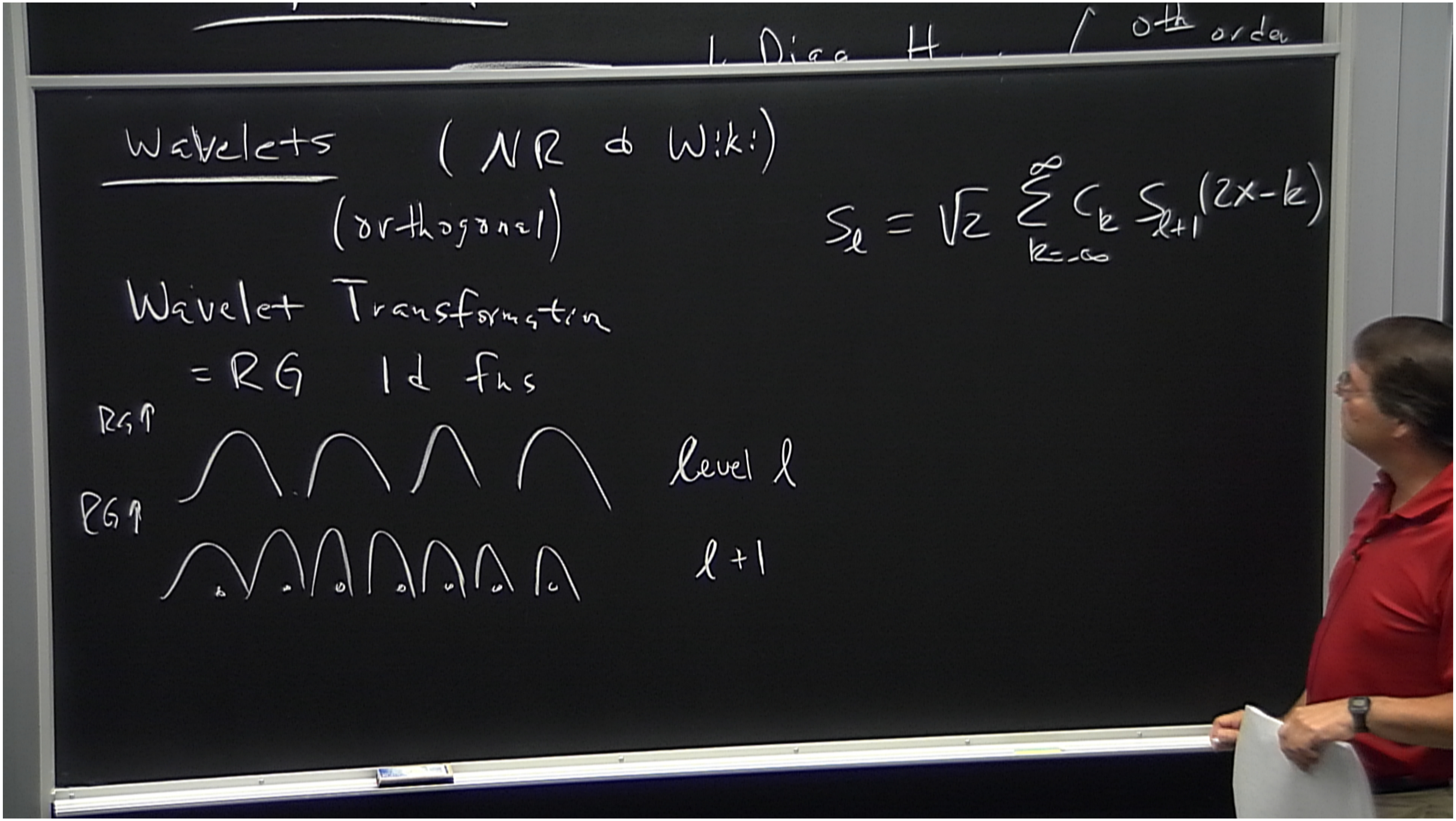

DMRG version of Real space RG
Block
Environment
Solve whole system : $|\psi^j\rangle$ j=1…m
$\rho_{\ell\ell'} = \sum_j \psi^j_\ell \psi^j_{\ell'}$
Diag ρ : Take m dominant eigvecs : keep
How does it work? Great!

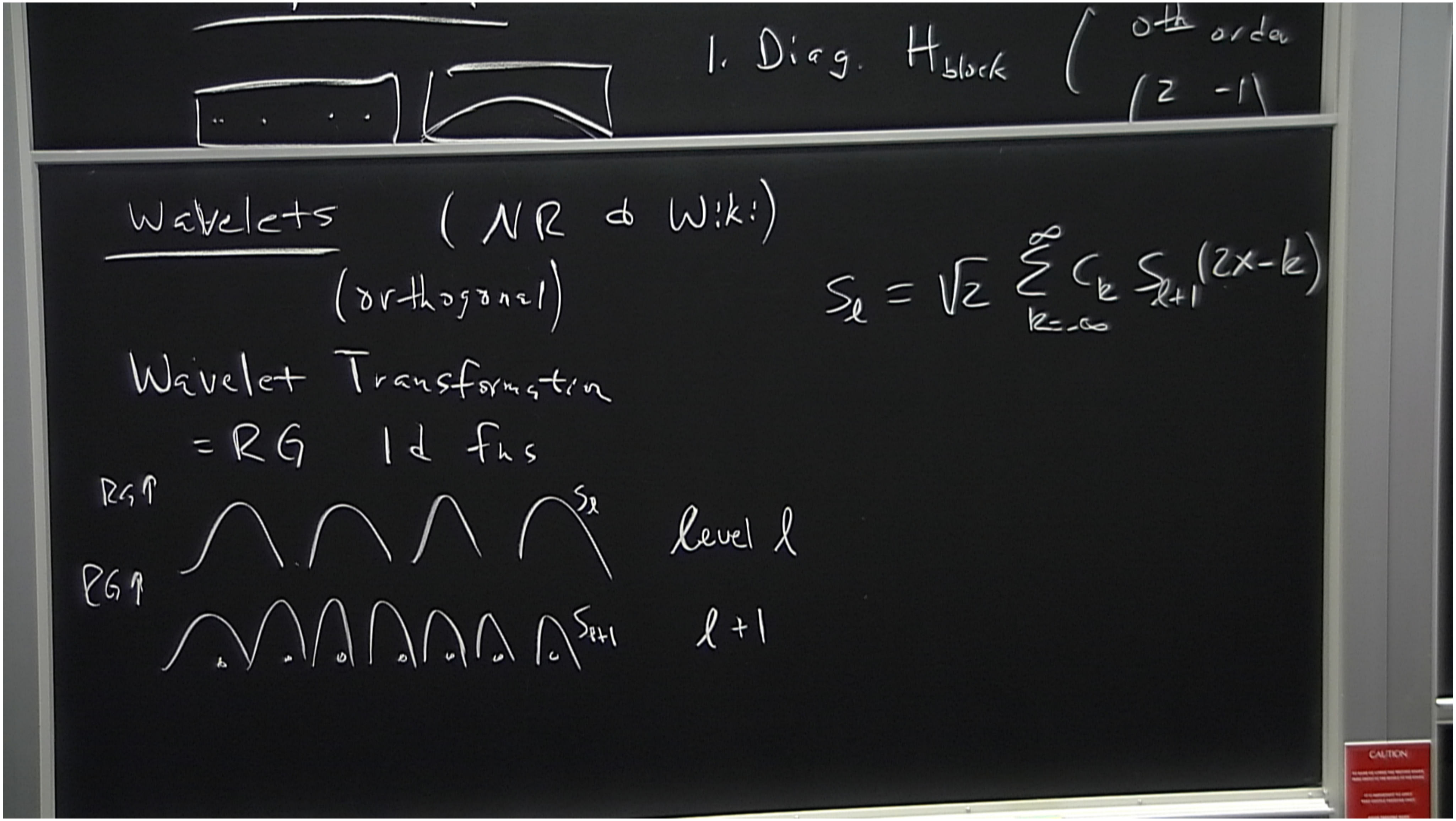

Length L
ψ ~ 1/√L

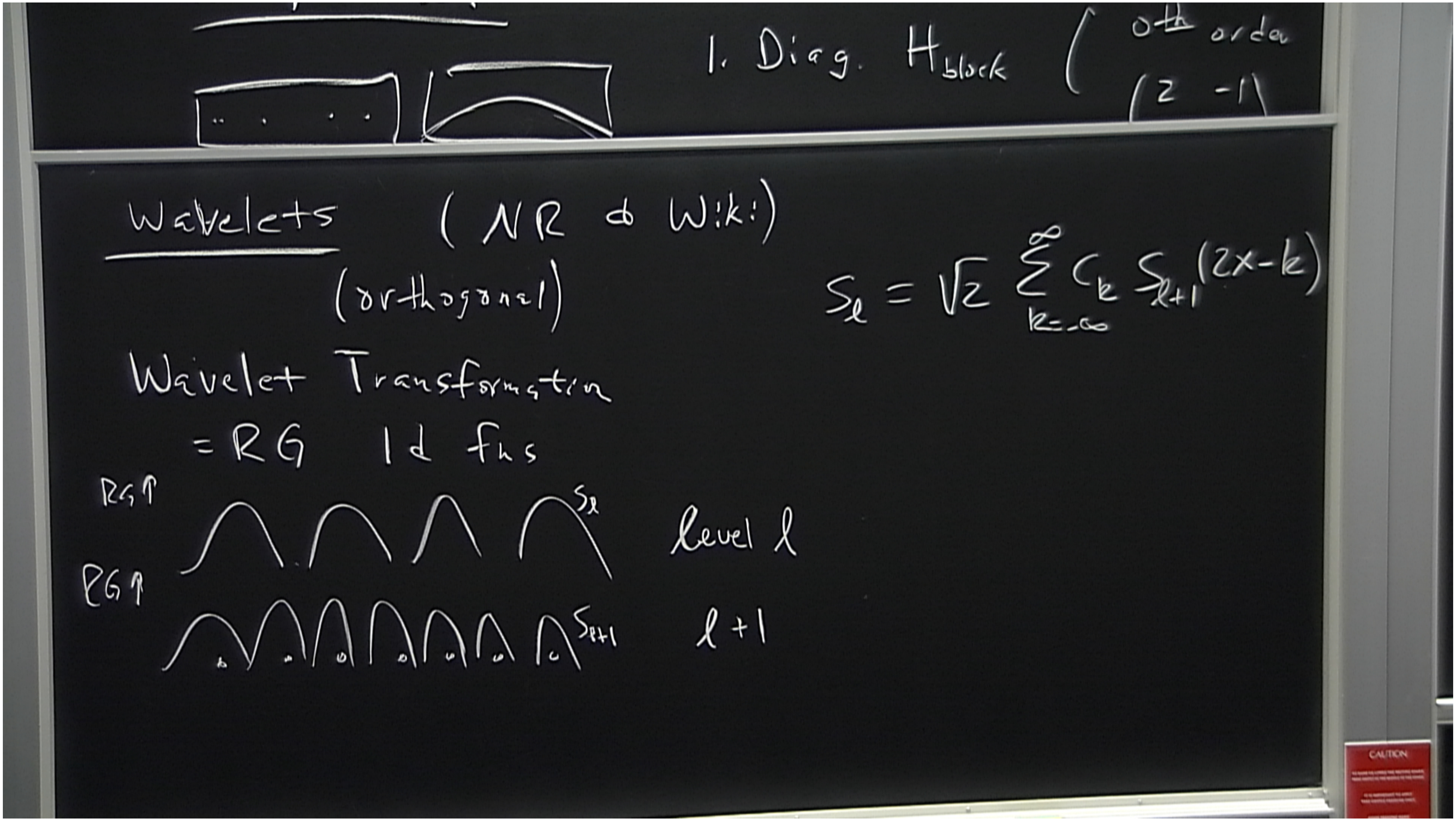

Length L
ψ ~ 1/√L
E ~ |ψ|² ~ O(1/L)
want O(1/L²)
dominant

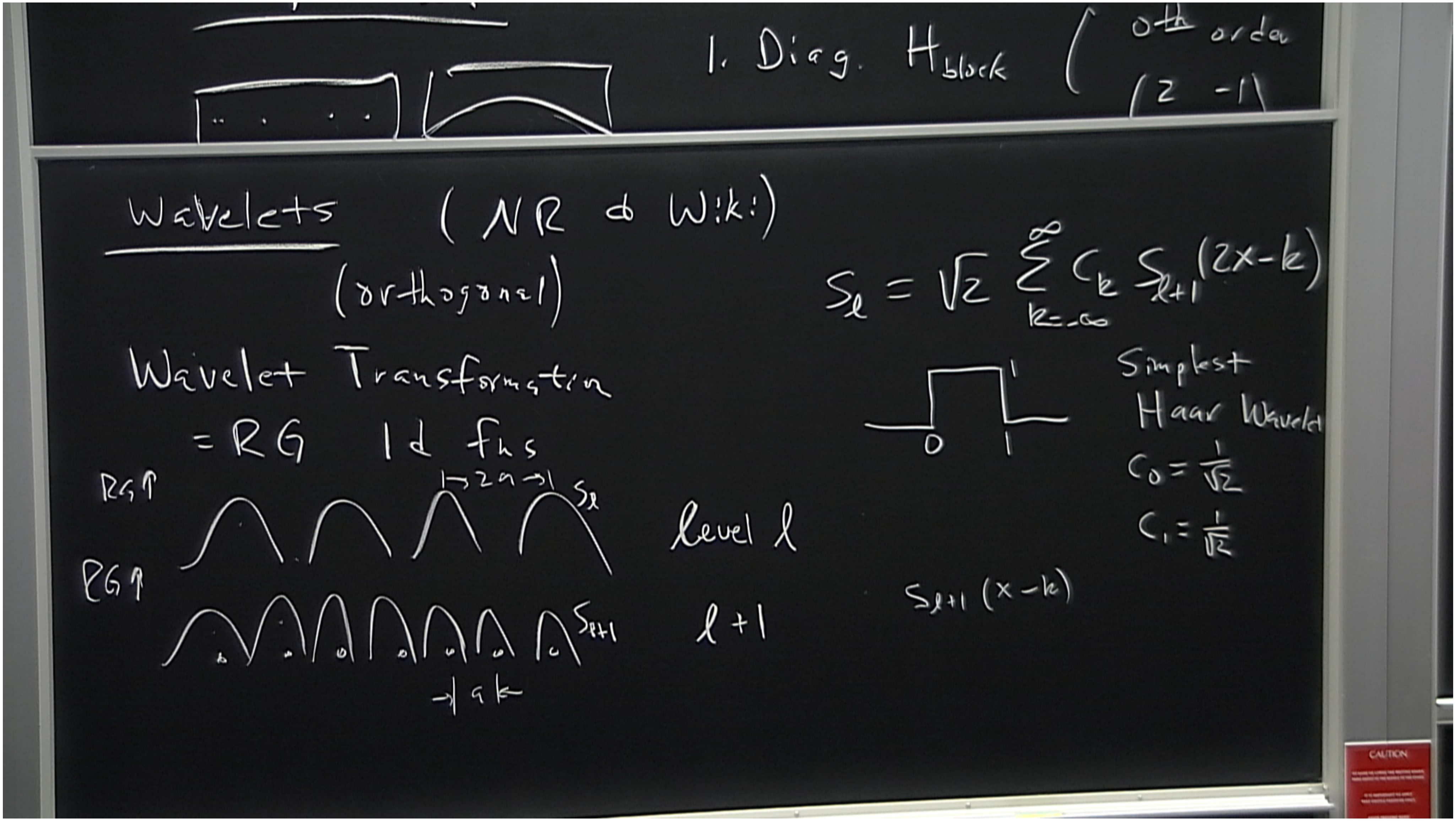

Length L
$\psi \sim \frac{1}{\sqrt{L}}$
dominant
$E \sim |\psi'|^2 \sim O\left(\frac{1}{L}\right)$ want $O\left(\frac{1}{L^2}\right)$
slope $\sim \frac{1}{a}$
$\int \psi'^2 dx \sim \frac{1}{a^2} a = O\left(\frac{1}{a}\right)$
a
1

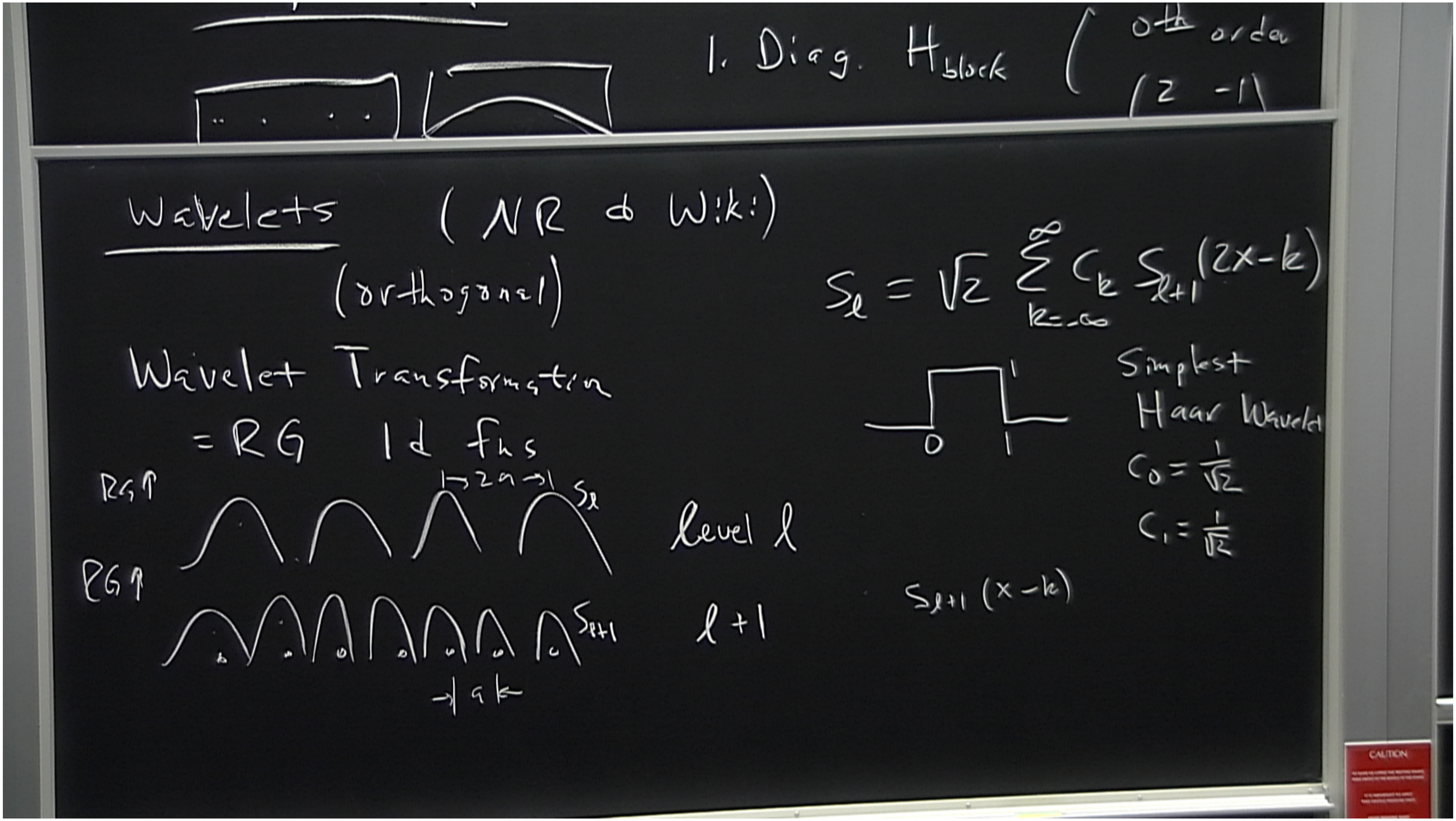

Length L
ψ ~ 1/√L
dominant
E ~ |ψ|² ~ O(1/L)
want O(1/L²)
slope ~ 1/a
∫ψ'² dx ~ 1/a² a = O(1/a)
Smear out the edges

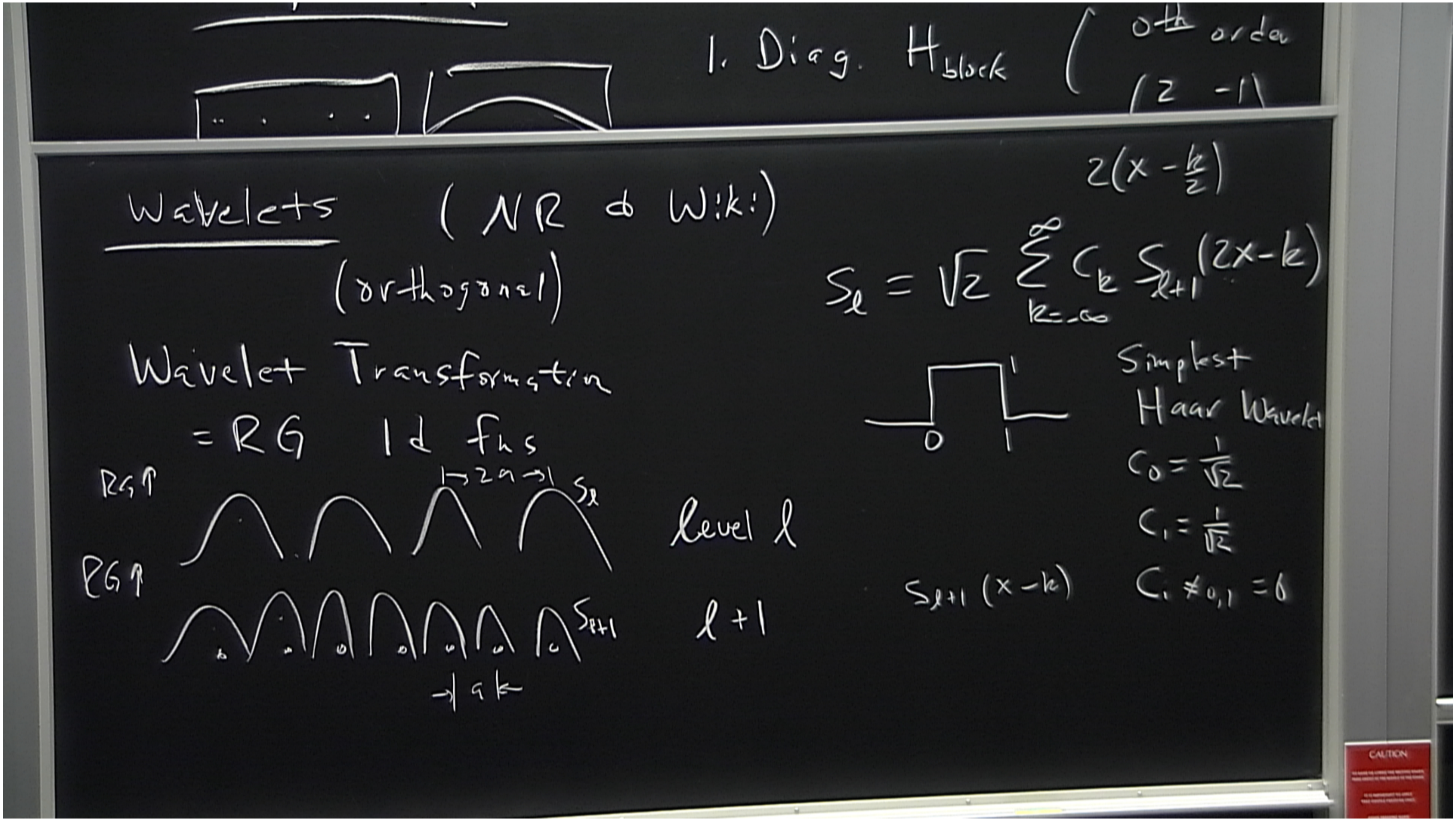

Wavelets (NR & Wiki)
(orthogonal)
Wavelet Transformation
= RG 1d fns
RG↑
RG↑
level l
l+1
S_l = √2 Σ_{k=-∞}^{∞} c_k S_{l+1}(2x-k)

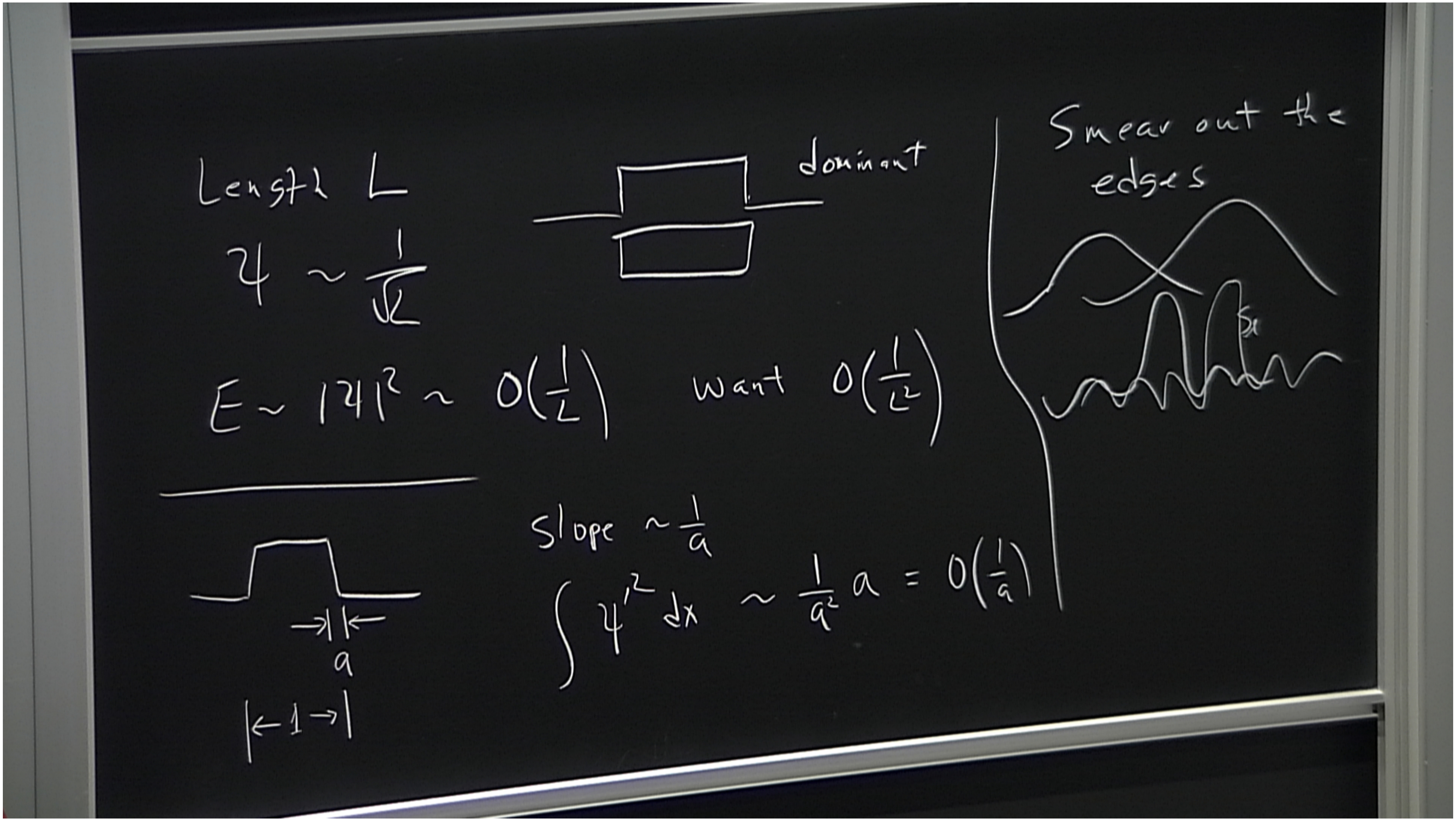

1. Diag. H block (0th order
(2 -1)
Wavelets (NR & Wiki)
(orthogonal)
Wavelet Transformation
= RG 1d fns
RG ↑
RG ↑
S_l
level l
S_l+1
l+1
S_l = √2 Σ_{k=-∞}^{∞} c_k S_l+1(2x-k)
CAUTION

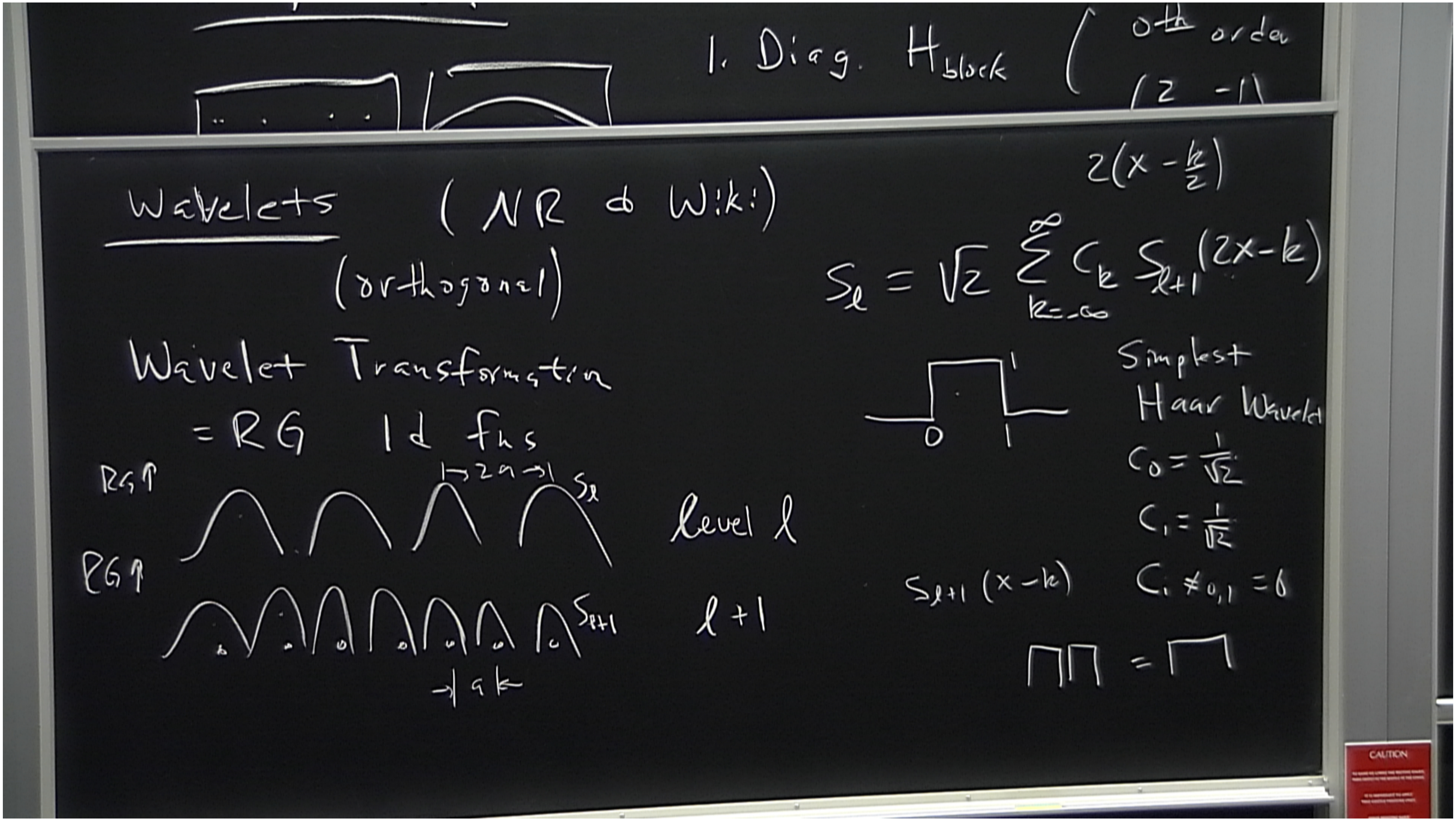

1. Diag. H block
0th order
(2 -1)
Wavelets (NR & Wiki)
(orthogonal)
S_l = √2 Σ_{k=-∞}^{∞} c_k S_{l+1}(2x-k)
Wavelet Transformation
= RG 1d fns
RG ↑
RG ↑
S_l
level l
S_{l+1}
l+1

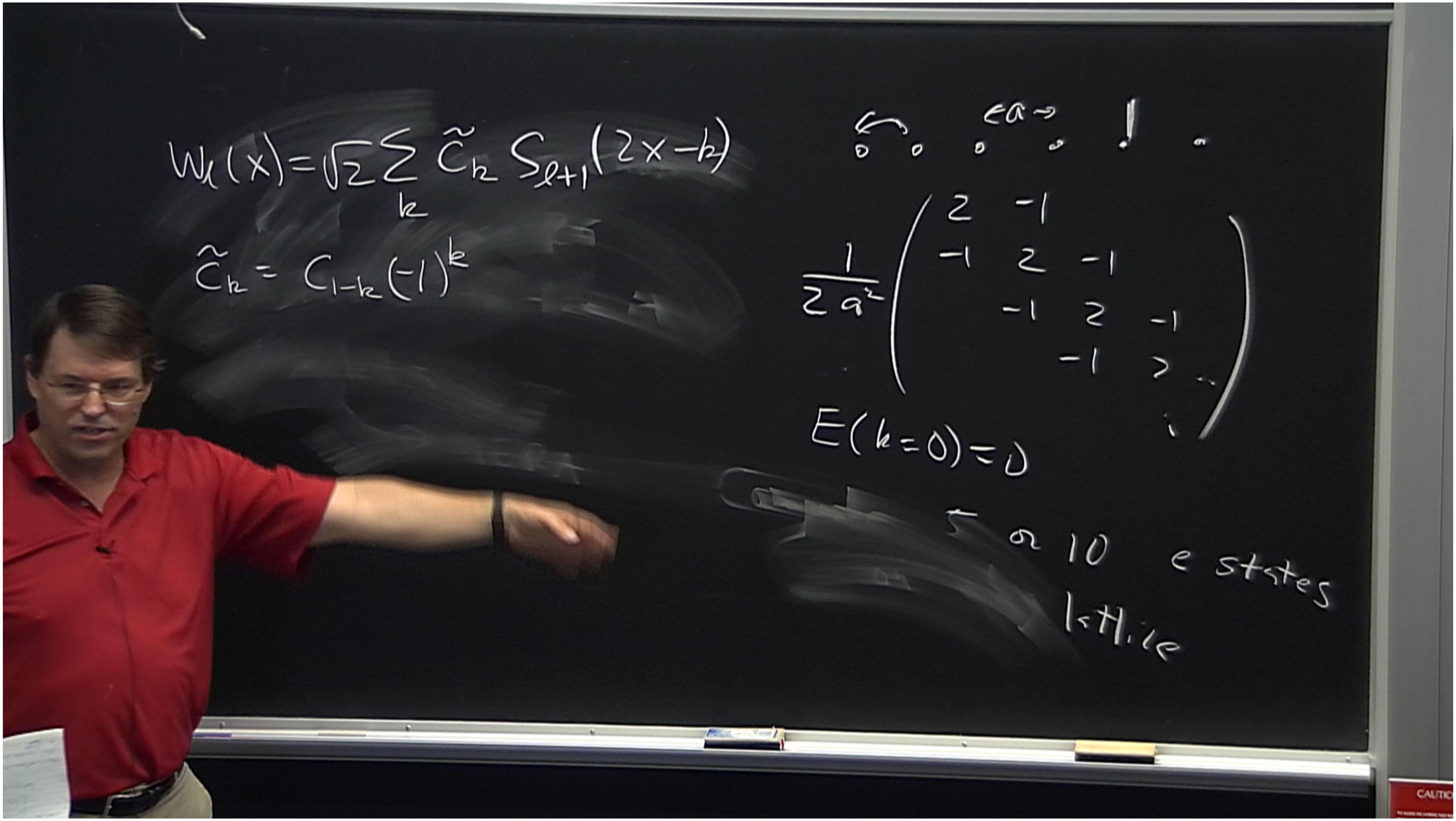

1. Diag. H block
0th order
Wavelets (NR & Wiki)
(orthogonal)
Wavelet Transformation
= RG 1d fns
RG ↑
RG ↑
level l
l + 1
$S_l = \sqrt{2} \sum_{k=-\infty}^{\infty} c_k S_{l+1}(2x-k)$
Simplest
Haar Wavelet
$c_0 = \frac{1}{\sqrt{2}}$
$c_1 = \frac{1}{\sqrt{2}}$
$S_{l+1}(x-k)$

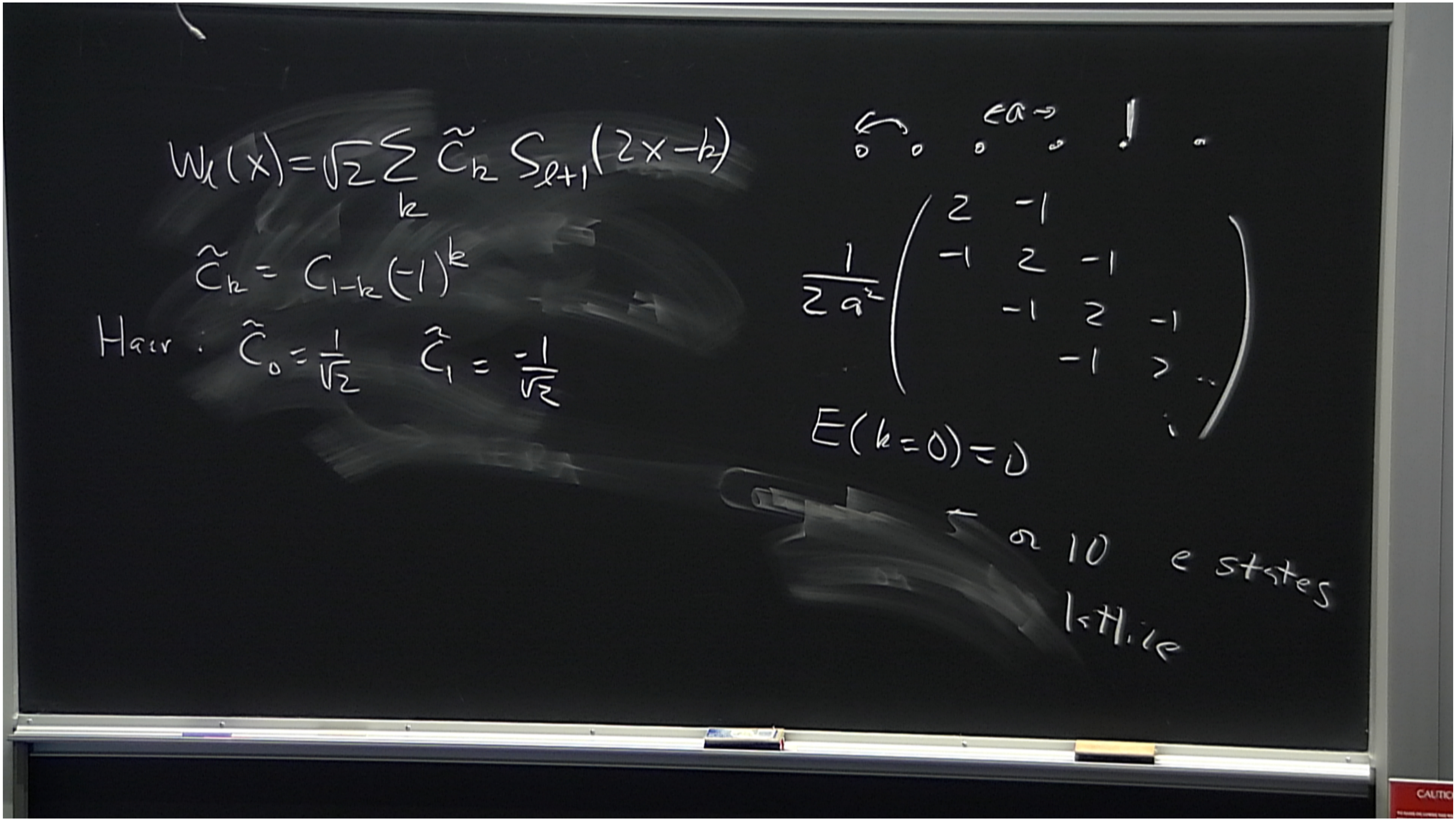

1. Diag. H block
0th order
Wavelets (NR & Wiki)
(orthogonal)
Wavelet Transformation
= RG 1d fns
RG ↑
RG ↑
level l
l + 1
$S_l = \sqrt{2} \sum_{k=-\infty}^{\infty} c_k S_{l+1}(2x-k)$
Simplest
Haar Wavelet
$c_0 = \frac{1}{\sqrt{2}}$
$c_1 = \frac{1}{\sqrt{2}}$
$S_{l+1}(x-k)$

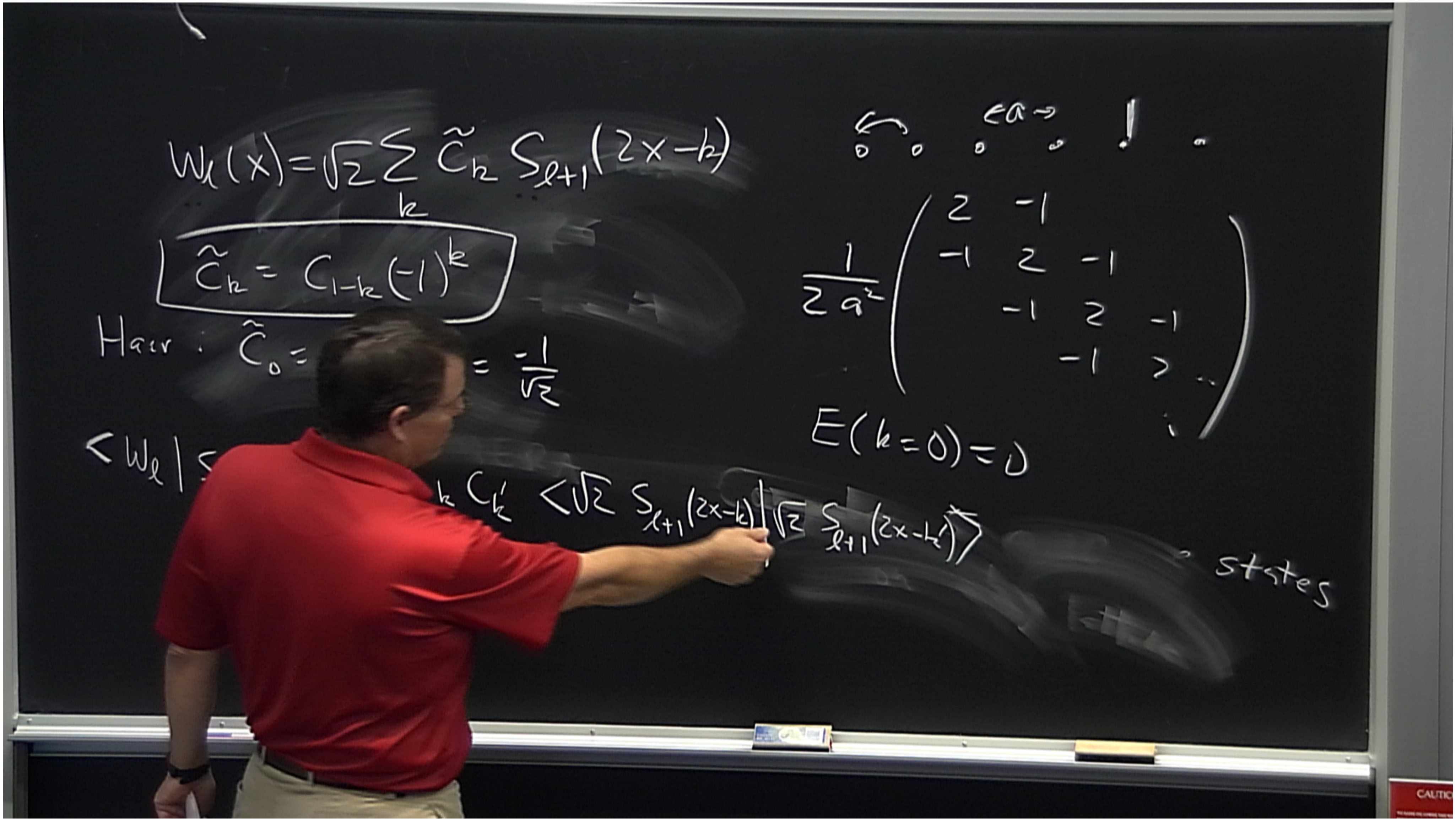

1. Diag. H block (0th order
(2 -1)
2(x - k/2)
Wavelets (NR & Wiki)
(orthogonal)
$S_\ell = \sqrt{2} \sum_{k=-\infty}^{\infty} C_k S_{\ell+1}(2x-k)$
Wavelet Transformation
= RG 1d fns
RG ↑
RG ↑
2a
S_ℓ
level ℓ
$S_{\ell+1}$
$\ell + 1$
a
Simplest
Haar Wavelet
$C_0 = \frac{1}{\sqrt{2}}$
$C_1 = \frac{1}{\sqrt{2}}$
$C_{i \neq 0,1} = 0$
$S_{\ell+1}(x-k)$
CAUTION

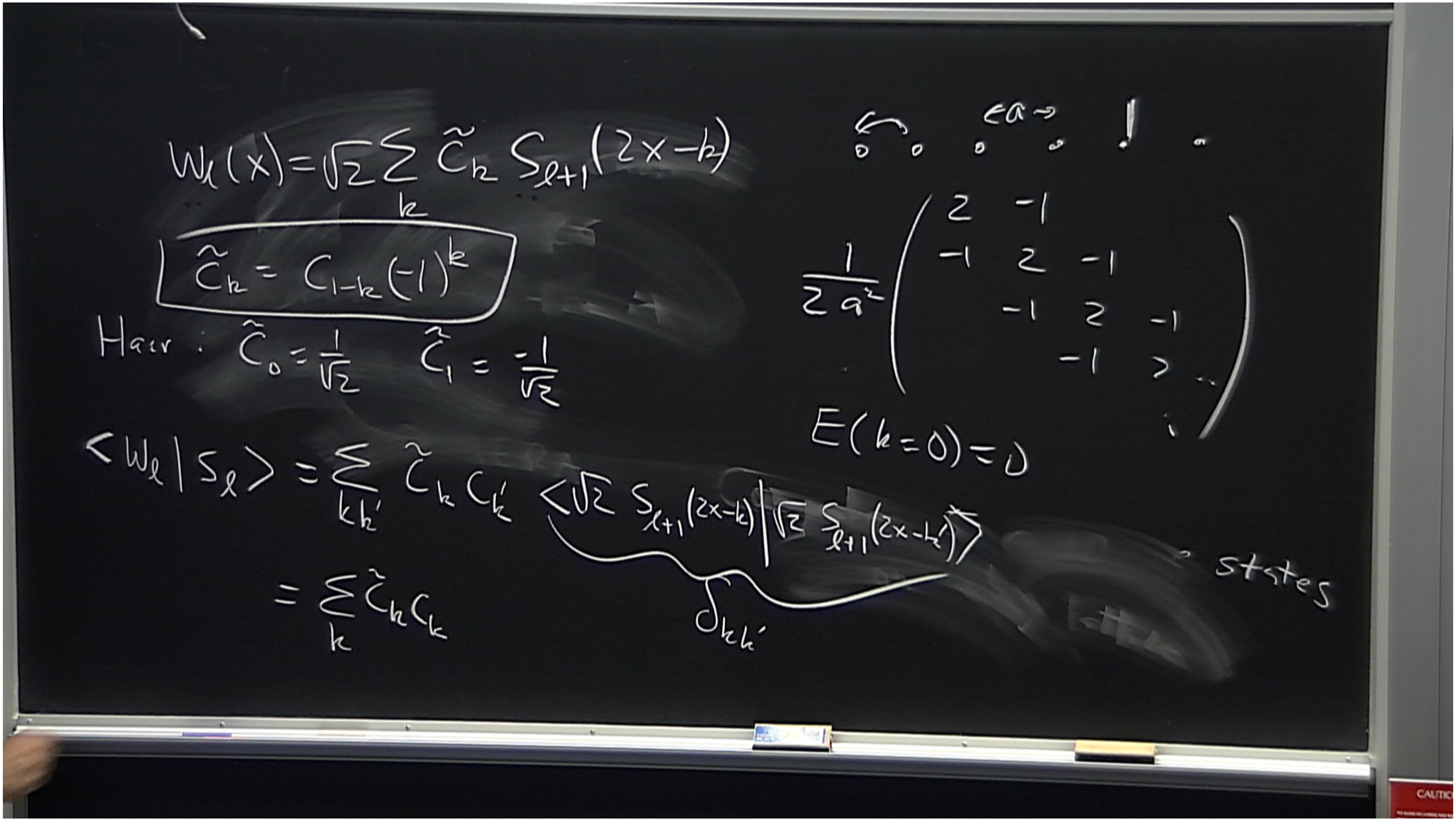

Length L
$\psi \sim \frac{1}{\sqrt{L}}$
$E \sim |\psi'|^2 \sim O(\frac{1}{L})$
want $O(\frac{1}{L^2})$
dominant
Smear out the edges
slope $\sim \frac{1}{a}$
$\int \psi'^2 dx \sim \frac{1}{a^2} a = O(\frac{1}{a})$
a
1

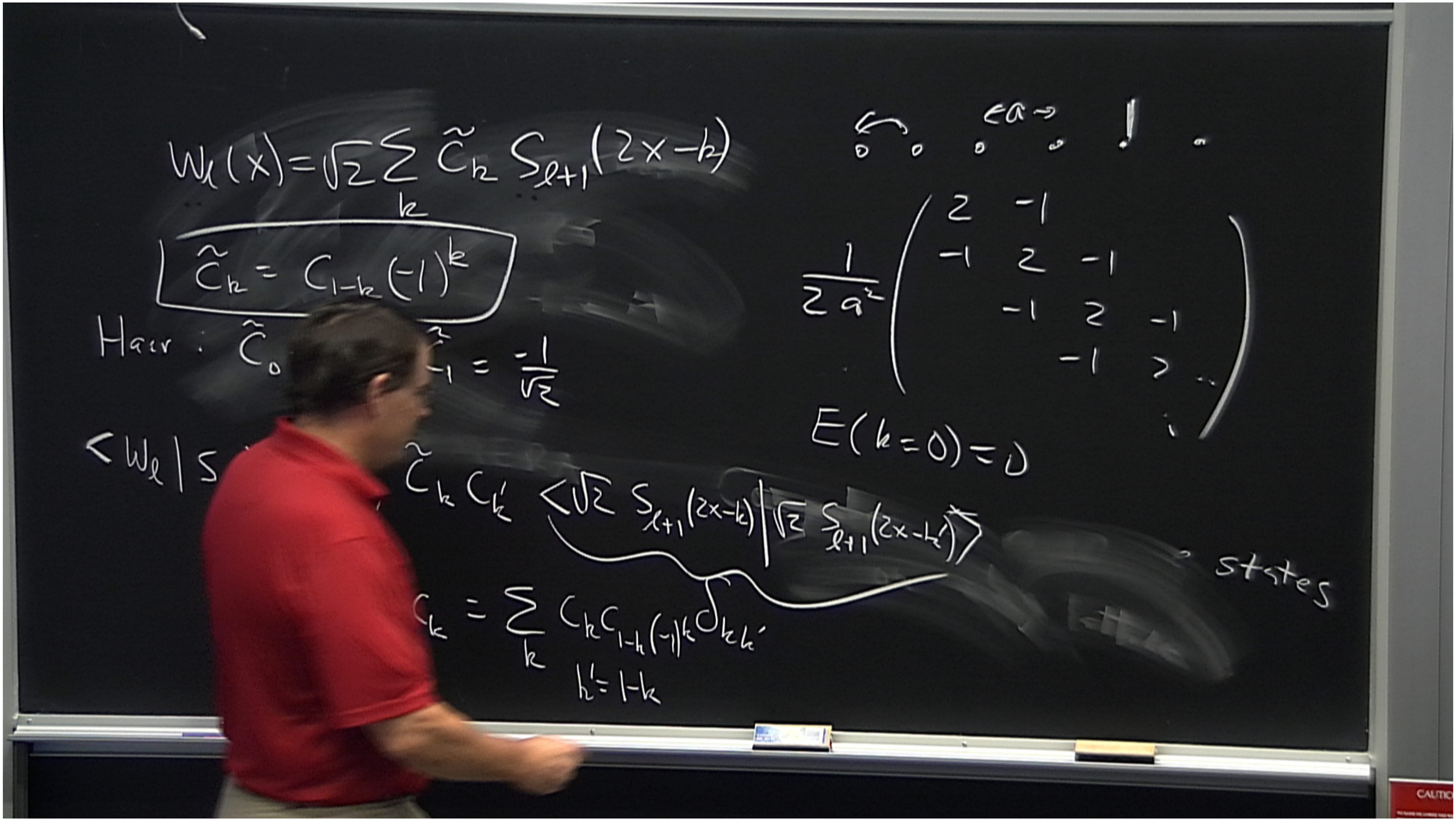

1. Diag. H block
0th order
Wavelets (NR & Wiki)
(orthogonal)
Wavelet Transformation
= RG 1d fns
RG ↑
RG ↑
2a
S_ℓ
level ℓ
$S_{\ell+1}$
$\ell + 1$
a
$2(x - \frac{k}{2})$
$S_\ell = \sqrt{2} \sum_{k=-\infty}^{\infty} C_k S_{\ell+1}(2x-k)$
Simplest
Haar Wavelet
0
1
1
$C_0 = \frac{1}{\sqrt{2}}$
$C_1 = \frac{1}{\sqrt{2}}$
$C_{i \neq 0,1} = 0$
$S_{\ell+1}(x-k)$
CAUTION

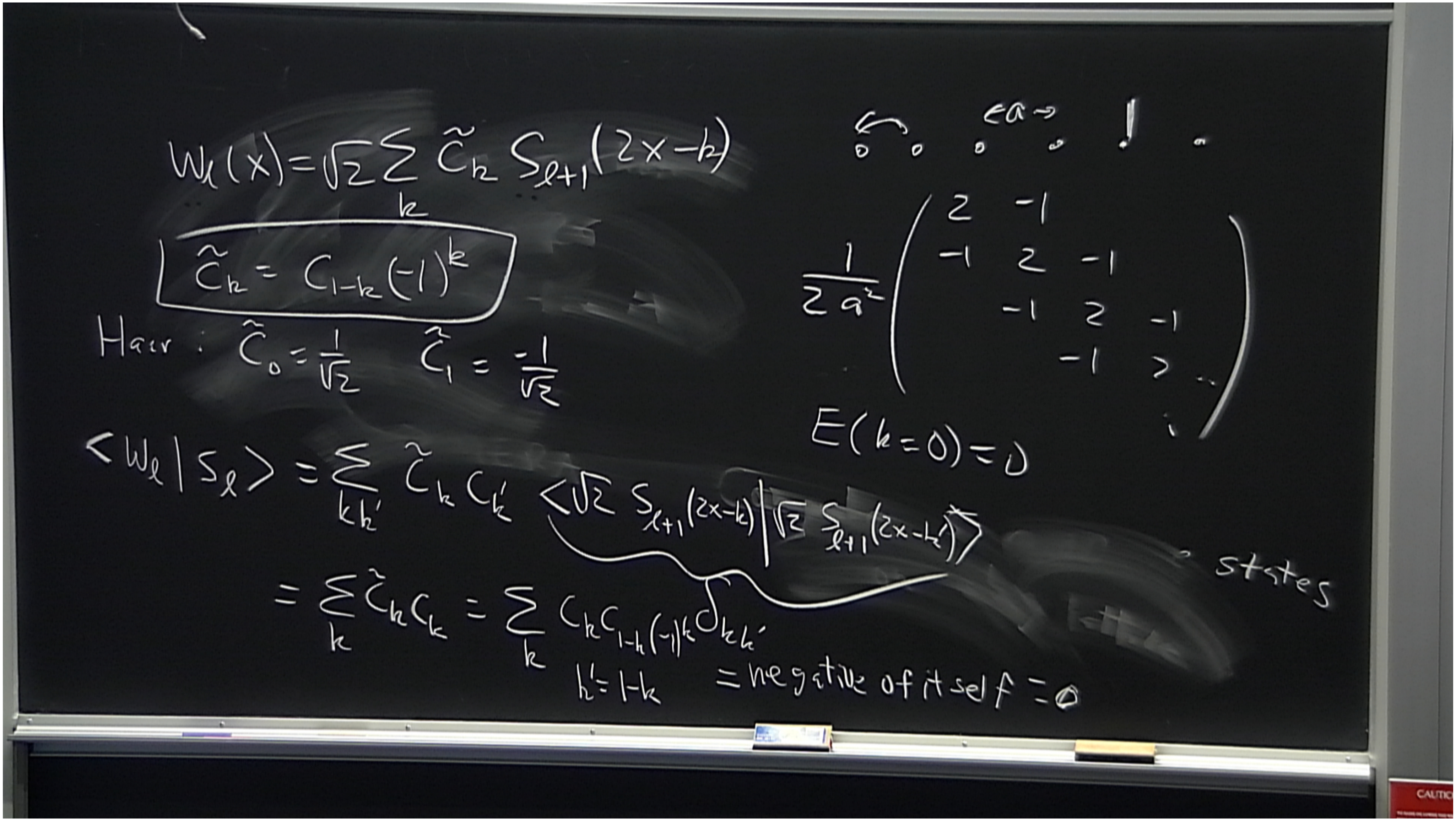

W_l(x) = √2 Σ_k C̃_k S_{l+1}(2x − k)
C̃_k = C_{1−k}(−1)^k
ea
1/(2a²) (2 −1 / −1 2 −1 / −1 2 −1 / −1 2 ...)
E(k=0)=0
or 10 e states
lattice

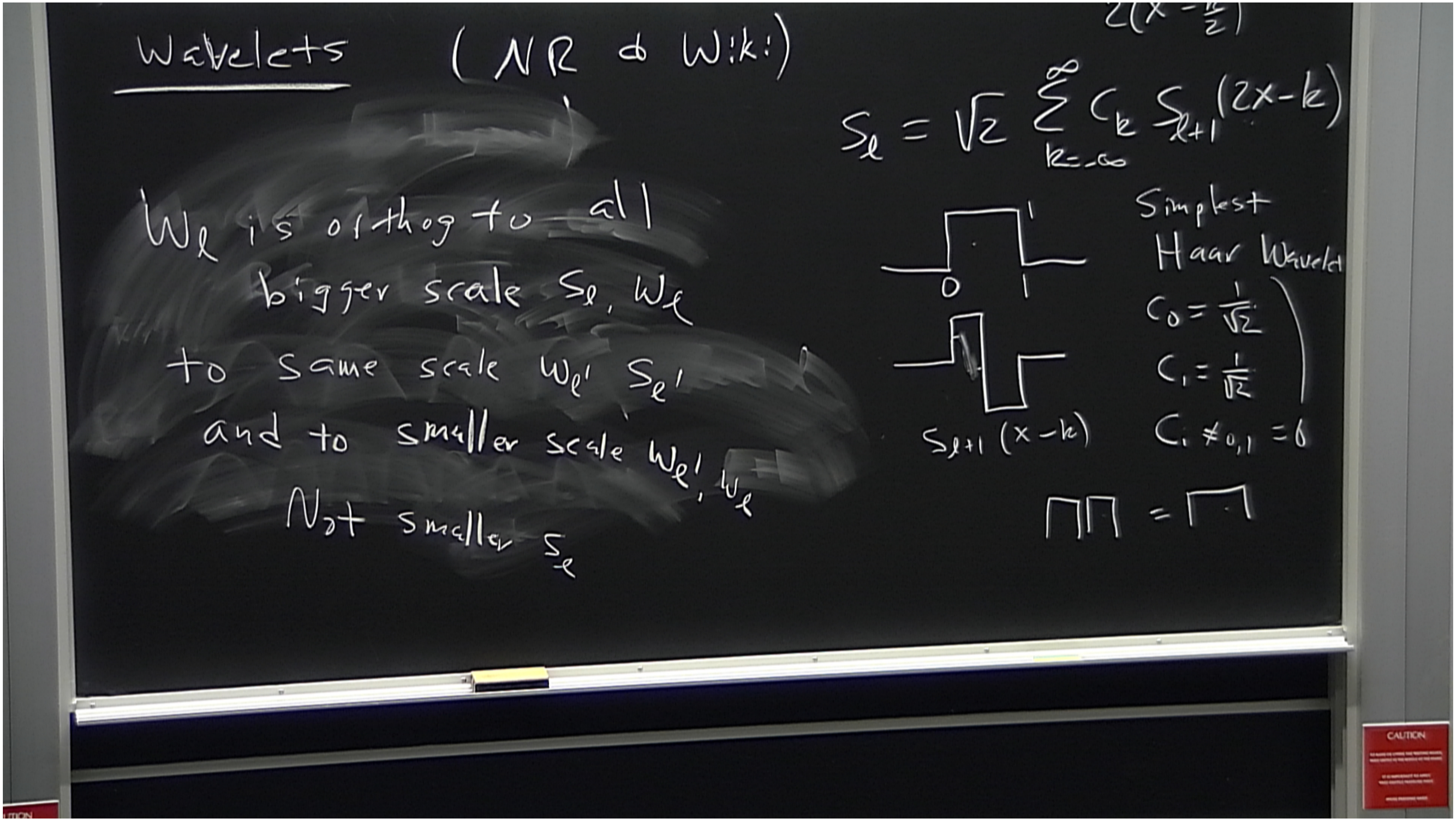

$W_l(x) = \sqrt{2} \sum_k \tilde{C}_k S_{l+1}(2x - k)$
$\tilde{C}_k = C_{1-k}(-1)^k$
Haar: $\tilde{C}_0 = \frac{1}{\sqrt{2}}$ $\tilde{C}_1 = \frac{-1}{\sqrt{2}}$
$\frac{1}{2a^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & \cdots \\ & & & \vdots & \end{pmatrix}$
$E(k=0)=0$
~ 10 e states
lattice


W_l(x) = √2 Σ_k C̃_k S_{l+1}(2x−k)
C̃_k = C_{1−k}(−1)^k
Haar
E(k=0)=0
states


$W_l(x) = \sqrt{2} \sum_k \tilde{c}_k S_{l+1}(2x-k)$
$\tilde{c}_k = c_{1-k}(-1)^k$
Haar: $\tilde{c}_0 = \frac{1}{\sqrt{2}}$ $\tilde{c}_1 = \frac{-1}{\sqrt{2}}$
$\langle W_l | S_l \rangle = \sum_{kk'} \tilde{c}_k c_{k'} \langle \sqrt{2} S_{l+1}(2x-k) | \sqrt{2} S_{l+1}(2x-k') \rangle$
$= \sum_k \tilde{c}_k c_k$
$\delta_{kk'}$
$\frac{1}{2a^2} \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$
$E(k=0)=0$
states


$W_l(x) = \sqrt{2}\sum_k \tilde{C}_k S_{l+1}(2x-k)$
$\tilde{C}_k = C_{1-k}(-1)^k$
Haar :
$E(k=0)=0$
states
$k' = 1-k$


W_l(x) = √2 Σ_k c̃_k S_{l+1}(2x−k)
c̃_k = c_{1−k}(−1)^k
Haar: c̃_0 = 1/√2   c̃_1 = −1/√2
<W_l | S_l> = Σ_{kk'} c̃_k c_{k'} <√2 S_{l+1}(2x−k) | √2 S_{l+1}(2x−k')>
= Σ_k c̃_k c_k = Σ_k c_k c_{1−k} (−1)^k δ_{kk'}
k' = 1−k   = negative of itself = 0
1/(2a²)
2 −1
−1 2 −1
−1 2 −1
−1 2
E(k=0) = 0
states


Wavelets (NR & Wiki)
W_ℓ is orthog to all
bigger scale S_ℓ, W_ℓ
to same scale $W_{\ell'}$ $S_{\ell'}$
and to smaller scale $W_{\ell'}$, W_ℓ
Not smaller S_ℓ
$S_\ell = \sqrt{2} \sum_{k=-\infty}^{\infty} C_k S_{\ell+1}(2x-k)$
0
1
$S_{\ell+1}(x-k)$
Simplest
Haar Wavelet
$C_0 = \frac{1}{\sqrt{2}}$
$C_1 = \frac{1}{\sqrt{2}}$
$C_{i \neq 0,1} = 0$

Wavelets (NR & Wiki)

$2(x - \frac{k}{2})$

$$S_\ell = \sqrt{2} \sum_{k=-\infty}^{\infty} c_k S_{\ell+1}(2x-k)$$

W_ℓ is orthog to all
bigger scale S_ℓ, W_ℓ

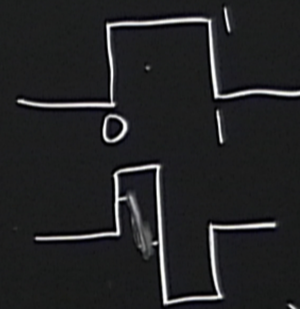
to same scale $W_{\ell'}\ S_{\ell'}$

and to smaller scale $W_{\ell'}, W_\ell$

Not smaller S_ℓ

Complete set

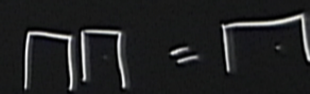
all W_ℓ, top level S_ℓ

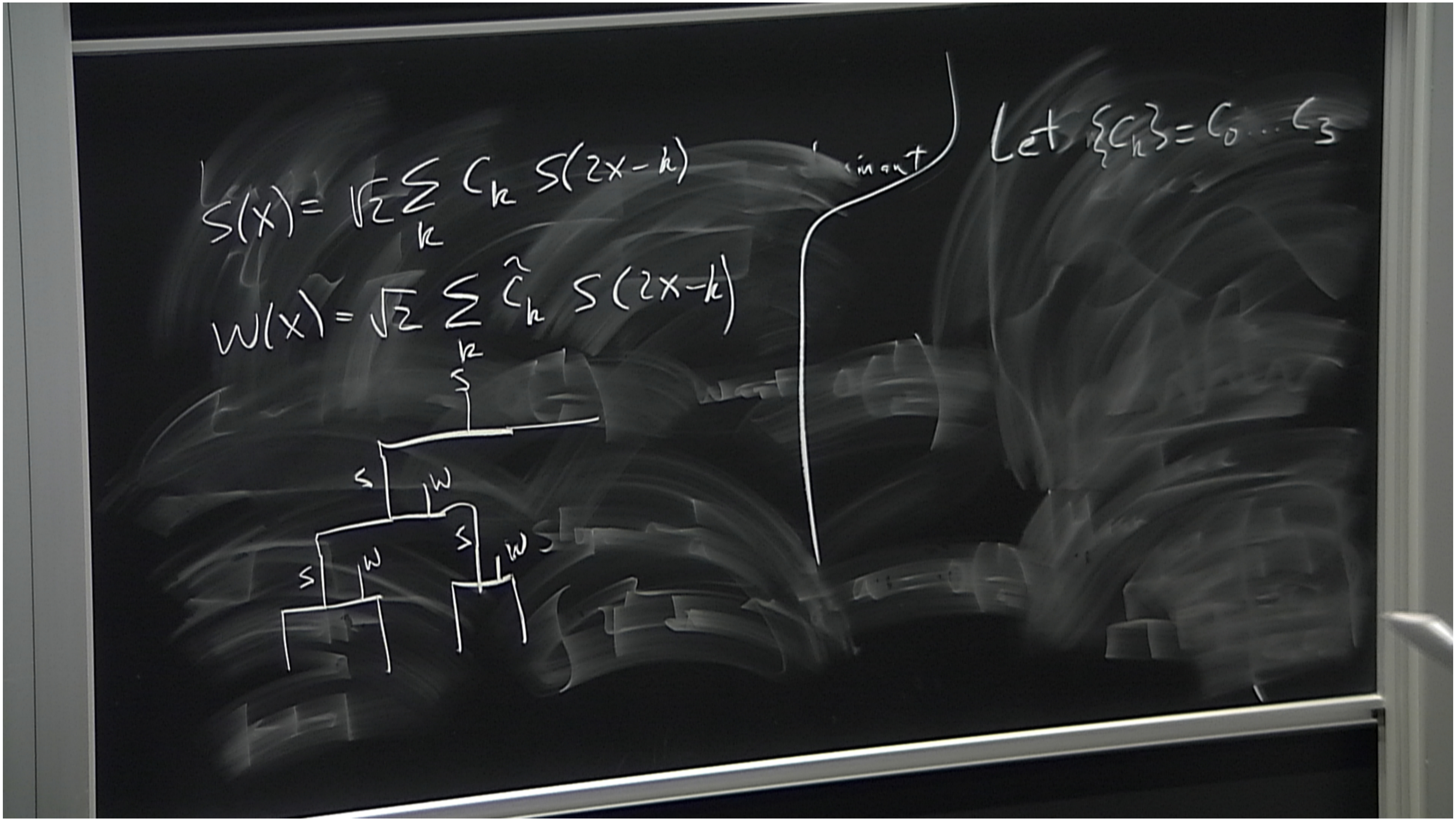


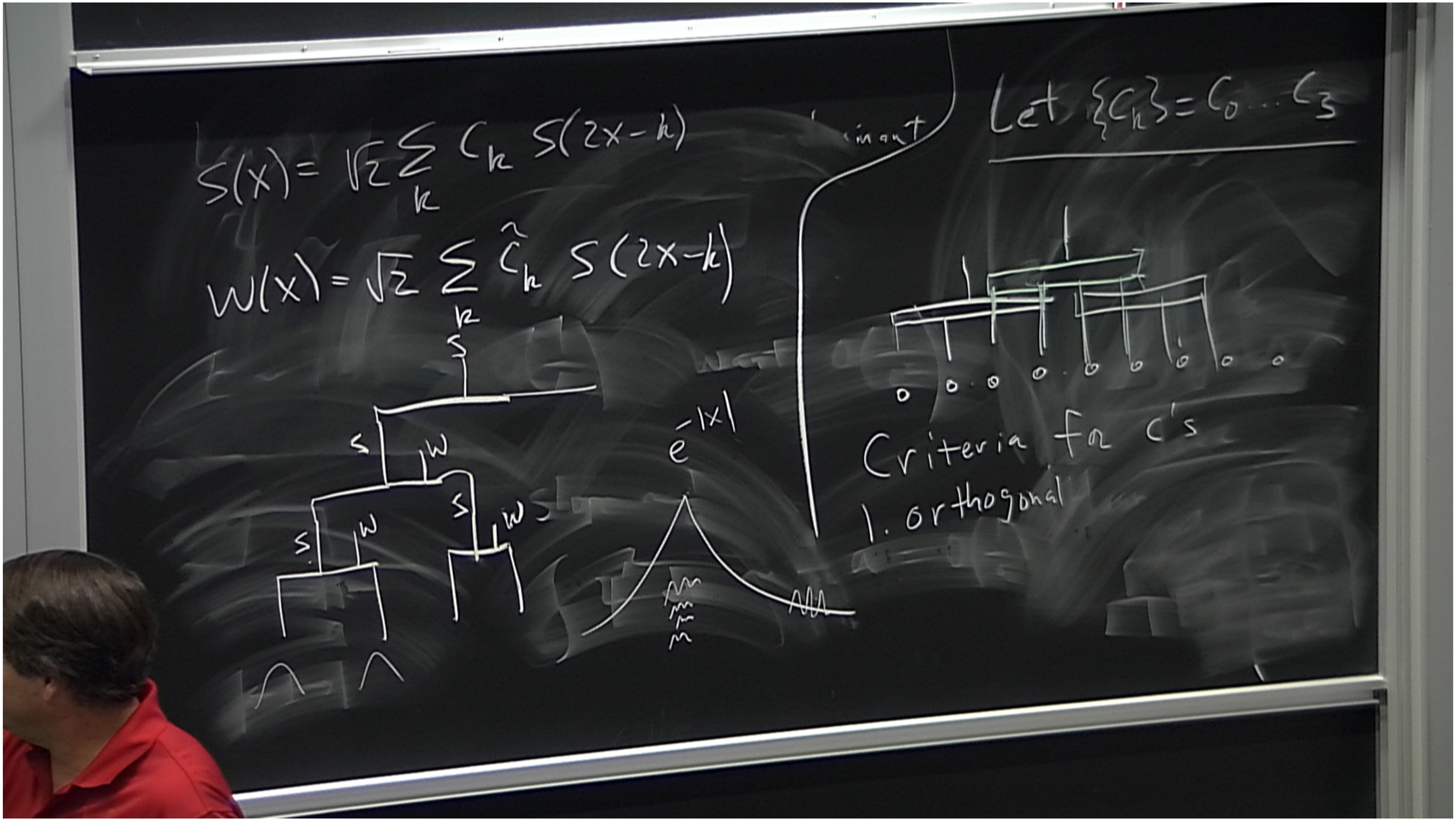
$S_{\ell+1}(x-k)$

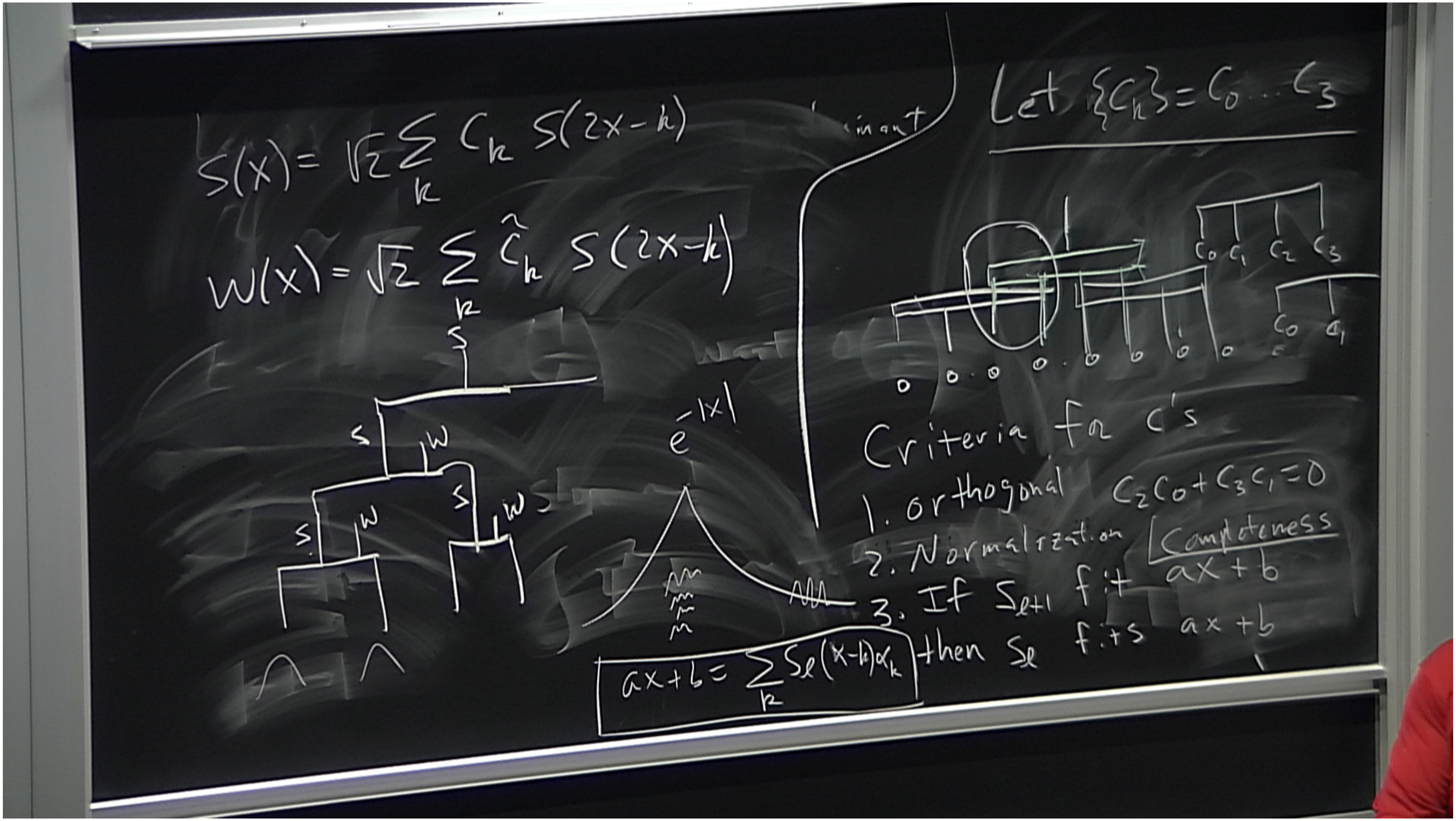
Simplest
Haar Wavelet

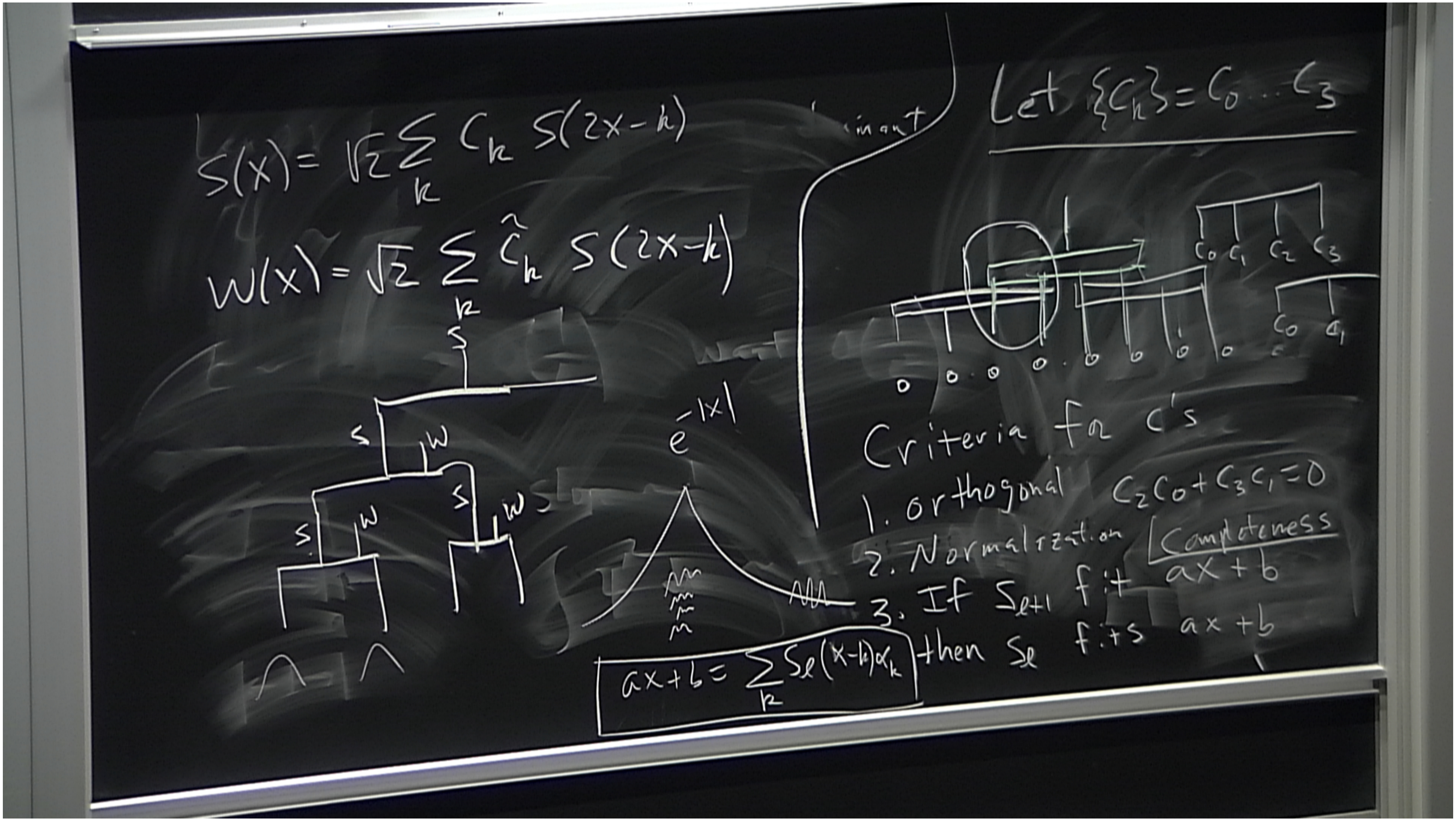
$$c_0 = \frac{1}{\sqrt{2}}, \quad c_1 = \frac{1}{\sqrt{2}}, \quad c_{i \neq 0,1} = 0$$

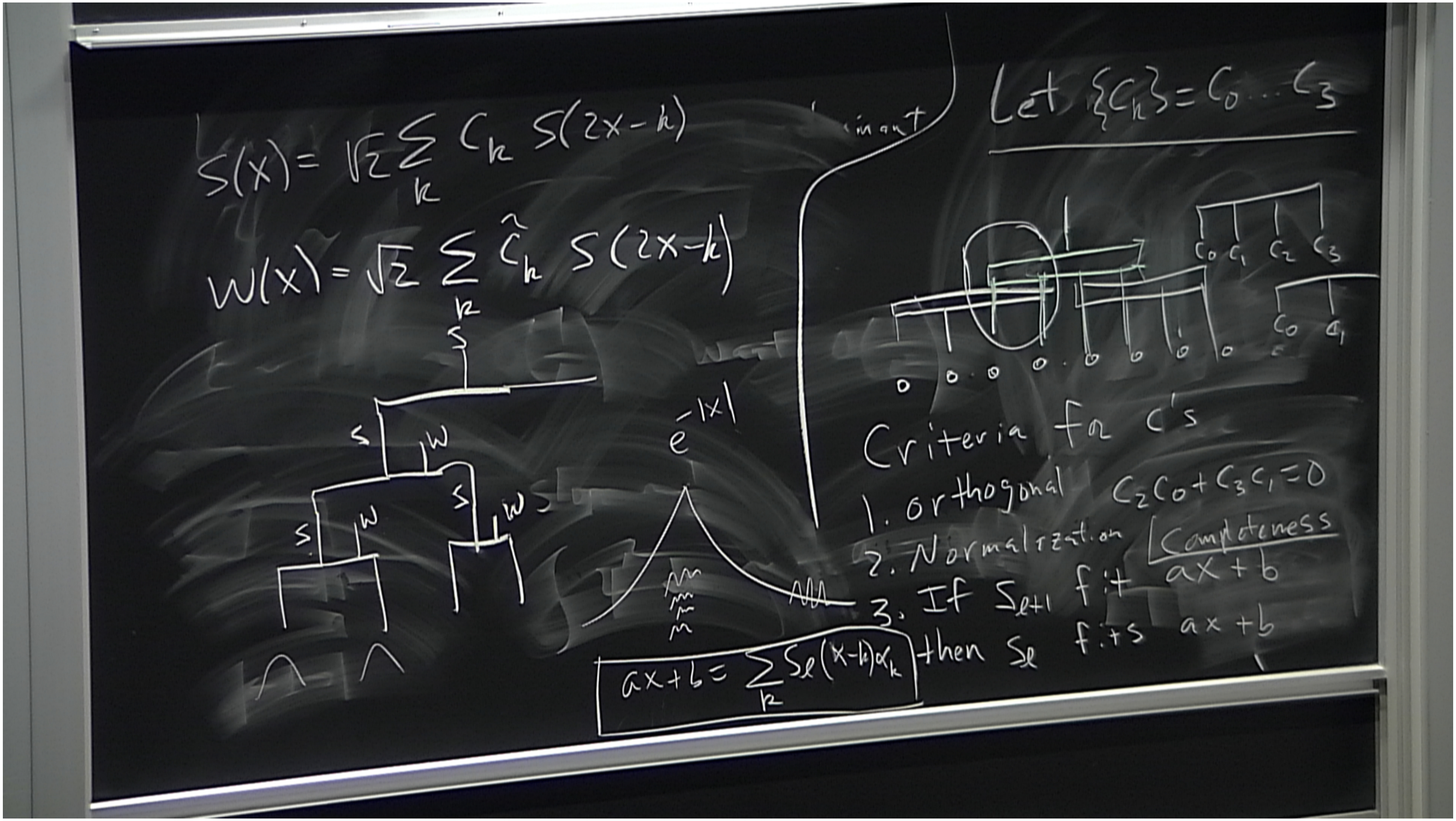


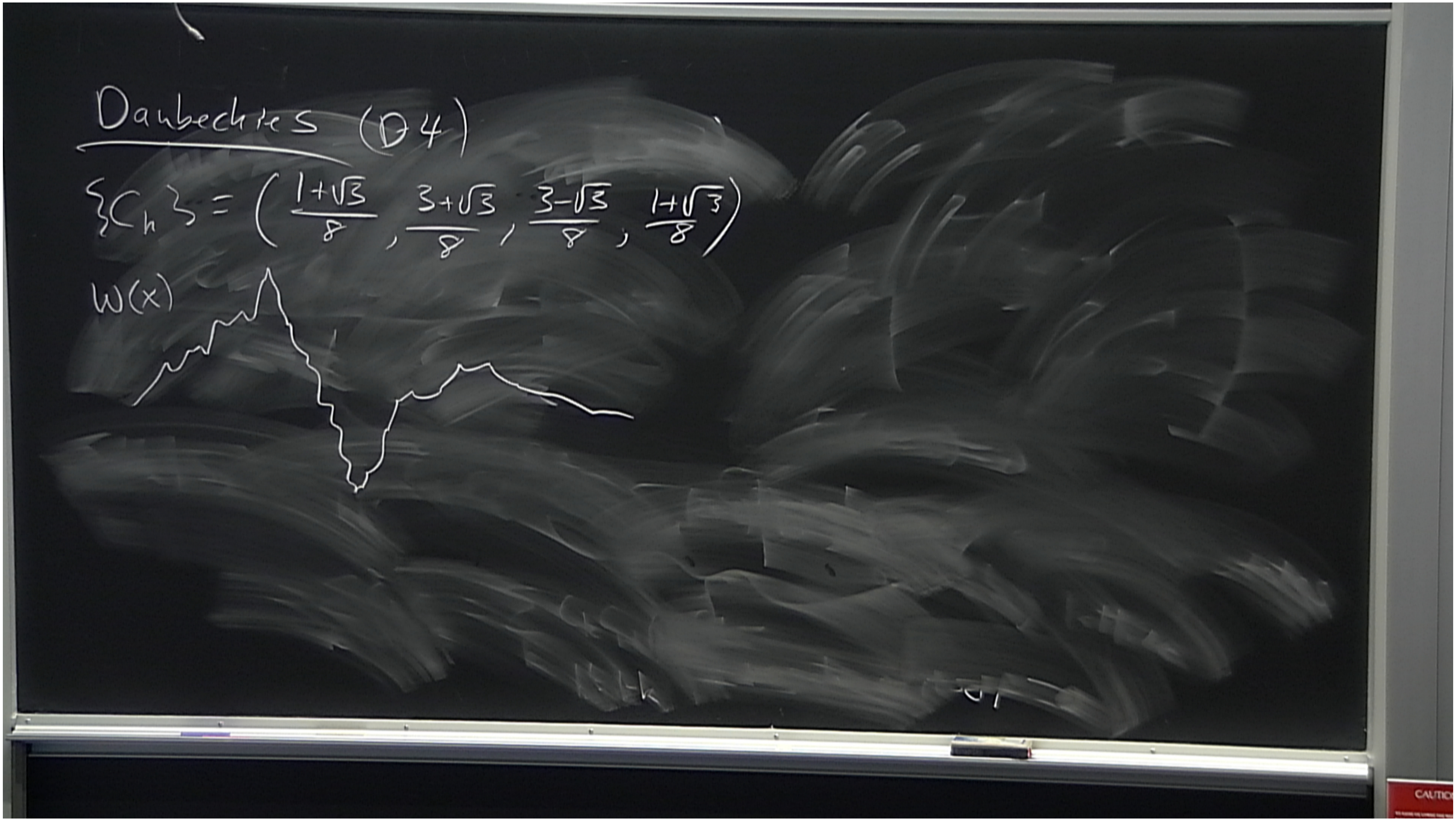

S(x) = √2 Σ_k c_k S(2x − k)
W(x) = √2 Σ_k ĉ_k S(2x − k)
Let {c_k} = c_0 … c_3
S
W
S
W
S
W

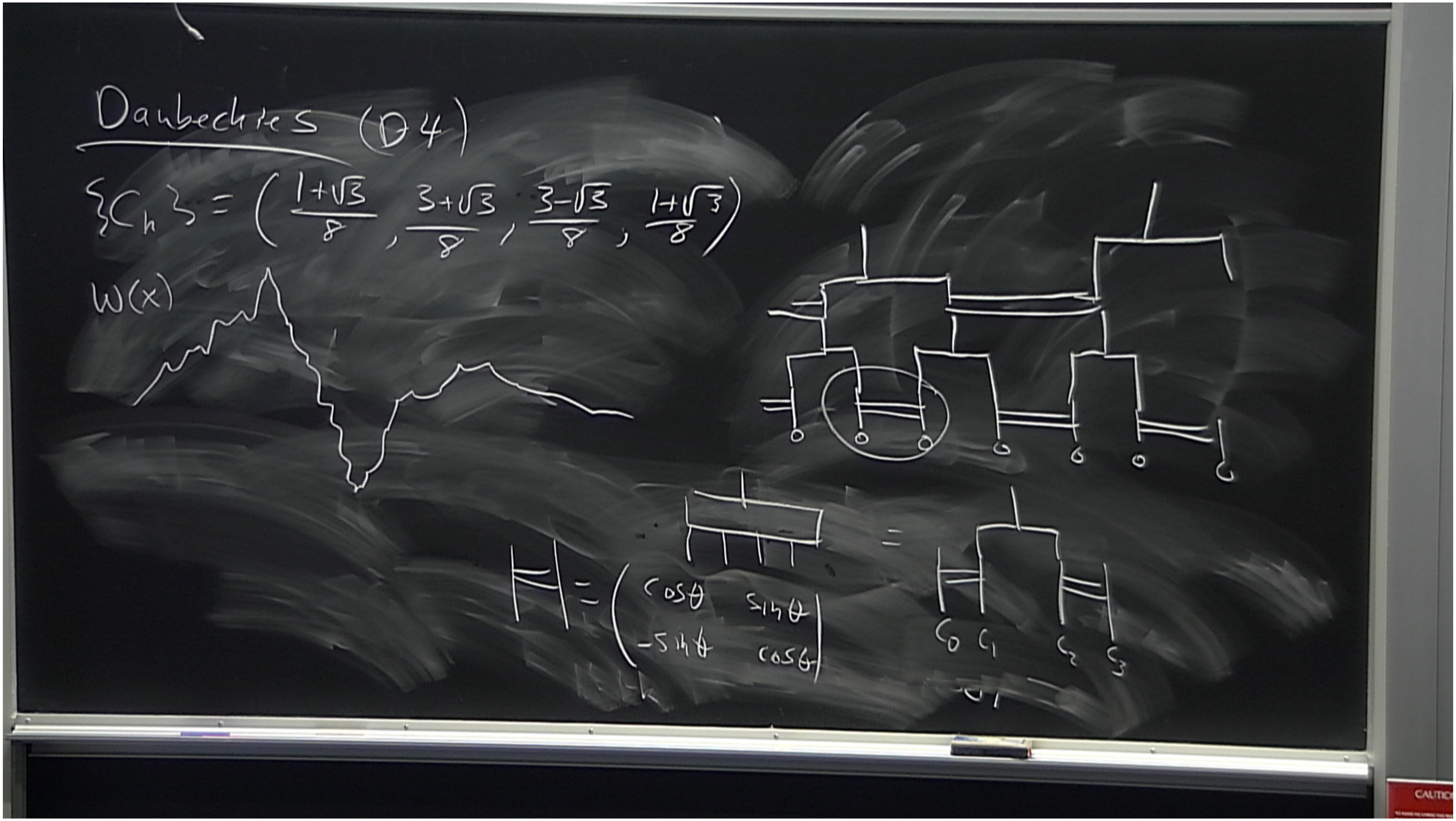

S(x) = √2 Σ_k c_k S(2x − k)
W(x) = √2 Σ_k ĉ_k S(2x − k)
e^{-|x|}
Let {c_k} = c_0 … c_3
Criteria for c's
1. orthogonal

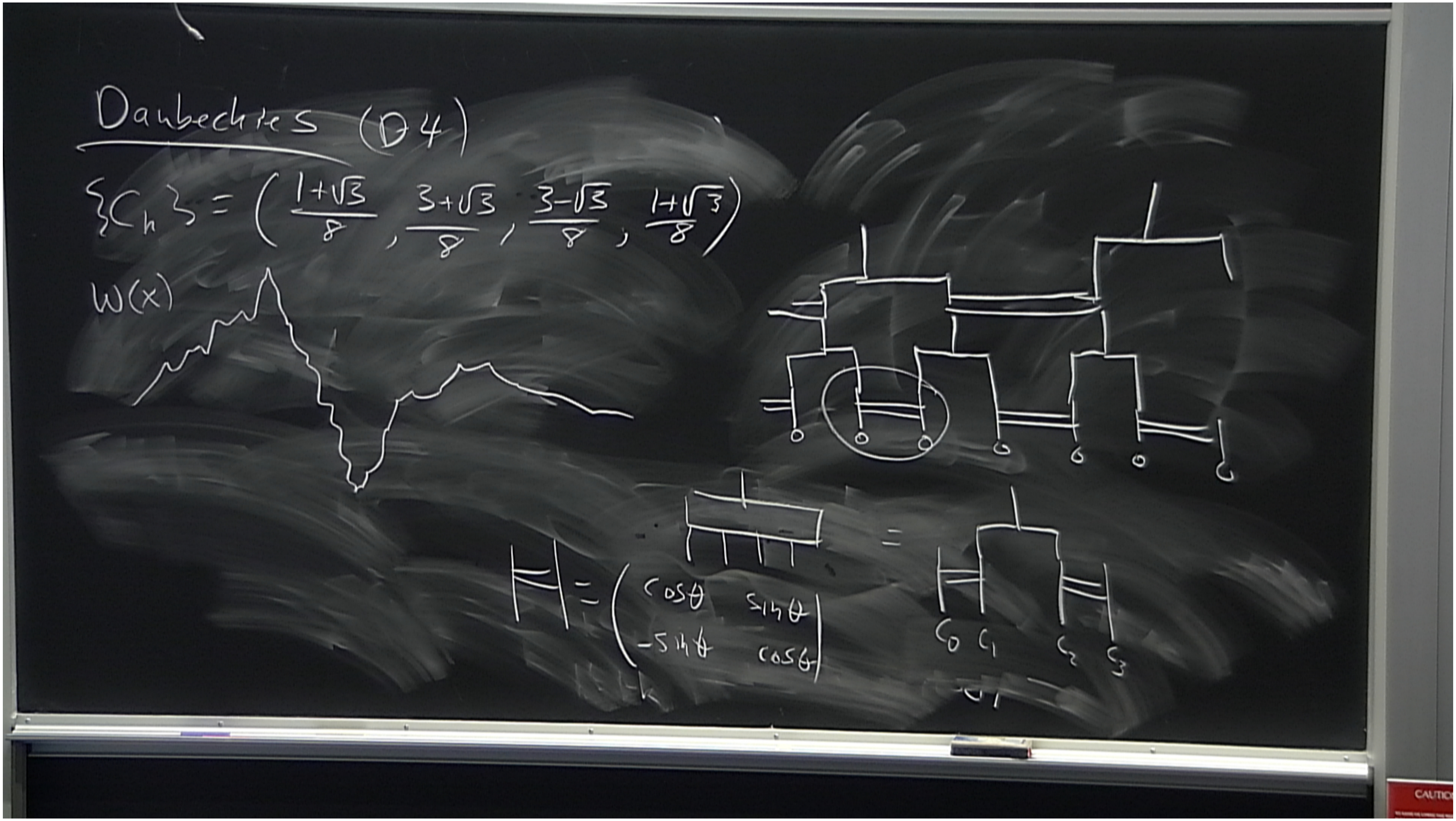

S(x) = √2 Σ_k c_k S(2x − k)
W(x) = √2 Σ_k ĉ_k S(2x − k)
e^{−|x|}
ax + b = Σ_k S_ℓ(x − k) α_k
Let {c_k} = c_0 … c_3
c_0 c_1 c_2 c_3
Criteria for c's
1. orthogonal   c_2c_0 + c_3c_1 = 0
2. Normalization   Completeness
3. If S_{ℓ+1} fit ax + b
then S_ℓ fits ax + b

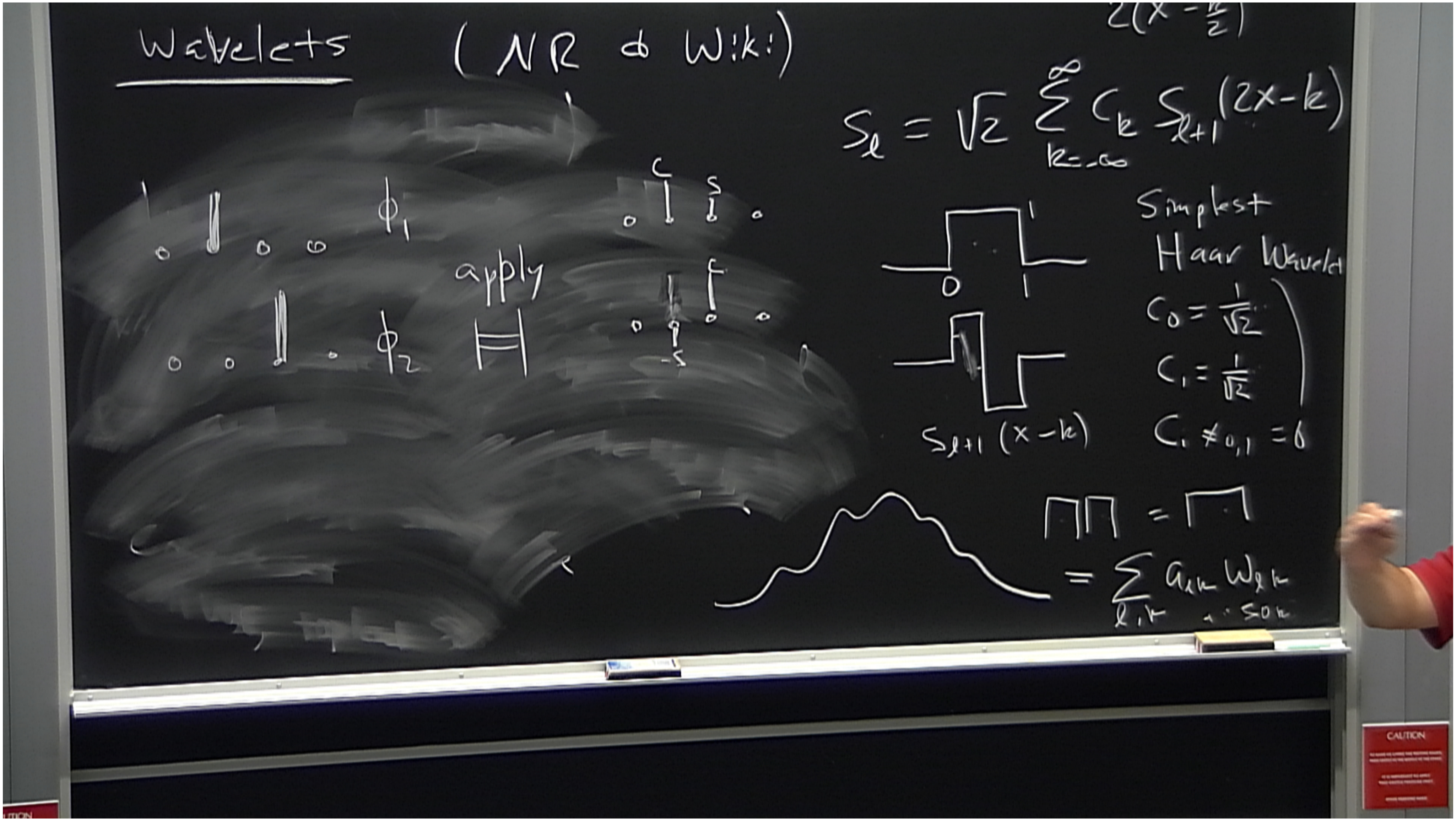

S(x) = √2 Σ_k C_k S(2x − k)
W(x) = √2 Σ_k Ĉ_k S(2x − k)
e^{-|x|}
Let {C_k} = C_0 … C_3
C_0 C_1 C_2 C_3
Criteria for C's
1. orthogonal C_2C_0 + C_3C_1 = 0
2. Normalization Completeness
3. If S_{l+1} fit ax + b
then S_l fits ax + b
ax + b = Σ_k S_l(x−k) α_k

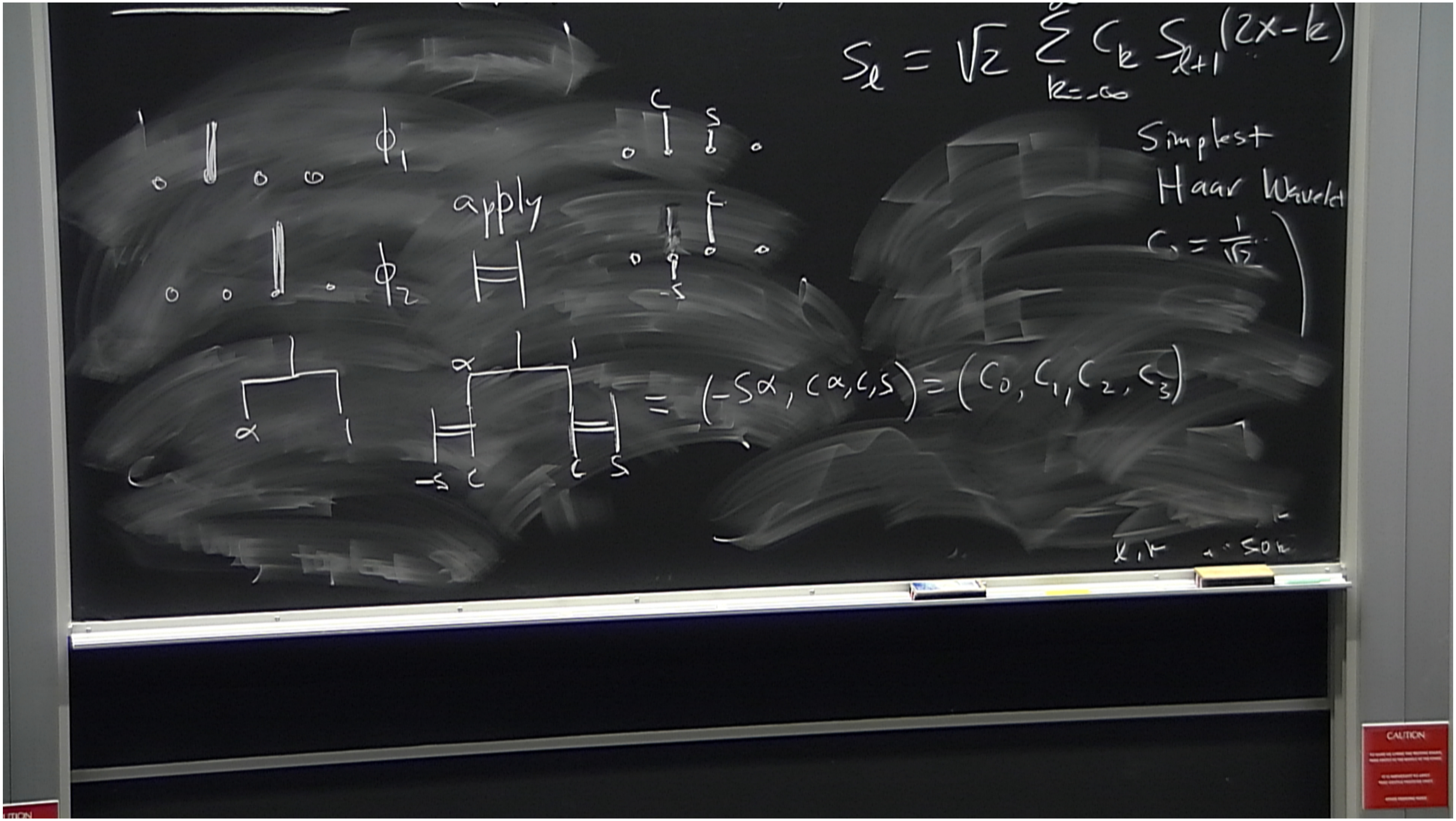

S(x) = √2 Σ c_k S(2x − k)
W(x) = √2 Σ ĉ_k S(2x − k)
e^{−|x|}
ax + b = Σ S_ℓ(x−k) α_k
Let {c_k} = c_0 … c_3
c_0 c_1 c_2 c_3
Criteria for c's
1. orthogonal c_2 c_0 + c_3 c_1 = 0
2. Normalization Completeness
3. If S_{ℓ+1} fit ax + b
then S_ℓ fits ax + b

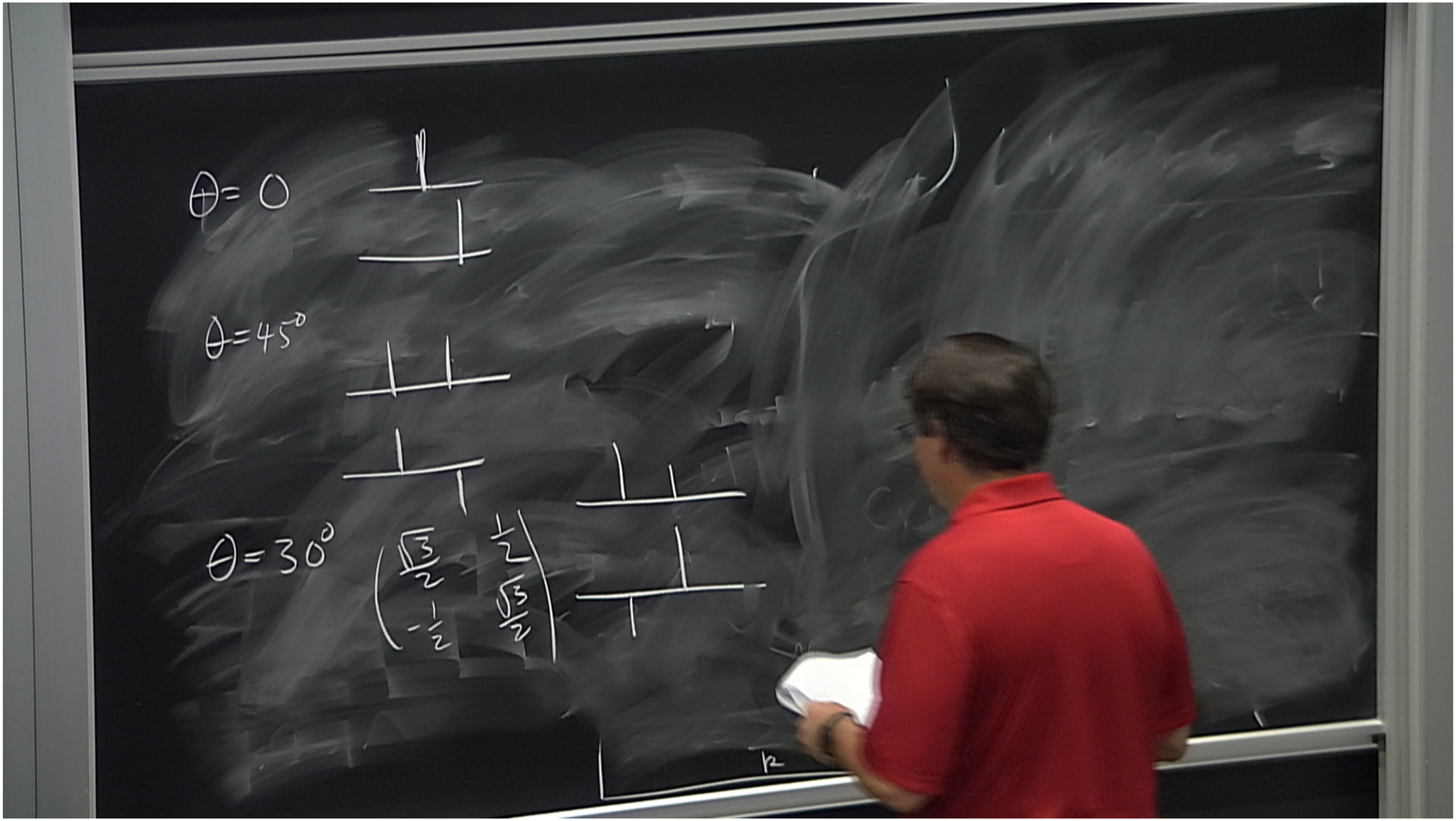

Daubechies (D4)
{c_n} = ((1+√3)/8 , (3+√3)/8 , (3−√3)/8 , (1+√3)/8)
W(x)

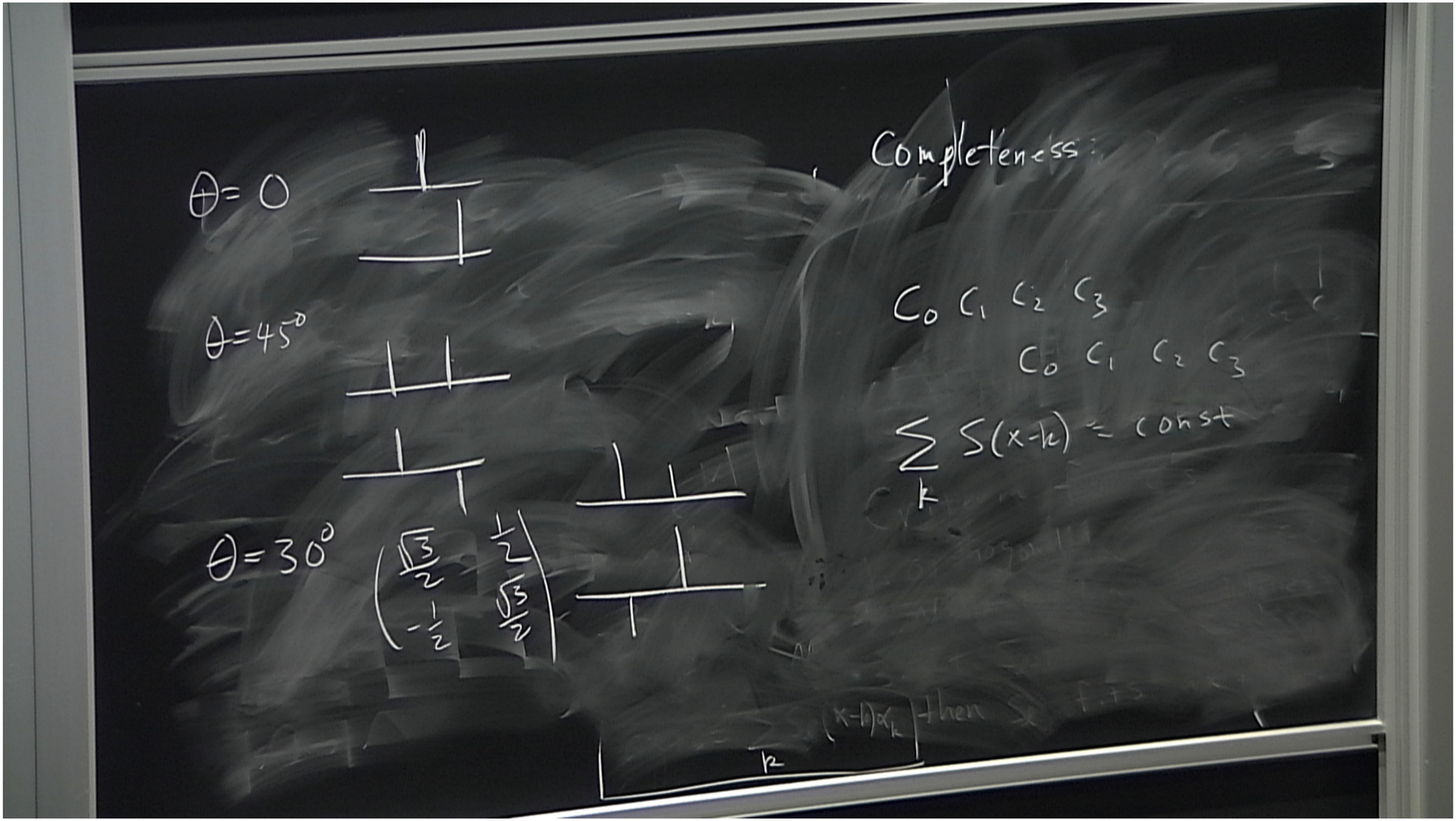

Daubechies (D4)
{c_n} = ((1+√3)/8 , (3+√3)/8 , (3-√3)/8 , (1+√3)/8)
W(x)
cosθ sinθ
-sinθ cosθ
c0 c1 c2 c3

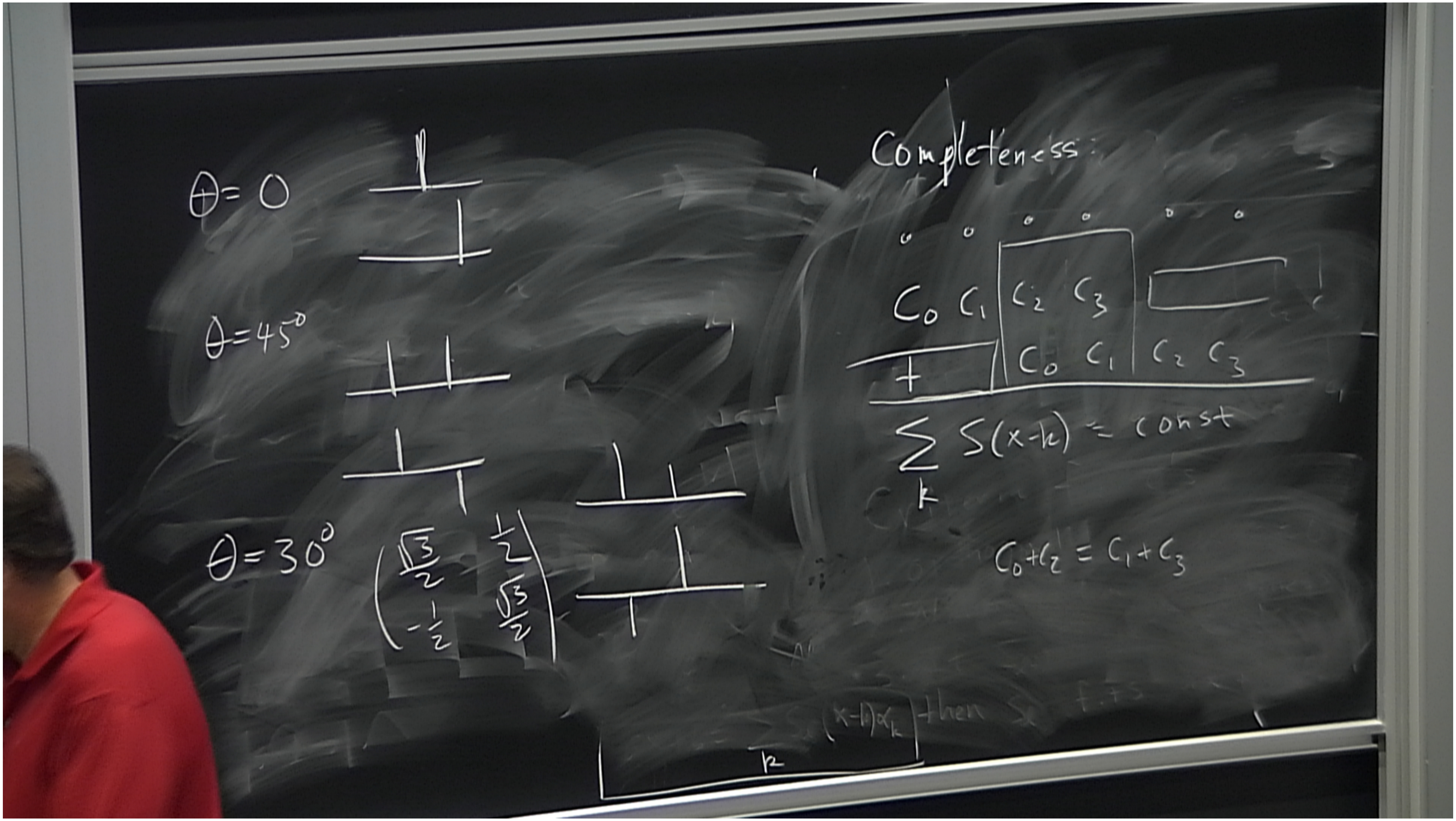

Daubechies (D4)
{c_n} = ((1+√3)/8, (3+√3)/8, (3-√3)/8, (1+√3)/8)
W(x)
cosθ sinθ
-sinθ cosθ
c0 c1 c2 c3

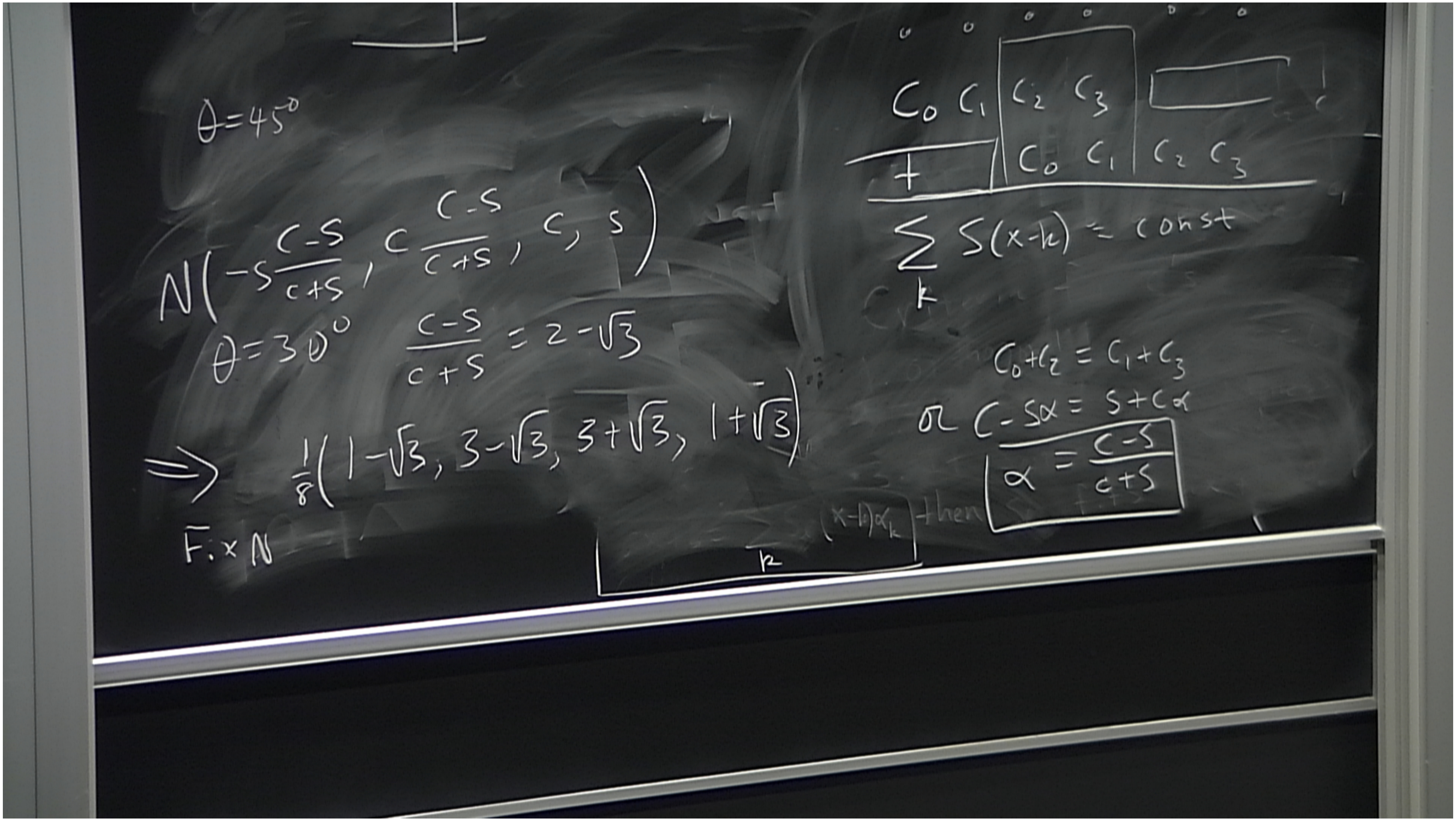

Wavelets (NR & Wiki)
apply
$S_\ell = \sqrt{2} \sum_{k=-\infty}^{\infty} c_k S_{\ell+1}(2x-k)$
Simplest
Haar Wavelet
$c_0 = \frac{1}{\sqrt{2}}$
$c_1 = \frac{1}{\sqrt{2}}$
$c_{i \neq 0,1} = 0$
$S_{\ell+1}(x-k)$
$= \sum_{\ell,k} a_{\ell k} W_{\ell k}$
CAUTION

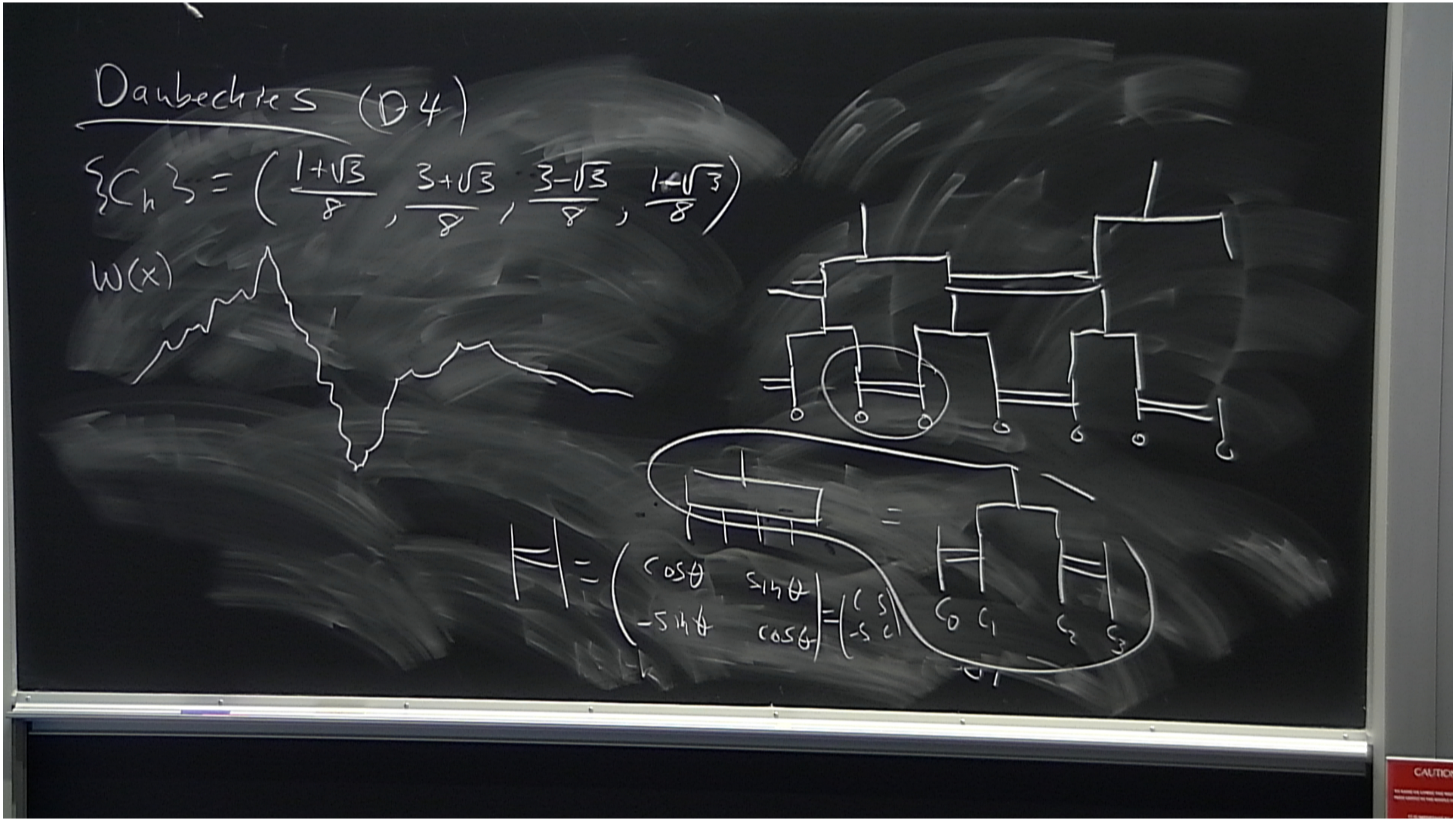

$S_\ell = \sqrt{2} \sum_{k=-\infty} c_k S_{\ell+1}(2x-k)$
Simplest
Haar Wavelet
$c_1 = \frac{1}{\sqrt{2}}$
ϕ_1
ϕ_2
apply
$= (-s\alpha, c\alpha, c, s) = (c_0, c_1, c_2, c_3)$


θ = 0
θ = 45°
θ = 30°
√3/2 1/2
-1/2 √3/2
k


θ = 0
θ = 45°
θ = 30°
√3/2 1/2
-1/2 √3/2
Completeness:
c0 c1 c2 c3
c0 c1 c2 c3
Σ_k S(x-k) = const
then


θ = 0
θ = 45°
θ = 30°
Completeness:
C0 C1 C2 C3
C0 C1 C2 C3
Σ S(x-k) = const
k
C0+C2 = C1+C3


θ=45°
N(−s (c−s)/(c+s), c (c−s)/(c+s), c, s)
θ=30° (c−s)/(c+s) = 2−√3
⇒ 1/8(1−√3, 3−√3, 3+√3, 1+√3)
F.×N
c0 c1 c2 c3
+ c0 c1 c2 c3
Σ_k S(x−k) = const
c0+c2 = c1+c3
or c−sα = s+cα
α = (c−s)/(c+s)
then


Daubechies (D4)
{c_n} = ((1+√3)/8 , (3+√3)/8 , (3−√3)/8 , (1−√3)/8)
W(x)
= (cosθ sinθ / −sinθ cosθ) = (c s / −s c)
=
c0 c1 c2 c3
CAUTION