Title: Orbifold partition function and their phases

Date: Aug 13, 2013 02:00 PM

URL: http://pirsa.org/13080044

Abstract: We discuss a partition function of 3d supersymmetric gauge theories on the (p, -1) Lens space.
In 3d the partition function is directly used to check dualities though the normalization is not seriously treated, especially, the phase is usually ignored. However, when we consider the partition function on the orbifold the partition function consists of the sum of factors labeled by holonomies

and their relative phases become crucial. We stress that the known formula for the partition function is incorrect and some relative phase factors are needed to identify the partition functions of dual theories.

Outline

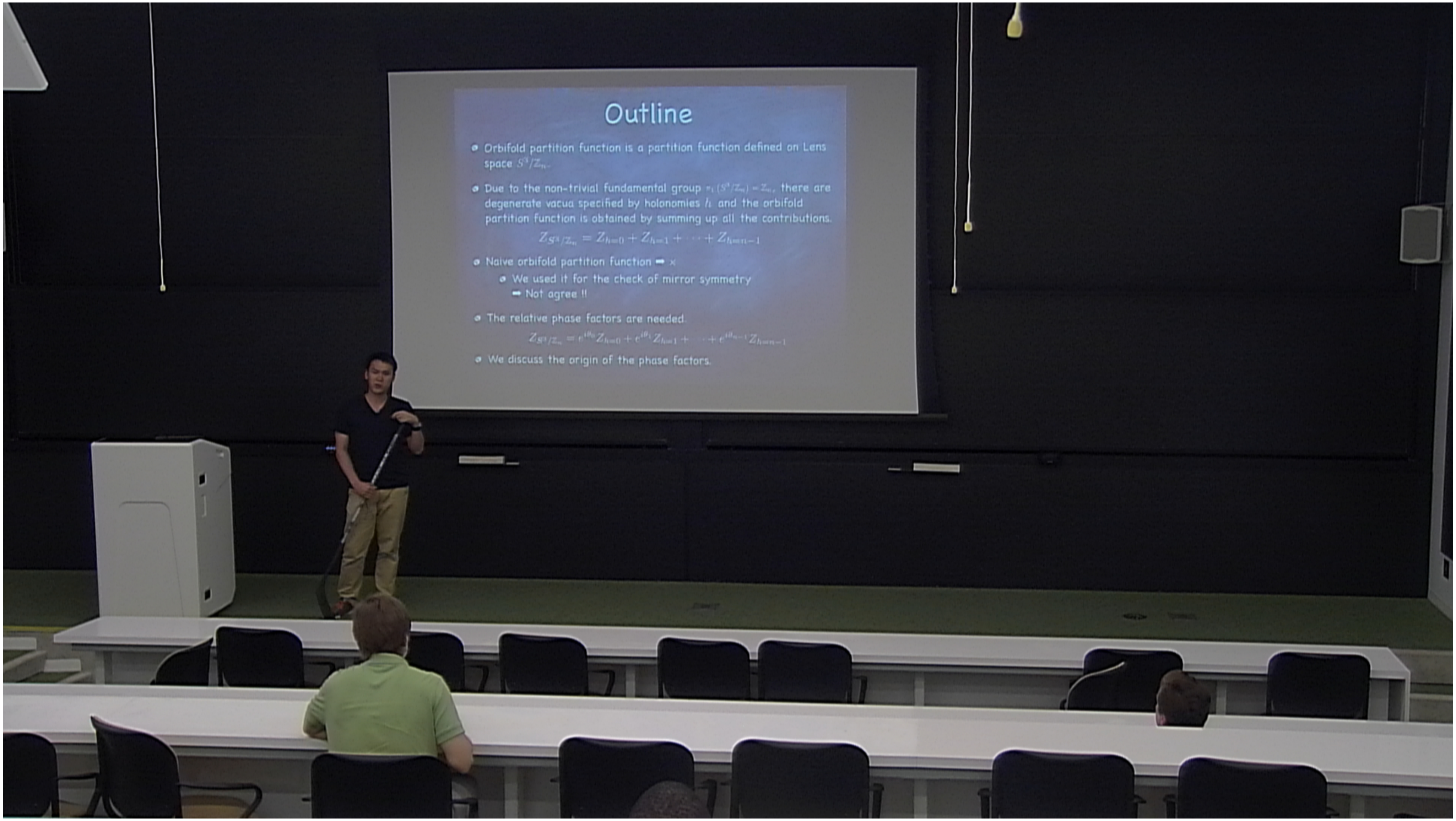
- Orbifold partition function is a partition function defined on Lens space $S^3/\mathbb{Z}_n$.
- Due to the non-trivial fundamental group $\pi_1(S^3/\mathbb{Z}_n) = \mathbb{Z}_n$, there are degenerate vacua specified by holonomies h and the orbifold partition function is obtained by summing up all the contributions.

$$Z_{S^3/\mathbb{Z}_n} = Z_{h=0} + Z_{h=1} + \cdots + Z_{h=n-1}$$

- Naive orbifold partition function ➡ $\times$
    - We used it for the check of mirror symmetry
      ➡ Not agree !!
- The relative phase factors are needed.

$$Z_{S^3/\mathbb{Z}_n} = e^{i\theta_0} Z_{h=0} + e^{i\theta_1} Z_{h=1} + \cdots + e^{i\theta_{n-1}} Z_{h=n-1}$$

- We discuss the origin of the phase factors.



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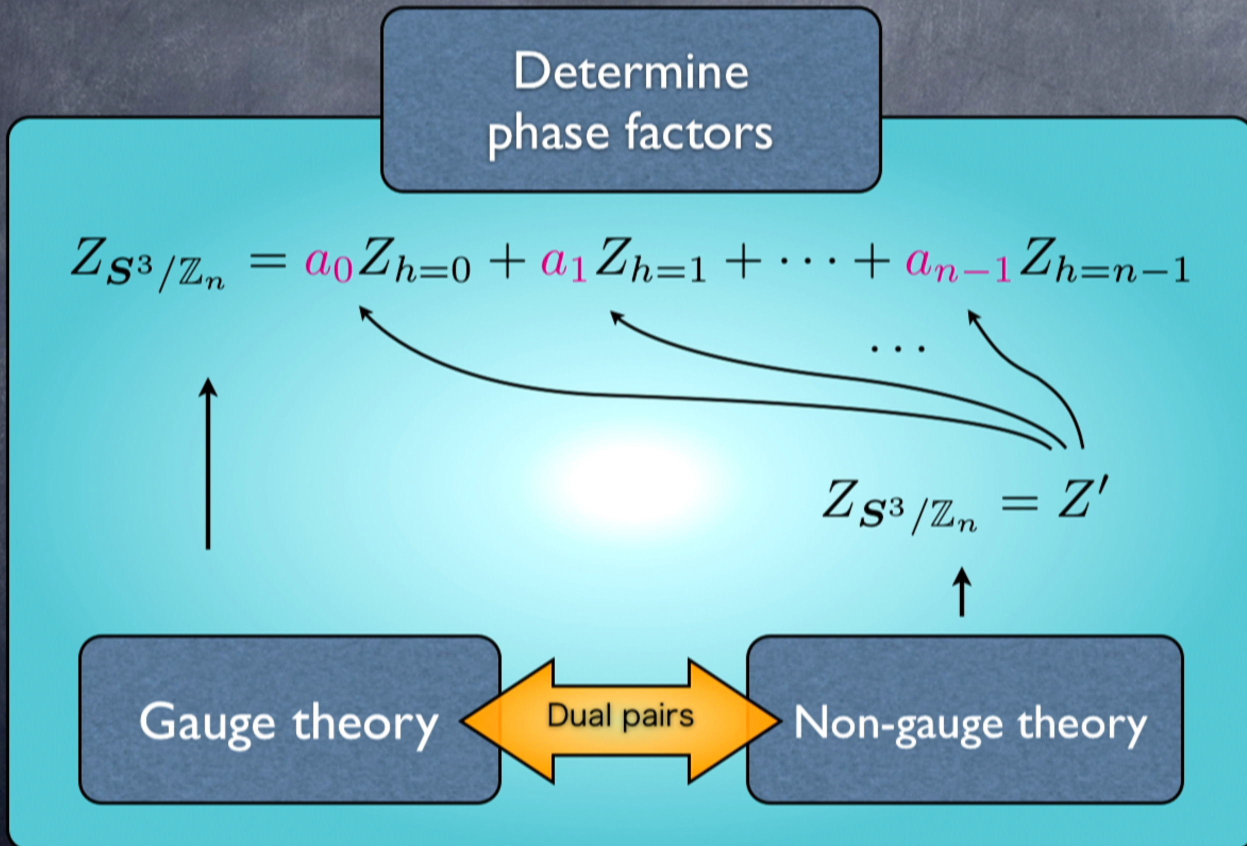
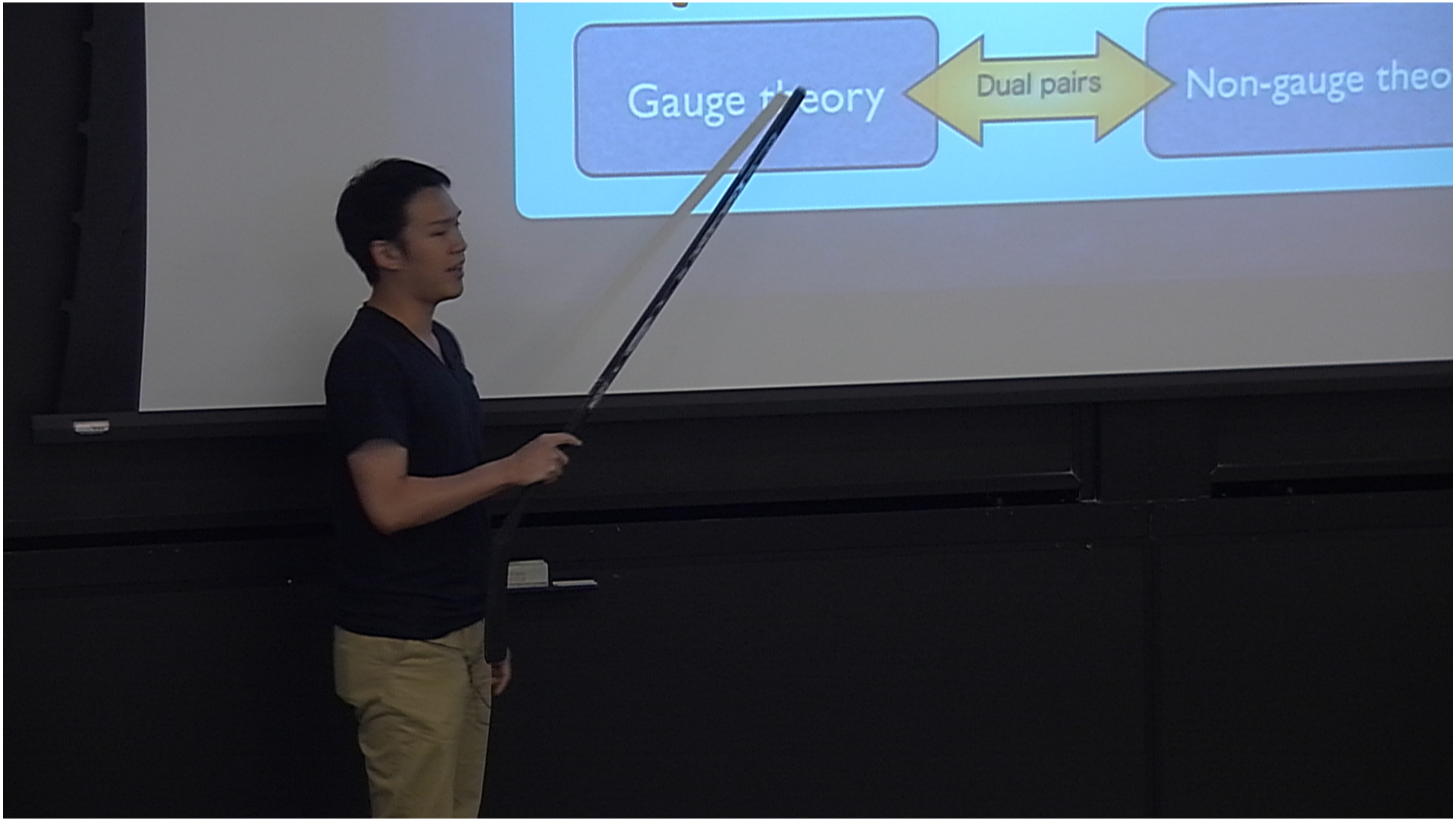
Criterion



Determine phase factors
$Z_{S^3/\mathbb{Z}_n} = a_0 Z_{h=0} + a_1 Z_{h=1} + \cdots + a_{n-1} Z_{h=n-1}$
$\cdots$
$Z_{S^3/\mathbb{Z}_n} = Z'$
Gauge theory
Dual pairs
Non-gauge theory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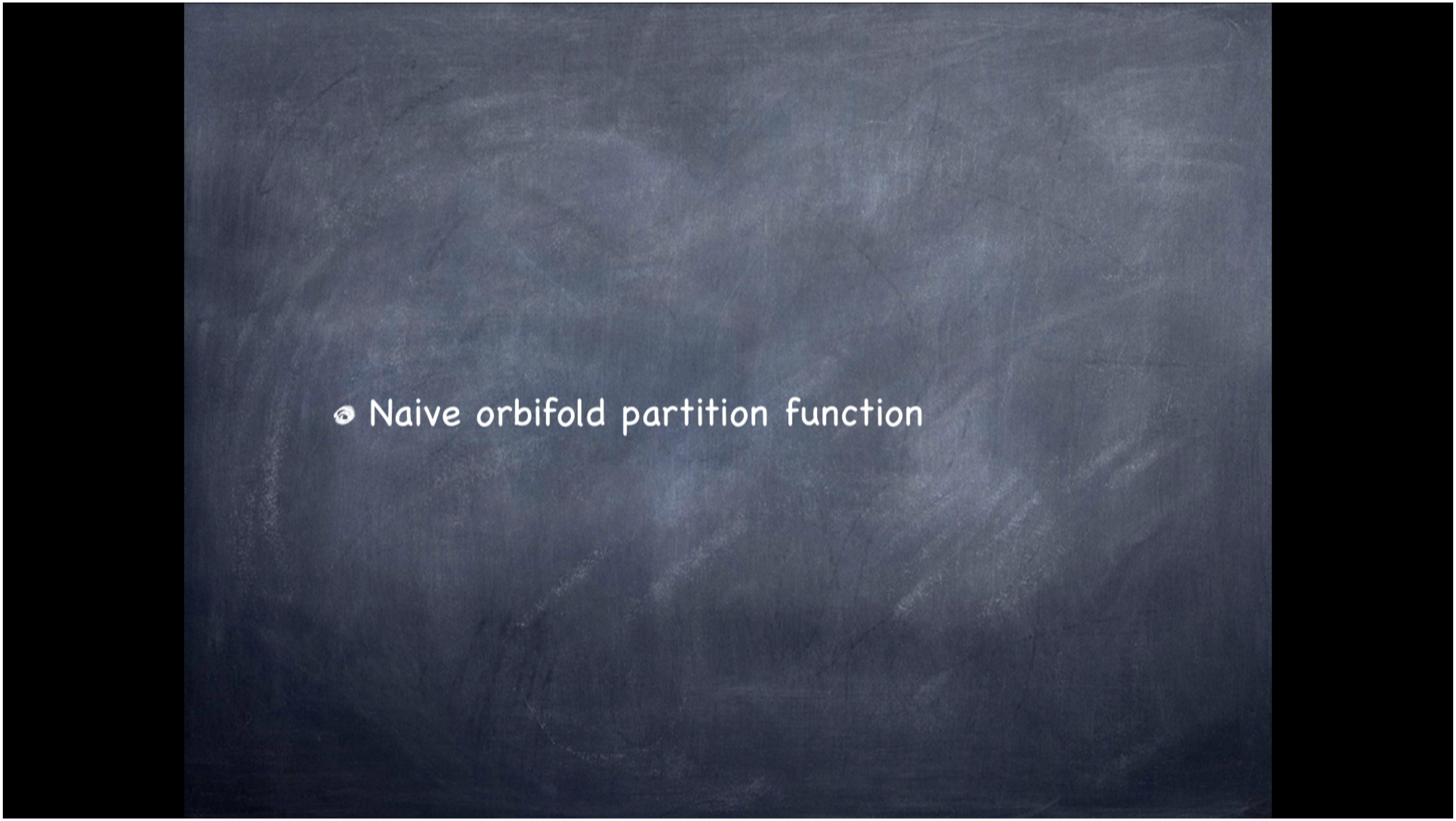
Contents

- Naive orbifold partition function
- Numerical check for dual theories
- Classical ambiguity of CS term
- Orbifolding procedure and improved orbifold partition function
- Conclusions


Naive orbifold partition function

Supersymmetry

Images	Isometries	Supercharge
S^3	$SU(2)_L \times SU(2)_R$	$(\mathbf{2},\mathbf{1})$ $(\mathbf{1},\mathbf{2})$
Squashed S^3	$SU(2)_L \times U(1)_r$	$\mathbf{2}_0$
~~Squashed S^3~~	$SU(2)_L \times U(1)_r/\mathbb{Z}_n$	$\mathbf{2}_0$

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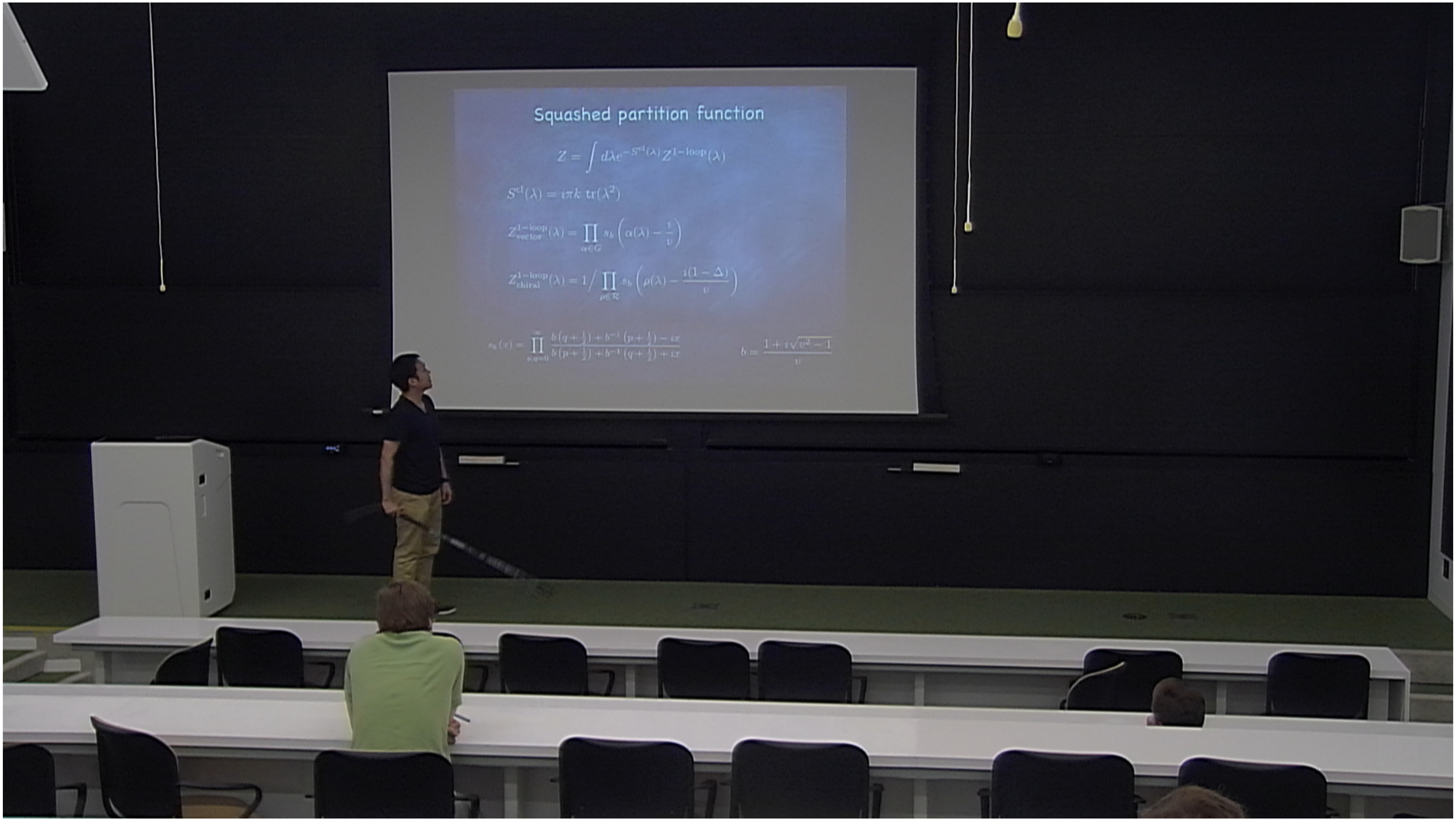
Background geometry

Images	Isometries	Parameters
S^3	$SU(2)_L \times SU(2)_R$	
	[Kapustin et al '09, Jafferis '10, Hama et al '10]	
Squashed S^3	$SU(2)_L \times U(1)_r$	v or $b = \frac{1 + i\sqrt{v^2-1}}{v}$
		[Imamura, D.Y. '11]
~~Squashed S^3~~	$SU(2)_L \times U(1)_r/\mathbb{Z}_n$	v and (n, h)
		[Benini et al '11, Imamura, D.Y. '12]



Squashed partition function
$Z = \int d\lambda e^{-S^{\mathrm{cl}}(\lambda)} Z^{1-\mathrm{loop}}(\lambda)$
$S^{\mathrm{cl}}(\lambda) = i\pi k \ \mathrm{tr}(\lambda^2)$
$Z^{1-\mathrm{loop}}_{\mathrm{vector}}(\lambda) = \prod_{\alpha \in G} s_b\left(\alpha(\lambda) - \frac{i}{v}\right)$
$Z^{1-\mathrm{loop}}_{\mathrm{chiral}}(\lambda) = 1 \Big/ \prod_{\rho \in \mathcal{R}} s_b\left(\rho(\lambda) - \frac{i(1-\Delta)}{v}\right)$
$s_b(x) = \prod_{p,q=0}^{\infty} \frac{b(q+\frac{1}{2}) + b^{-1}(p+\frac{1}{2}) - ix}{b(p+\frac{1}{2}) + b^{-1}(q+\frac{1}{2}) + ix}$
$b = \frac{1 + i\sqrt{v^2-1}}{v}$

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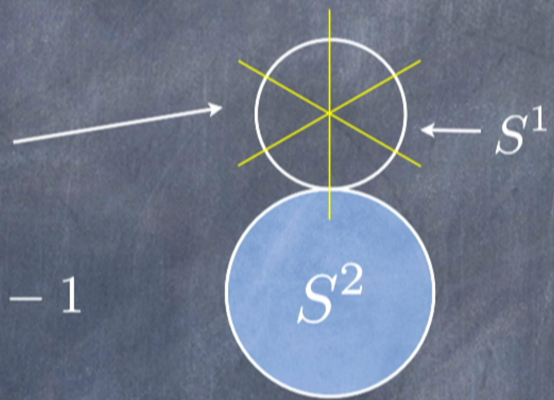
$S^3/\mathbb{Z}_n$ partition function

- Holonomy

$$h = \frac{n}{2\pi}\oint_C A$$

$$h = 0, 1, \cdots, n-1$$



- Due to the non-trivial fundamental group $\pi_1(S^3/\mathbb{Z}_n) = \mathbb{Z}_n$ one can introduce the Wilson line along the circle.

- Holonomy changes the boundary conditions (AB effect)

$$\Phi\left(\psi + \frac{2\pi}{n}\right) = e^{2\pi i \frac{\rho(h)}{n}}\Phi(\psi)$$

Orbifold partition function

- Holonomy

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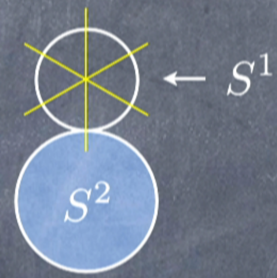


- One can introduce holonomy not only for gauge symmetry but also global symmetry.
- Holonomy for global symmetry becomes a parameter
- Holonomy for gauge symmetry should be summed up in the path integral.

$$Z(h_{\text{global}}) = \sum_{h_{\text{local}}} \int [d\lambda] e^{-S_0(\lambda,h)} Z^{1-\text{loop}}(\lambda, h) \qquad [d\lambda] = \frac{1}{|W|}\prod_{a=1}^{\text{rank}G} \frac{d\lambda_a}{n}$$

$\mathbb{Z}_n$ orbifolding

- Classical part (CS term)

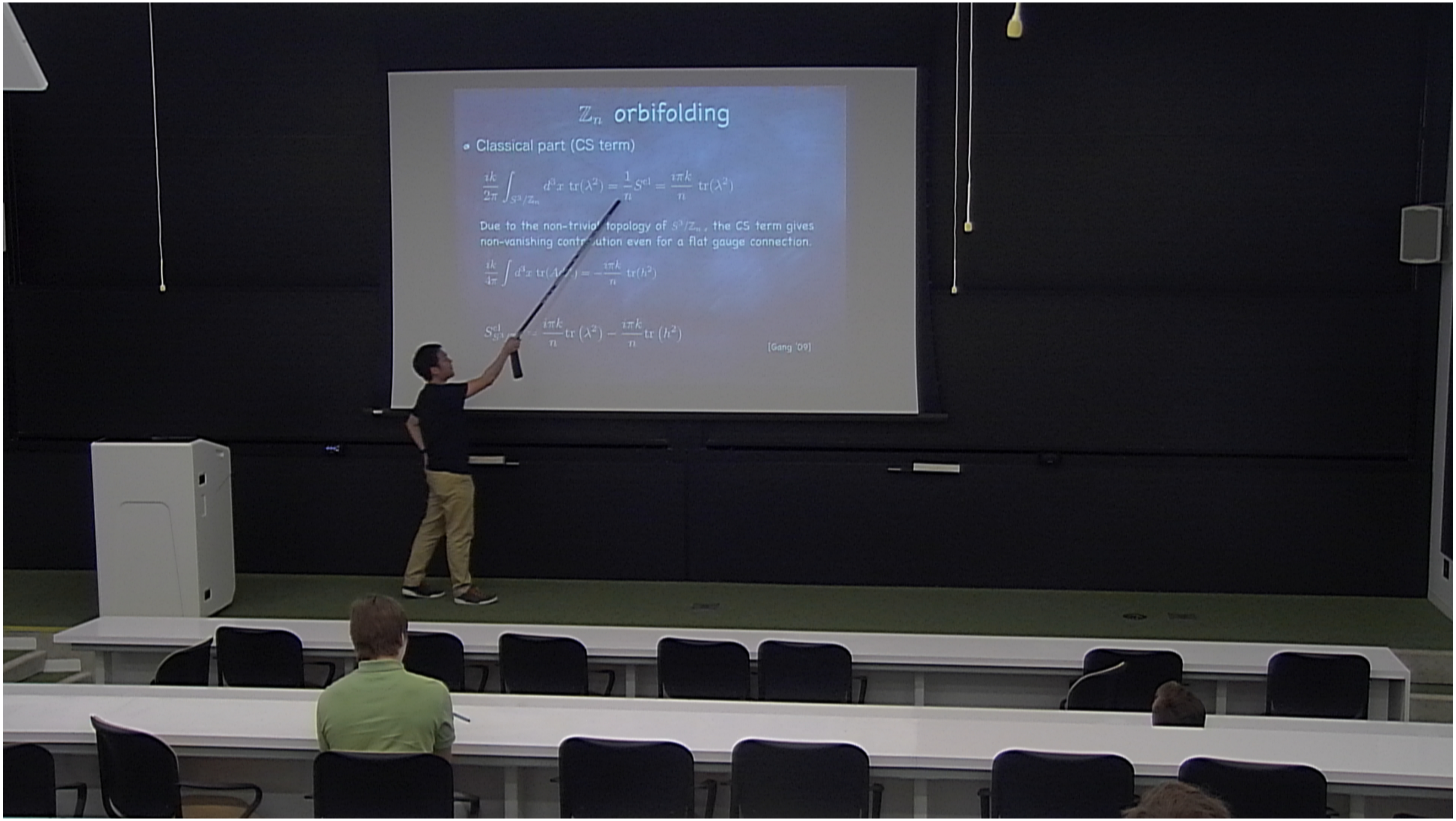
$$\frac{ik}{2\pi}\int_{S^3/\mathbb{Z}_n} d^3x\ \mathrm{tr}(\lambda^2) = \frac{1}{n}S^{\mathrm{cl}} = \frac{i\pi k}{n}\ \mathrm{tr}(\lambda^2)$$

Due to the non-trivial topology of $S^3/\mathbb{Z}_n$, the CS term gives non-vanishing contribution even for a flat gauge connection.

$$\frac{ik}{4\pi}\int d^3x\ \mathrm{tr}(AdA) = -\frac{i\pi k}{n}\ \mathrm{tr}(h^2)$$

$$S^{\mathrm{cl}}_{S^3/\mathbb{Z}_n} = \frac{i\pi k}{n}\mathrm{tr}\left(\lambda^2\right) - \frac{i\pi k}{n}\mathrm{tr}\left(h^2\right)$$

[Gang '09]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$\frac{i\pi k}{n}\mathrm{tr}(\lambda^2) - \frac{i\pi k}{n}\mathrm{tr}(h^2)$
[Gang '09]

$\mathbb{Z}_n$ orbifolding

- Squash

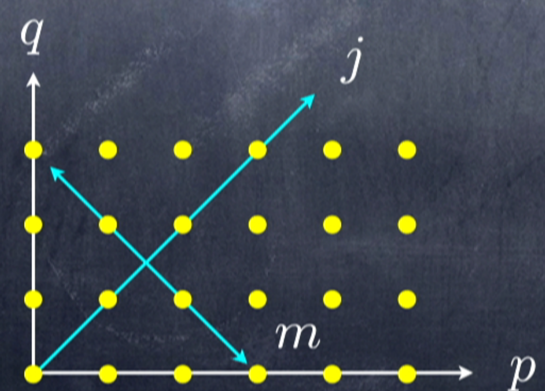
$$Z^{1-\text{loop}} = \frac{\text{Det}\mathcal{D}_F}{\text{Det}\mathcal{D}_B} \longrightarrow \frac{2j - 2imu - iv\alpha(\lambda)}{2j + 2 + 2imu + iv\alpha(\lambda)}$$

$$j = \frac{p+q}{2}$$

$$m = \frac{p-q}{2}$$

$$s_b(x) = \prod_{p,q=0}^{\infty} \frac{b\left(q+\frac{1}{2}\right) + b^{-1}\left(p+\frac{1}{2}\right) - ix}{b\left(p+\frac{1}{2}\right) + b^{-1}\left(q+\frac{1}{2}\right) + ix}$$

$$b = \frac{1 + i\sqrt{v^2 - 1}}{v}$$



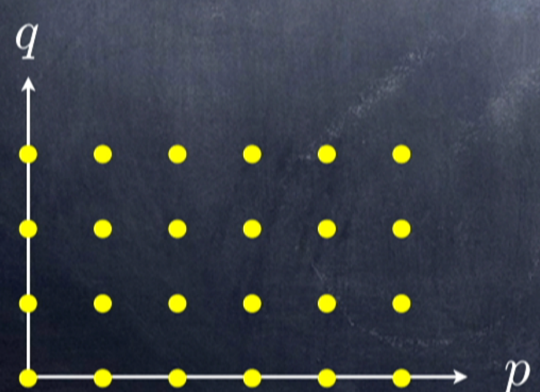
$\mathbb{Z}_n$ orbifolding

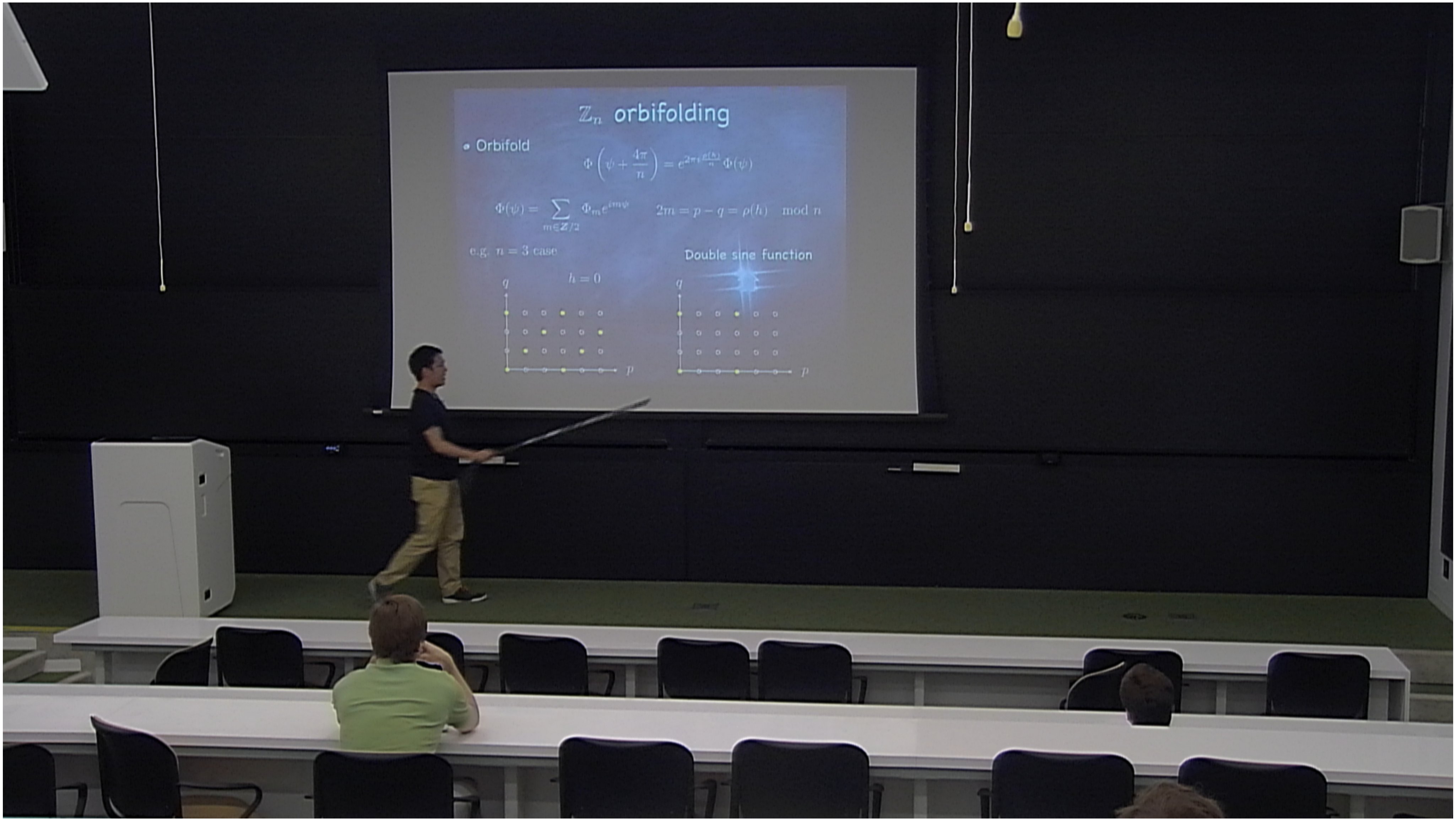
- Orbifold

$$\Phi\left(\psi + \frac{4\pi}{n}\right) = e^{2\pi i \frac{\rho(h)}{n}} \Phi(\psi)$$

$$\Phi(\psi) = \sum_{m \in \boldsymbol{Z}/2} \Phi_m e^{im\psi} \qquad 2m = p - q = \rho(h) \mod n$$

e.g. $n = 3$ case





$\mathbb{Z}_n$ orbifolding
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$2m=p-q=\rho(h) \mod n$
e.g. $n=3$ case
$h=0$
Double sine function
q
p
q
p

$\mathbb{Z}_n$ orbifolding

- Orbifold

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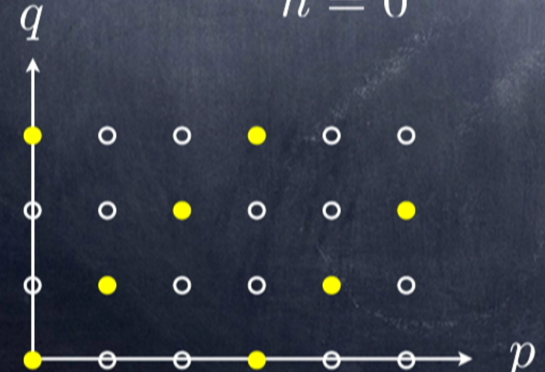
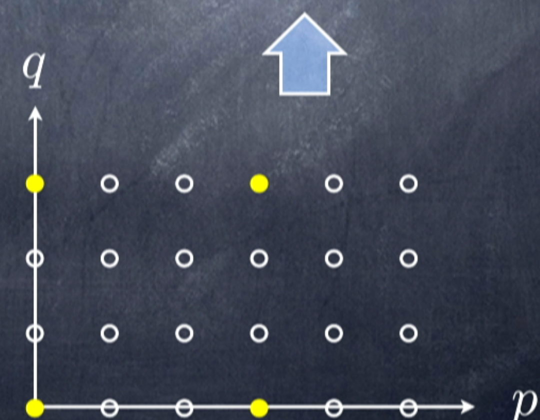
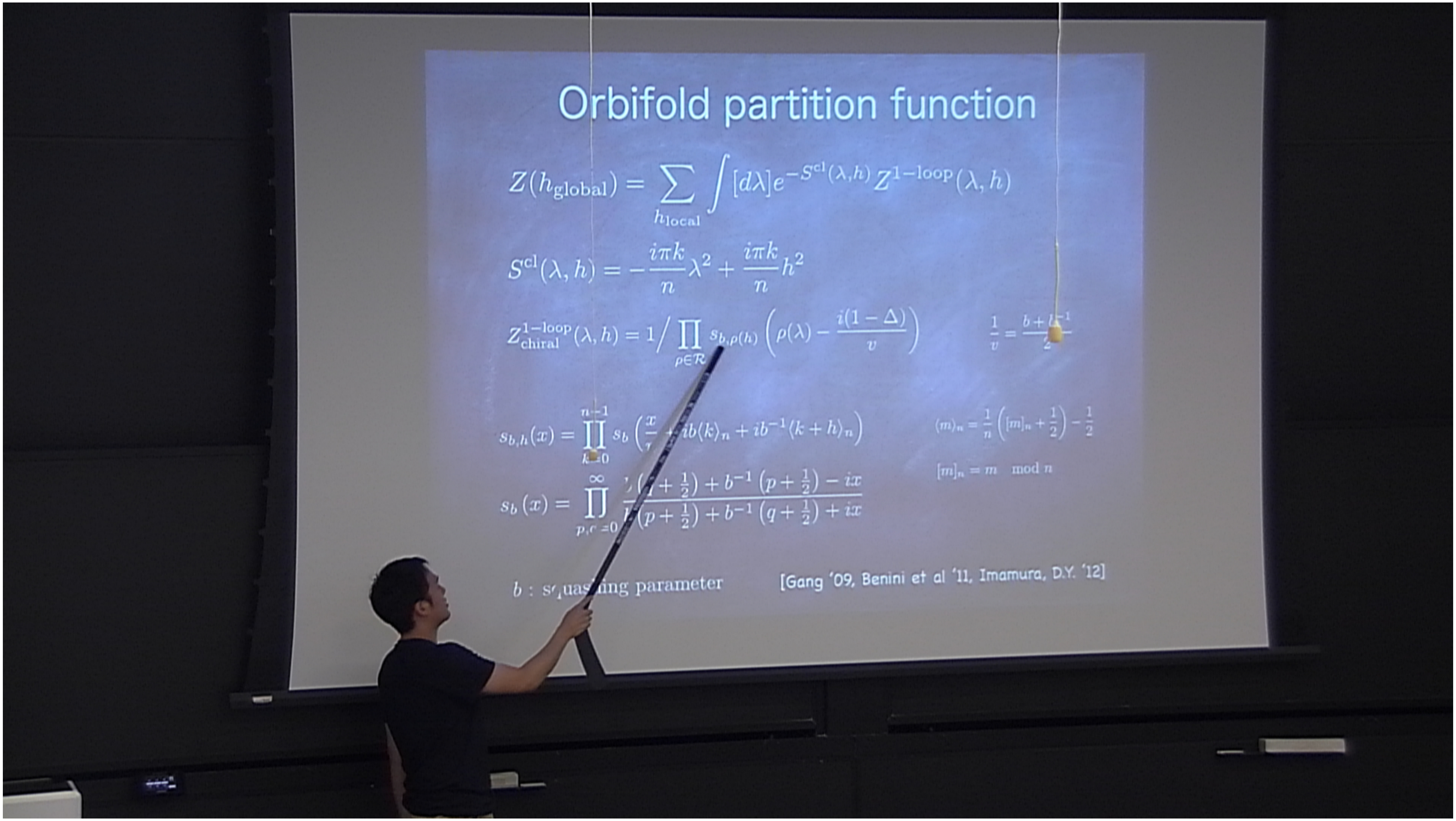


Double sine function

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$Z^{1-\text{loop}}_{\text{chiral}}(\lambda, h) = 1 \Big/ \prod_{\rho \in \mathcal{R}} s_{b,\rho(h)}\left(\rho(\lambda) - \frac{i(1-\Delta)}{v}\right)$
$\langle m \rangle_n = \frac{1}{n}\left([m]_n + \frac{1}{2}\right) - \frac{1}{2}$
$[m]_n = m \mod n$
b : squashing parameter
[Gang '09, Benini et al '11, Imamura, D.Y. '12]

- Numerical check for dual theories

Numerical evaluation

$$s_b(\lambda) = \prod_{p,q=0}^{\infty} \frac{b\left(p+\frac{1}{2}\right) + b^{-1}(q+\frac{1}{2}) - i\lambda}{b(p+\frac{1}{2}) + b^{-1}(q+\frac{1}{2}) + i\lambda}$$

$$= \exp\left[-\frac{i\pi}{2}\left(\lambda^2 + \frac{b^2+b^{-2}}{12}\right)\right] \frac{(-q^{1/2}x;q)_\infty}{(-\widetilde{q}^{1/2}\widetilde{x};\widetilde{q})_\infty}$$

$$q = e^{2i\pi b^2}\,, \qquad \widetilde{q} = e^{2i\pi b^{-2}}, \qquad x = e^{2\pi b\lambda}\,, \qquad \widetilde{x} = e^{2\pi b^{-1}\lambda}\,,$$

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x)$$ called q-deformed Pochhammer function.

Numerical evaluation

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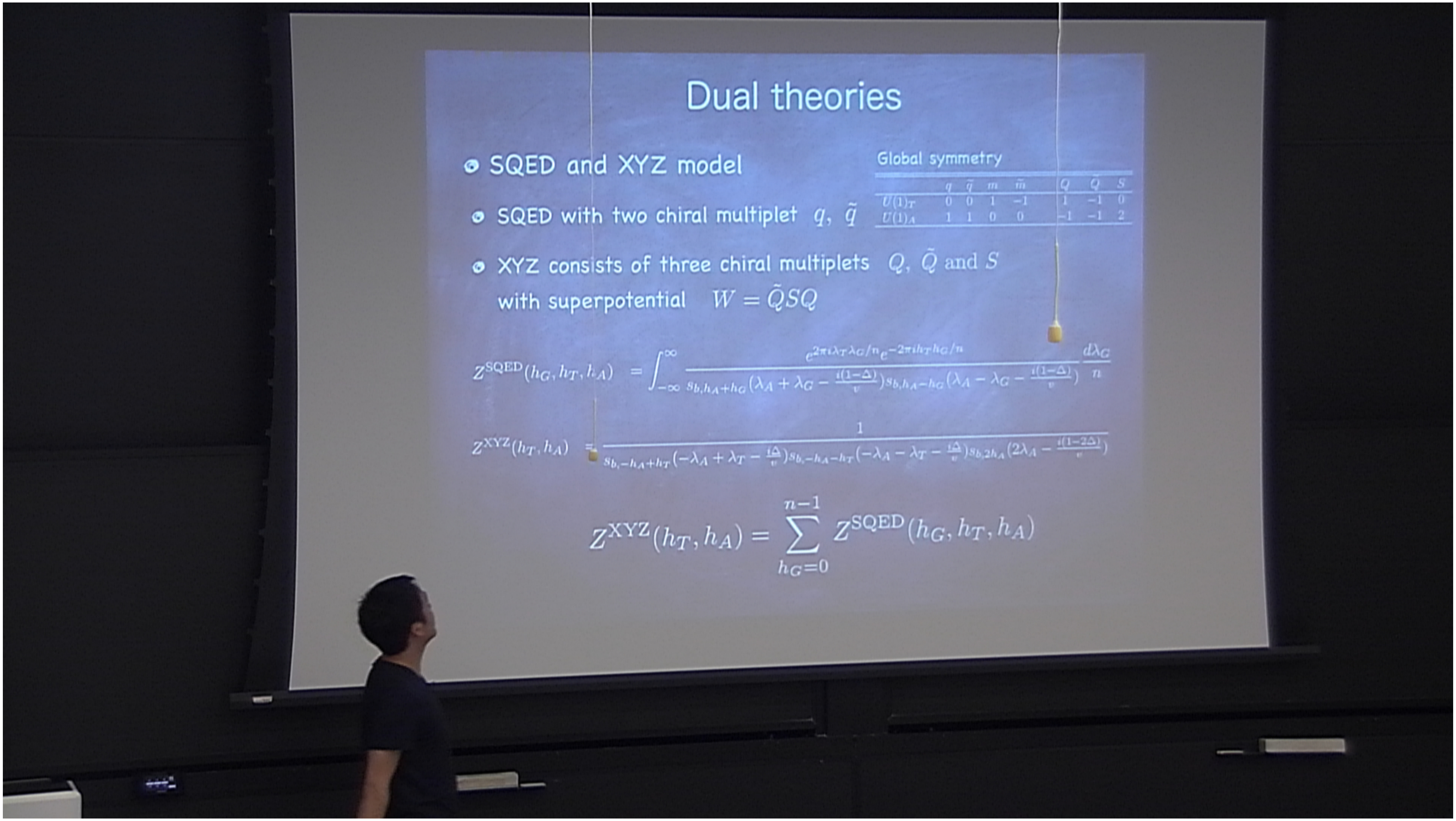
Point is that Mathematica knows this function !


Dual theories
SQED and XYZ model
Global symmetry
	q	$\tilde{q}$	m	$\tilde{m}$	Q	$\tilde{Q}$	S
$U(1)_T$	0	0	1	−1	1	−1	0
$U(1)_A$	1	1	0	0	−1	−1	2
SQED with two chiral multiplet $q,\ \tilde{q}$
XYZ consists of three chiral multiplets $Q,\ \tilde{Q}$ and S
with superpotential $W = \tilde{Q}SQ$
$Z^{\text{SQED}}(h_G, h_T, h_A) = \int_{-\infty}^{\infty} \frac{e^{2\pi i \lambda_T \lambda_G/n} e^{-2\pi i h_T h_G/n}}{s_{b,h_A+h_G}(\lambda_A + \lambda_G - \frac{i(1-\Delta)}{v}) s_{b,h_A-h_G}(\lambda_A - \lambda_G - \frac{i(1-\Delta)}{v})} \frac{d\lambda_G}{n}$
$Z^{\text{XYZ}}(h_T, h_A) = \frac{1}{s_{b,-h_A+h_T}(-\lambda_A + \lambda_T - \frac{i\Delta}{v}) s_{b,-h_A-h_T}(-\lambda_A - \lambda_T - \frac{i\Delta}{v}) s_{b,2h_A}(2\lambda_A - \frac{i(1-2\Delta)}{v})}$
$Z^{\text{XYZ}}(h_T, h_A) = \sum_{h_G=0}^{n-1} Z^{\text{SQED}}(h_G, h_T, h_A)$

Dual theories

- SQED and XYZ model
  - SQED with two chiral multiplet $q,\ \tilde{q}$
  - XYZ consists of three chiral multiplets $Q,\ \tilde{Q}$ and S with superpotential $W = \tilde{Q}SQ$

Global symmetry

	q	$\tilde{q}$	m	$\widetilde{m}$	Q	$\tilde{Q}$	S
$U(1)_T$	0	0	1	−1	1	−1	0
$U(1)_A$	1	1	0	0	−1	−1	2

$$Z^{\rm SQED}(h_G, h_T, h_A) = \int_{-\infty}^{\infty} \frac{e^{2\pi i \lambda_T \lambda_G/n} e^{-2\pi i h_T h_G/n}}{s_{b,h_A+h_G}(\lambda_A + \lambda_G - \frac{i(1-\Delta)}{v}) s_{b,h_A-h_G}(\lambda_A - \lambda_G - \frac{i(1-\Delta)}{v})} \frac{d\lambda_G}{n}$$

$$Z^{\rm XYZ}(h_T, h_A) = \frac{1}{s_{b,-h_A+h_T}(-\lambda_A + \lambda_T - \frac{i\Delta}{v}) s_{b,-h_A-h_T}(-\lambda_A - \lambda_T - \frac{i\Delta}{v}) s_{b,2h_A}(2\lambda_A - \frac{i(1-2\Delta)}{v})}$$

$$Z^{\rm XYZ}(h_T, h_A) = \sum_{h_G=0}^{n-1} Z^{\rm SQED}(h_G, h_T, h_A)$$

Dual theories

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	q	$\tilde{q}$	m	$\tilde{m}$	Q	$\tilde{Q}$	S
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$$Z^{\mathrm{XYZ}}(h_T, h_A) = \sum_{h_G=0}^{n-1} Z^{\mathrm{SQED}}(h_G, h_T, h_A)$$

Dual theories

- Numerical check

$\mathcal{N}=2$ SQED $\longleftrightarrow$ XYZ model

$n=3$ case

$$Z^{SQED}_{h=0} = 0.394$$

$$Z^{SQED}_{h=1} = -0.298$$

$$Z^{SQED}_{h=2} = -0.125 \quad (+$$

$$Z^{SQED} = -0.031 \longleftarrow$$

$$Z^{XYZ} = 0.817$$

$\uparrow$

Not match !?

Dual theories

- Numerical check

$\mathcal{N}=2$ SQED $\longleftrightarrow$ XYZ model

$n=3$ case

$Z_{h=0}^{SQED} = +(\ 0.394\ )$

$Z_{h=1}^{SQED} = -(-0.298\ )$

$Z_{h=2}^{SQED} = -(-0.125\ )$ (+

$Z^{SQED} = \quad 0.817$ $\longleftarrow$ Appropriate signs lead to coincidence !

$Z^{XYZ} = 0.817$

Dual theories

$\mathcal{N}=2$ SQED $\longleftrightarrow$ XYZ model

~~$Z^{\rm XYZ}(h_T,h_A)=\sum_{h_G=0}^{n-1} Z^{\rm SQED}(h_G,h_T,h_A)$~~ ⇒ $Z^{\rm XYZ}(h_T,h_A)=\sum_{h_G=0}^{n-1}\sigma(h_G,h_T,h_A)Z^{\rm SQED}(h_G,h_T,h_A)$

Trivial case

$$\sigma(h_G,h_T,0)=1$$

Non-trivial case (ex.)

$$\sigma(h_G,h_T,h_A)=(\sigma^{(n)}_{h_A})_{h_G,h_T}$$

$$\sigma_1^{(2)}=\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}$$

$$\sigma_1^{(3)}=\sigma_2^{(3)}=\begin{pmatrix}-1 & 1 & 1\\ 1 & -1 & -1\\ 1 & -1 & -1\end{pmatrix}$$

$$\sigma_1^{(4)}=\sigma_3^{(4)}=\begin{pmatrix}-1 & 1 & 1 & 1\\ 1 & -1 & -1 & -1\\ 1 & -1 & -1 & -1\\ 1 & -1 & -1 & -1\end{pmatrix}$$

$$\sigma_2^{(4)}=\begin{pmatrix}1 & -1 & 1 & -1\\ -1 & 1 & -1 & 1\\ 1 & -1 & 1 & -1\\ -1 & 1 & -1 & 1\end{pmatrix}$$

General formula (proposal)

$$\sigma(h_G,h_T,h_A)=(-1)^{f(h_A)+g(h_A,h_G)+g(h_A,h_T)}$$

$$f(h)=\min(|h+n\mathbb{Z}|),\quad g(h,h')=\min(f(h),f(h'))$$

Dual theories

$\mathcal{N}=2$ SQED $\longleftrightarrow$ XYZ model

~~$Z^{\mathrm{XYZ}}(h_T,h_A)=\sum_{h_G=0}^{n-1} Z^{\mathrm{SQED}}(h_G,h_T,h_A)$~~ ⇒ $Z^{\mathrm{XYZ}}(h_T,h_A)=\sum_{h_G=0}^{n-1} \sigma(h_G,h_T,h_A) Z^{\mathrm{SQED}}(h_G,h_T,h_A)$

Trivial case

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Non-trivial case (ex.)

$$\sigma(h_G,h_T,h_A)=(\sigma^{(n)}_{h_A})_{h_G,h_T}$$

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$$f(h)=\min(|h+n\mathbb{Z}|),\quad g(h,h')=\min(f(h),f(h'))$$


Dual theories
$\mathcal{N}=2$ SQED
XYZ model
Trivial case
$\sigma(h_G, h_T, 0) = 1$
Non-trivial case (ex.)
General formula (proposal)

Dual theories

- $S^3/\mathbb{Z}_{2k+1}$ case

Define the following function

$$\sigma_h = (-1)^{[h]_n([h]_n-(-1)^{(n-1)/2})/2} \qquad \sigma_h = \sigma_{-h}$$

Then, the general formula can be rewritten as follows.

$$(-1)^{f(h)} = \sigma_{2h} \qquad (-1)^{g(h,h')} = \sigma_{h+h'}\sigma_{h-h'}$$

$$\sigma(h, h_V, h_A) = \sigma_{h_A+h}\sigma_{h_A-h}\sigma_{-h_A+h_V}\sigma_{-h_A-h_V}\sigma_{2h_A}$$

Phase factor can be absorbed into the definition of the orbifolded double sine function !

$$\widehat{s}_{b,h}(z) = \sigma_h s_{b,h}(z)$$

$$\widehat{Z}^{\mathrm{XYZ}}(h_V, h_A) = \sum_{h=0}^{n-1} \widehat{Z}^{\mathrm{SQED}}(h, h_V, h_A)$$

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- Classical ambiguity of CS term

Classical ambiguity

- Niarchos duality (Seiberg like duality) [Niarchos '08]
  - $U(2)_1$ vector + an adjoint X with $W = \mathrm{tr} X^3$
  - Trivial theory $\to Z = 1$

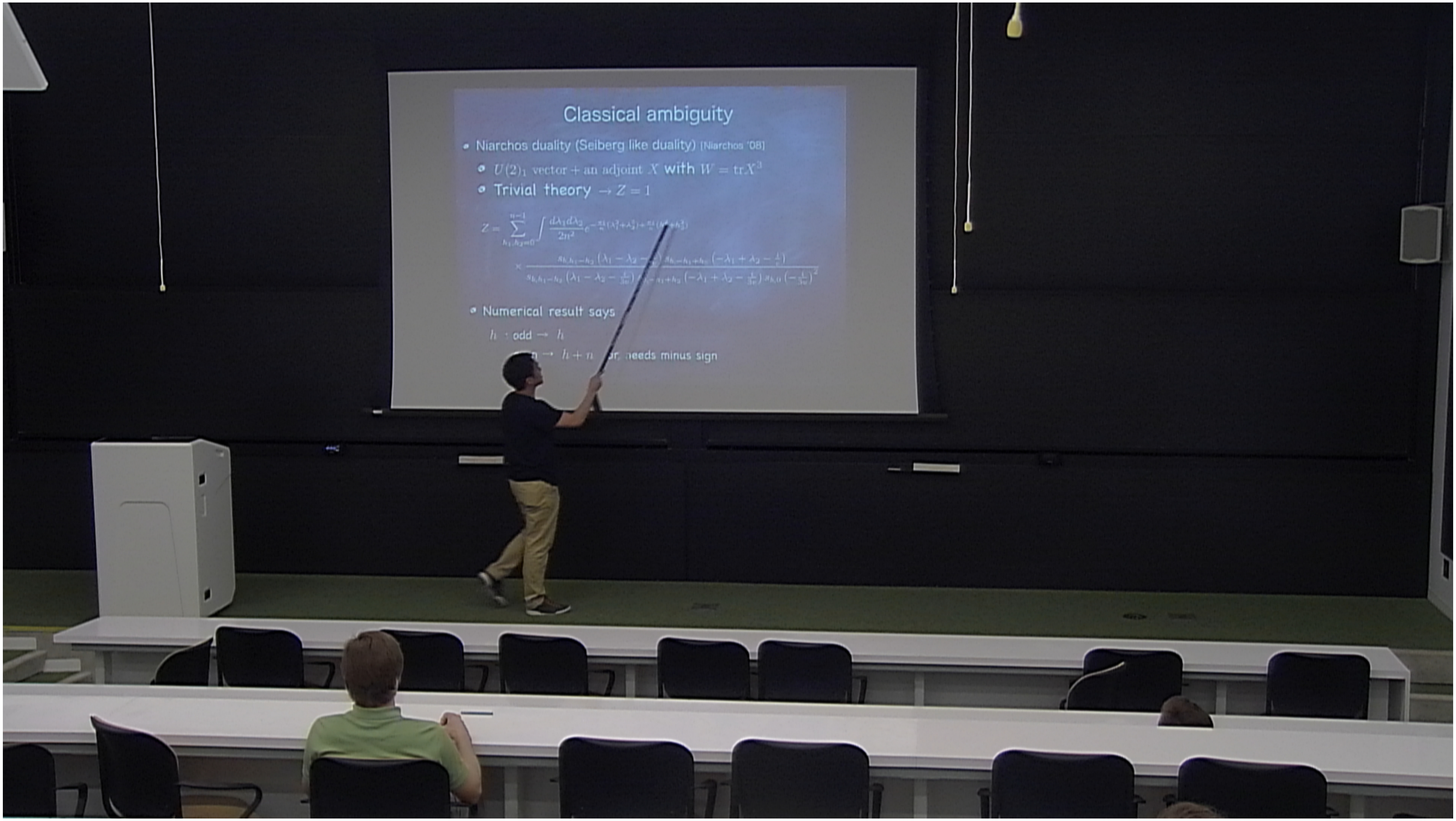
$$Z = \sum_{h_1,h_2=0}^{n-1} \int \frac{d\lambda_1 d\lambda_2}{2n^2} e^{-\frac{\pi i}{n}(\lambda_1^2+\lambda_2^2)+\frac{\pi i}{n}(h_1^2+h_2^2)}$$

$$\times \frac{s_{b,h_1-h_2}\left(\lambda_1-\lambda_2-\frac{i}{v}\right) s_{b,-h_1+h_2}\left(-\lambda_1+\lambda_2-\frac{i}{v}\right)}{s_{b,h_1-h_2}\left(\lambda_1-\lambda_2-\frac{i}{3v}\right) s_{b,-h_1+h_2}\left(-\lambda_1+\lambda_2-\frac{i}{3v}\right) s_{b,0}\left(-\frac{i}{3v}\right)^2}$$

- Numerical result says

h : odd $\to$ h

even $\to$ $h+n$ or, needs minus sign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This means that the CS term is not well-defined.
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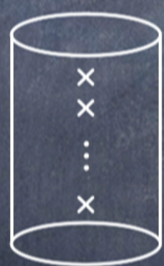
$$\int_M AdA = \int_X F \wedge F \qquad M = \partial X = S^3/\mathbb{Z}_n$$

$X \sim \mathbb{C}^2/\mathbb{Z}_n$ (n-centered Taub-NUT)



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$n=3, k=1$ case	$h=0$	$h=1$	$h=2$
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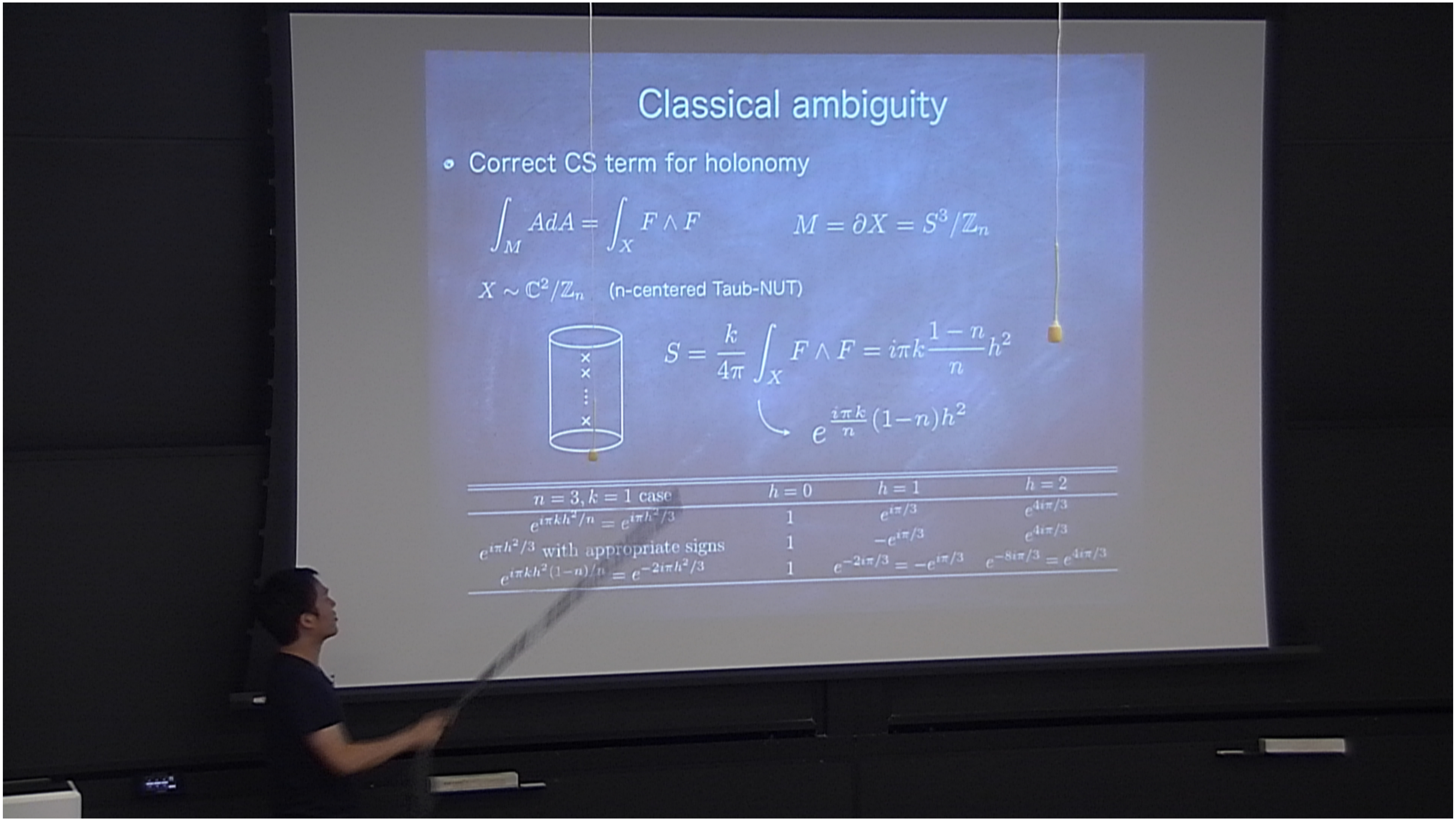
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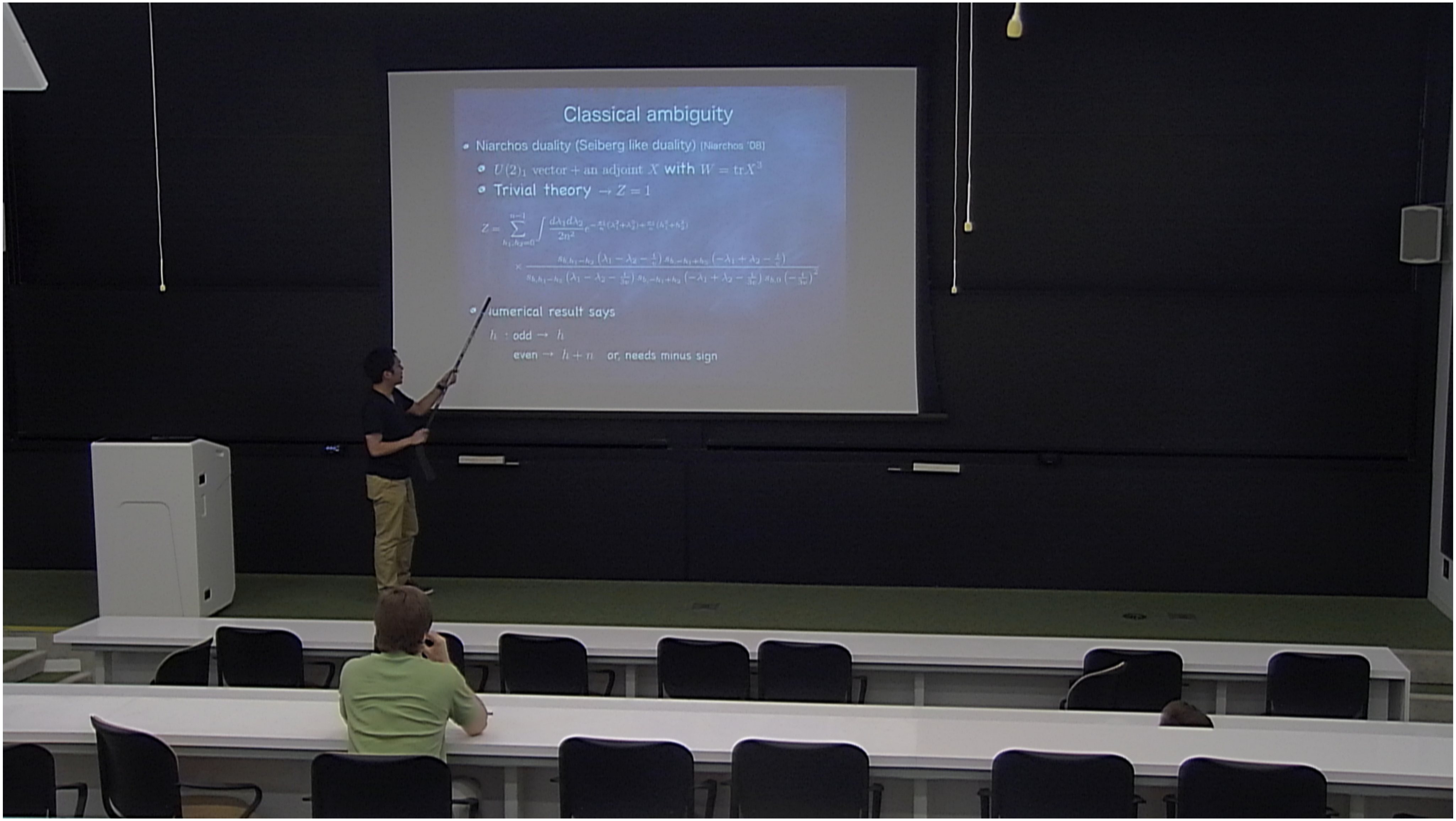

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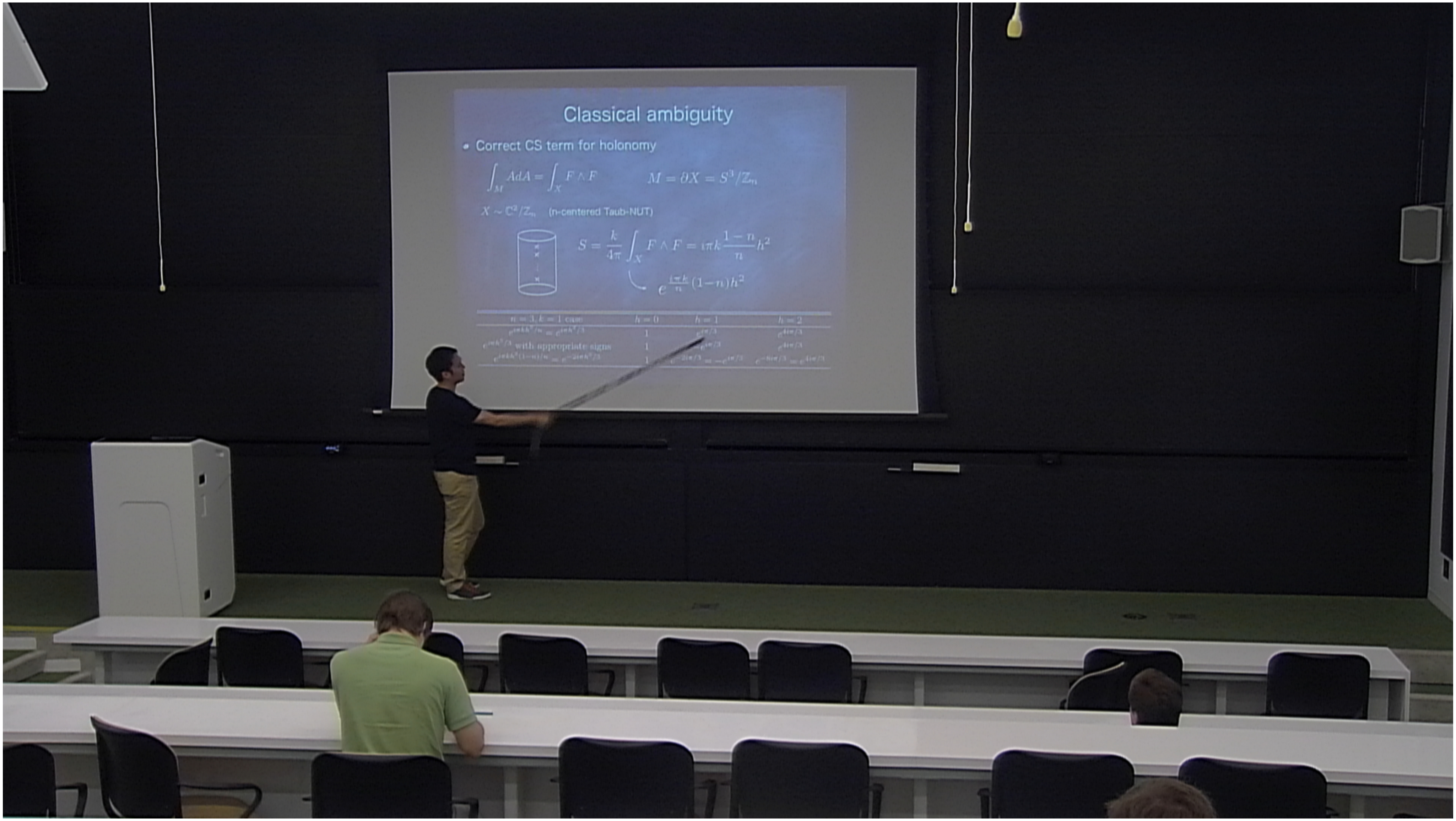
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- Orbifolding procedure and improved orbifold partition function

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Orbifolding procedure

- Orbifolding of a double sine function

$$s_b\left(\lambda - \frac{i}{v}\right) \quad \to \quad s_{b,h}\left(\lambda - \frac{i}{v}\right)$$

- Replace $\boldsymbol{\lambda}$ as $\lambda \to \dfrac{1}{n}\left(\lambda + ibp + ib^{-1}q\right)$
- Take a product over $0 \le p, q < n$ with $q - p = h \mod n$

$$s_{b,h}(x) = \prod_{k=0}^{n-1} s_b\left(\frac{x}{n} + ib\langle k\rangle_n + ib^{-1}\langle k+h\rangle_n\right) \qquad \langle m\rangle_n = \frac{1}{n}\left([m]_n + \frac{1}{2}\right) - \frac{1}{2}$$

- The same procedure for the CS term

$$e^{i\pi k\lambda^2} \to e^{\frac{i\pi k}{n}\left(\lambda^2 + [h]_n(n - [h]_n)\right)}$$

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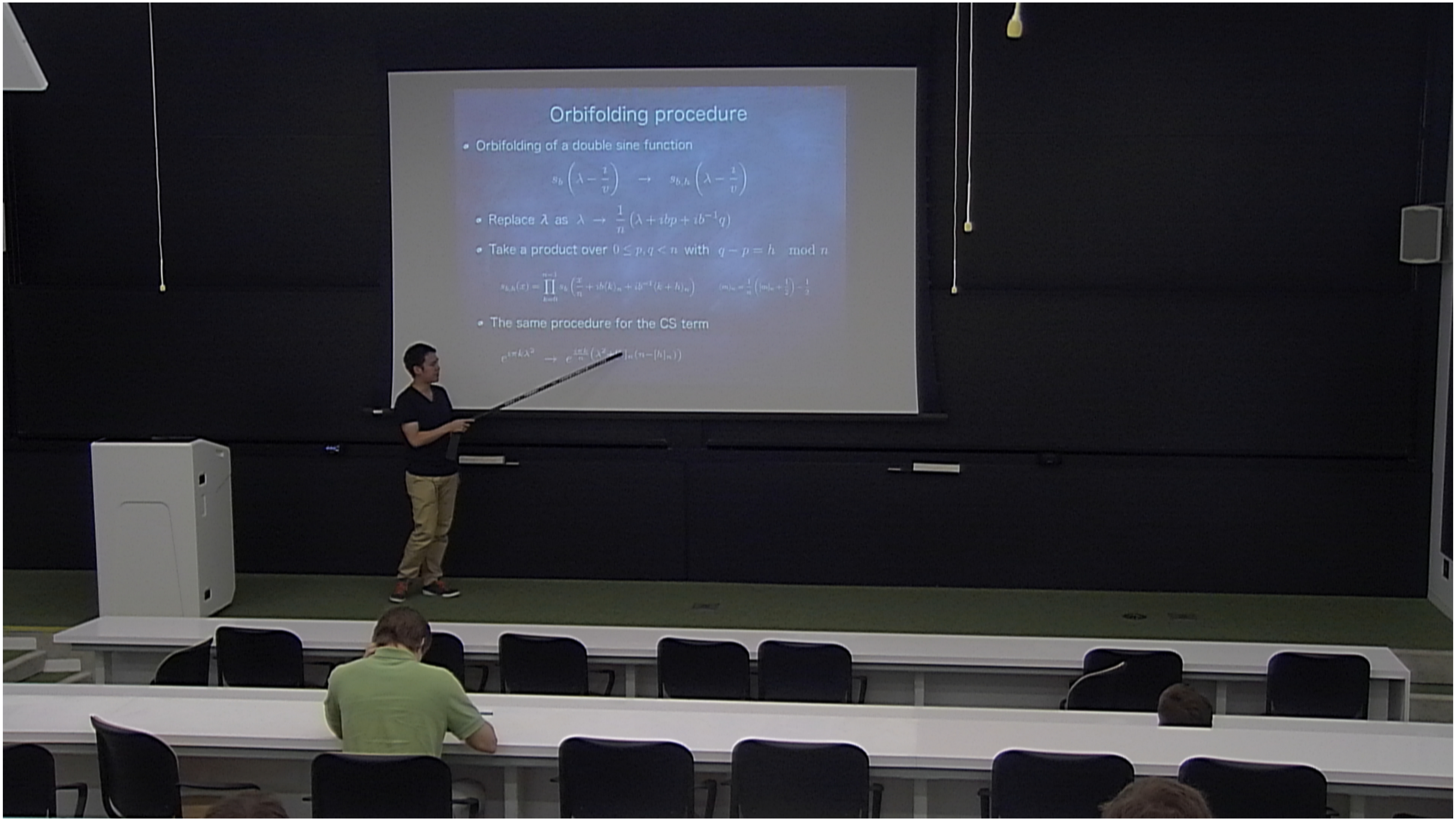
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Orbifolding procedure

- Fractional CS term from fermion 1-loop

	$U(1)_R$	$U(1)_F$
Φ	0	$+1$
ψ	-1	$+1$

$$\frac{1/2}{4\pi}\int (A_F - A_R)d(A_F - A_R)$$

$$s_b(\lambda) = \prod_{p,q=0}^{\infty} \frac{b\left(p+\frac{1}{2}\right) + b^{-1}(q+\frac{1}{2}) - i\lambda}{b(p+\frac{1}{2}) + b^{-1}(q+\frac{1}{2}) + i\lambda}$$

$$= \exp\left[-\frac{i\pi}{2}\left(\lambda^2 + \frac{b^2+b^{-2}}{12}\right)\right]\frac{(-q^{1/2}x;q)_\infty}{(-\widetilde{q}^{1/2}\widetilde{x};\widetilde{q})_\infty}$$

$$\to e^{-\frac{i\pi}{2n}\left(\lambda^2+[h]_n(n-[h]_n)\right)}$$

Tetrahedron theory (Single chiral multiplet with the bare CS term)

$$1/Z_\Delta(\lambda,h) = 1/\left(e^{\frac{i\pi}{2n}\left(\lambda^2+[h]_n(n-[h]_n)\right)} s_{b,h}(\mu - iQ/2)\right)$$

Orbifolding procedure

- Correct orbifolded double sine function

$$k = k' + k_{\rm bare} \longrightarrow$$ used to cancel the fractional CS term

$\downarrow$

must be integer

$$Z(\lambda,h) = e^{-\frac{i\pi k'}{n}(\lambda^2+(n-1)h^2)}\frac{\prod_A Z_\Delta(\lambda_A,h_A)}{\prod_I Z_\Delta(\lambda_I,h_I)}$$

$$= e^{-\frac{i\pi k}{n}(\lambda^2+(n-1)h^2)}\frac{\prod_A \sigma_n(h_A)s_{b,h_A}(\lambda_A)}{\prod_I \sigma_n(h_I)s_{b,h_I}(\lambda_I)}$$

$$\sigma_n(h) = e^{-\frac{i\pi}{2n}(n-1)h^2}e^{\frac{i\pi}{2n}[h]_n(n-[h]_n)}$$

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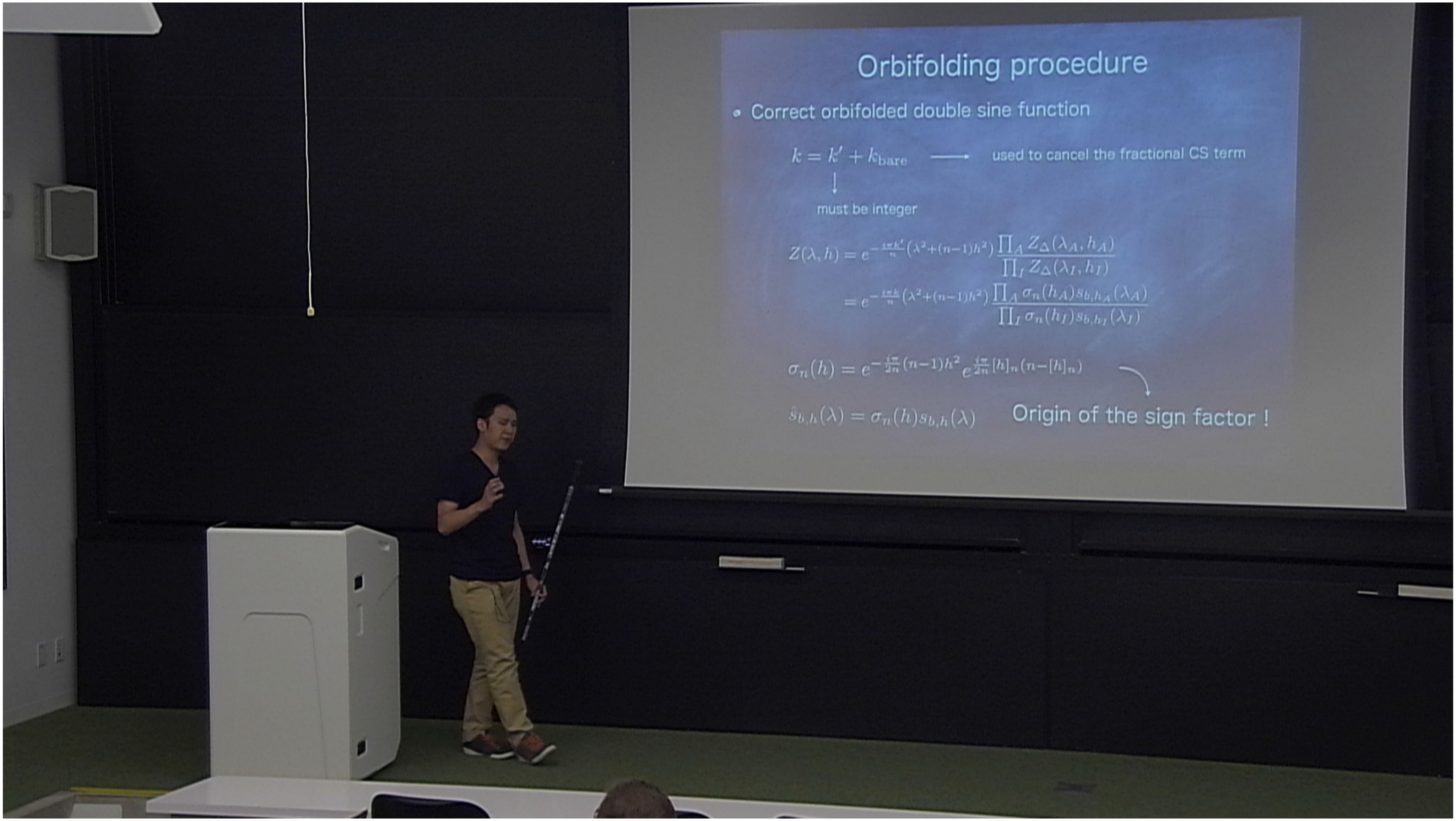
Origin of the sign factor !



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Dual theories

- SQED and XYZ model

Global symmetry

	q	$\tilde{q}$	m	$\tilde{m}$	Q	$\tilde{Q}$	S
$U(1)_T$	0	0	1	−1	1	−1	0
$U(1)_A$	1	1	0	0	−1	−1	2

$$Z^{\mathrm{SQED}}(h_G,h_T,h_A) = \int_{-\infty}^{\infty} \frac{e^{2\pi i\lambda_T\lambda_G/n}e^{-2\pi i h_T h_G/n}}{s_{b,h_A+h_G}(\lambda_A+\lambda_G-\frac{i(1-\Delta)}{v})s_{b,h_A-h_G}(\lambda_A-\lambda_G-\frac{i(1-\Delta)}{v})}\frac{d\lambda_G}{n}$$

$$Z^{\mathrm{XYZ}}(h_T,h_A) = \frac{1}{s_{b,-h_A+h_T}(-\lambda_A+\lambda_T-\frac{i\Delta}{v})s_{b,-h_A-h_T}(-\lambda_A-\lambda_T-\frac{i\Delta}{v})s_{b,2h_A}(2\lambda_A-\frac{i(1-2\Delta)}{v})}$$

$$Z^{\mathrm{XYZ}}(h_T,h_A) = \sum_{h_G=0}^{n-1}\sigma_n(h_G,h_T,h_A)Z^{\mathrm{SQED}}(h_G,h_T,h_A)$$

$$\sigma_n(h_G,h_T,h_A) = \frac{\sigma_n(-h_A+h_T)\sigma_n(-h_A-h_T)\sigma_n(2h_A)}{\sigma_n(h_A+h_G)\sigma_n(h_A-h_G)}$$

Conclusions

- We found that the relative phase factors become very important on Lens space.
- We understood the origin of the phase factors.
- Factorisation of the orbifold partition function
- Relation to the 4d Lens space index ??


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