

Title: 13/14 PSI - Integral Transformations & Green's Function - Lecture 3

Date: Aug 29, 2013 09:00 AM

URL: <http://pirsa.org/13080030>

Abstract:

II) CLASSICAL ELECTRODYNAMICS

a)

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi \mathbf{j}$$
$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$
$$\nabla \cdot \mathbf{E} = 4\pi \rho$$

II) CLASSICAL ELECTRODYNAMICS

a)

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi \mathbf{j} \quad (1)$$
$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (2)$$
$$\nabla \cdot \mathbf{E} = 4\pi \rho \quad (3)$$
$$\nabla \cdot \mathbf{B} = 0 \quad (4)$$

ICS

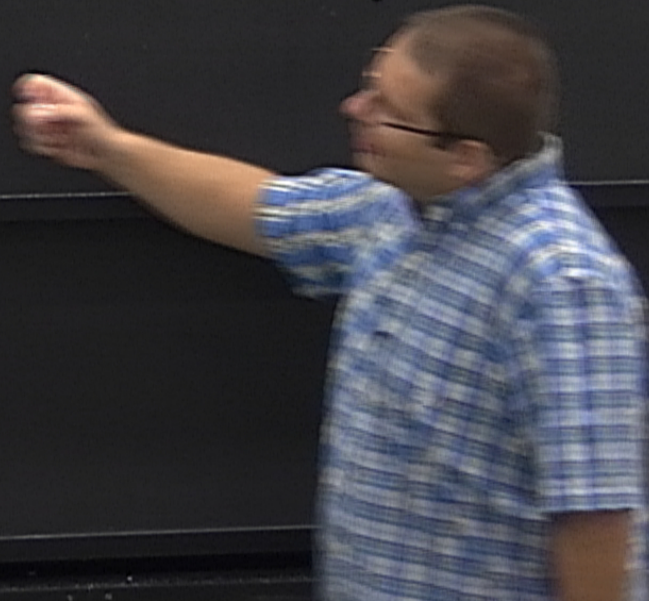
∴ DOF: 8 EQS... FOR 6 UNKNOWN (E, B) ?

ICS

1) DOF: 8 EQS... FOR 6 UNKNOWN (E, B) ?

$$\nabla \cdot (2) \quad \underbrace{\nabla \cdot (\nabla \times \mathbf{E})}_{0} + \frac{\partial \nabla \cdot \mathbf{B}}{\partial t} = 0 \quad \Leftrightarrow (4) \quad \text{INITIAL COND}$$

$$\nabla \cdot (1) \quad - \frac{\partial \nabla \cdot \mathbf{E}}{\partial t} = 4\pi \nabla \cdot \mathbf{j}$$



λ) DOF: 8 EQS .. FOR 6 UNKNOWN (E, B) ?

$$\nabla \cdot (2) \quad \underbrace{\nabla \cdot (\nabla \times \mathbf{E})}_{0} + \frac{\partial \nabla \cdot \mathbf{B}}{\partial t} = 0 \quad \Leftrightarrow (4) \quad \text{INITIAL COND.}$$

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$$\nabla \cdot (2) \quad \underbrace{\nabla \cdot (\nabla \times \mathbf{E})}_0 + \frac{\partial \nabla \cdot \mathbf{B}}{\partial t} = 0 \quad \Leftrightarrow (4) \dots \text{INITIAL COND.}$$

$$\nabla \cdot (1) \quad - \frac{\partial \nabla \cdot \mathbf{E}}{\partial t} = 4\pi \underbrace{\nabla \cdot \mathbf{j}}_{-\partial_t \rho}$$

$$\frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) = 4\pi \frac{\partial}{\partial t} \rho$$

$$\nabla \cdot (2) \quad \underbrace{\nabla \cdot (\nabla \times \mathbf{E})}_0 + \frac{\partial \rho}{\partial t} = 0 \quad \leftrightarrow (4) \dots \text{INITIAL COND.}$$

$$\nabla \cdot (1) \quad - \frac{\partial \nabla \cdot \mathbf{E}}{\partial t} = 4\pi \underbrace{\nabla \cdot \mathbf{j}}_{-\partial_t \rho}$$

$$\frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) = 4\pi \frac{\partial \rho}{\partial t}$$

(3) ... INITIAL COND.

INTERPRETATION: { EVOLUTIONARY POINT OF VIEW.
RELATIVITY.

D FIELD = SOURCE

EVOLUTIONARY POINT OF VIEW.

RELATIVITY.

$$D \text{ FIELD} = \text{SOURCE} \quad (G_{\mu\nu} = 8\pi T_{\mu\nu})$$

RELATIVITY

D FIELD = SDC

iii) OTHER FORMS

$E, B \leftrightarrow F$

$$\begin{aligned} \nabla_\mu F^{\mu\nu} &= 4\pi g^\mu \\ F_{[\mu\nu]\rho}{}_{;\sigma} &= 0 \end{aligned}$$

$\Leftrightarrow$

(SEPTEMBER)

RELATIVITY.

D FIELD = SOURCE

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

MS
F

$$\begin{aligned} \nabla_\mu F^{\mu\nu} &= 4\pi j^\nu \\ F_{[\mu\nu]} &= 0 \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} d * F &= j \\ dF &= 0 \end{aligned}$$

SEPTEMBER)

(JANUARY)

RELATIVITY.

D FIELD = SOURCE

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MS
F

$$\nabla_\mu F^{\mu\nu} = 4\pi j^\mu$$

$$F_{[\mu\nu]j}{}_{;\alpha} = 0$$

$\Leftrightarrow$

$$d * F = j$$

$$dF = 0$$

SEPTEMBER)

(JANUARY)

6) ELECTROMAGNETIC POTENTIALS

$$(4) \Leftrightarrow \boxed{B = \nabla \times A} \quad \checkmark$$

b) ELECTROMAGNETIC POTENTIALS

$$(4) \Leftrightarrow \boxed{B = \nabla \times A} \quad \checkmark$$

$$(2) \quad \nabla \times E + \frac{\partial \nabla \times A}{\partial t} = 0 = \nabla \times \left(E + \frac{\partial A}{\partial t} \right) = 0$$

6) ELECTROMAGNETIC POTENTIALS

$$(4) \Leftrightarrow \boxed{B = \nabla \times A} \quad \checkmark$$

$$(2) \quad \nabla \times E + \frac{\partial \nabla \times A}{\partial t} = 0 = \nabla \times \left(E + \underbrace{\frac{\partial A}{\partial t}}_{\nabla \phi} \right) = 0$$

POTENTIALS



$$0 = \nabla \times \left(\underbrace{E + \frac{\partial A}{\partial t}}_{-\nabla \phi} \right) = 0$$

$$E = -\nabla \phi - \frac{\partial A}{\partial t}$$

MAGNETIC POTENTIALS

$$\boxed{B = \nabla \times A} \quad \checkmark$$

$$\mathbf{E} + \frac{\partial \nabla \times \mathbf{A}}{\partial t} = 0 = \nabla \times \left(\underbrace{\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}}_{-\nabla \phi} \right) = 0$$

$$\boxed{\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}} \quad \checkmark$$

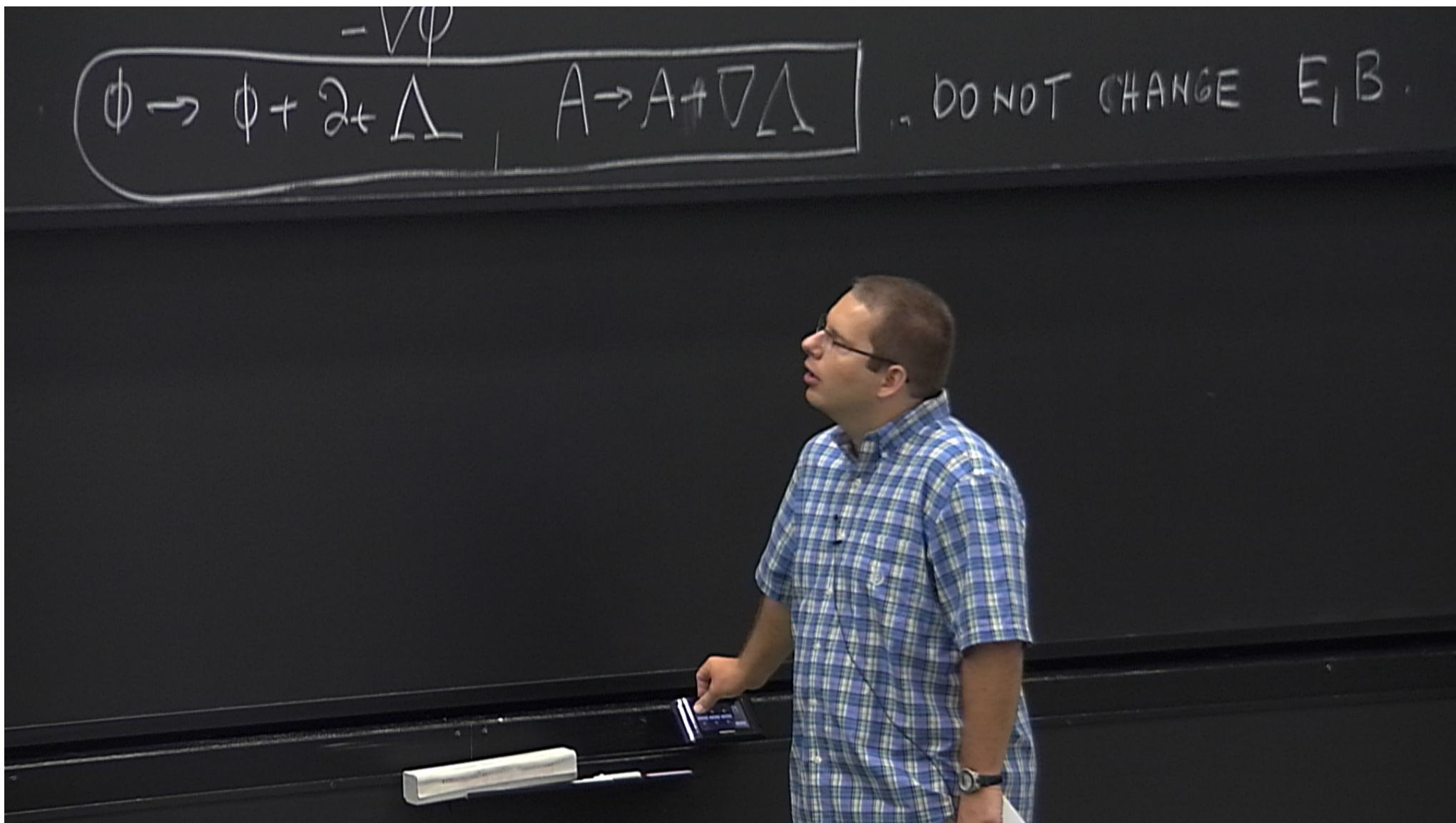
$$\Rightarrow \boxed{B = \nabla \times A} \quad \checkmark$$

$$\nabla \times E + \frac{\partial \nabla \times A}{\partial t} = 0 = \nabla \times \left(\underbrace{E + \frac{\partial A}{\partial t}}_{-\nabla \phi} \right) = 0$$

$$\boxed{E = -\nabla \phi - \frac{\partial A}{\partial t}} \quad \checkmark$$

GAUGE FREEDOM

$$\boxed{\phi \rightarrow \phi + \partial_t \Lambda, \quad A \rightarrow A + \nabla \Lambda}$$



CAN IMPOSE GAUGE CONDITION:

$$\partial_t \phi + \nabla \cdot \mathbf{A} = 0$$

LORENTZ

$$A^\mu = (\phi, \mathbf{A}) \quad \left[\partial_\mu A^\mu = 0 \right]$$

N IMPOSE GAUGE CONDITION.

$$\boxed{\partial_t \phi + \nabla \cdot \mathbf{A} = 0}$$

LORENTZ

$$A^\mu = (\phi, \mathbf{A})$$

$$\partial_\mu = (\partial_t, \partial_x)$$

$$\boxed{\partial_\mu A^\mu = 0}$$

LORENTZ

$$\partial_\mu = (\partial_t, \partial_x)$$

C) WAVE EQ. REMAIN (1) & (3)

$$(3): \nabla \cdot \left(-\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \right) = 4\pi \rho = -\Delta \phi - \frac{\partial}{\partial t} \nabla \cdot \mathbf{A}$$

LORENTZ

$$\partial_\mu = (\partial_t, \partial_x)$$

C) WAVE EQ. REMAIN (1) & (3)

$$(3): \nabla \cdot \left(-\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \right) = 4\pi \rho = -\Delta \phi - \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) =$$
$$-\frac{\partial}{\partial t} \phi$$

$$A=0$$

$$A^\mu = (\phi, A)$$

$$\partial_\mu A^\mu = 0$$

$$\partial_\mu = (\partial_t, \partial_x)$$

D'ALEMBERT OP

$$\square = -\frac{\partial^2}{\partial t^2} + \Delta$$

(3)

$$= -\Delta \phi - \frac{\partial}{\partial t} \left(\underbrace{\nabla \cdot A}_{-\frac{\partial}{\partial t} \phi} \right) = \left(-\Delta + \frac{\partial^2}{\partial t^2} \right) \phi = 4\pi g$$

$$A^\mu = (\phi, \mathbf{A}) \quad \boxed{\partial_\mu A^\mu = 0}$$

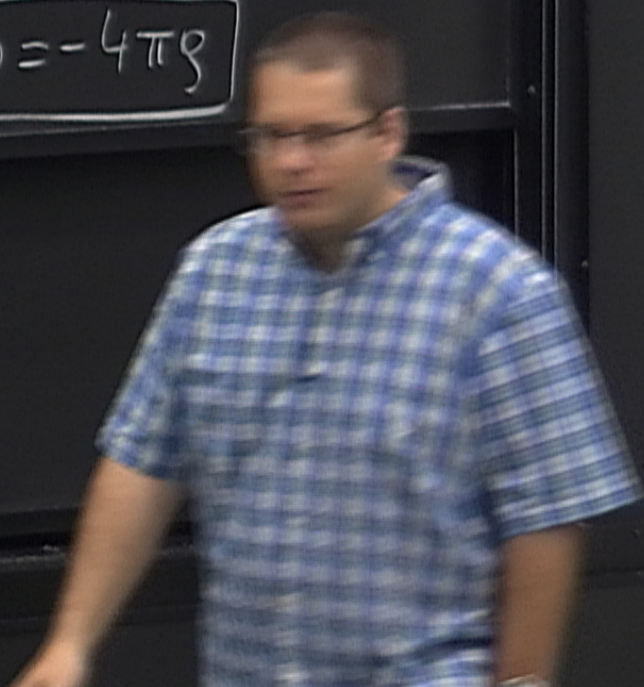
$$\partial_\mu = (\partial_t, \nabla)$$

D'ALEMBERT OP

$$\boxed{\square = -\frac{\partial^2}{\partial t^2} + \Delta}$$

$$\Delta \phi - \frac{\partial}{\partial t} \left(\nabla \cdot \mathbf{A} \right) = \left(-\Delta + \frac{\partial^2}{\partial t^2} \right) \phi = 4\pi \rho$$

$$\boxed{\square \phi = -4\pi \rho}$$



$$(1) : \underbrace{\nabla \times (\nabla \times A)} - \frac{\partial}{\partial t} \left(-\nabla \phi - \frac{\partial A}{\partial t} \right) = 4\pi j$$

$$\nabla \nabla \cdot A - \Delta A$$

$$(1): \underbrace{\nabla \times (\nabla \times A)} - \frac{\partial}{\partial t} \left(-\nabla \phi - \frac{\partial A}{\partial t} \right) = 4\pi j$$

$$\nabla \nabla \cdot A - \Delta A + \frac{\partial^2 A}{\partial t^2} + \nabla \frac{\partial \phi}{\partial t} = 4\pi j$$

$$(1): \nabla \times (\nabla \times A) - \frac{\partial}{\partial t} \left(-\nabla \phi - \frac{\partial A}{\partial t} \right) = 4\pi j$$

$$\nabla \nabla \cdot A - \Delta A + \frac{\partial^2 A}{\partial t^2} + \nabla \frac{\partial \phi}{\partial t} = 4\pi j$$

$$\Leftrightarrow \square A = -4\pi j$$

$$\square \phi = -4\pi \rho$$

$$\Leftrightarrow \square A = -4\pi j \quad \dots \text{SAME WAVE EQ.}$$

$$\square A^\mu = -4\pi j^\mu \quad j^\mu = (\rho, \vec{j})$$

$$\boxed{\square \phi = -4\pi \rho}$$

$$\Leftrightarrow \boxed{\square A = -4\pi j} \quad \dots \text{SAME WAVE EQ.}$$

$$\boxed{\square A^\mu = -4\pi j^\mu} \quad j^\mu = (\rho, \vec{j})$$

a) GREEN'S FUNCTION

$$\square G(\vec{R}, T) = -\delta(\vec{R})\delta(T)$$

$$\frac{1}{(2\pi)^4} \int d\vec{k} d\omega G_{\vec{k}\omega} e^{i\vec{k}\cdot\vec{R} - i\omega T}$$

a) GREEN'S FUNCTION

$$\square G(\vec{R}, T) = -\delta(\vec{R})\delta(T)$$

$$\frac{1}{(2\pi)^4} \int d\vec{k} d\omega G_{\vec{k}\omega} \square e^{i\vec{k}\cdot\vec{R} - i\omega T} =$$

$$(\vec{R}, T) = -\delta(\vec{R})\delta(T)$$

$$d\omega \underbrace{G_{\vec{k}\omega} \square}_{i\vec{k}\cdot\vec{R}-i\omega T} = \frac{-1}{(2\pi)^4} \int d\vec{k} d\omega \, e^{i\vec{k}\cdot\vec{R}-i\omega T}$$

$$\frac{(i\vec{k})^2 - (-i\omega)^2}{(-|\vec{k}|^2 + \omega^2)} e^{i\vec{k}\cdot\vec{R}-i\omega T}$$

$\Rightarrow$

• GOING BACK TO POSITION SPACE:

$$G(\vec{R}, T) = \frac{1}{(2\pi)^4} \int \frac{1}{k^2 - \omega^2} e^{i\vec{k} \cdot \vec{R} - i\omega T} d\vec{k} d\omega$$

$k^2 d\vec{k} d\omega$

$$\vec{k} \cdot \vec{R} - i\omega T \int \frac{d^3k}{(2\pi)^4} \frac{e^{i\vec{k} \cdot \vec{R} - i\omega T}}{k^2 - \omega^2} = \frac{1}{(2\pi)^4} \int \frac{e^{i\vec{k} \cdot \vec{R} - i\omega T}}{k^2 - \omega^2} k^2 dk d(\cos\theta) d\varphi$$

$$\vec{k} \cdot \vec{R} - i\omega T \int \frac{d^3k}{(2\pi)^4} \frac{e^{i\vec{k} \cdot \vec{R} \cos\theta - i\omega T}}{k^2 - \omega^2} k^2 dk d(\cos\theta) d\omega$$

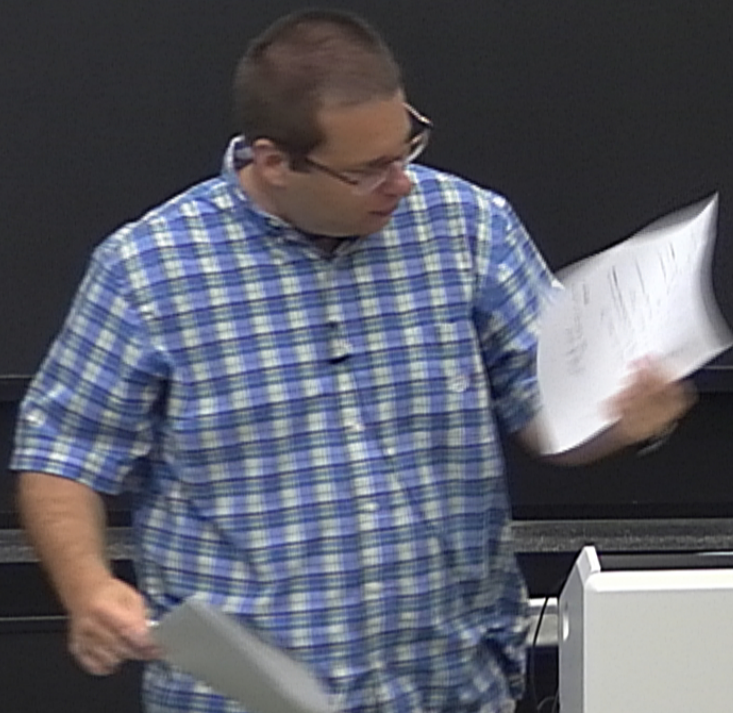
$$\begin{aligned}
 \vec{R} - i\omega T \int \frac{d\vec{R}}{k^2} d\omega &= \frac{2\pi}{(2\pi)^4} \int \frac{i k R \cos\theta - i\omega T}{k^2 - \omega^2} k^2 dk d(\cos\theta) d\omega \\
 &= \frac{1}{(2\pi)^3} \frac{1}{ikR} \int
 \end{aligned}$$

$$\begin{aligned}
 \vec{R} - i\omega T \int \vec{d\vec{r}} d\omega &= \frac{2\pi}{(2\pi)^4} \int \frac{i k R \cos\theta - i\omega T}{k^2 - \omega^2} k^2 dk d(\cos\theta) d\omega \\
 k^2 dk d(\cos\theta) d\varphi &= \frac{1}{(2\pi)^3} \frac{1}{R} \int \frac{e^{i k R} - e^{-i k R}}{k(k^2 - \omega^2)} k^2 dk e^{-i\omega T} d\omega
 \end{aligned}$$

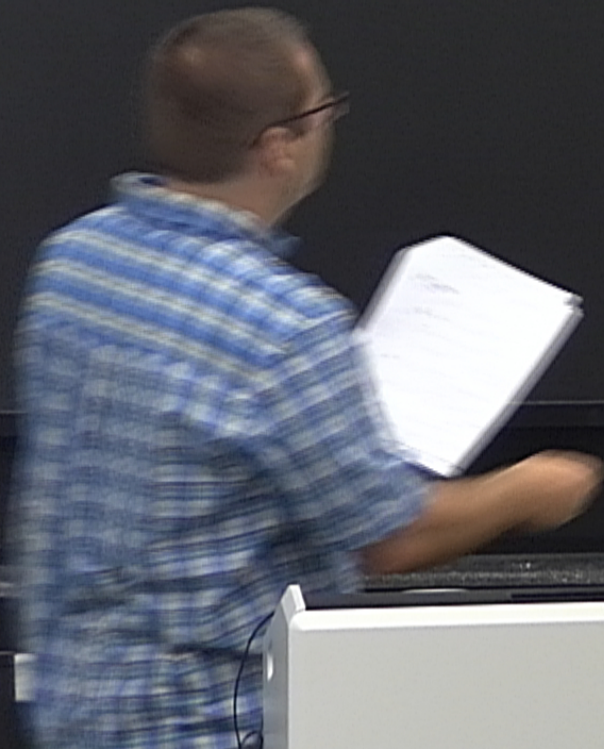
$$I = \int_{-\infty}^{\infty} \frac{e^{ikR}}{k^2 - \omega^2} k dk$$

$$\int_{-\infty}^{\infty} \frac{e^{ikR}}{k^2 - \omega^2} k dk$$

$$I = I_1 + I_2$$



$$I = I_1 + I_2 = i\pi \left(\text{Res}_{-\omega} \frac{e^{ikR}}{k^2 - \omega^2} + \text{Res}_{\omega} \frac{e^{ikR}}{k^2 - \omega^2} \right)$$



$$\begin{aligned}
 I = I_1 + I_2 &= i\pi \left(\operatorname{Res}_{-\omega} \frac{e^{ikR}}{k^2 - \omega^2} + \operatorname{Res}_{\omega} \frac{e^{ikR}}{k^2 - \omega^2} \right) \\
 &= i\pi \left(\frac{e^{-i\omega R}}{2(-\omega)} + \frac{e^{i\omega R}}{2\omega} \right)
 \end{aligned}$$

$$\begin{aligned}
 I_1 + I_2 &= i\pi \left(\text{Res}_{-\omega} \frac{e^{ikR}}{k^2 - \omega^2} + \text{Res}_{\omega} \frac{e^{ikR}}{k^2 - \omega^2} \right) \\
 &= i\pi \left(\frac{e^{-i\omega R}}{2(-\omega)} + \frac{e^{i\omega R}}{2\omega} \right) = \frac{i\pi}{2}
 \end{aligned}$$

$$\pi \left(\text{Res}_{-w} \frac{e^{ikR}}{k^2 - w^2} + \text{Res}_w \frac{e^{ikR}}{k^2 - w^2} \right)$$

$$\pi \left(\frac{e^{-iwR}}{2(-w)} + \frac{e^{iwR}}{2w} \right) = \frac{i\pi}{2} \left(e^{-iwR} + e^{iwR} \right)$$

$$G(\vec{R}, T) = \frac{1}{(2\pi)^3} \frac{1}{R} \frac{\pi}{2} \int d\omega \left(e^{-i\omega(R+T)} + e^{i\omega(R-T)} \right)$$

$$= \frac{1}{8\pi R} \left(\frac{1}{2\pi} \int d\omega (\dots) \right)$$

$$G(\vec{R}, T) = \frac{1}{(2\pi)^3 R} \frac{\pi}{2} \int d\omega \left(e^{-i\omega(R+T)} + e^{i\omega(R-T)} \right)$$

$$= \frac{1}{8\pi R} \left[\frac{1}{2\pi} \int d\omega (\dots) \right] = \frac{1}{8\pi R} \left(\delta(R+T) \right)$$

$$\begin{aligned}
 & \left(R+T \right) + e^{i\omega(R-T)} \\
 & \left(\right) = \frac{1}{8\pi R} \left(\delta(R+T) + \delta(R-T) \right) = \frac{1}{4\pi} \delta(R^2 - T^2) \\
 & \text{ONLY NONZERO ON THE LIGHT CONE}
 \end{aligned}$$

• IN CLASSICAL ELDYN. TAKE ONLY RETARDED PART.

$$\phi(r, t) = \int G(r, r', t, t') 4\pi \rho(r', t') dr' dt' = \int \frac{\rho(r', t')}{|r - r'|} dr' dt'$$

ONLY RETARDED PART.

$$g(\mathbf{r}', t') d\mathbf{r}' dt' = \int \frac{\delta(|\mathbf{r} - \mathbf{r}'| - |t - t'|)}{|\mathbf{r} - \mathbf{r}'|} g(\mathbf{r}', t') d\mathbf{r}' dt'$$

TAKE ONLY RETARDED PART.

$$e) 4\pi \oint(\mathbf{r}', t') d\mathbf{r}' dt' = \int \frac{\delta(|\mathbf{r}-\mathbf{r}'| - (t-t'))}{|\mathbf{r}-\mathbf{r}'|} \oint(\mathbf{r}', t') d\mathbf{r}' dt'$$

$$\frac{\oint(\mathbf{r}, t - |\mathbf{r}-\mathbf{r}'|, \mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

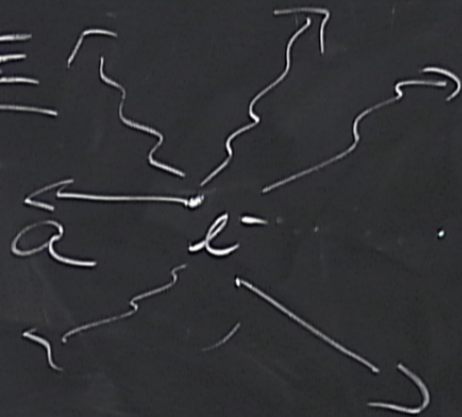
RETARDED POTENTIAL

2) FEYNMAN'S STORY

• RADIATION FORCE

... STORY

... IONIZATION FORCE



... BACK REACTION

$$f' = g(E + \vec{v} \times \vec{B}) + f$$

$\propto (-\omega)$

$\propto \omega$