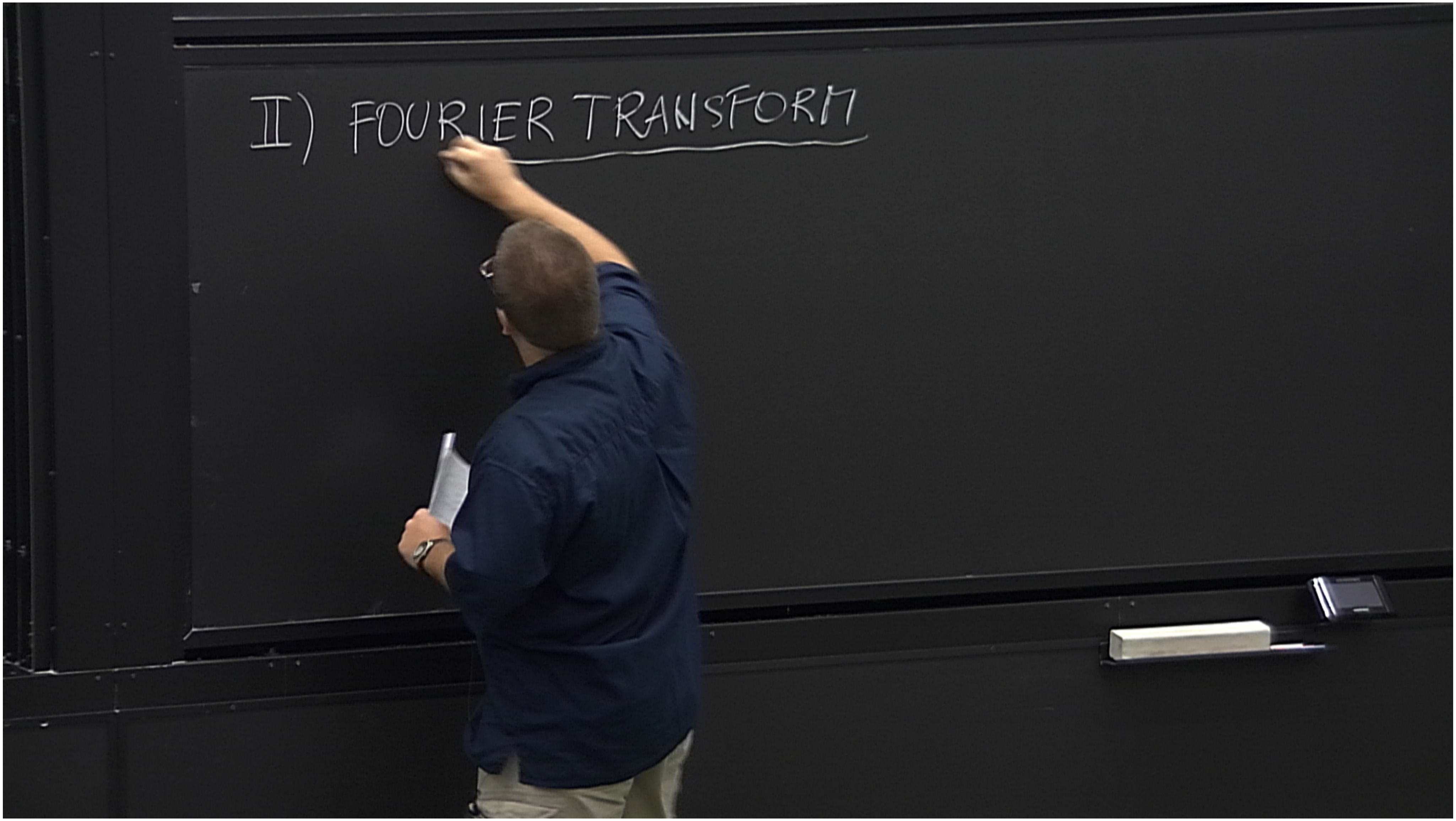
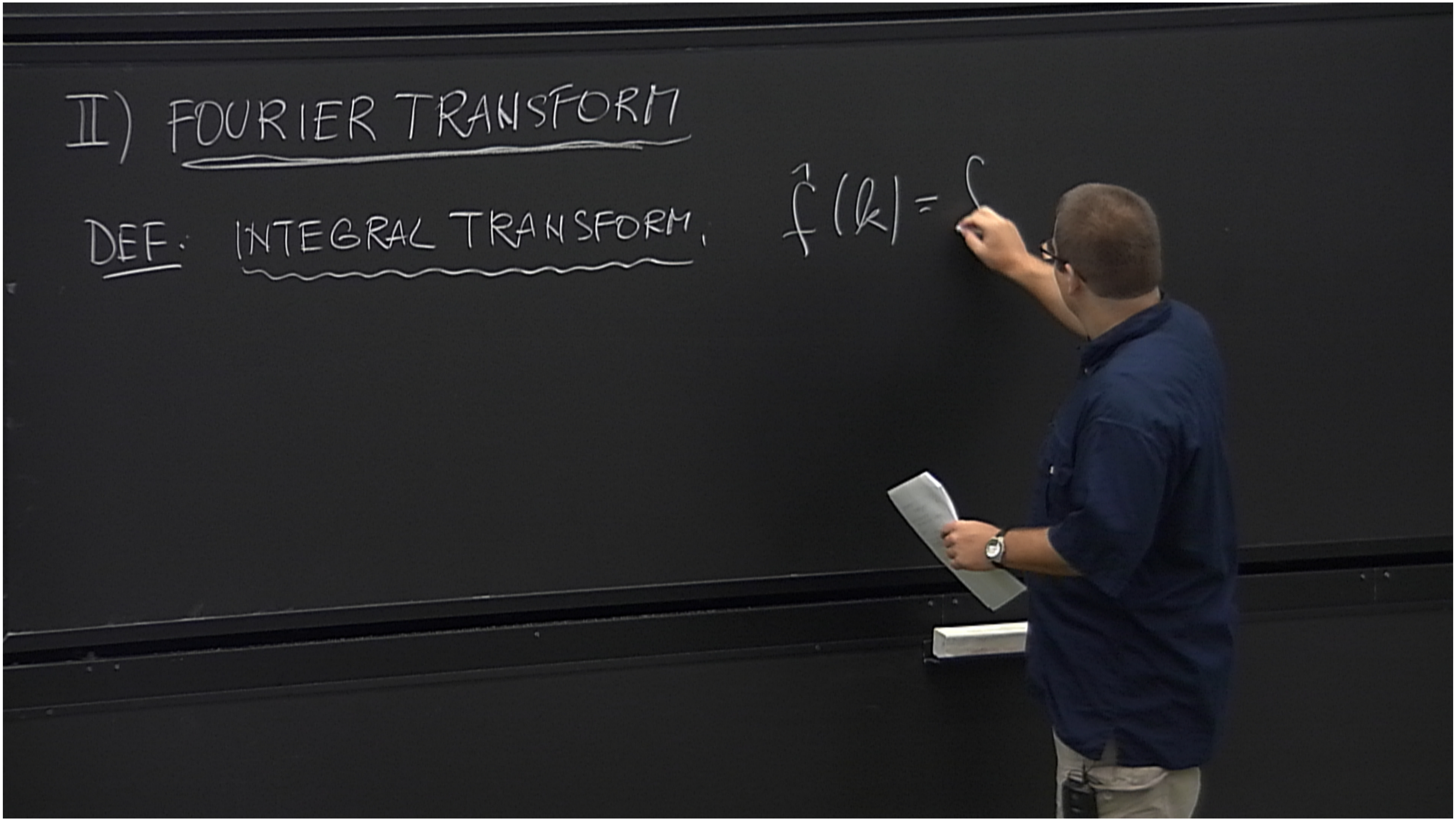
Title: 13/14 PSI - Integral Transformations & Green's Function - Lecture 2

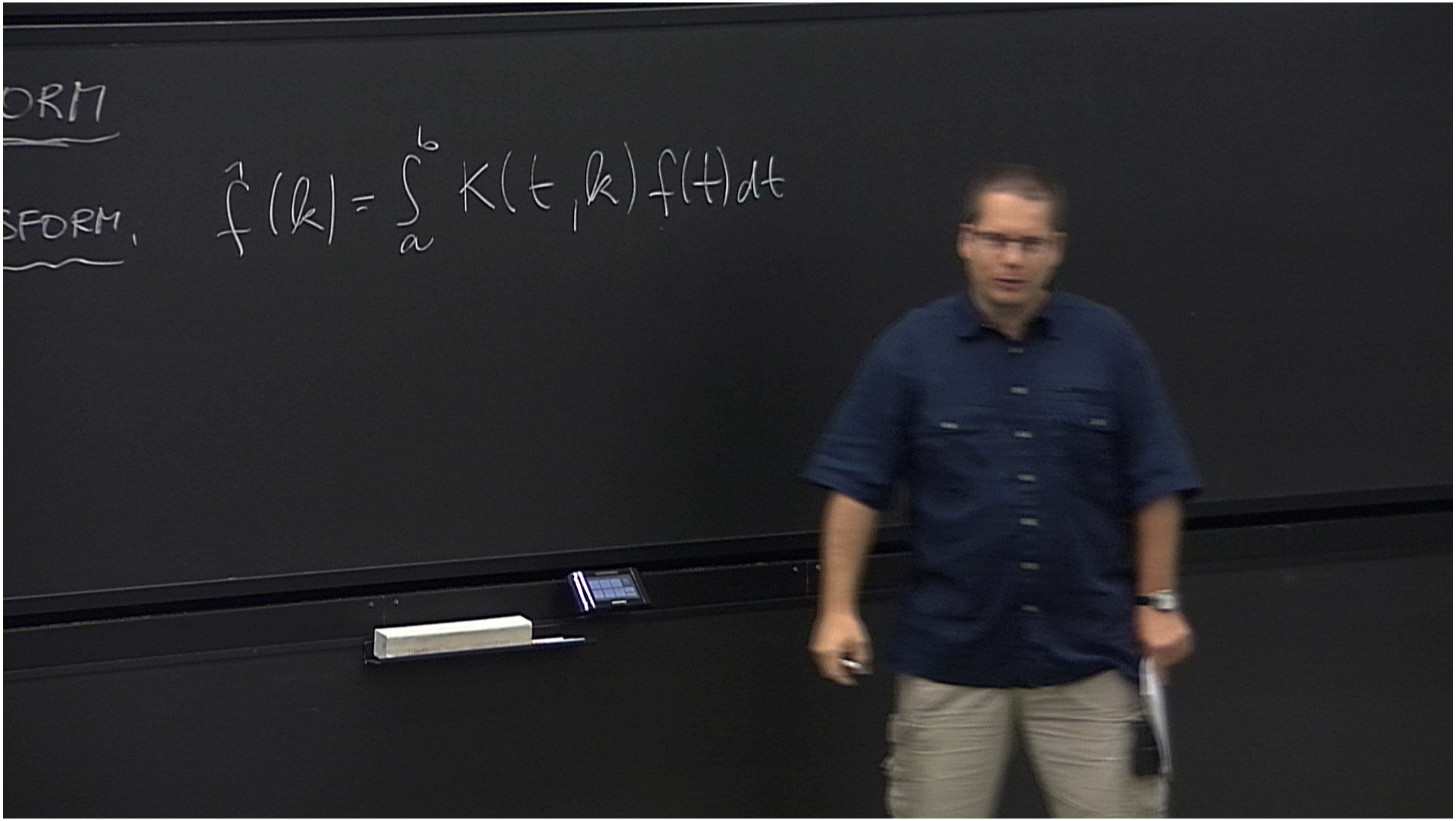
Date: Aug 28, 2013 09:00 AM

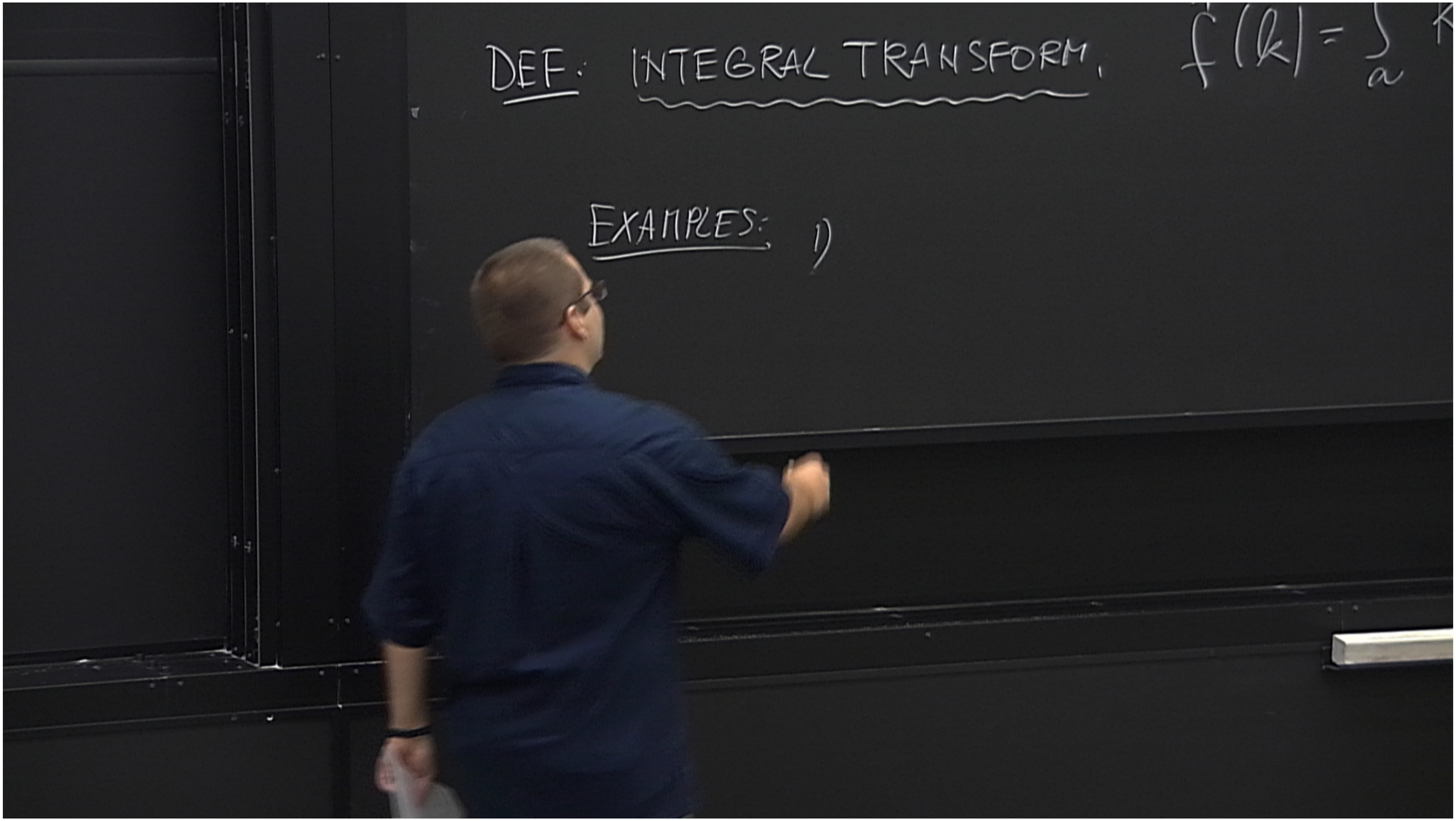
URL: http://pirsa.org/13080029

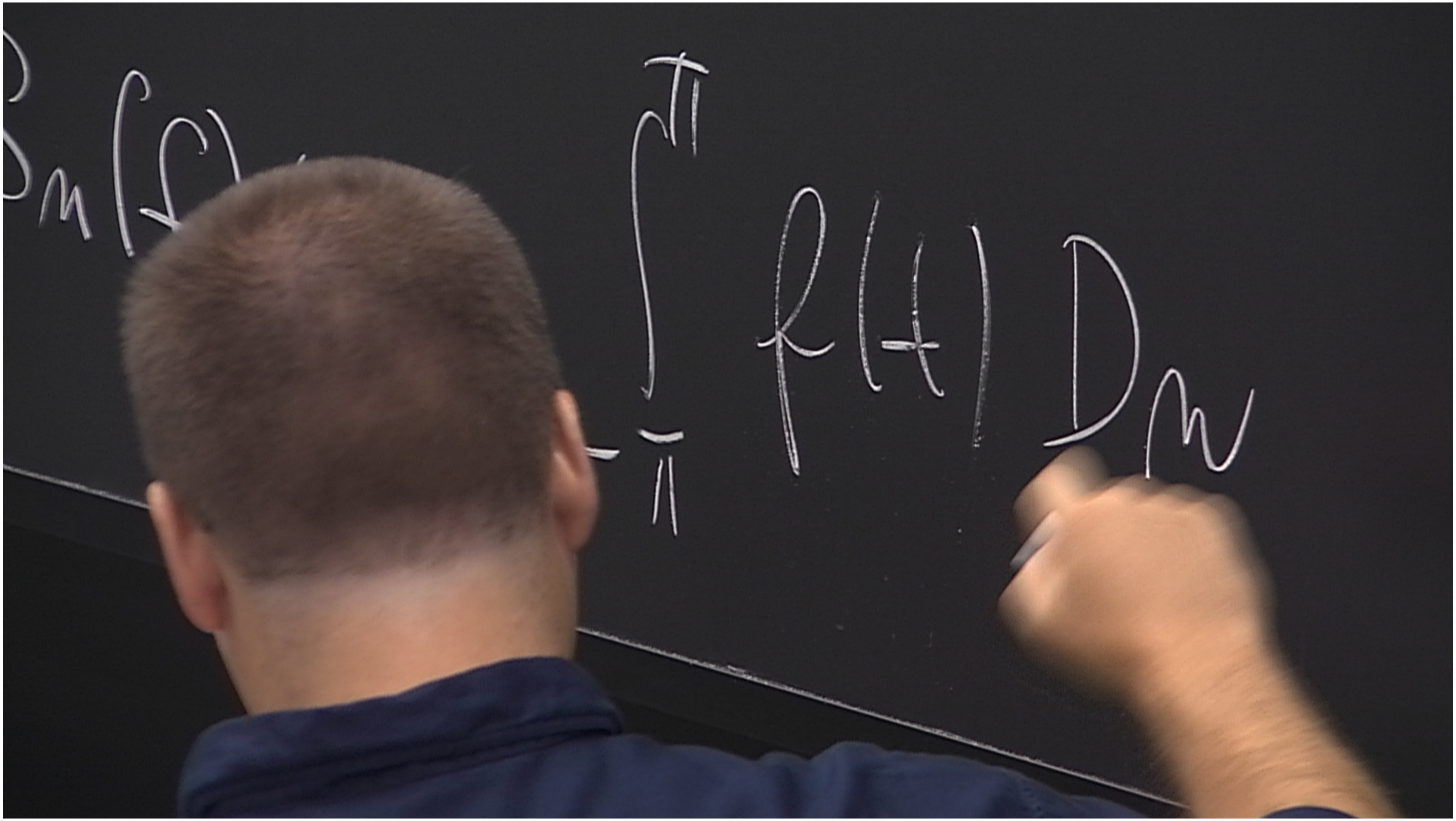
Abstract:

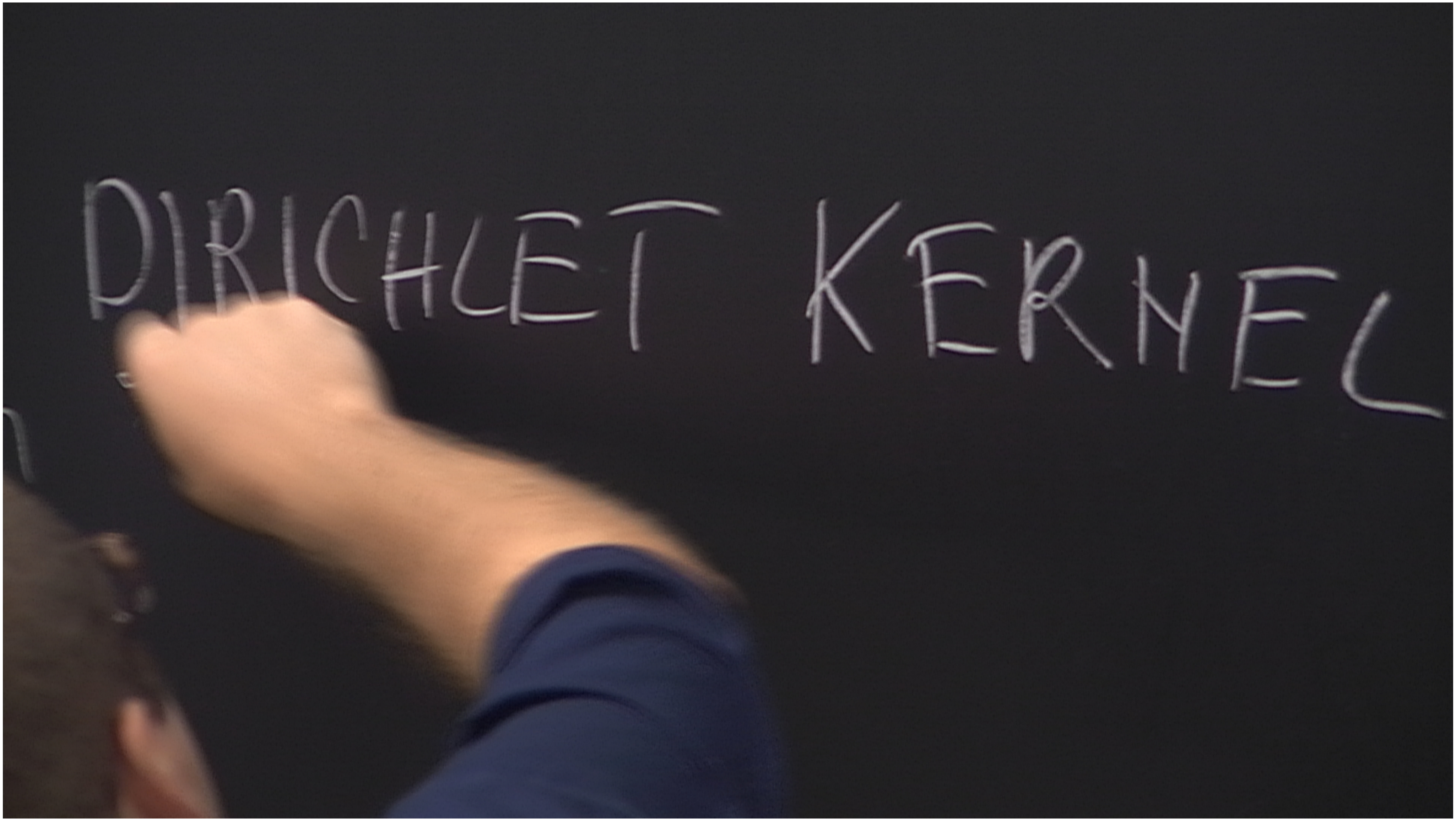

II) FOURIER TRANSFORM

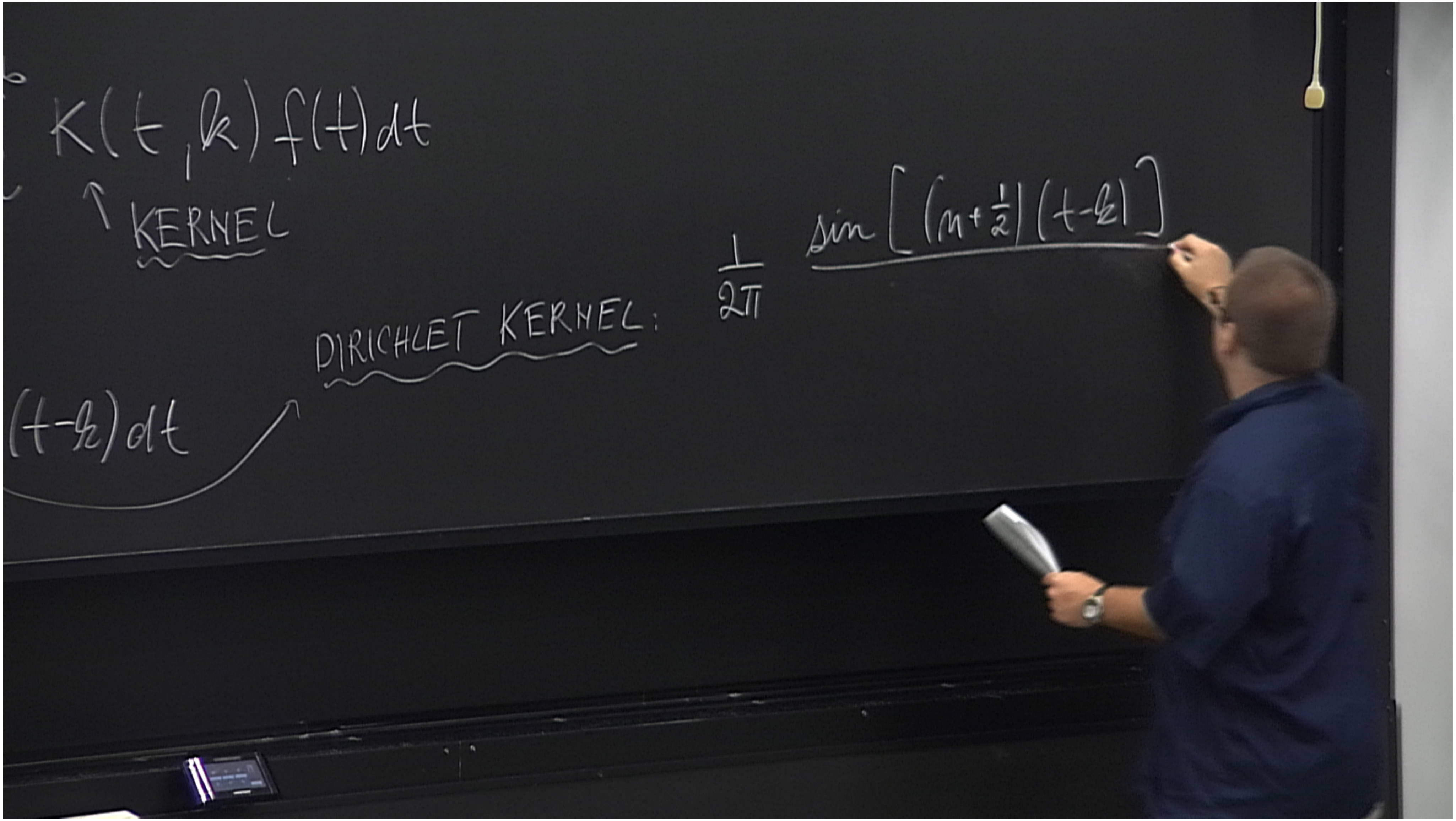

II) FOURIER TRANSFORM
DEF: INTEGRAL TRANSFORM.
f̃(k) = ∫

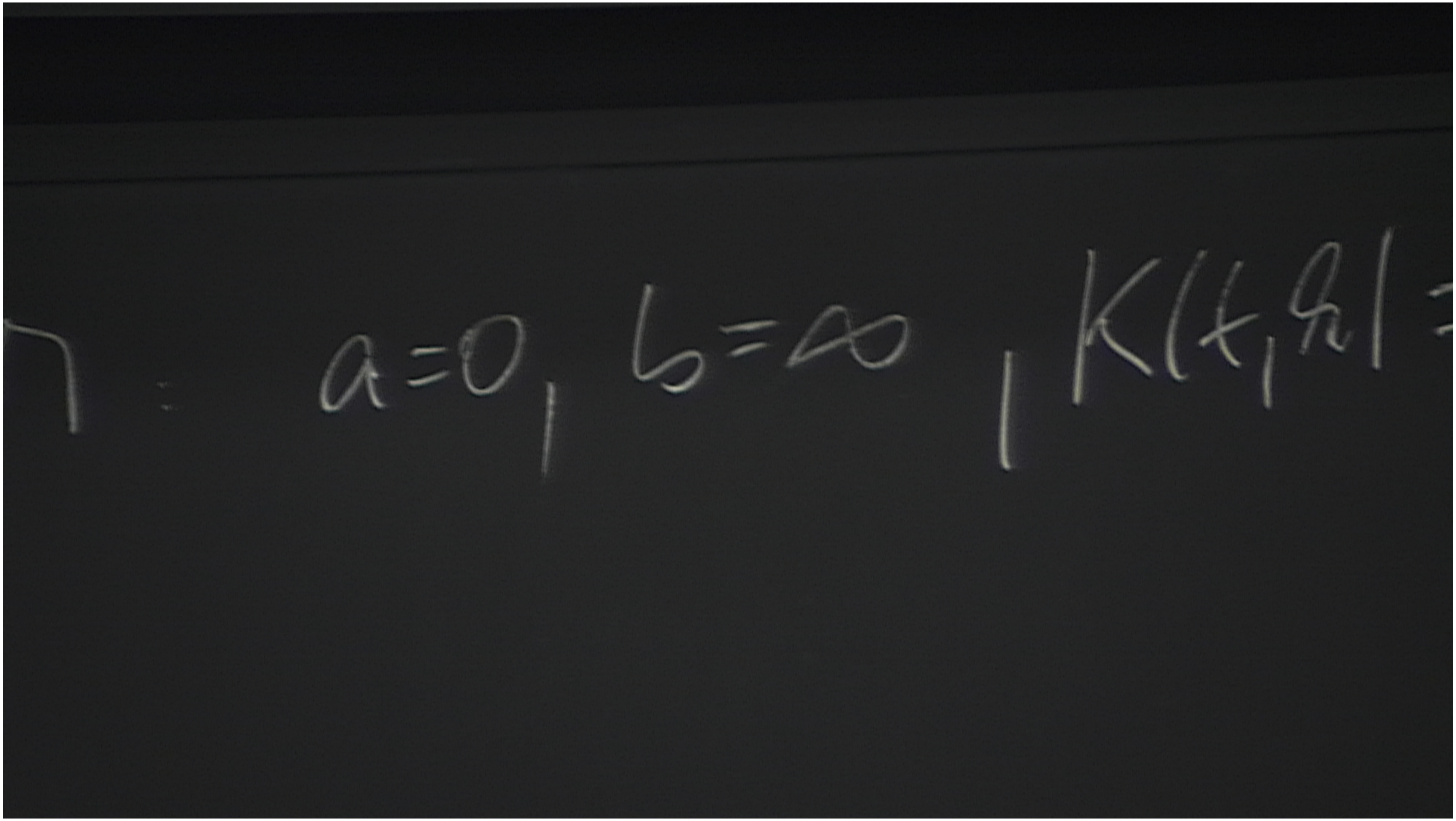

ORM
SFORM.
$\tilde{f}(k) = \int_a^b K(t,k) f(t) dt$

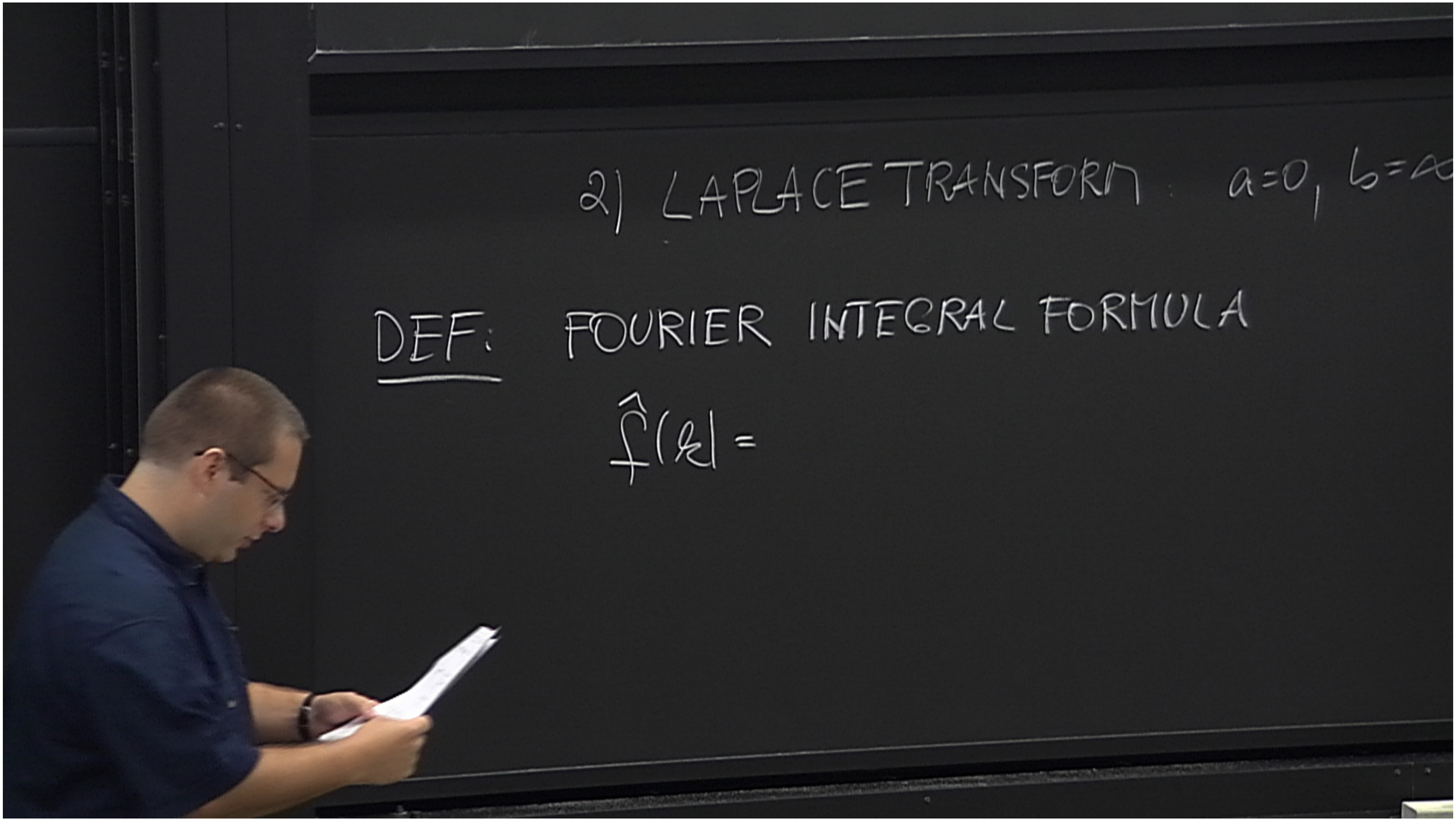

DEF: INTEGRAL TRANSFORM.
EXAMPLES: 1)

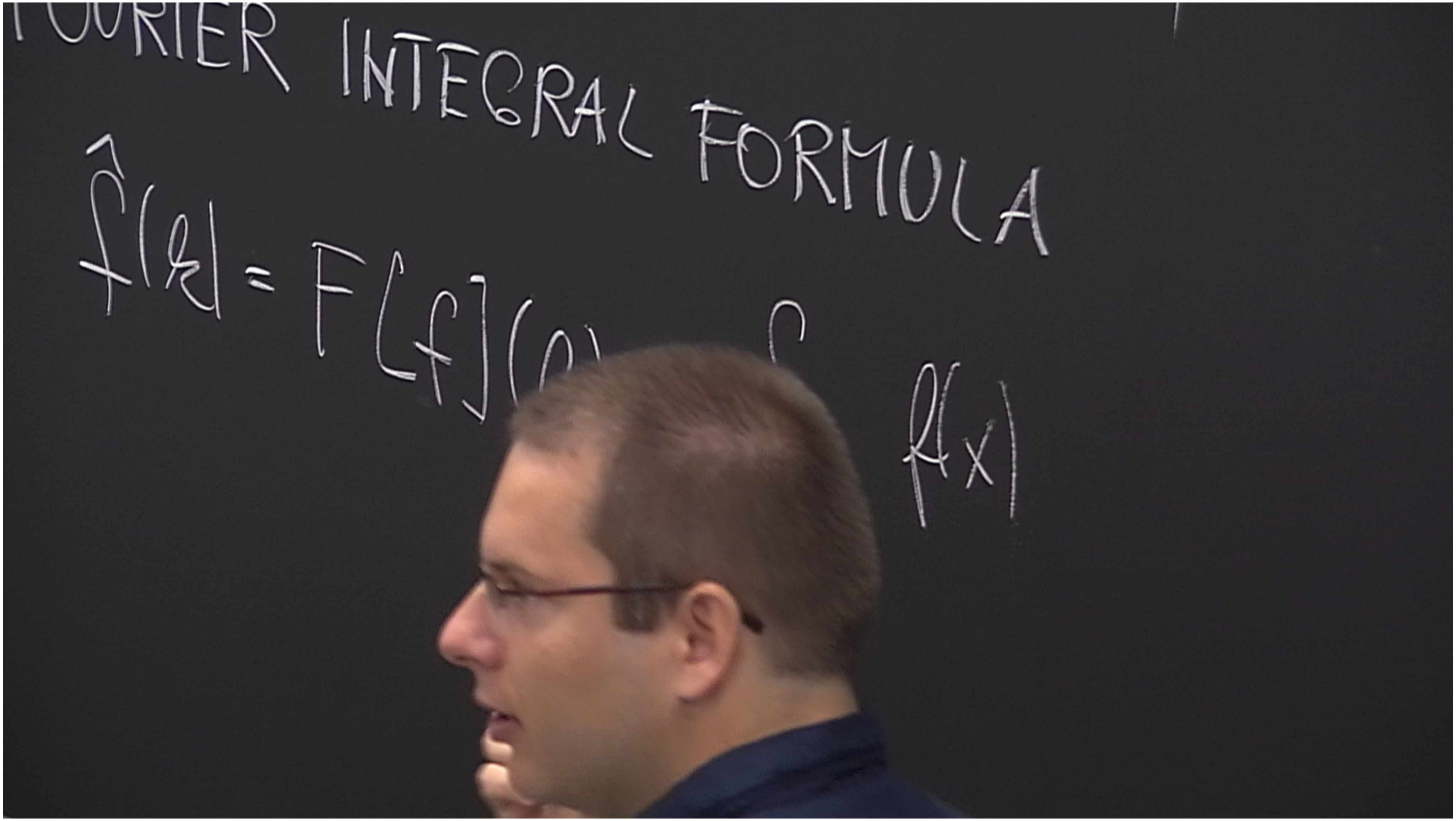


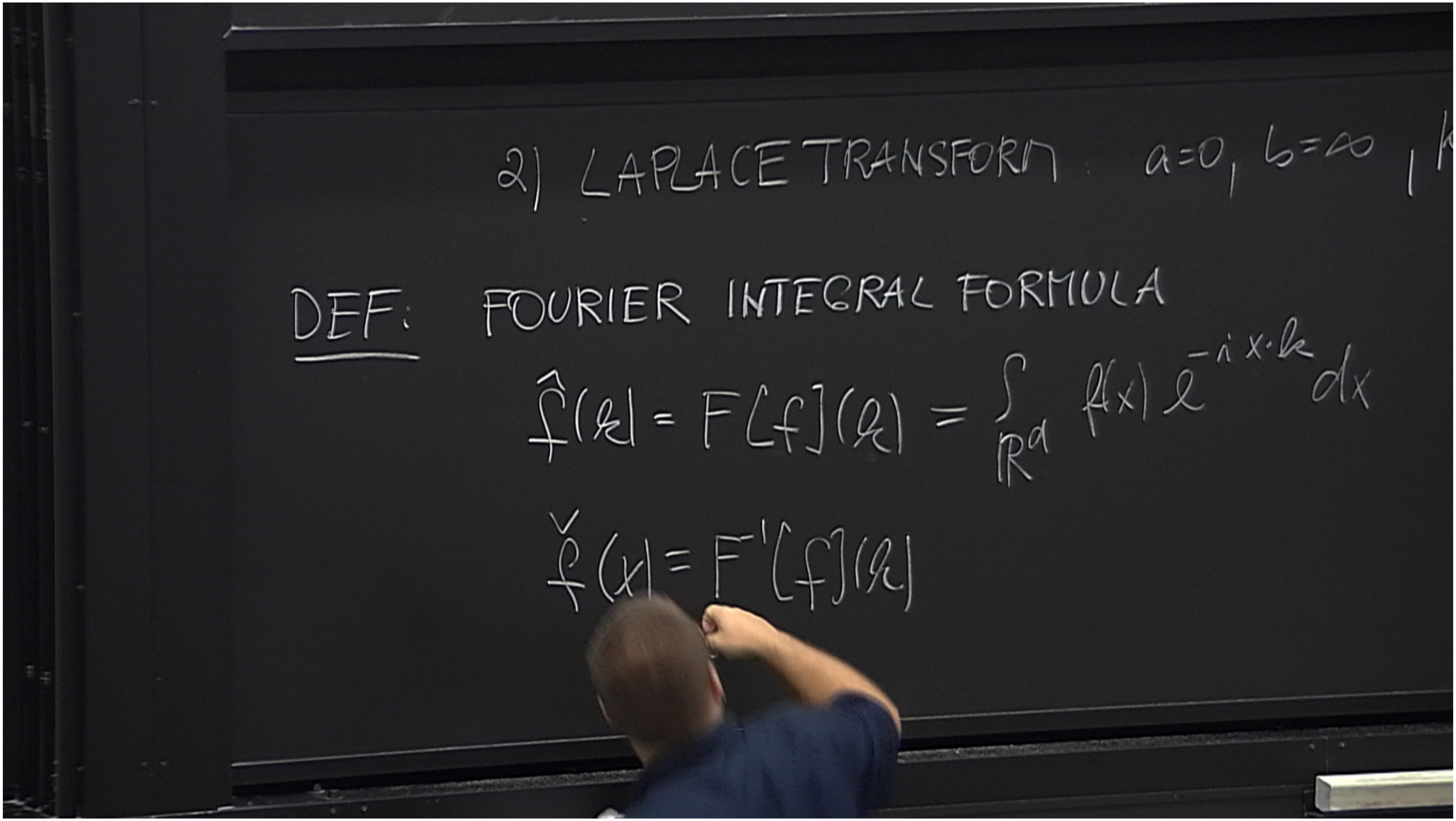

DIRICHLET KERNEL

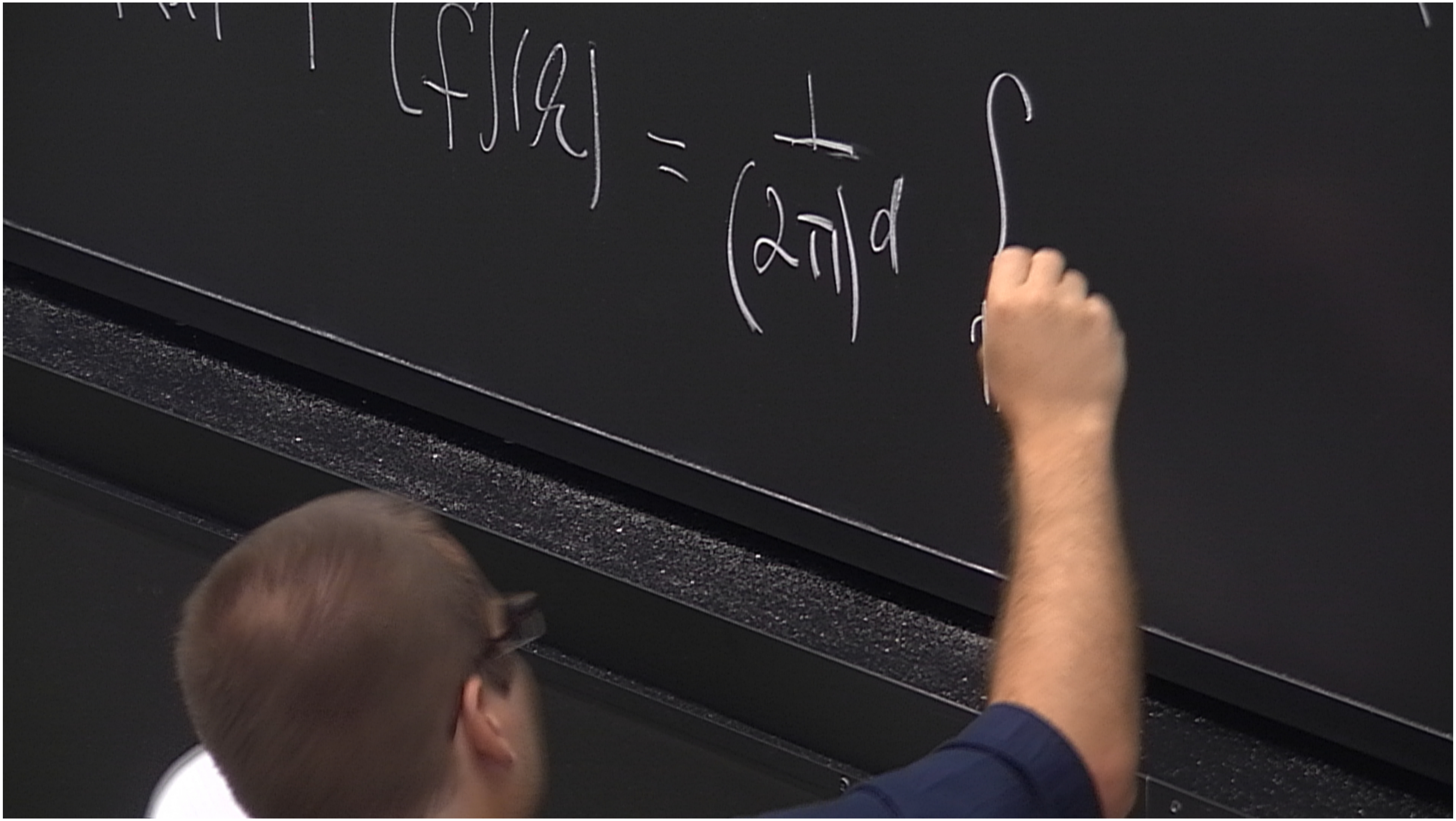

K(t, k) f(t) dt
KERNEL
DIRICHLET KERNEL:
1/2π sin[(n+1/2)(t-k)]
(t-k) dt


a=0, b=∞, K(t,


2) LAPLACE TRANSFORM : a=0, b=
DEF: FOURIER INTEGRAL FORMULA
f̂(k) =


INTEGRAL FORMULA

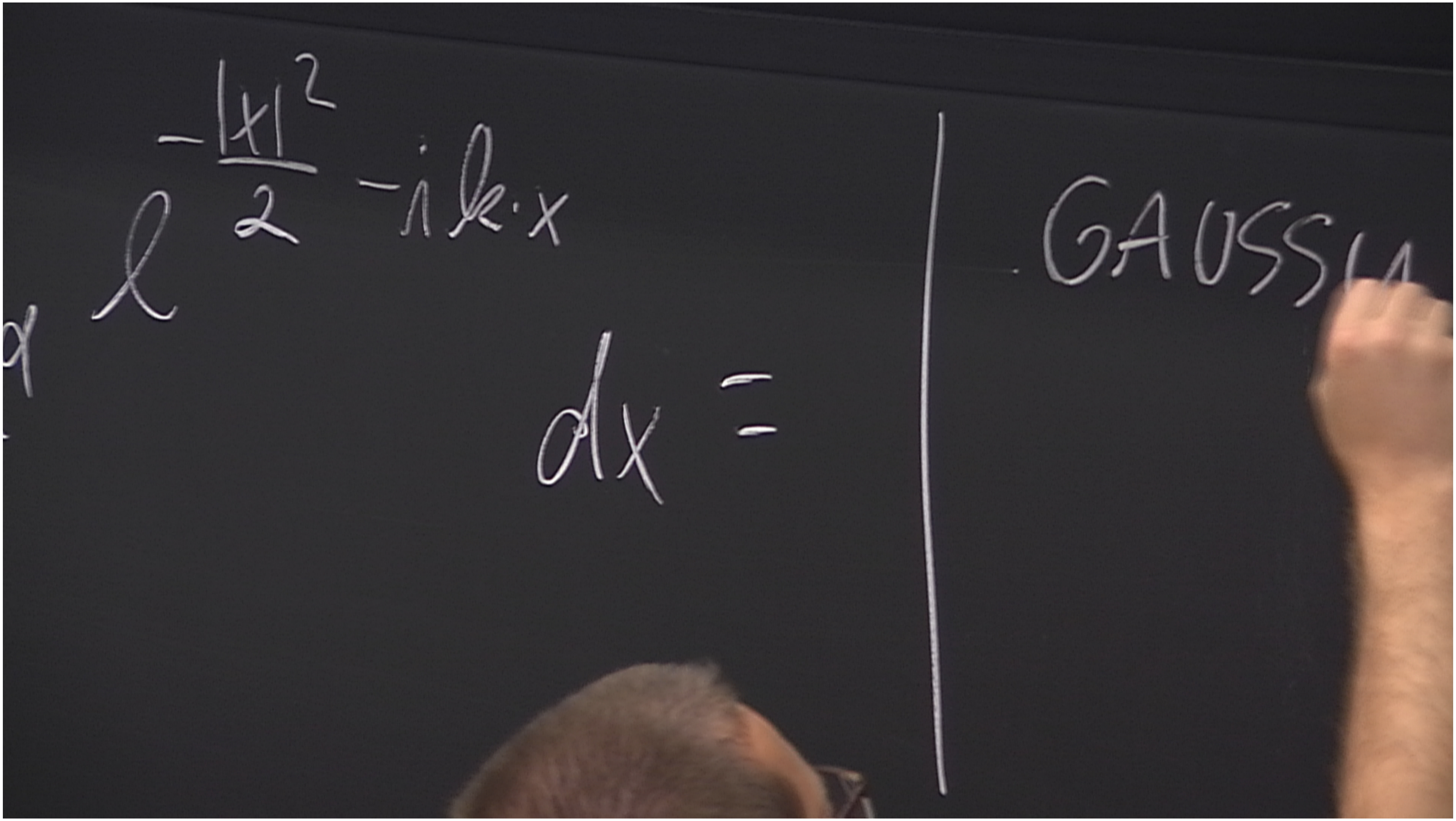

2) LAPLACE TRANSFORM : $a=0$, $b=\infty$,
DEF: FOURIER INTEGRAL FORMULA
$\hat{f}(k) = F[f](k) = \int_{\mathbb{R}^d} f(x)\, e^{-i x \cdot k}\, dx$
$\check{f}(x) = F^{-1}[f](k)$

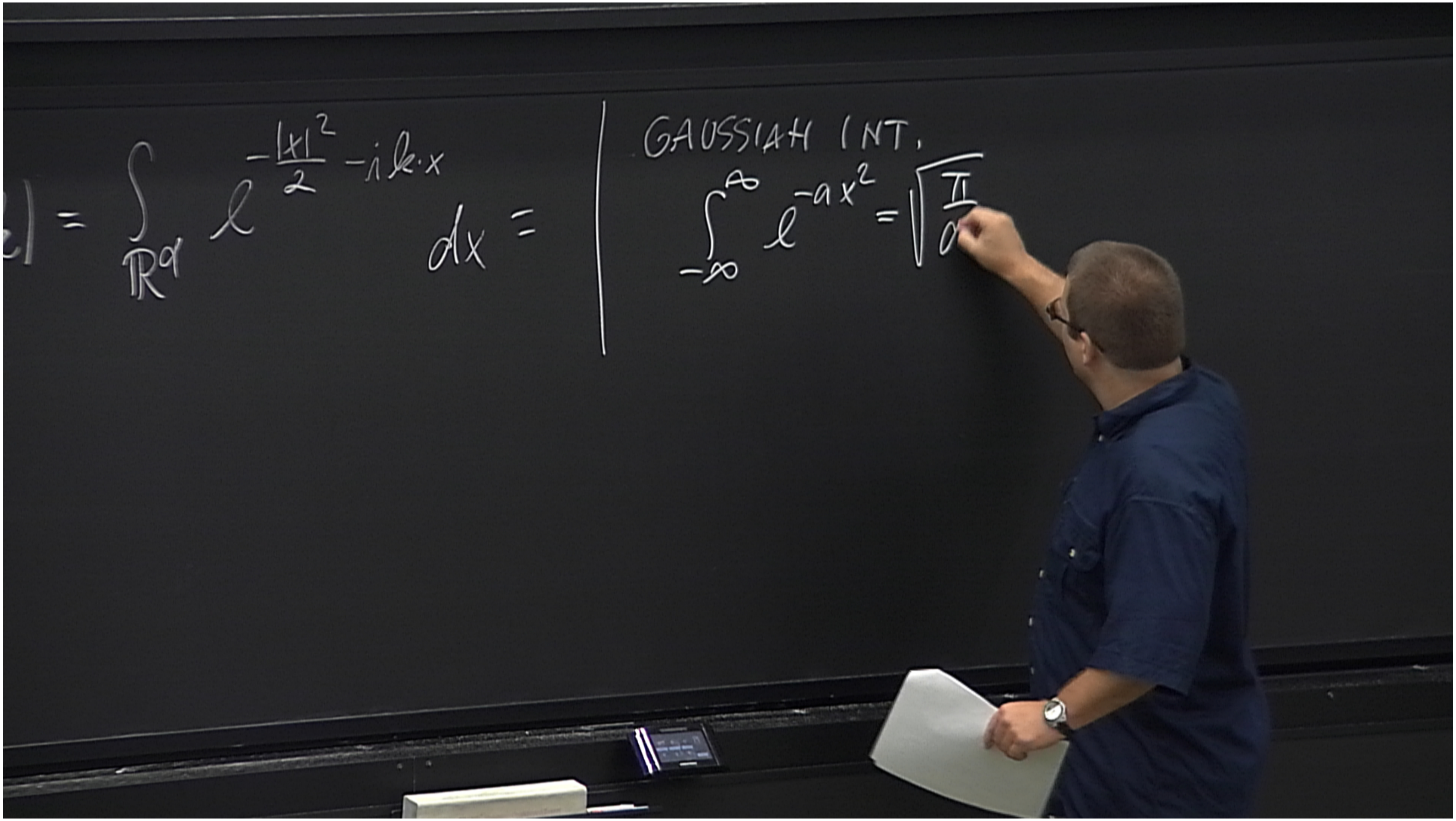

$[f](k) = \frac{1}{(2\pi)^d} \int$


ACE TRANSFORM
INTEGRAL FORMULA
(FT)

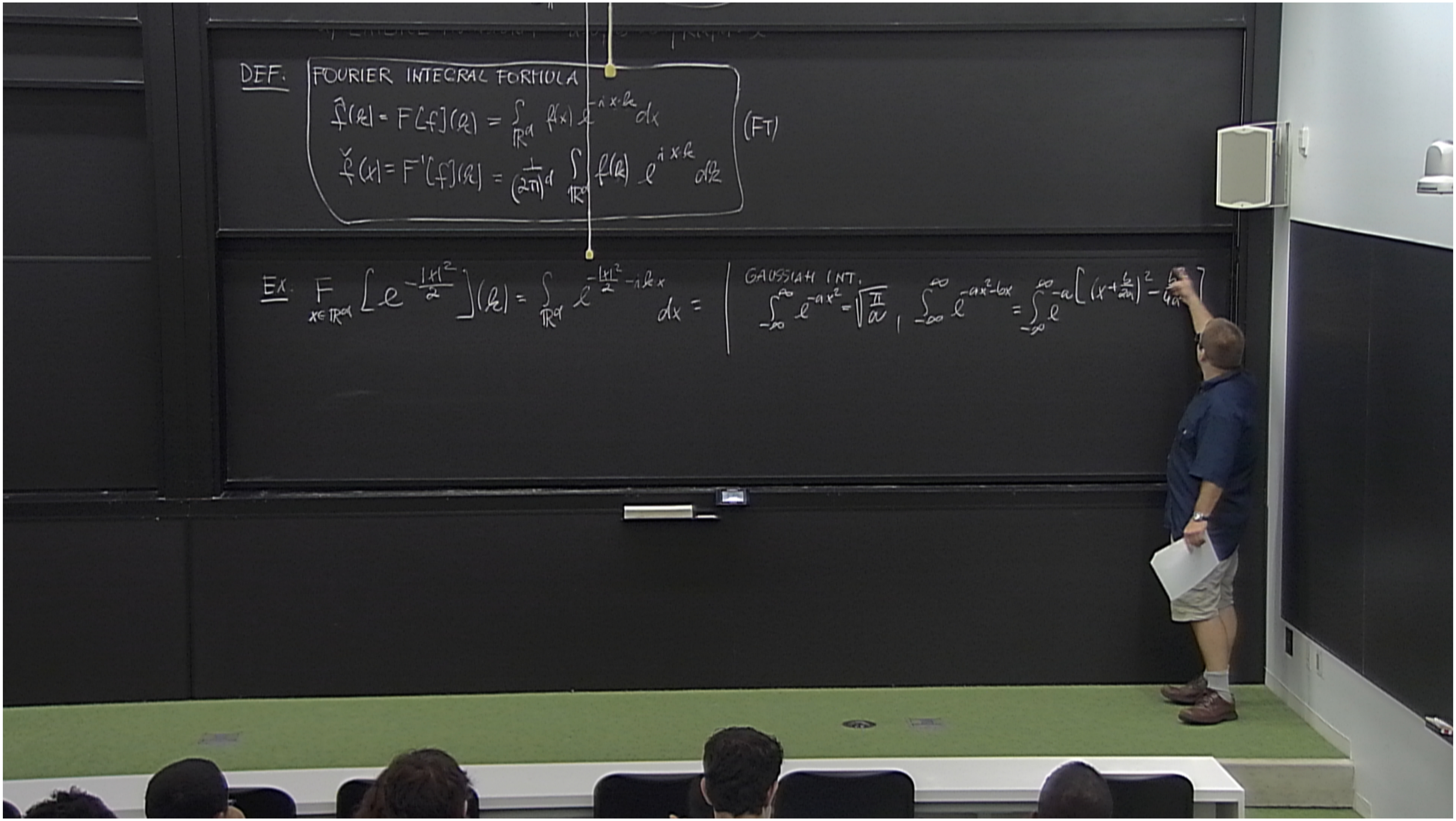
Ex: $\mathcal{F}_{x \in \mathbb{R}^d}\left[e^{-\frac{|x|^2}{2}}\right](k) =$

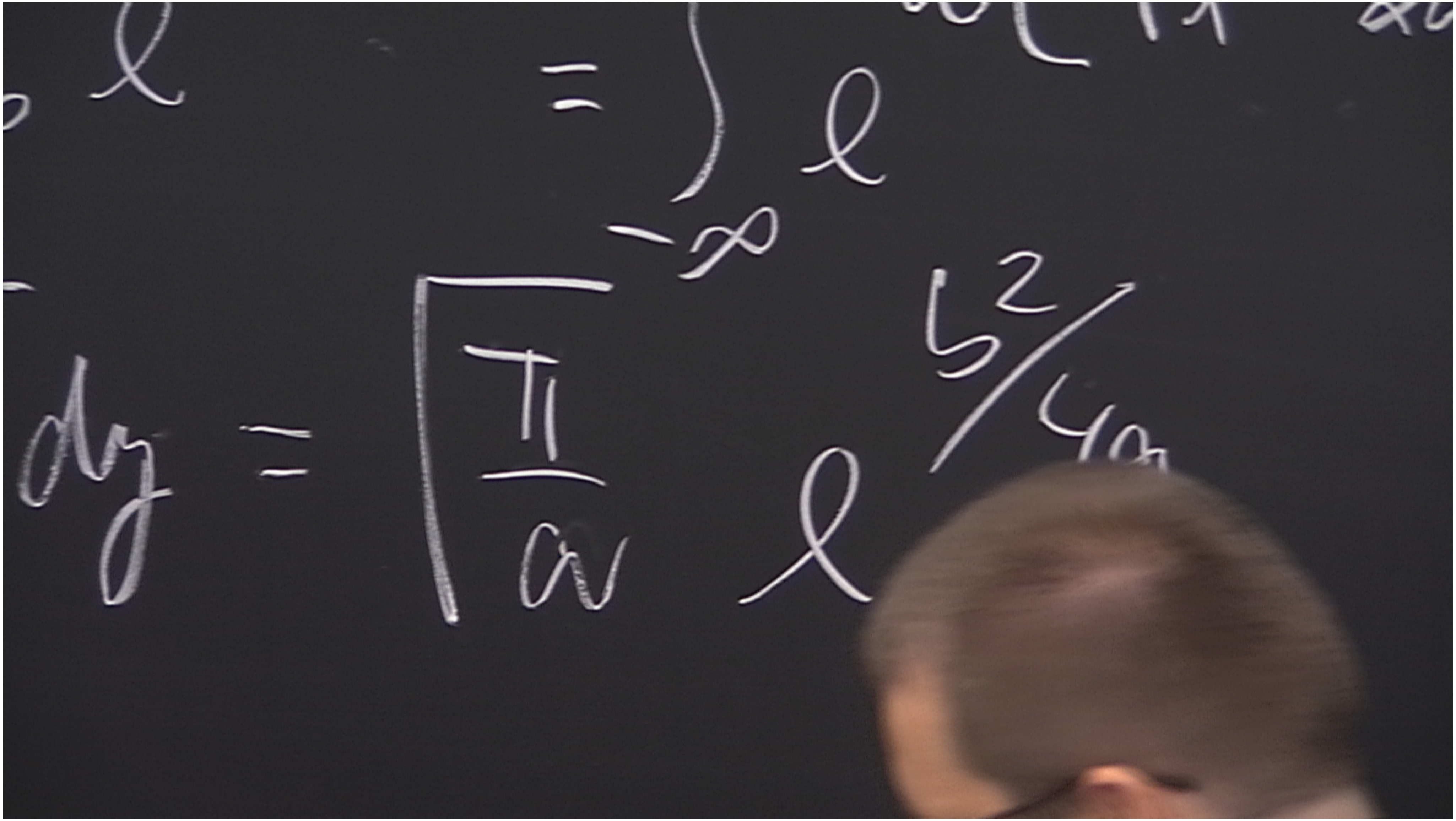
$$\mathcal{J}(k) = \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2} - ik\cdot x}\, dx =$$

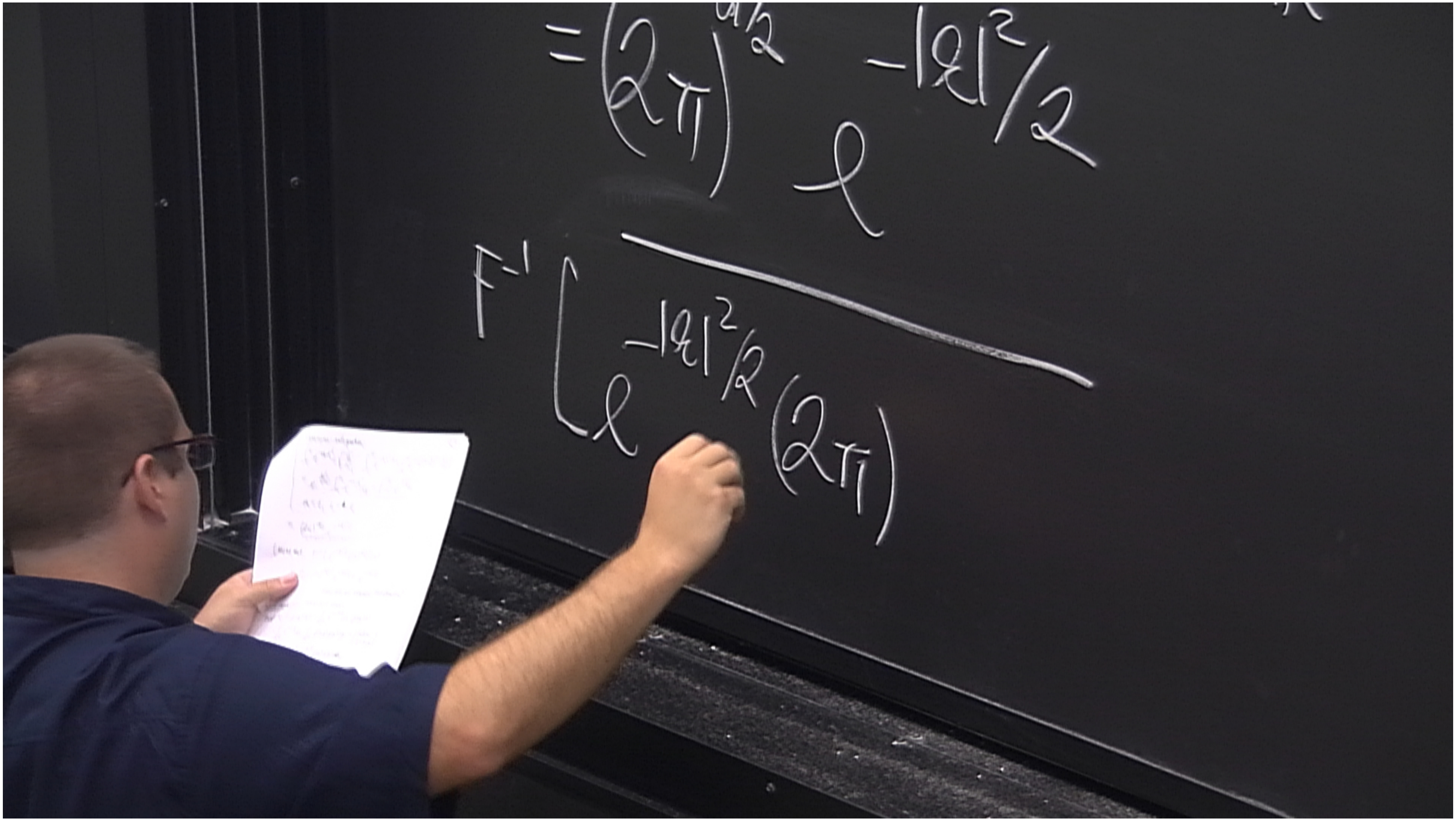

$e^{-\frac{|x|^2}{2} - ik\cdot x}$
$dx =$
GAUSS

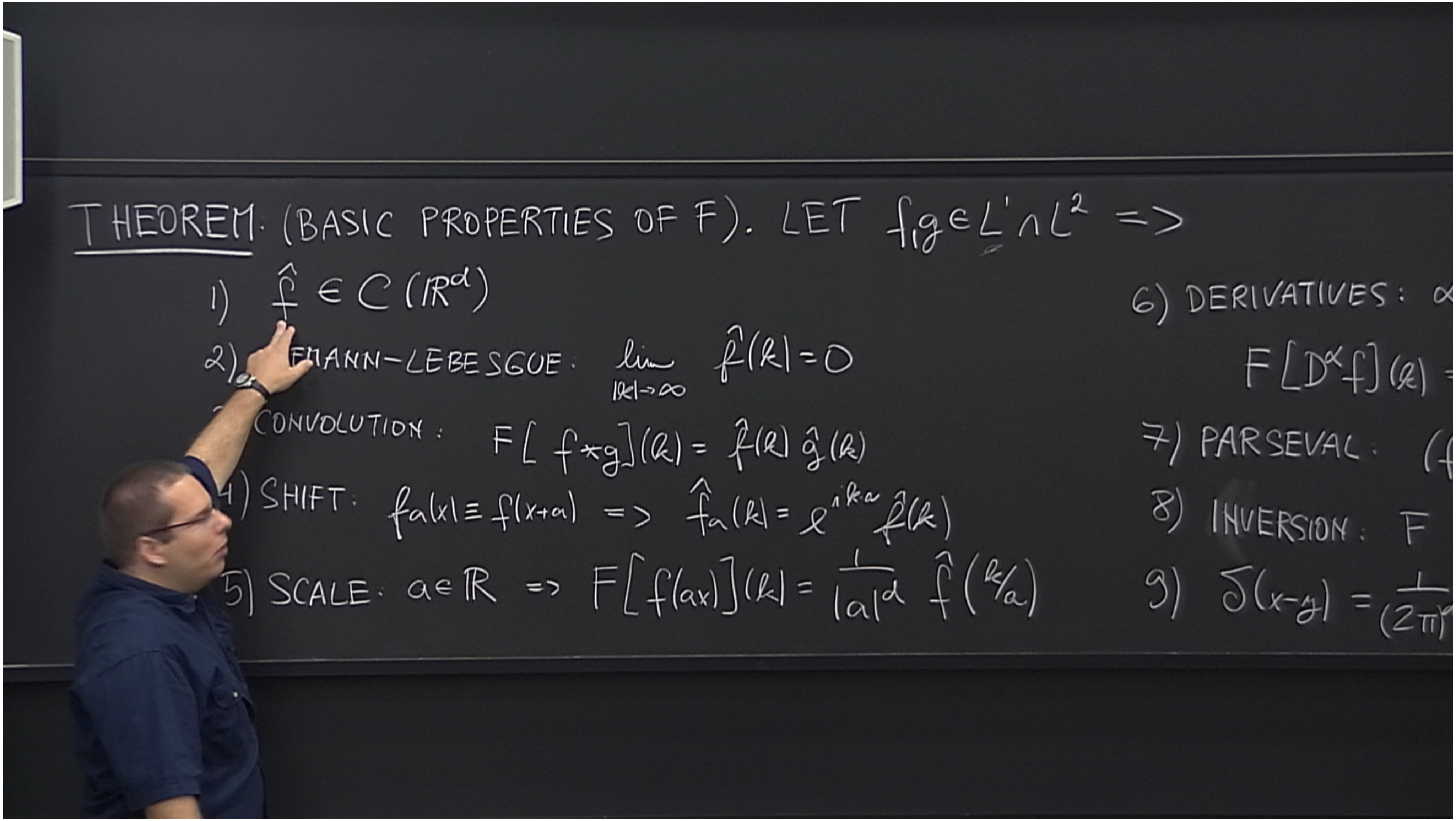


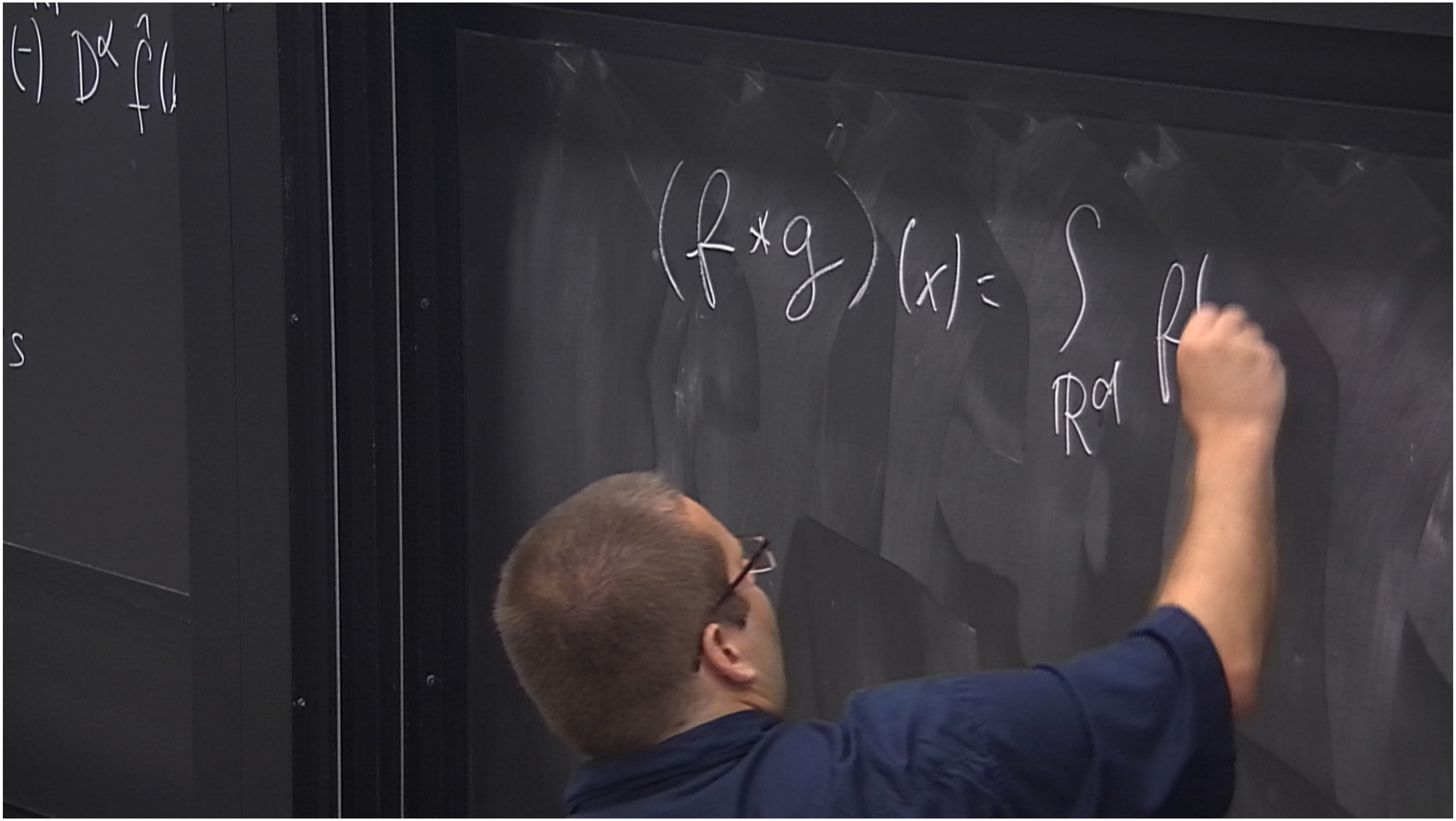
$= \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2} - ik\cdot x} dx =$
GAUSSIAN INT.
$\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}$

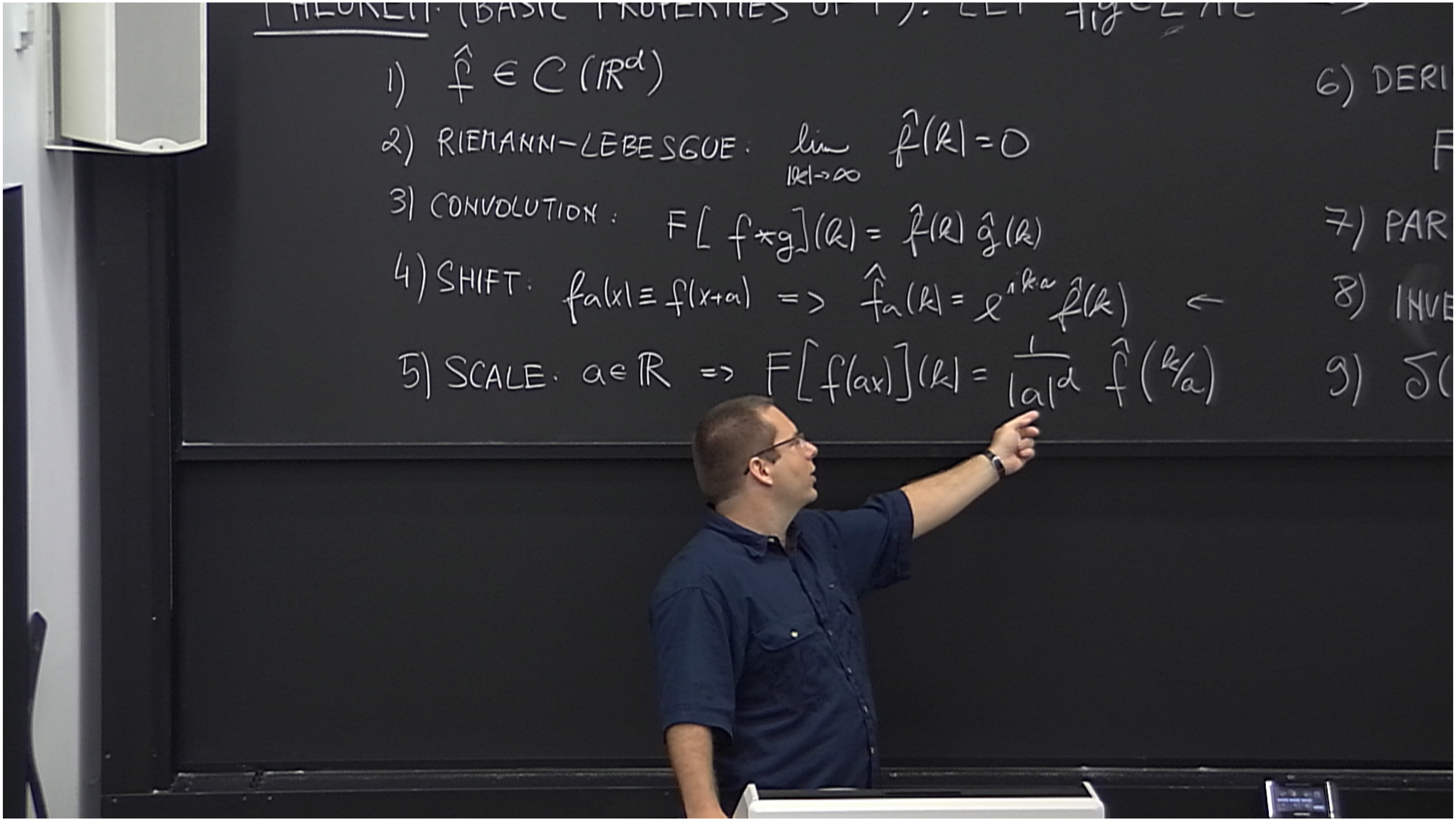

DEF.
FOURIER INTEGRAL FORMULA
$\hat{f}(k) = F[f](k) = \int_{\mathbb{R}^d} f(x)\, e^{-i x\cdot k} dx$
$\check{f}(x) = F^{-1}[f](k) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(k)\, e^{i x\cdot k} dk$
(FT)
EX.
$F_{x\in\mathbb{R}^d}\left[e^{-\frac{|x|^2}{2}}\right](k) = \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2} - i k\cdot x} dx =$
GAUSSIAN INT.
$\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}$, $\int_{-\infty}^{\infty} e^{-ax^2-bx} = \int_{-\infty}^{\infty} e^{-a\left[(x+\frac{b}{2a})^2 - \frac{b^2}{4a^2}\right]}$



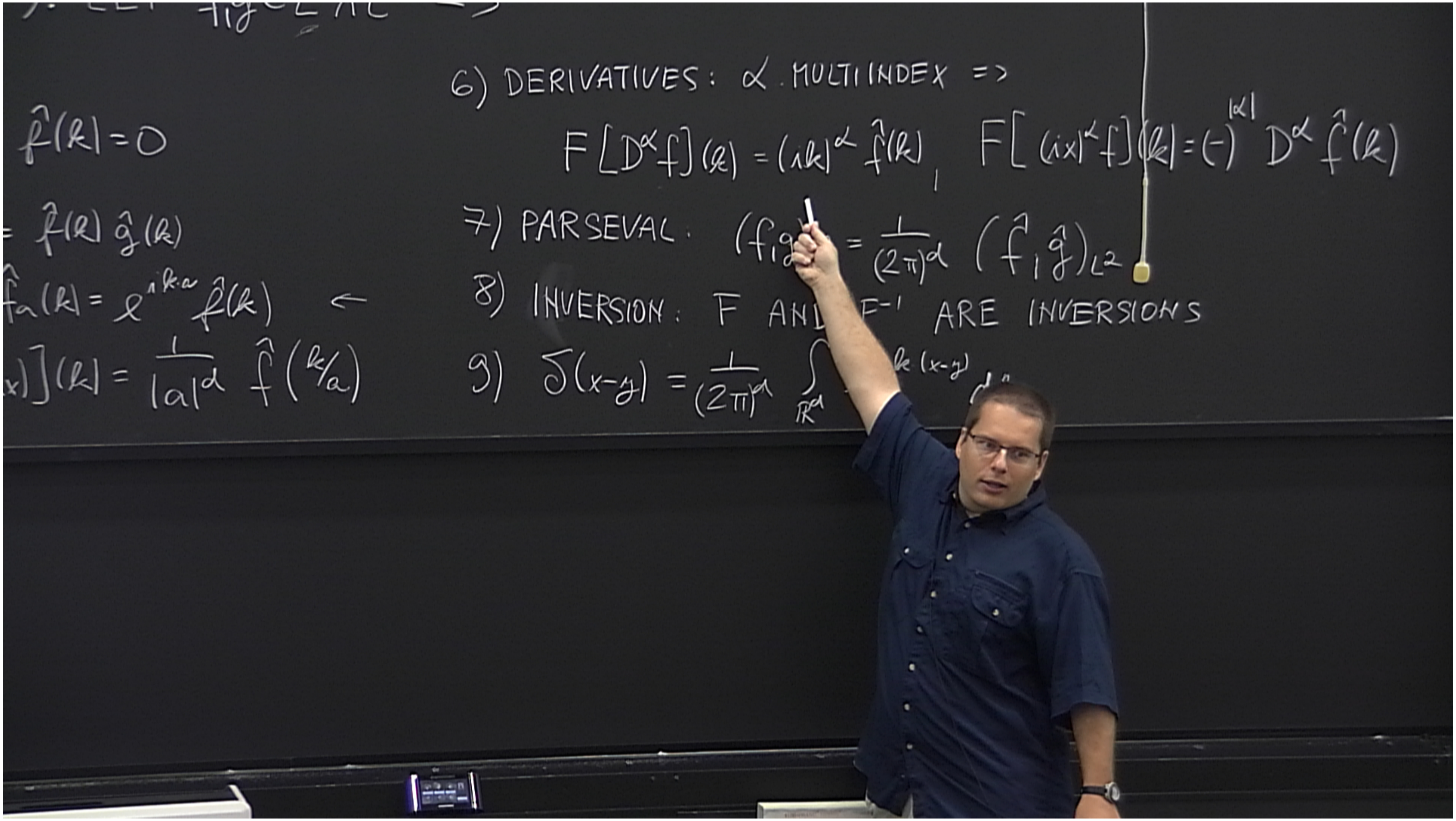

$= (2\pi)^{d/2} e^{-|\xi|^2/2}$
$F^{-1}\left[e^{-|\xi|^2/2} (2\pi) \right.$

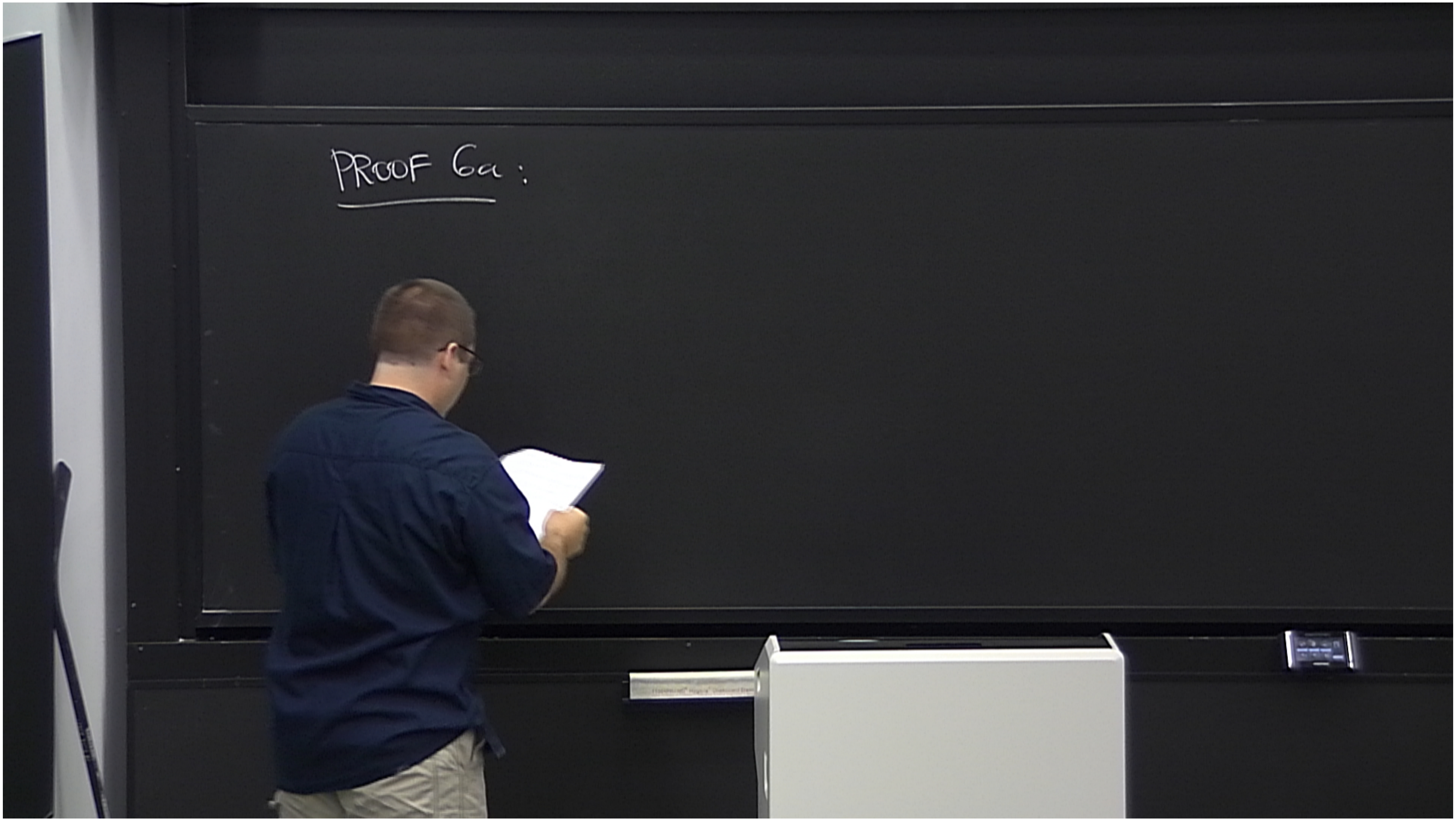

THEOREM. (BASIC PROPERTIES OF F). LET f,g ∈ L¹ ∩ L² =>
1) f̂ ∈ C(ℝ^d)
2) MANN-LEBESGUE: lim_{|k|→∞} f̂(k) = 0
CONVOLUTION: F[f*g](k) = f̂(k) ĝ(k)
4) SHIFT: f_a(x) ≡ f(x+a) => f̂_a(k) = e^{ik·a} f̂(k)
5) SCALE: a ∈ ℝ => F[f(ax)](k) = 1/|a|^d f̂(k/a)
6) DERIVATIVES:
F[D^α f](k)
7) PARSEVAL:
8) INVERSION: F
9) δ(x-y) = 1/(2π)

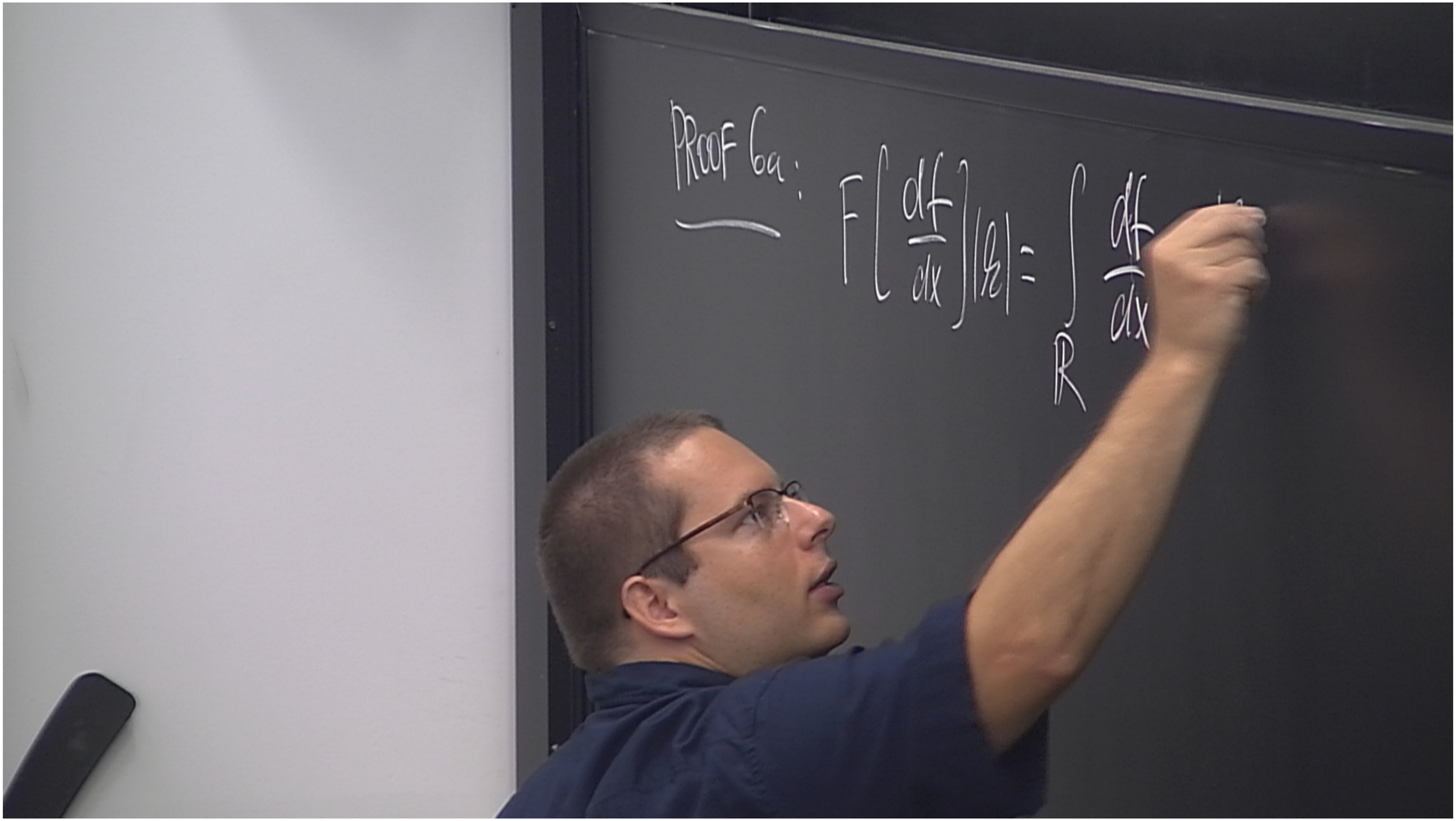

(f*g)(x) = ∫ f(
ℝd

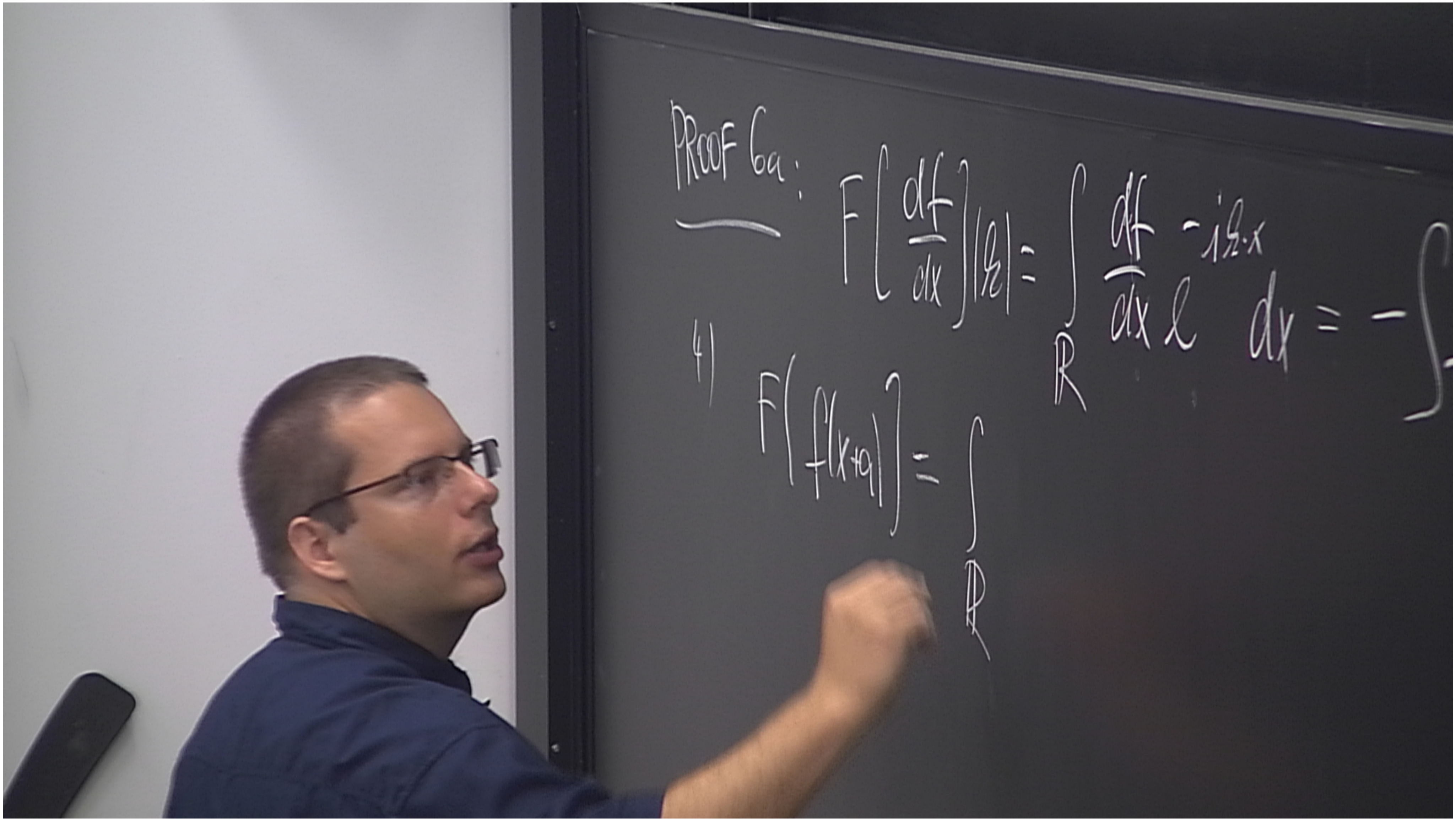


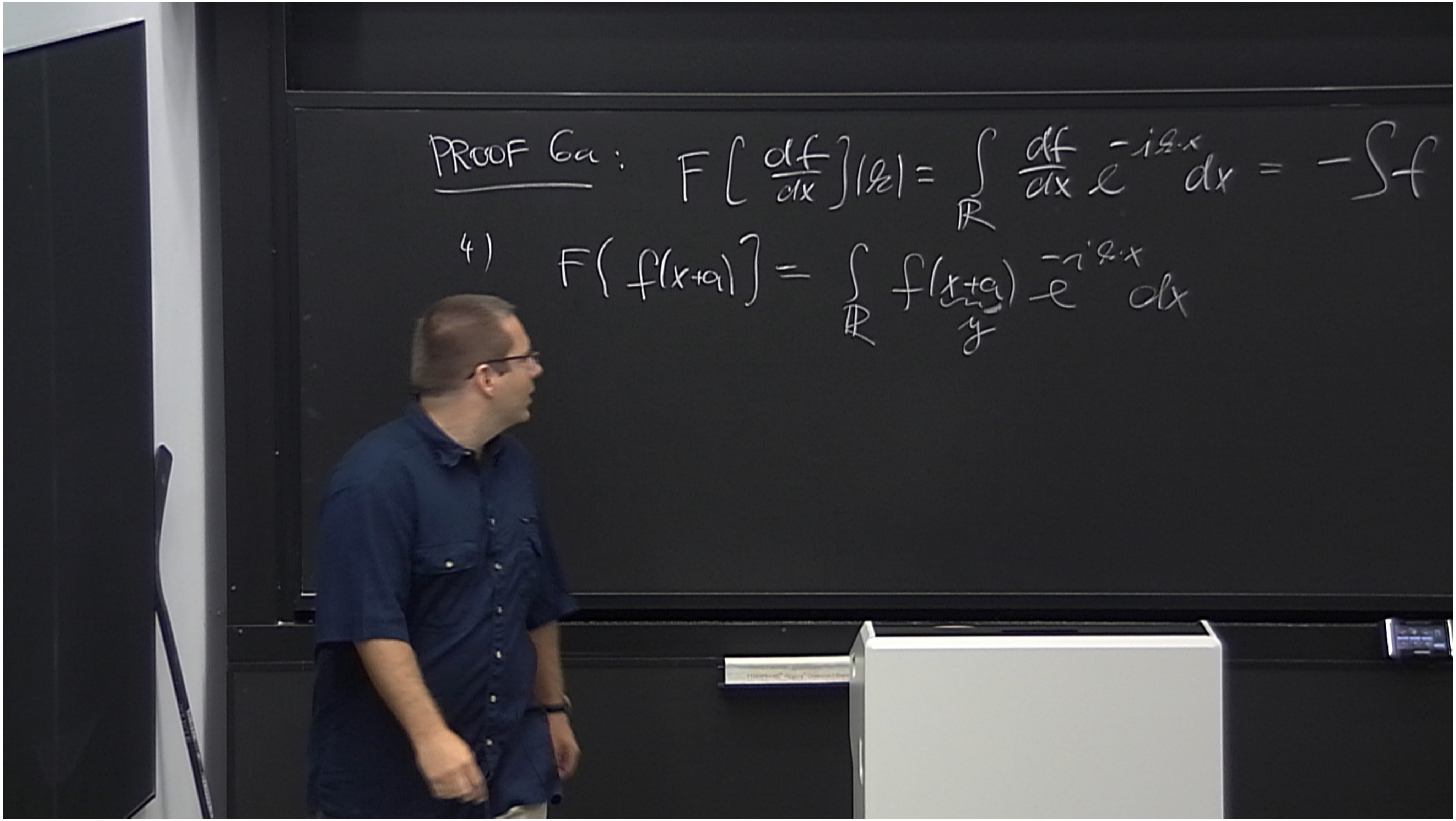
1) $\hat{f} \in C(\mathbb{R}^d)$
2) RIEMANN-LEBESGUE: $\lim_{|k|\to\infty} \hat{f}(k) = 0$
3) CONVOLUTION: $F[f*g](k) = \hat{f}(k)\,\hat{g}(k)$
4) SHIFT: $f_a(x) \equiv f(x+a) \Rightarrow \hat{f}_a(k) = e^{ik\cdot a}\hat{f}(k)$
5) SCALE. $a \in \mathbb{R} \Rightarrow F[f(ax)](k) = \frac{1}{|a|^d}\hat{f}(k/a)$
6) DERI
7) PAR
8) INVE

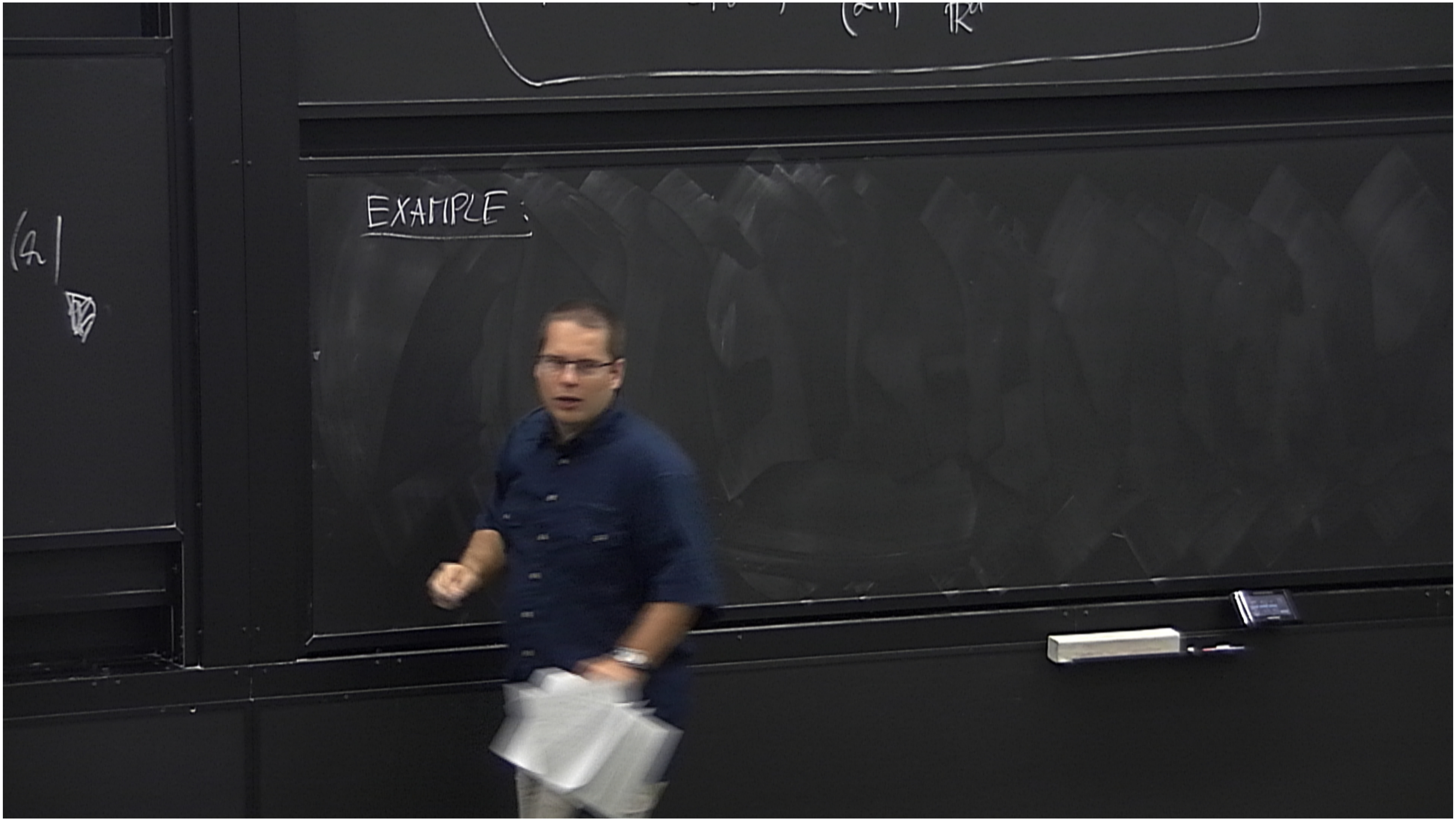

6) DERIVATIVES: α MULTIINDEX =>
7) PARSEVAL:
8) INVERSION: F AND F⁻¹ ARE INVERSIONS
9)

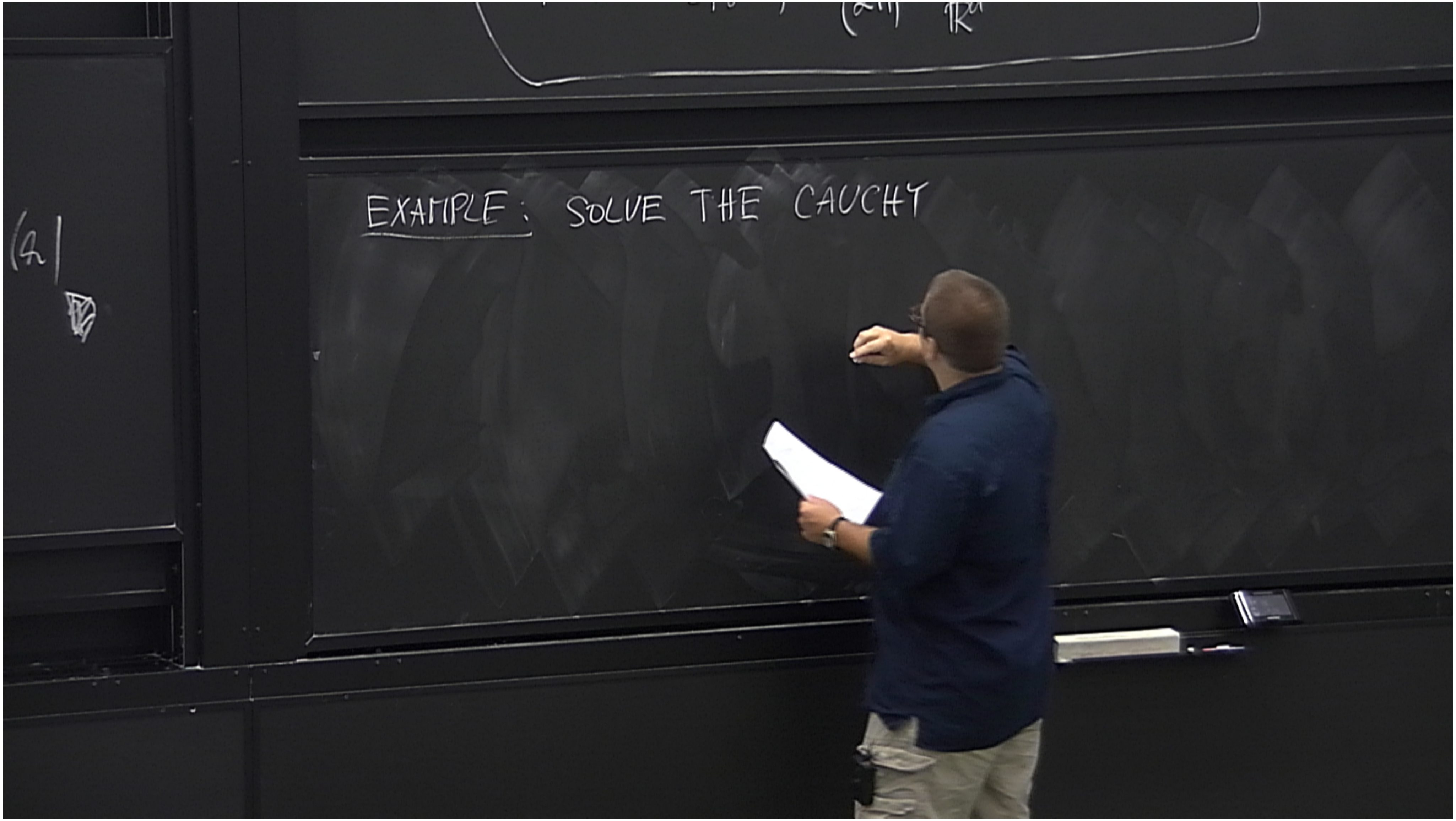

PROOF 6a :

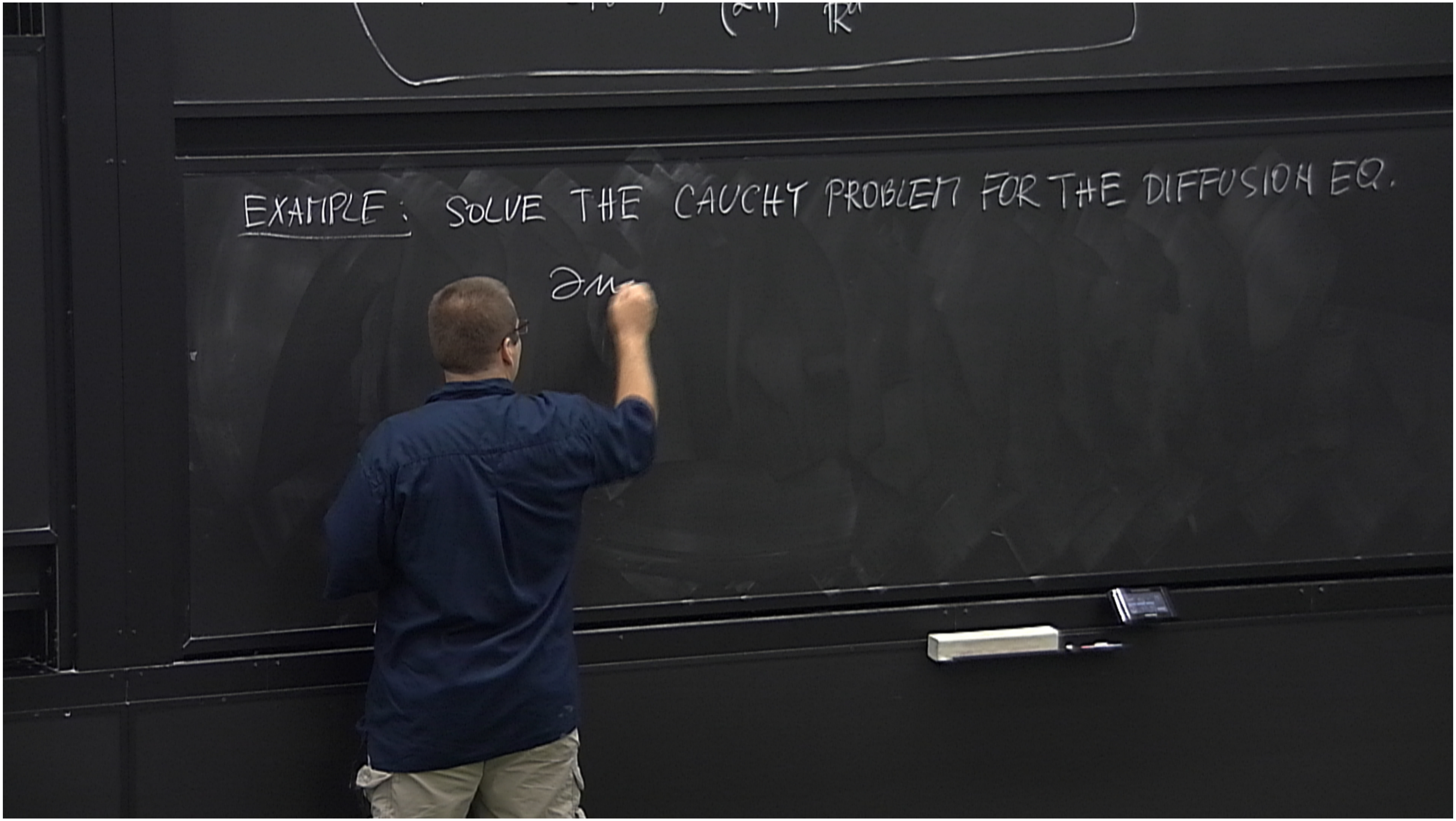

PROOF 6a:
F[df/dx](k) = ∫_R df/dx

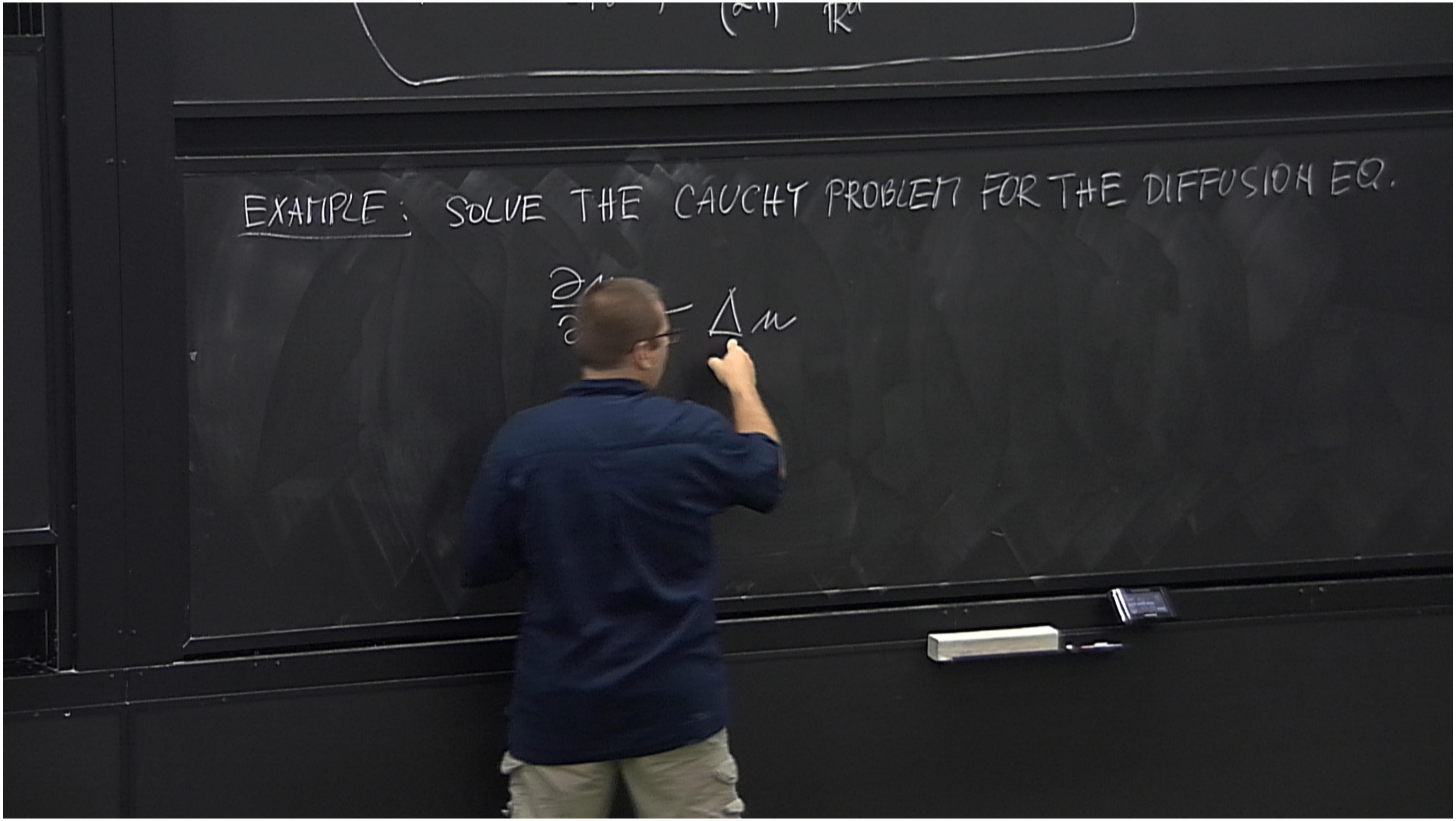

PROOF 6a:
F[df/dx](k) = ∫_R df/dx e^{-ik·x} dx = −∫
4) F(f(x+a)) = ∫_R

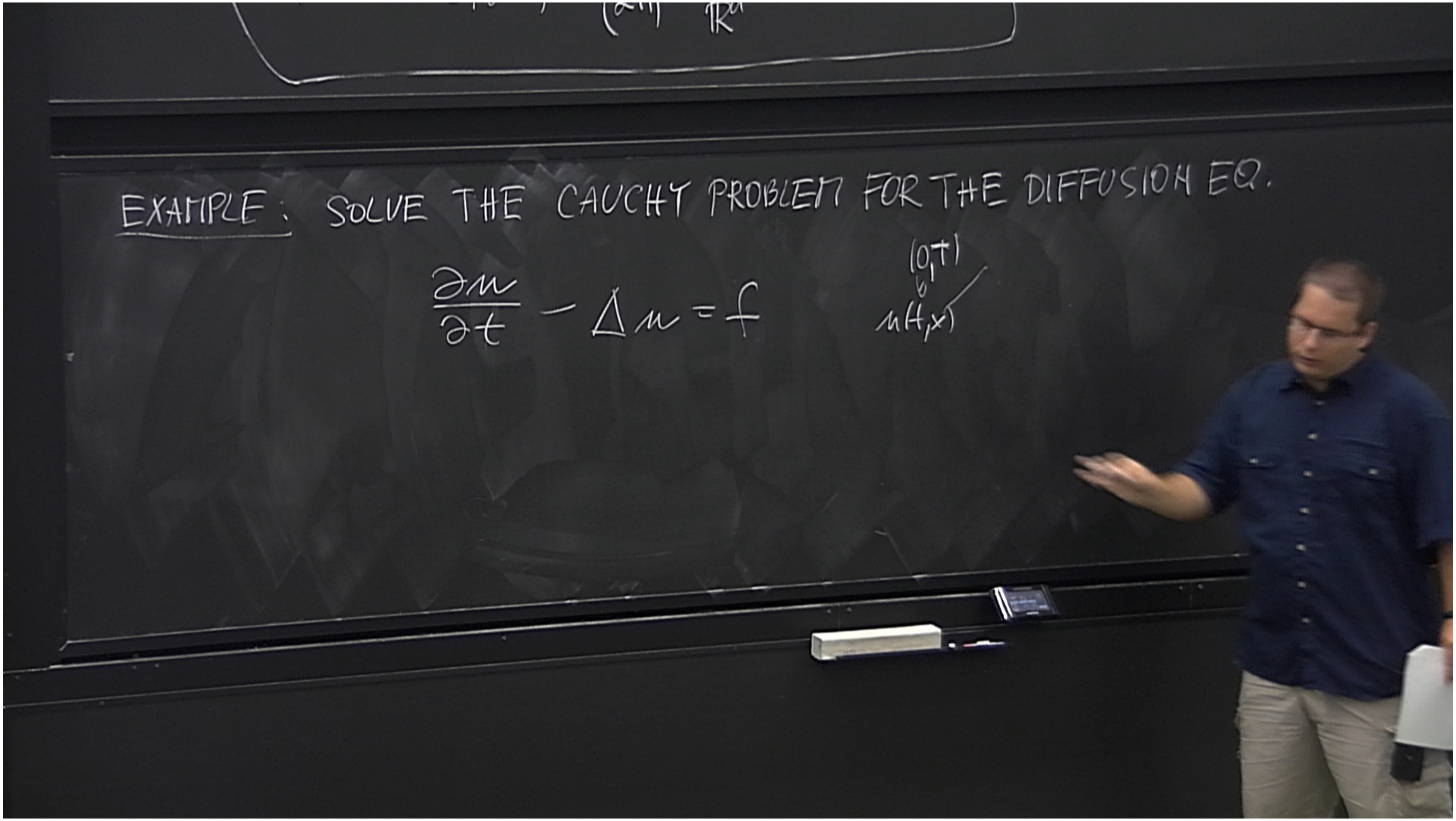

PROOF 6a: $F\left[\frac{df}{dx}\right](k) = \int_{\mathbb{R}} \frac{df}{dx} e^{-ik\cdot x} dx = -\int f$
4) $F[f(x+a)] = \int_{\mathbb{R}} f(x+a) e^{-ik\cdot x} dx$
y

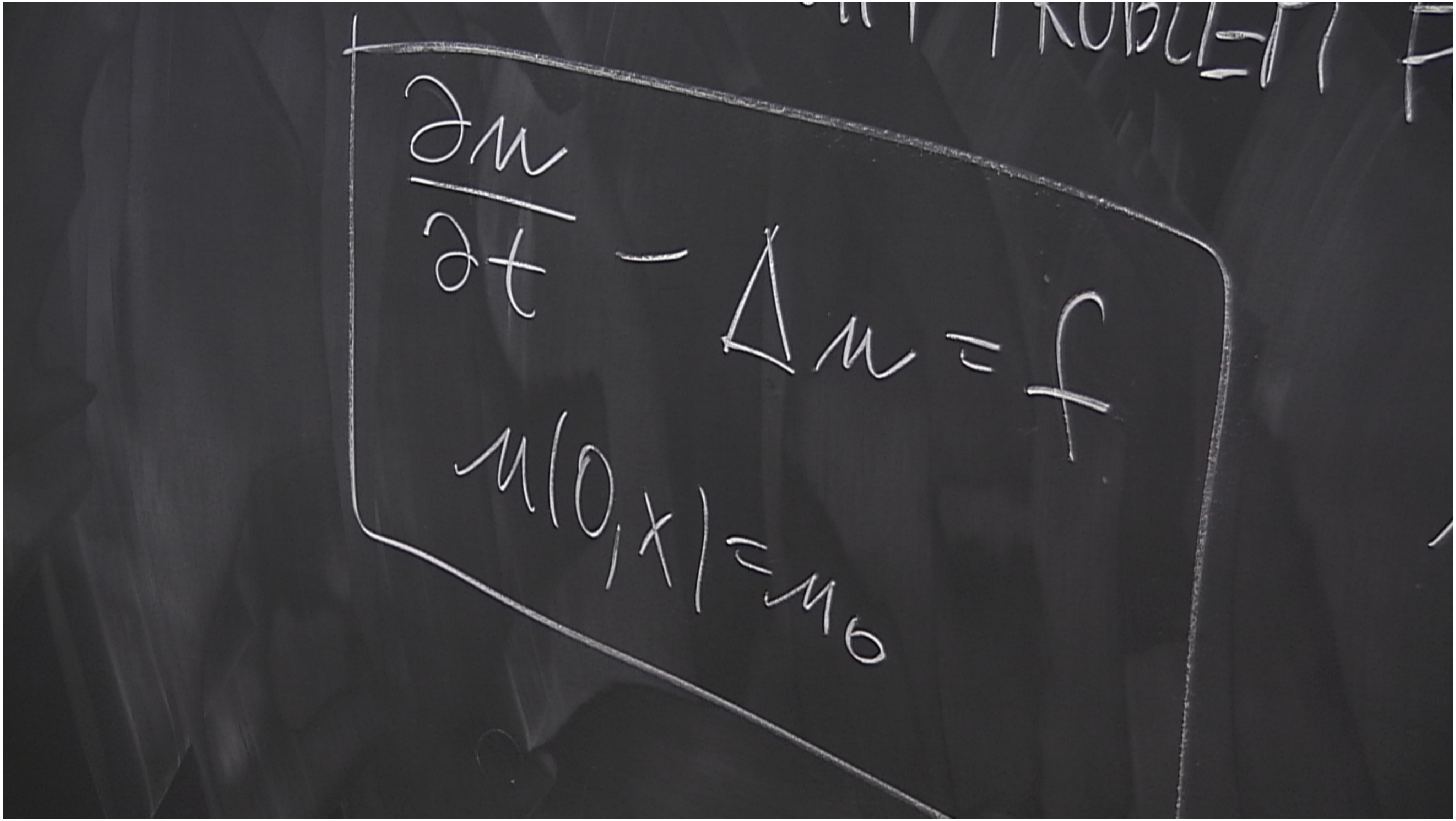

EXAMPLE :


EXAMPLE: SOLVE THE CAUCHY

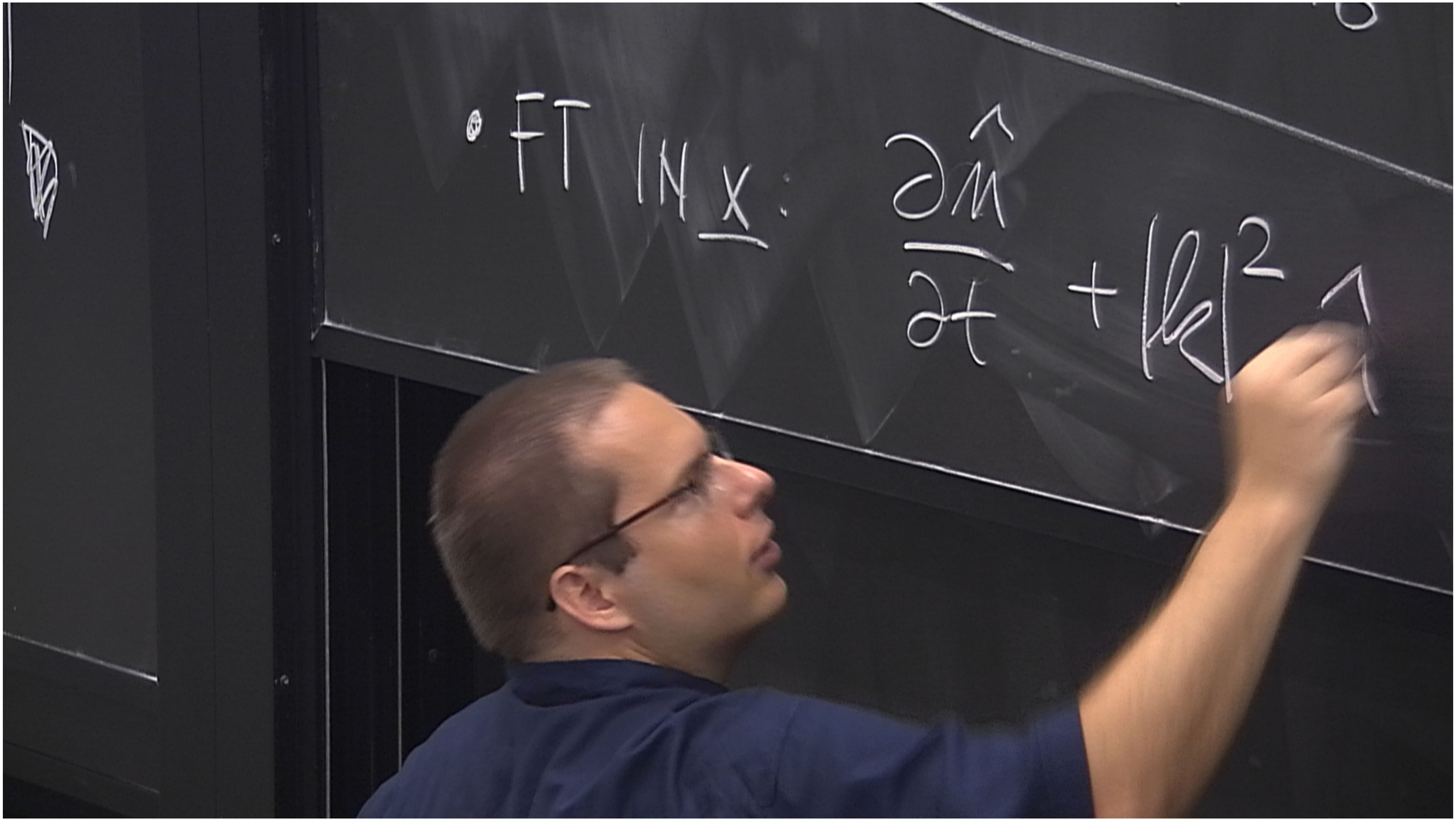

EXAMPLE: SOLVE THE CAUCHY PROBLEM FOR THE DIFFUSION EQ.


EXAMPLE: SOLVE THE CAUCHY PROBLEM FOR THE DIFFUSION EQ.

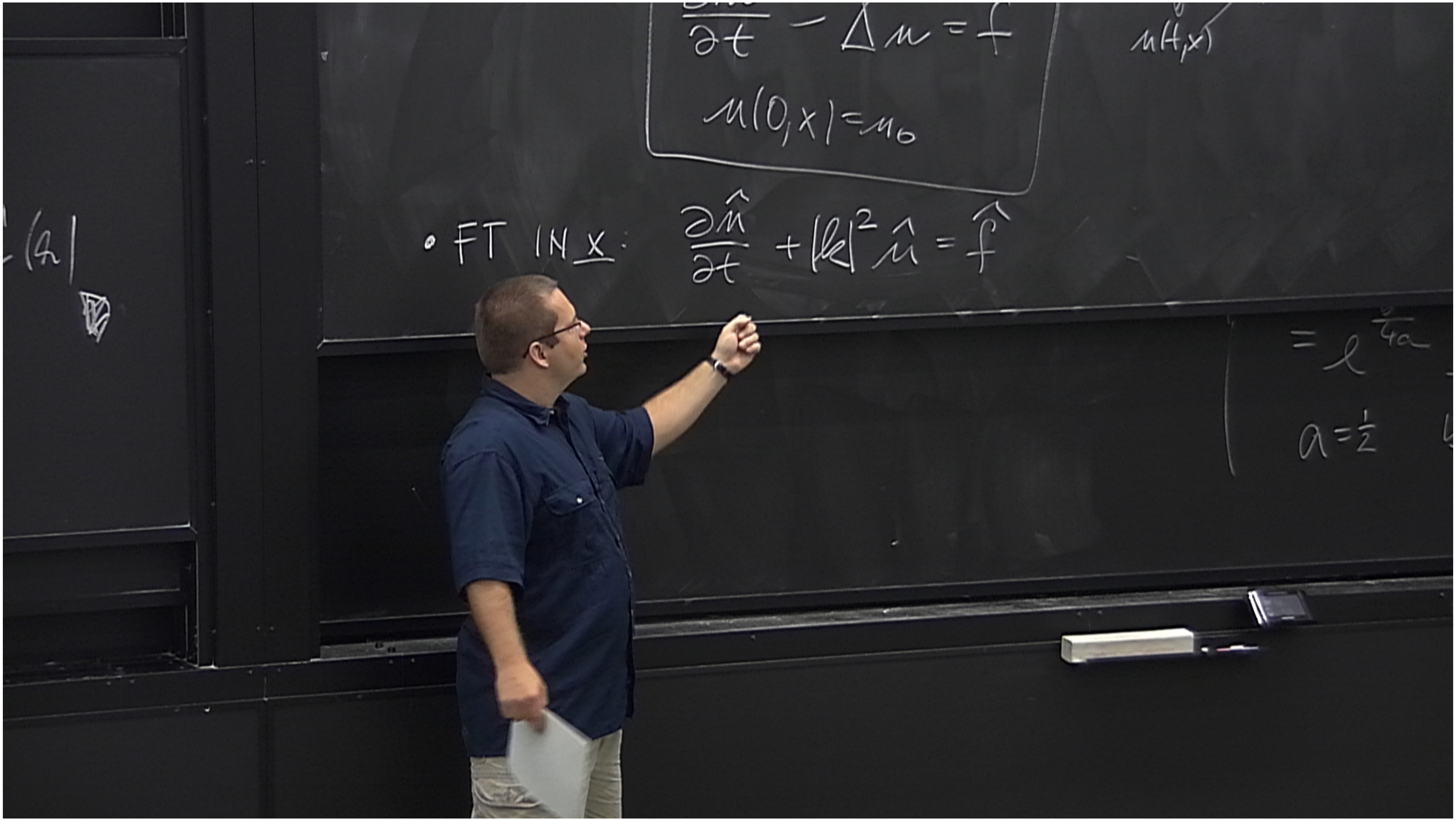

EXAMPLE: SOLVE THE CAUCHY PROBLEM FOR THE DIFFUSION EQ.
∂u/∂t − Δu = f
u(t,x)
t ∈ (0,T)

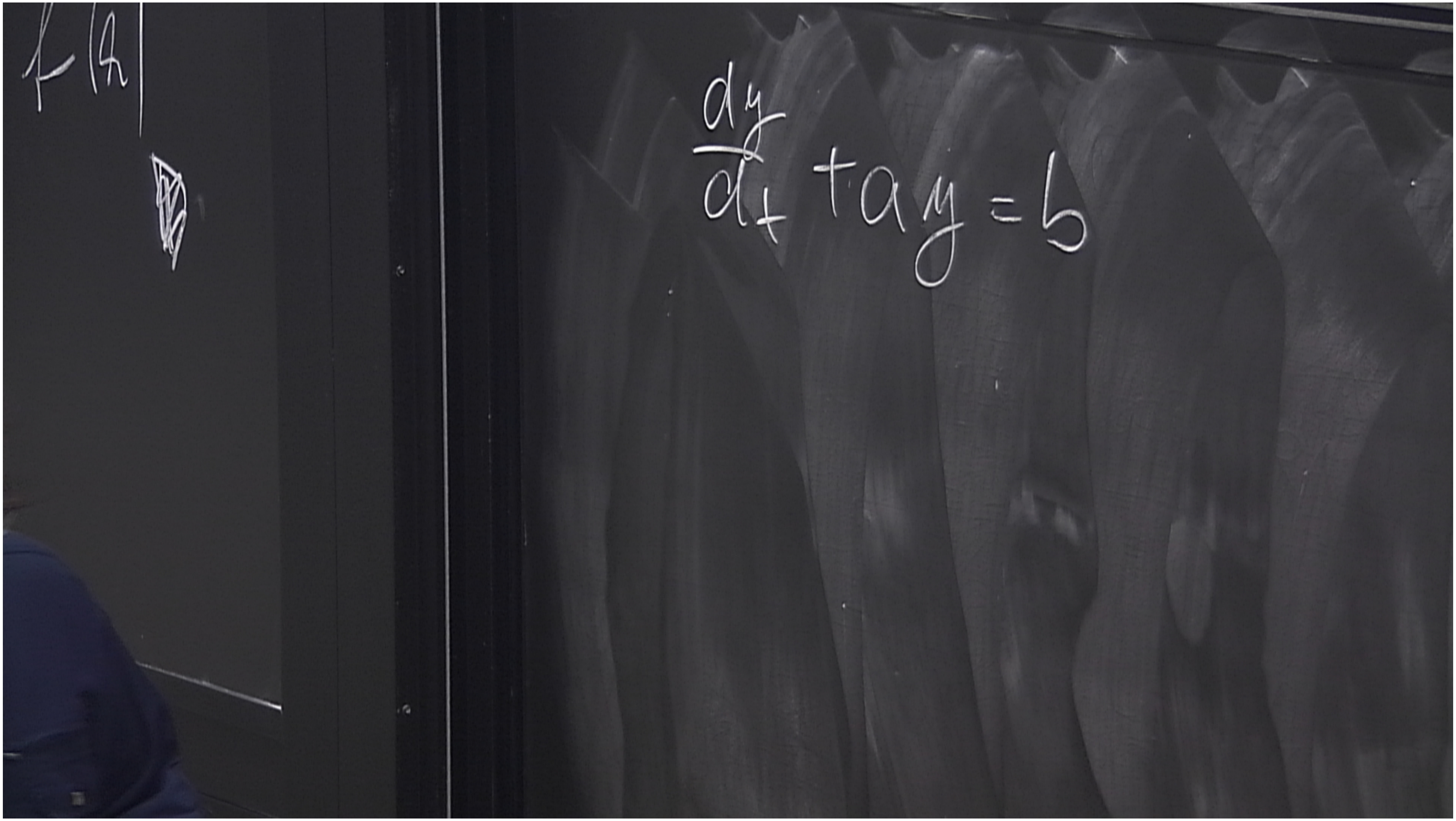

PROBLEM
$\frac{\partial u}{\partial t} - \Delta u = f$
$u(0,x) = u_0$

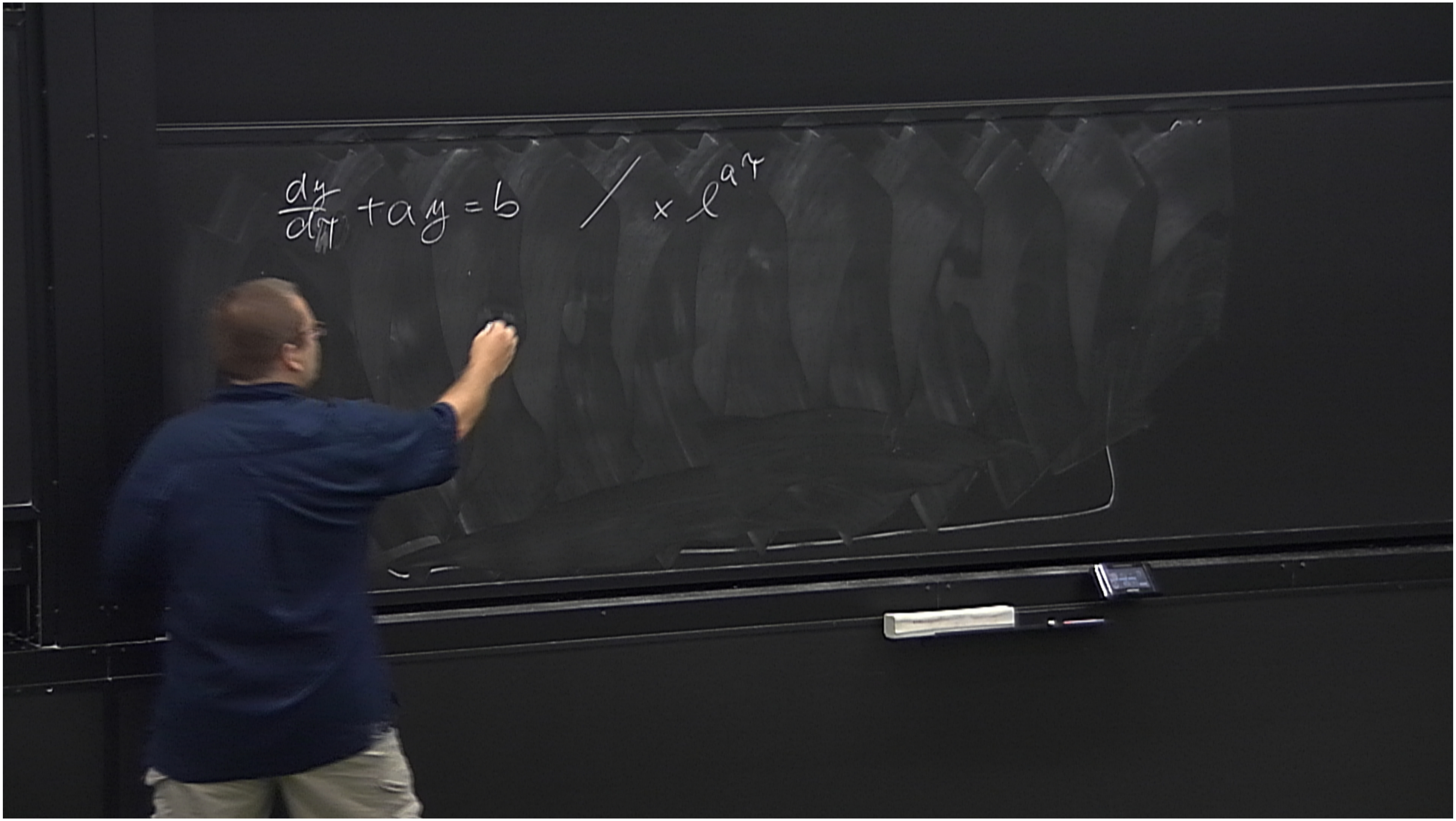

$\frac{\partial u}{\partial t} - \Delta u = f$
$u(0,x) = u_0$
$u(t,x)$
• FT IN x :
$a = \frac{1}{2}$

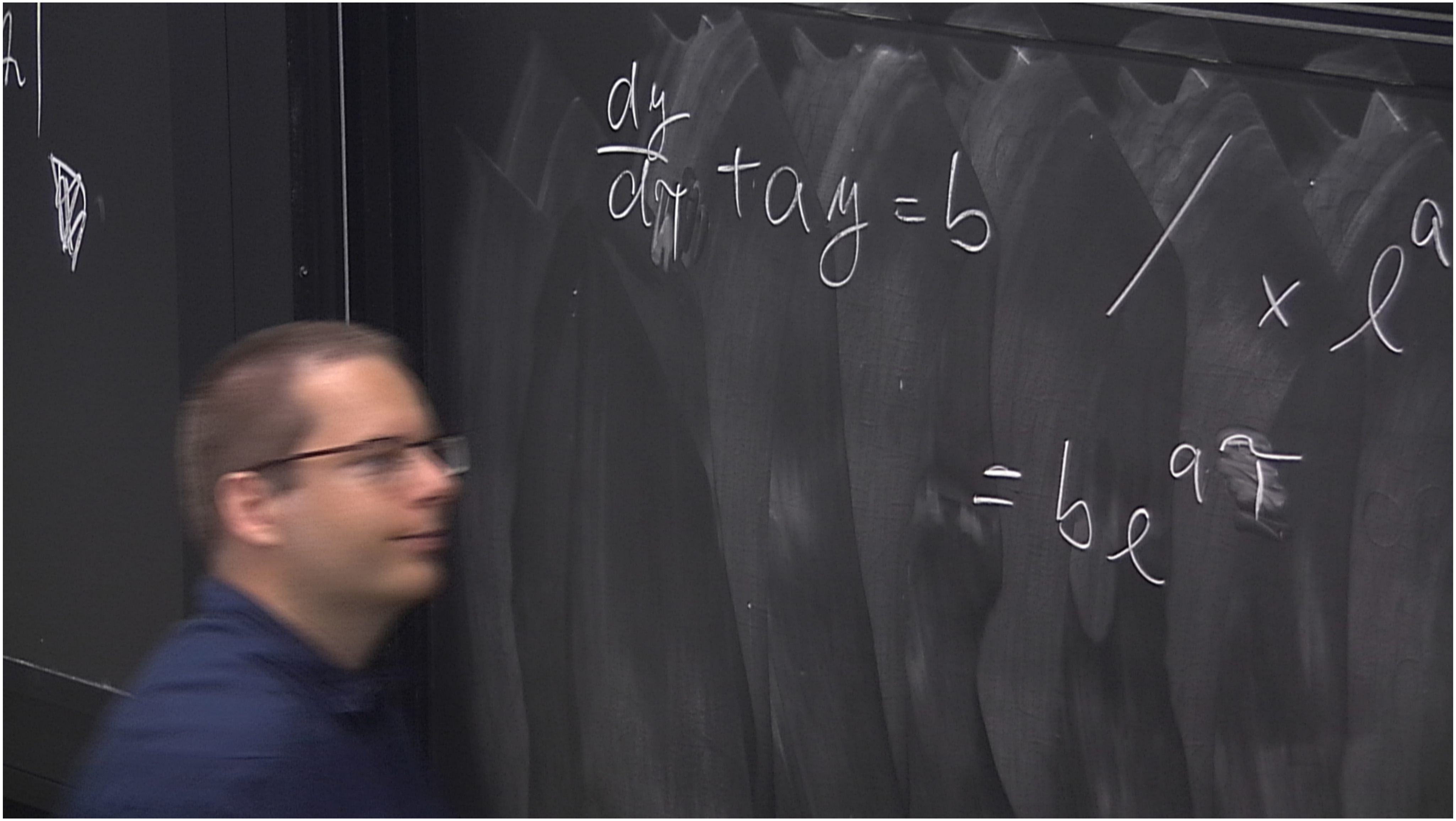


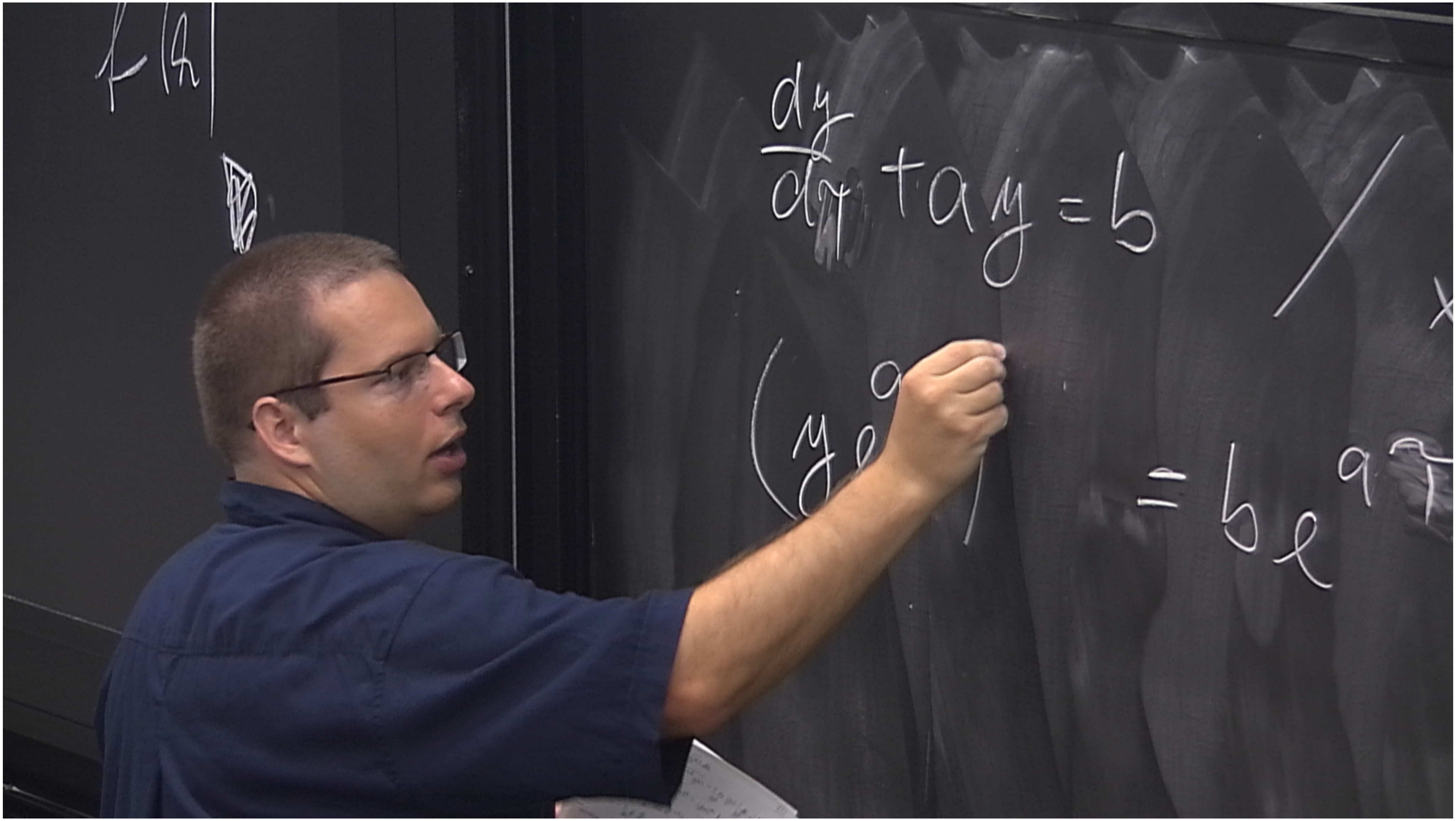
FT IN x :
$\frac{\partial \hat{u}}{\partial t} + |k|^2$

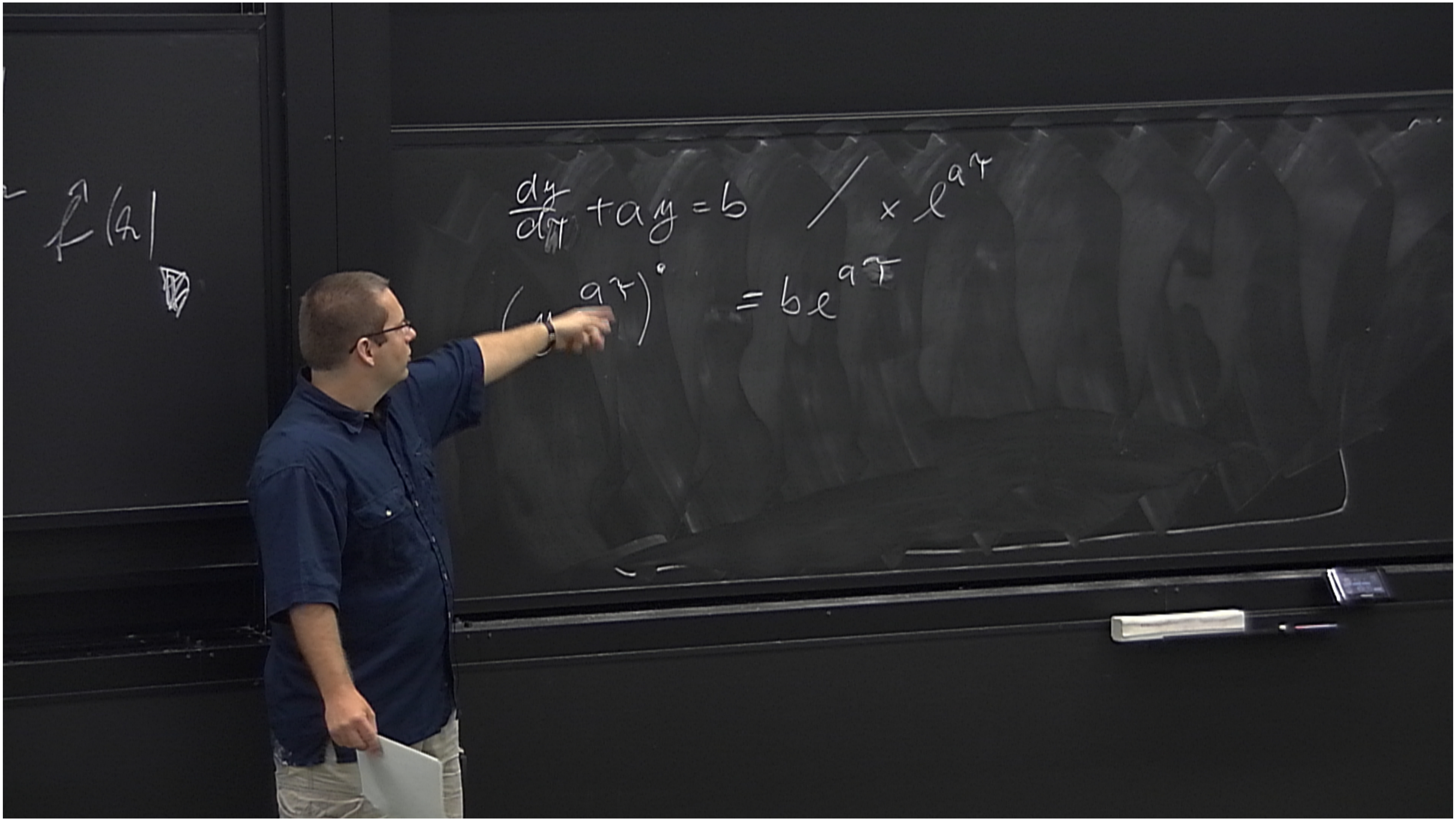

∂t − Δu = f
u(t,x)
u(0,x) = u0
• FT IN x:
∂û/∂t + |k|² û = f̂
a = 1/2

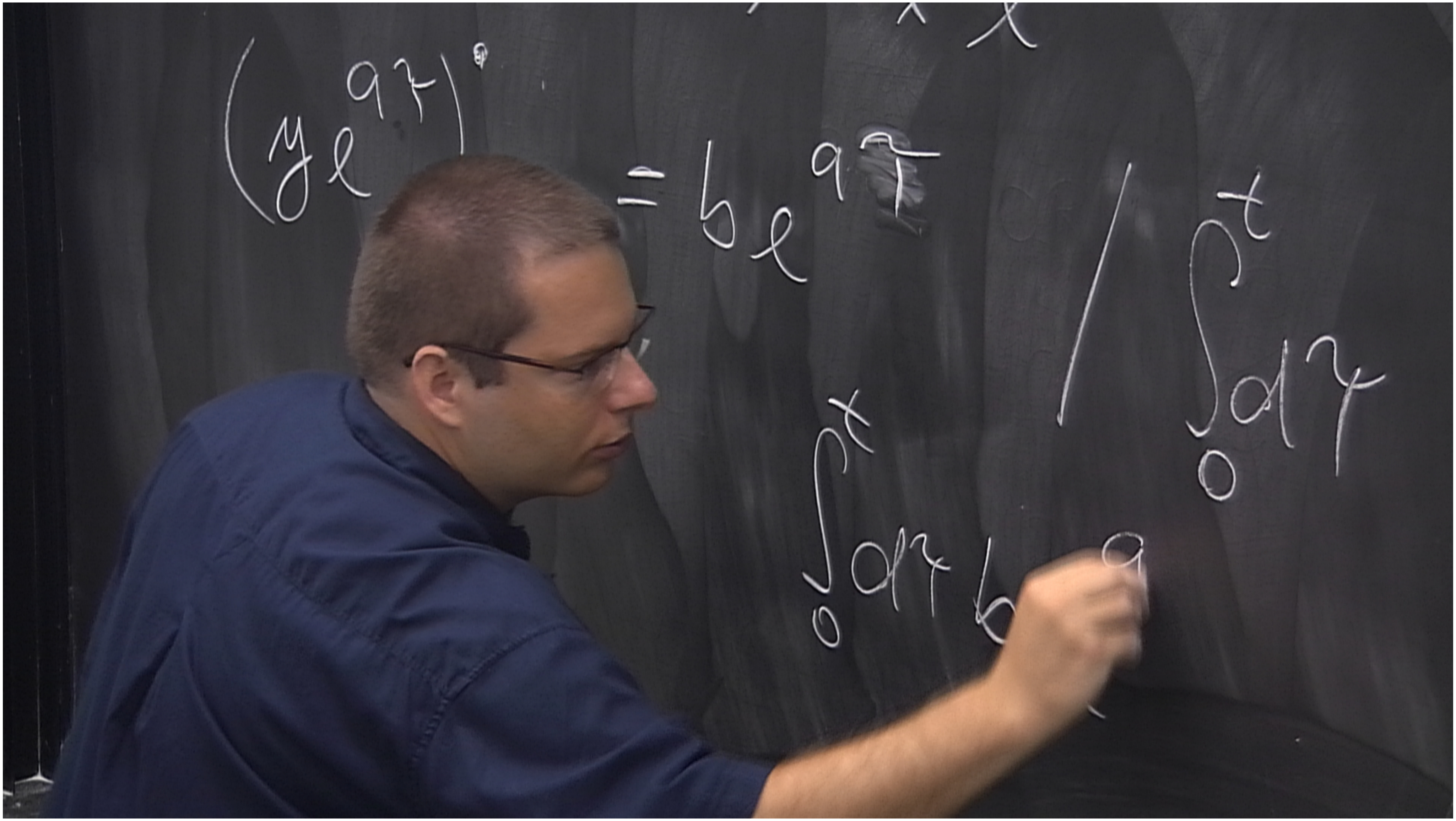

dy/dt + ay = b

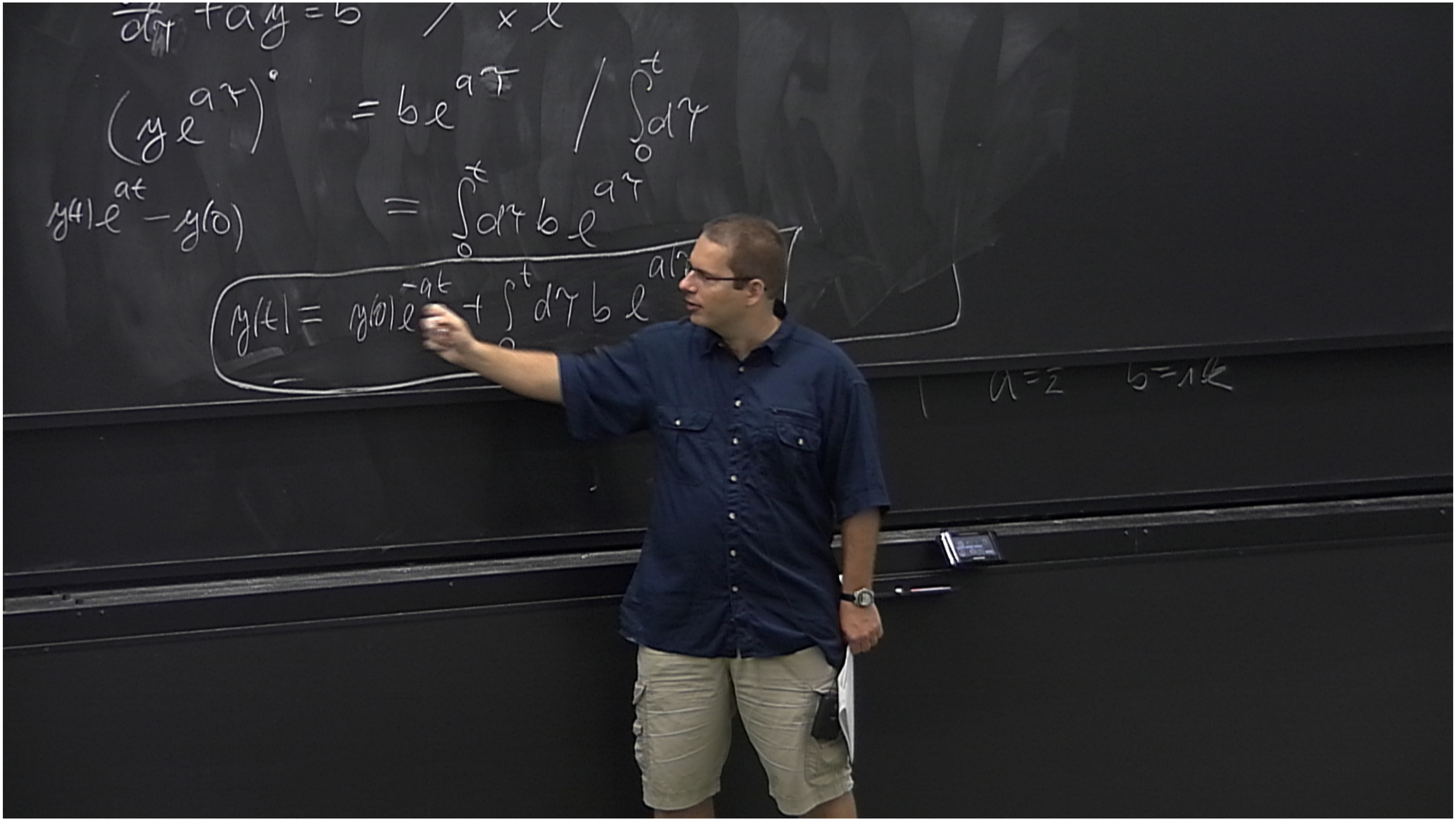

dy/dτ + ay = b
× e^{aτ}


dy/dτ + ay = b
× e^a
= b e^aτ

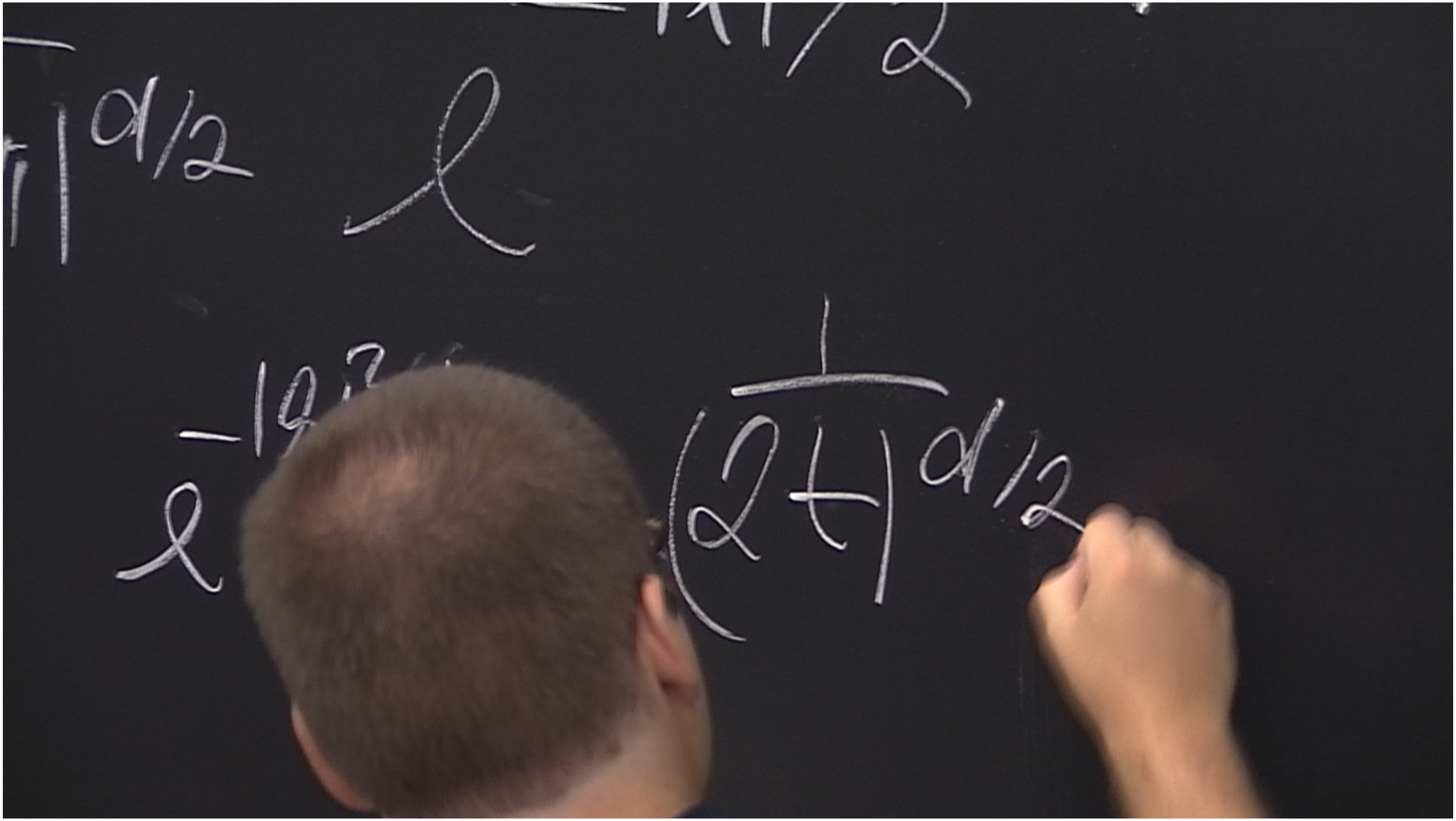

dy/dτ + ay = b

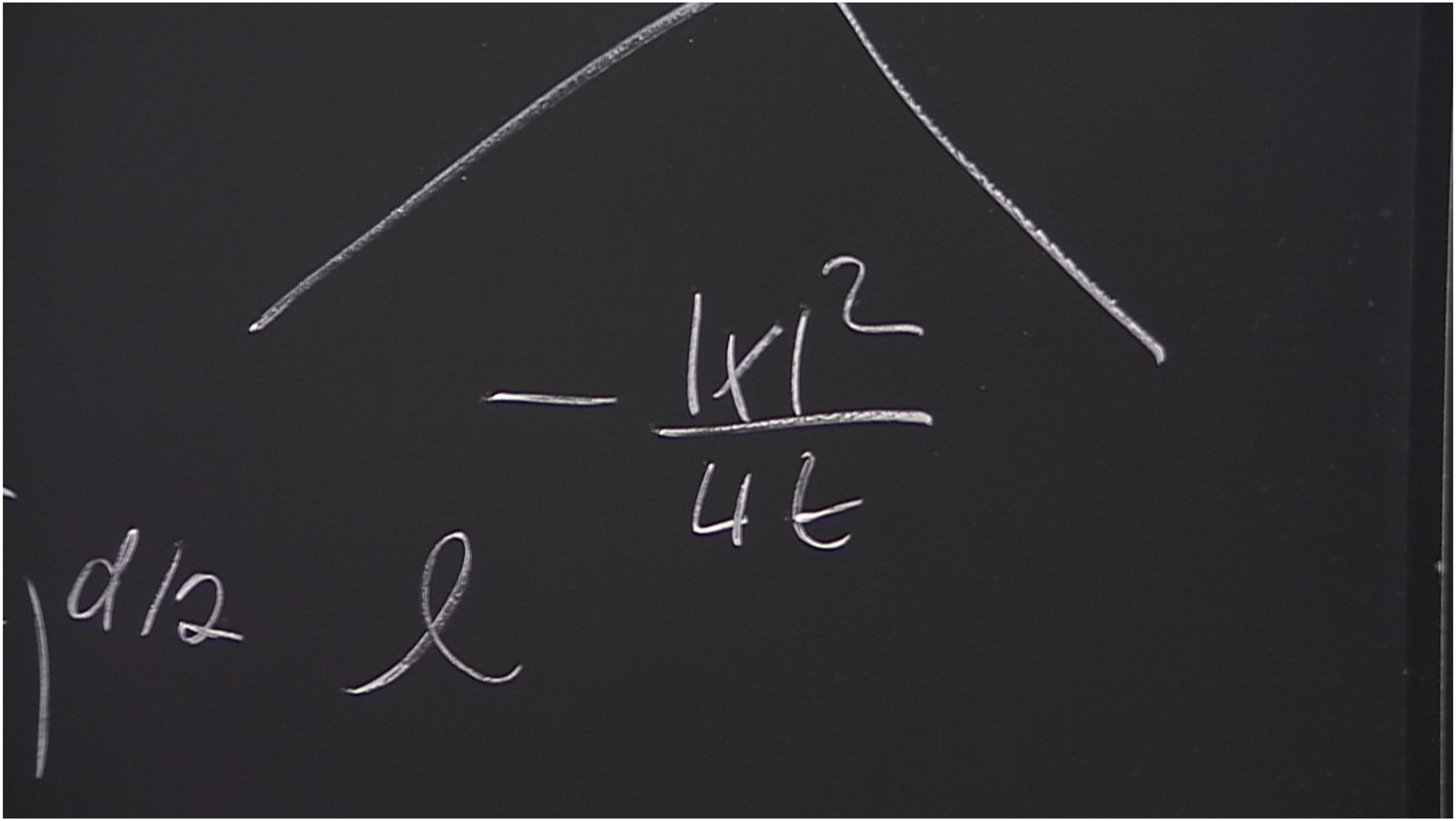


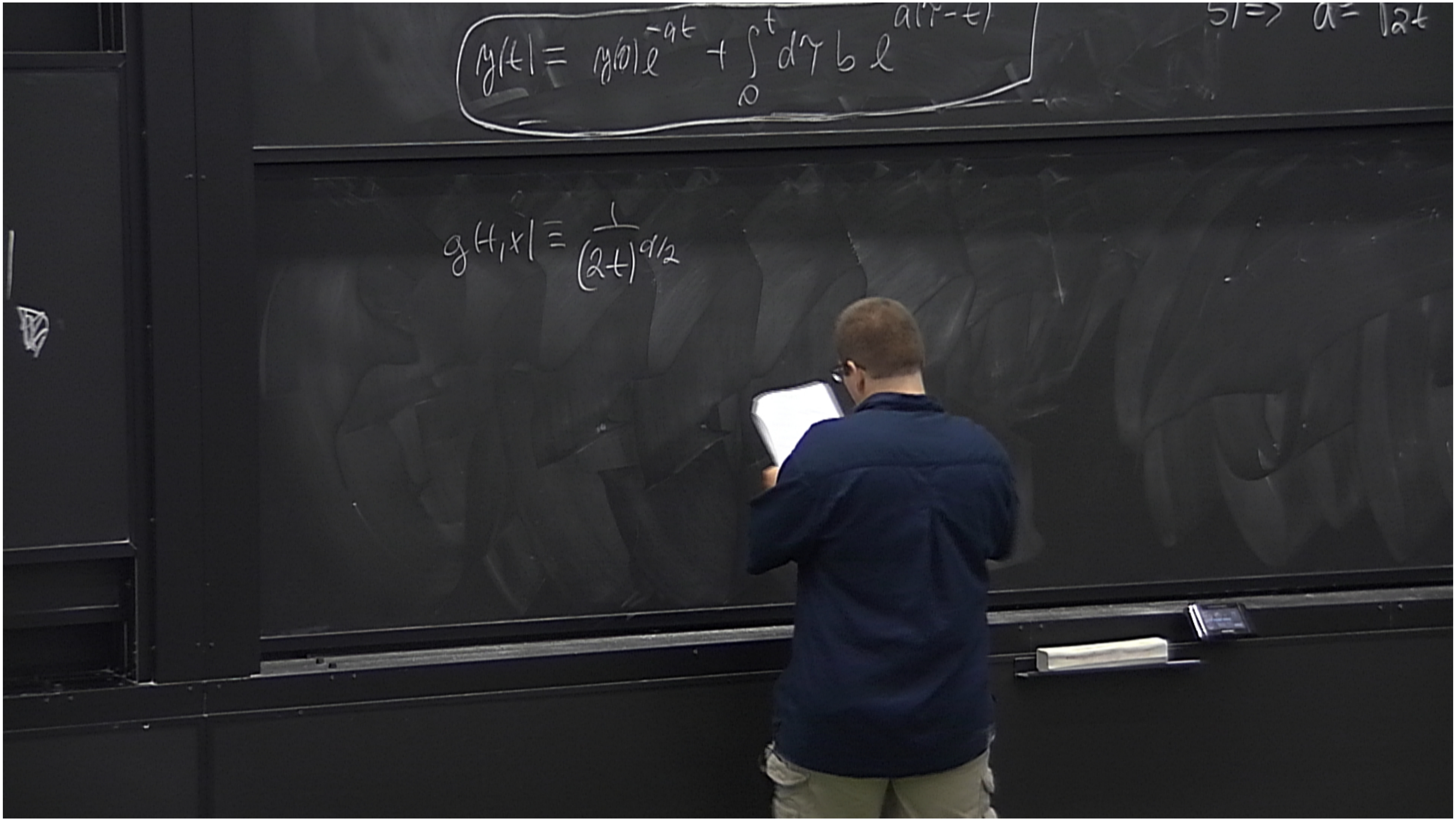


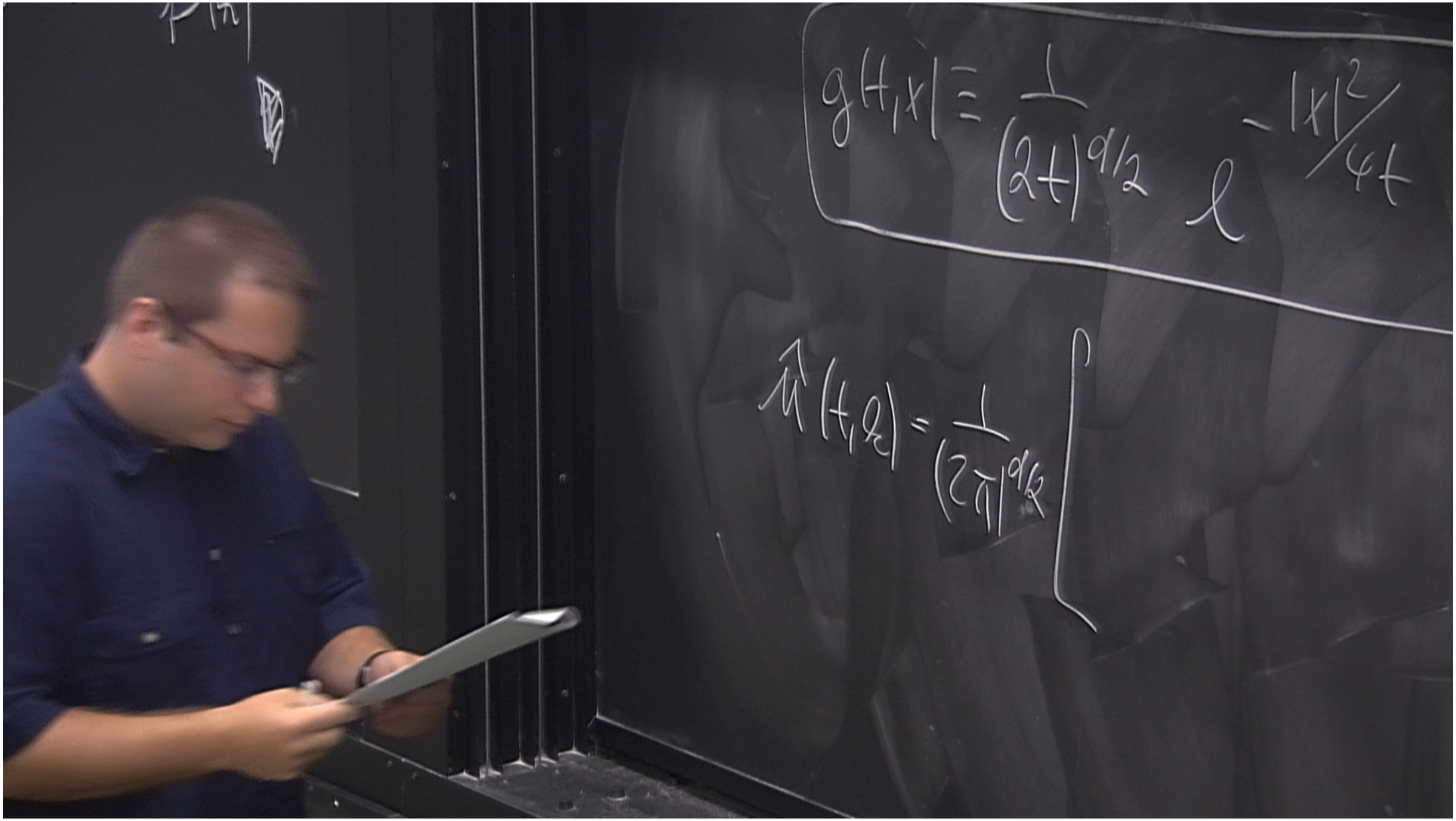

(y e^{aτ})˙ = b e^{aτ} / ∫_0^t dτ
y(t) e^{at} − y(0) = ∫_0^t dτ b e^{aτ}
y(t) = y(0) e^{−at} + ∫_0^t dτ b e^{a(τ
a = 2   b = 1k

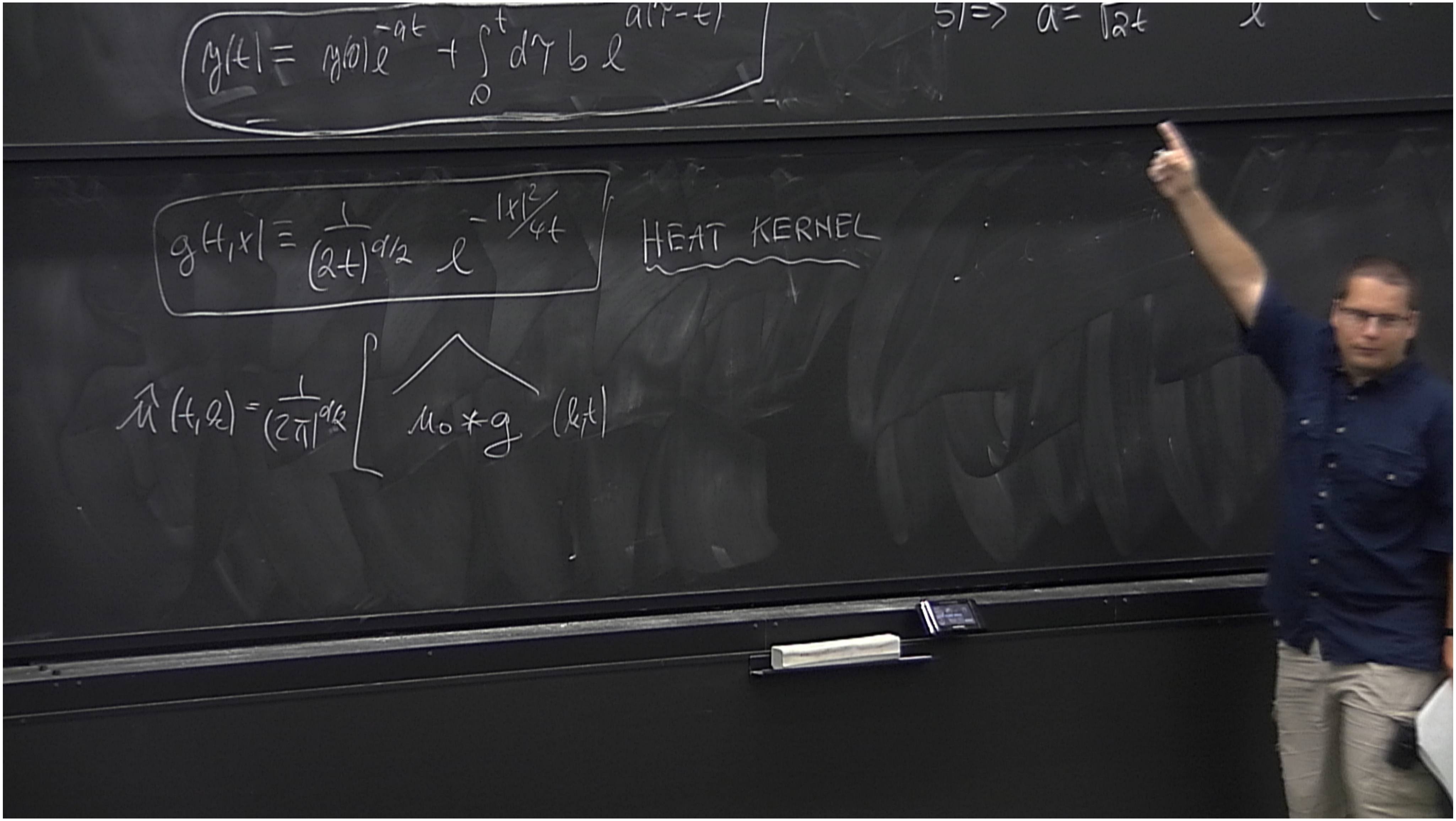
$$\frac{1}{|\pi|^{d/2}} e^{-|x|^2/2}$$

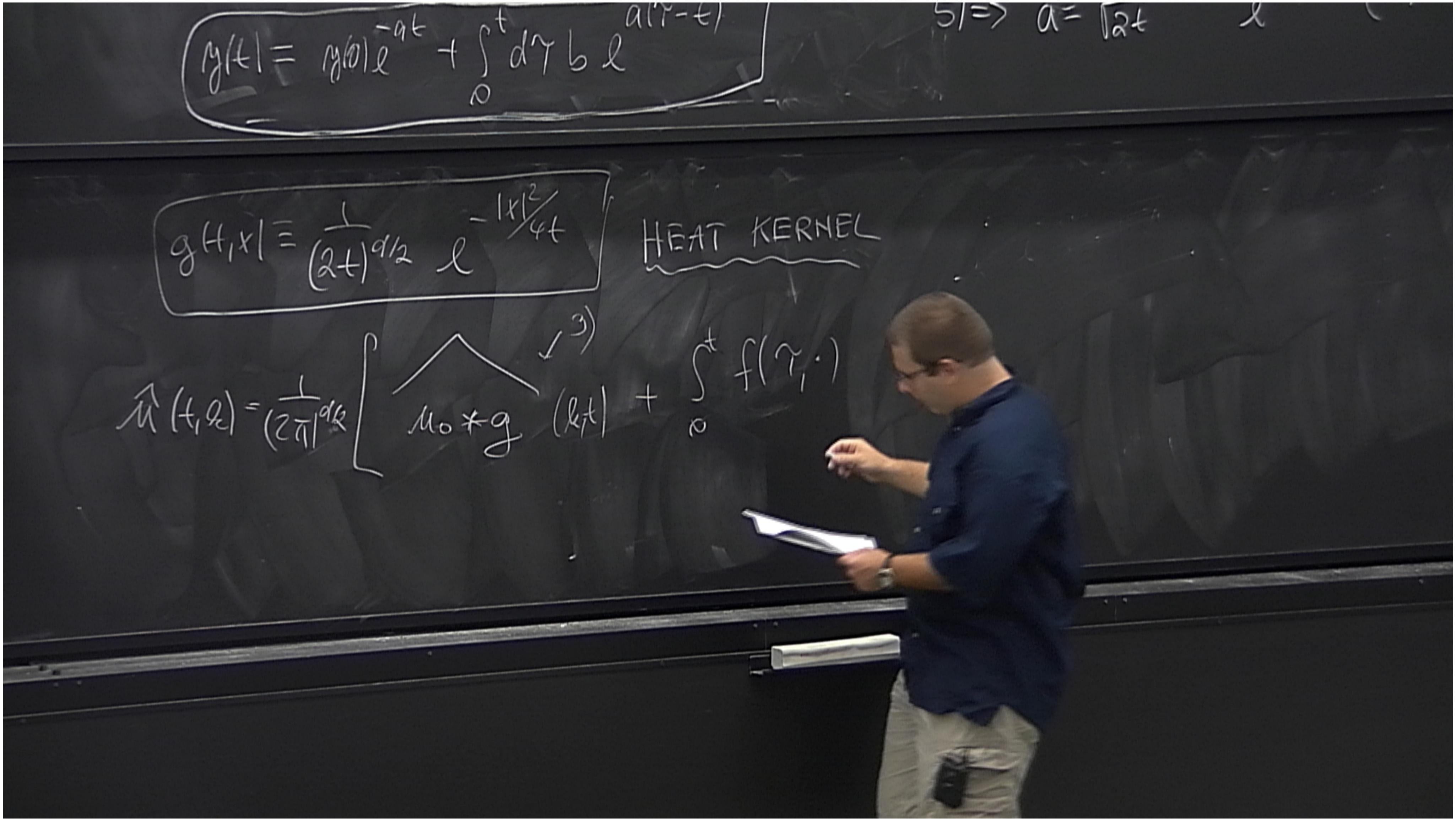


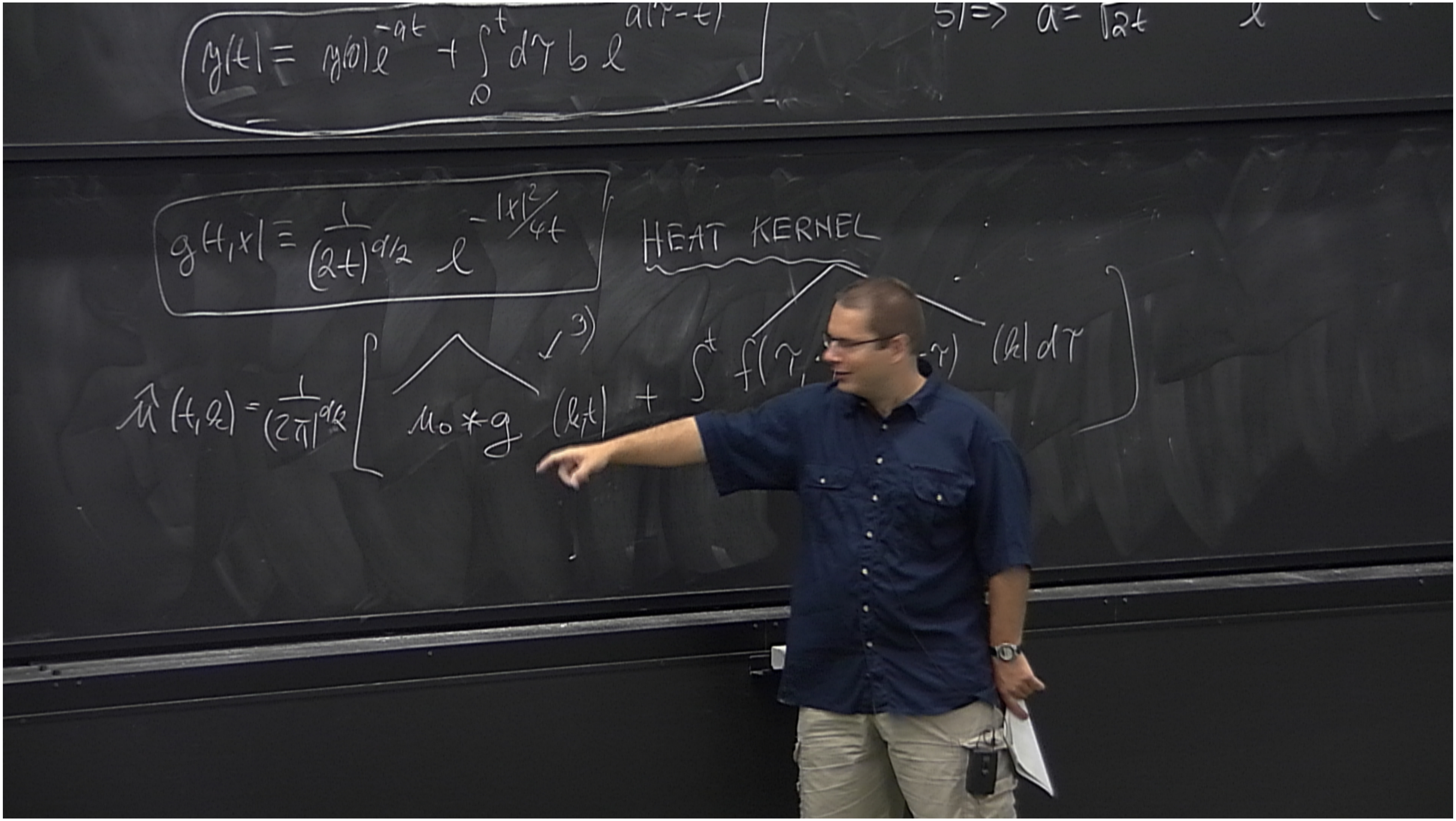

$)^{d/2} \, e^{-\frac{|x|^2}{4t}}$

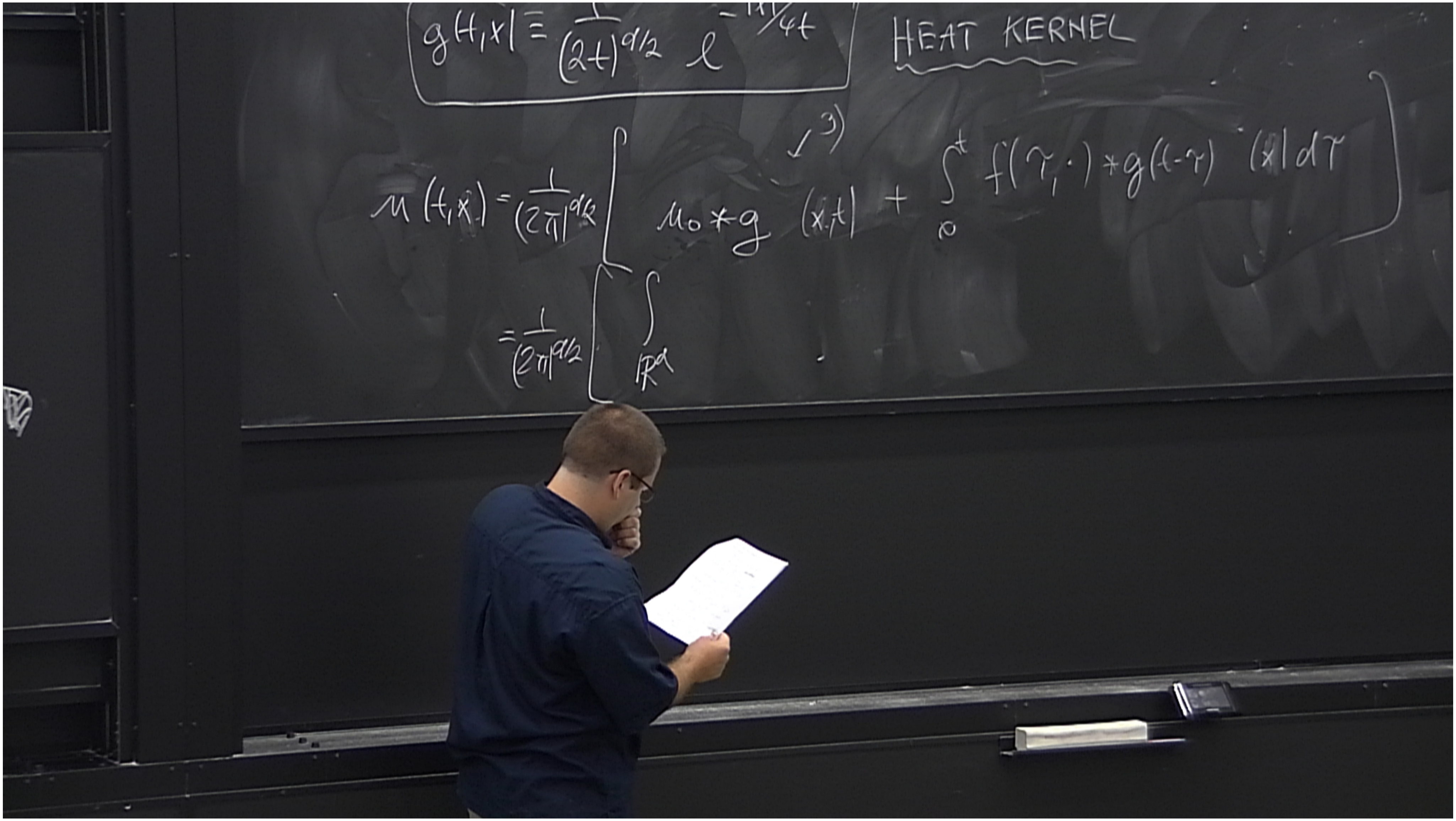

y(t) = y(0)e^{-at} + ∫_0^t dτ b e^{a(τ-t)}
g(t,x) ≡ 1/(2t)^{d/2}

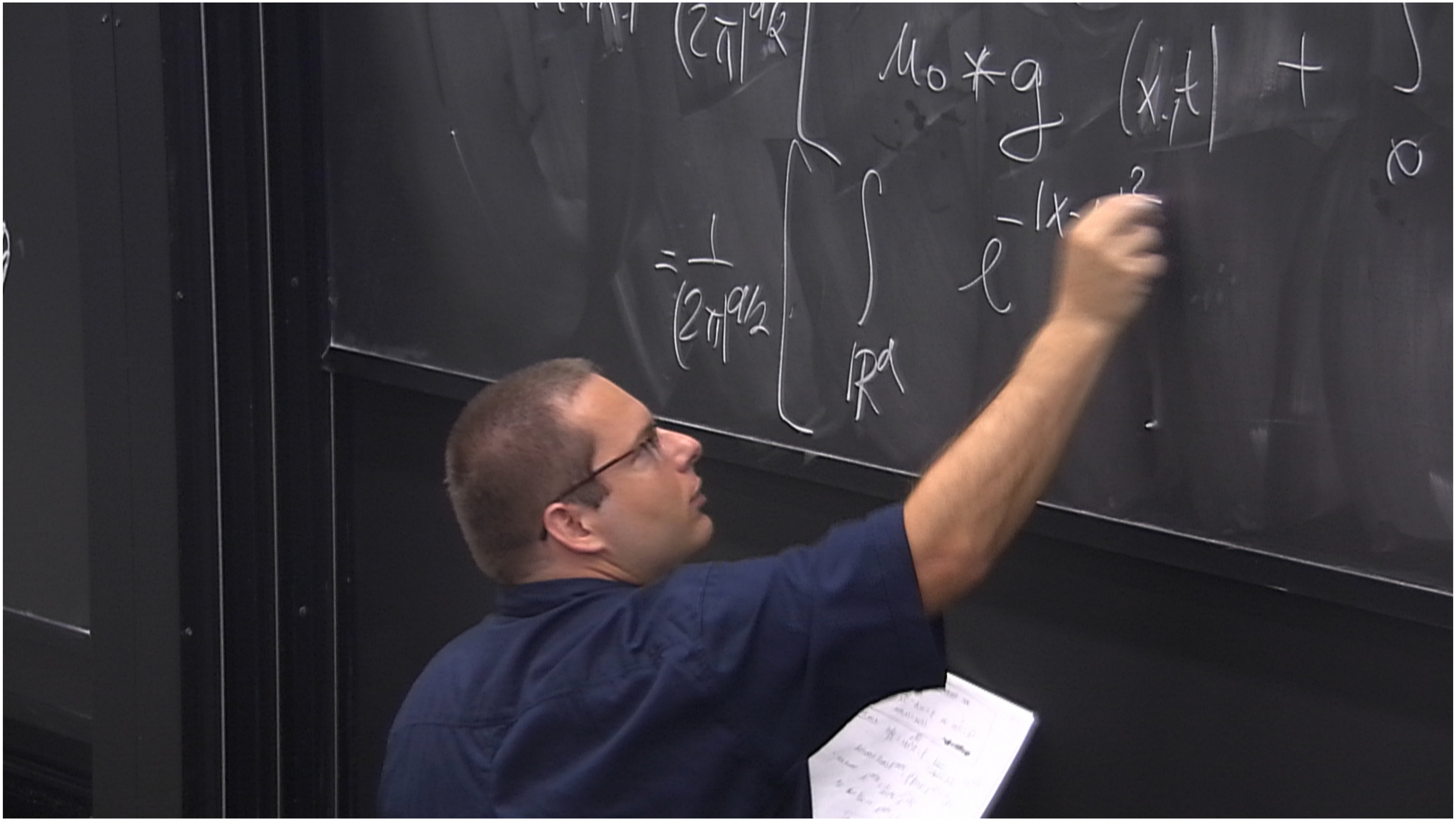

$g(t,x) \equiv \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/4t}$
$\hat{u}(t,k) = \frac{1}{(2\pi)^{d/2}} \int$

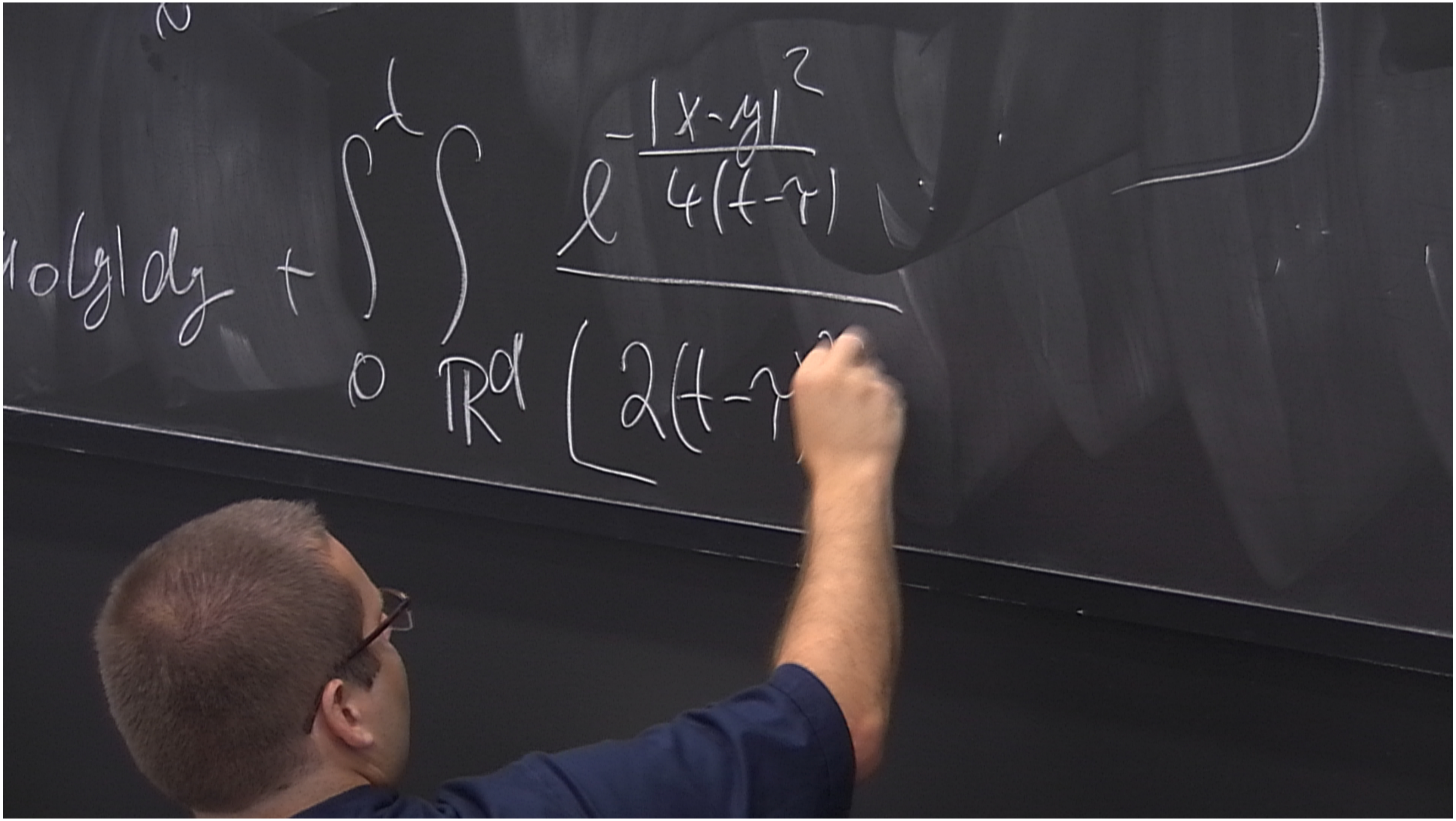

$y(t) = y(0)\, e^{-at} + \int_0^t d\tau\, b\, e^{a(\tau - t)}$
$5) \Rightarrow a = \sqrt{2t}$
$g(t,x) \equiv \frac{1}{(2t)^{d/2}} e^{-|x|^2/4t}$
HEAT KERNEL
$\hat{u}(t,k) = \frac{1}{(2\pi)^{d/2}} \left[\widehat{u_0 * g} \right] (k,t)$

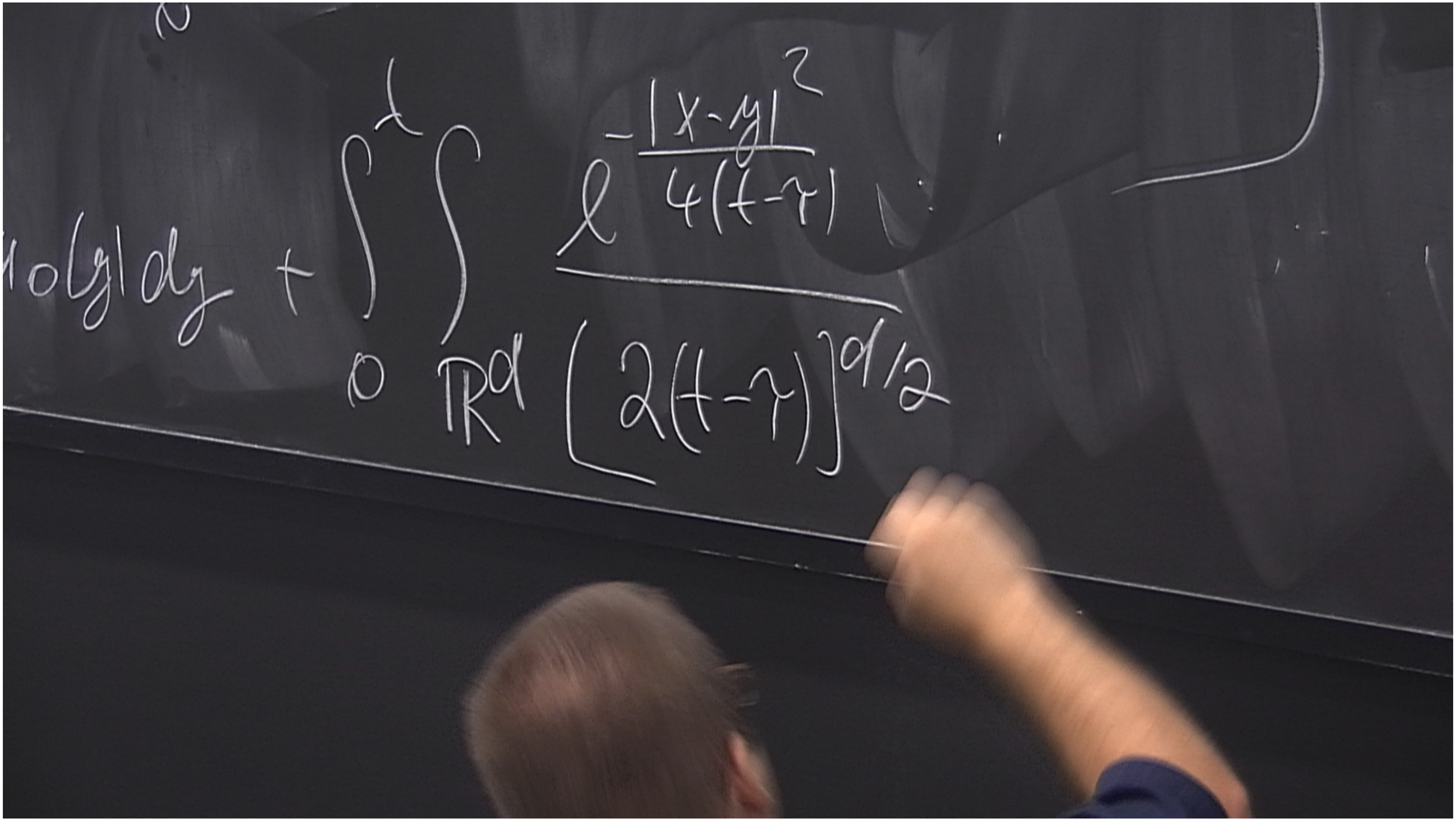

HEAT KERNEL

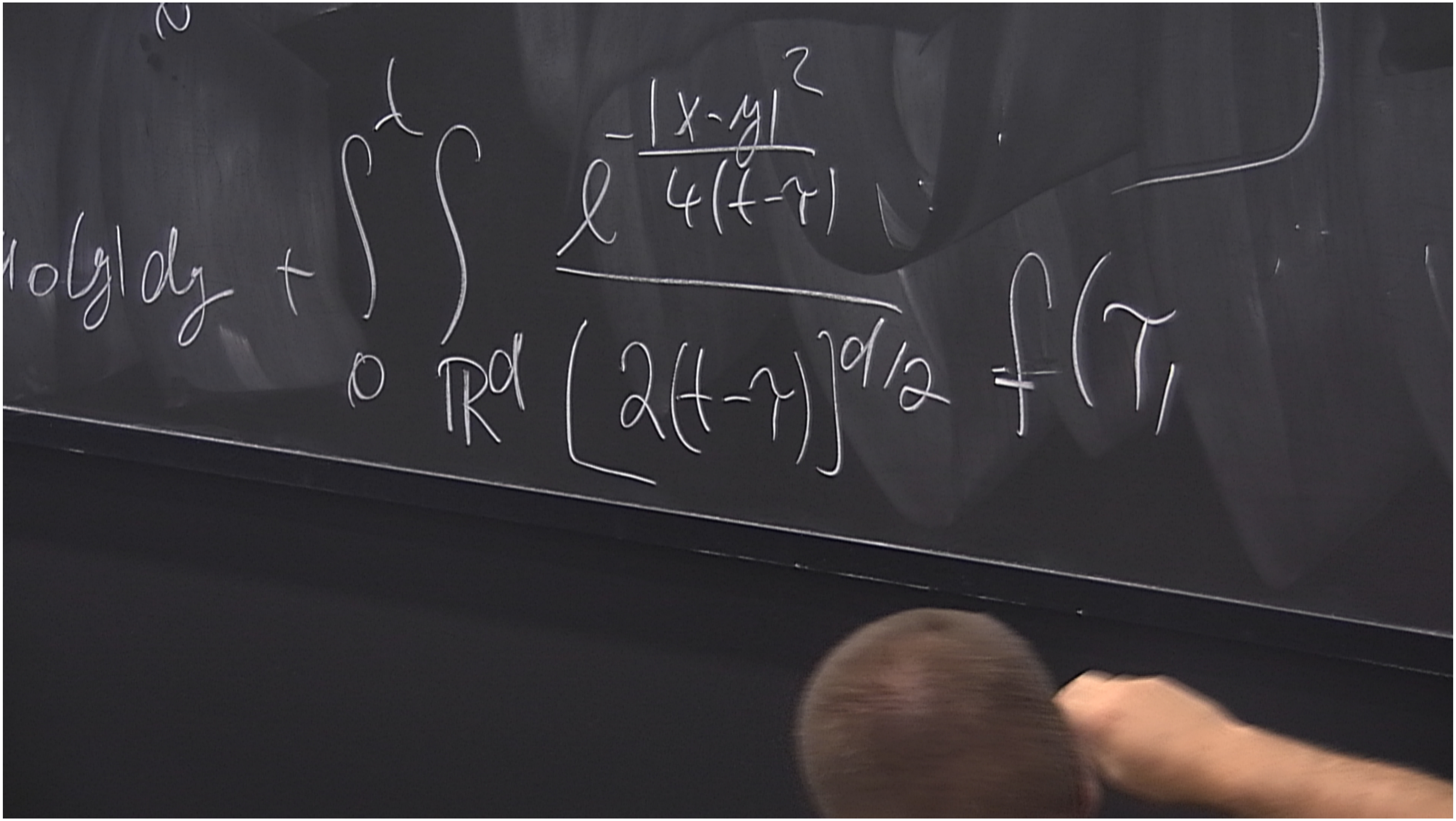

HEAT KERNEL

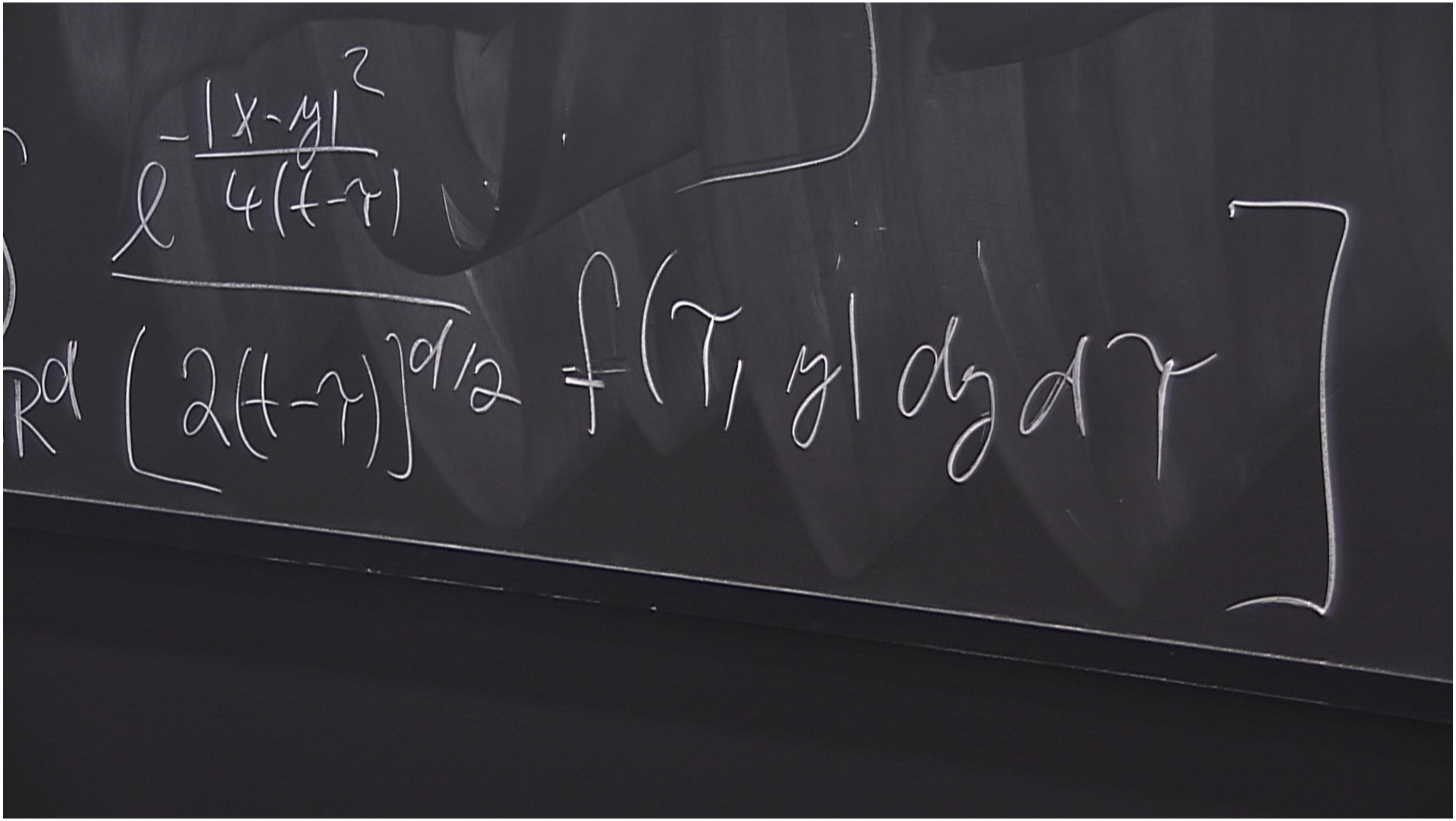

HEAT KERNEL



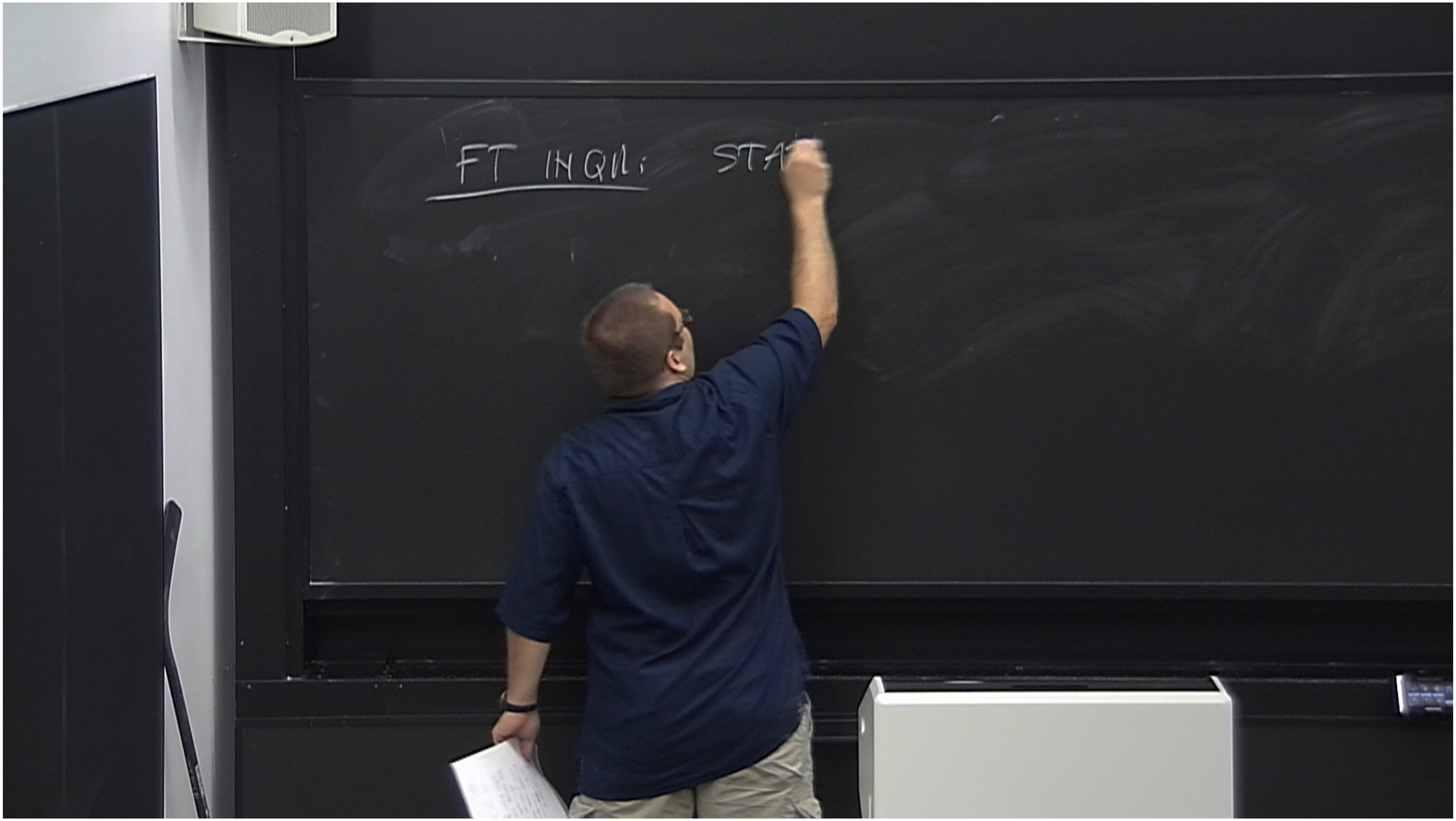


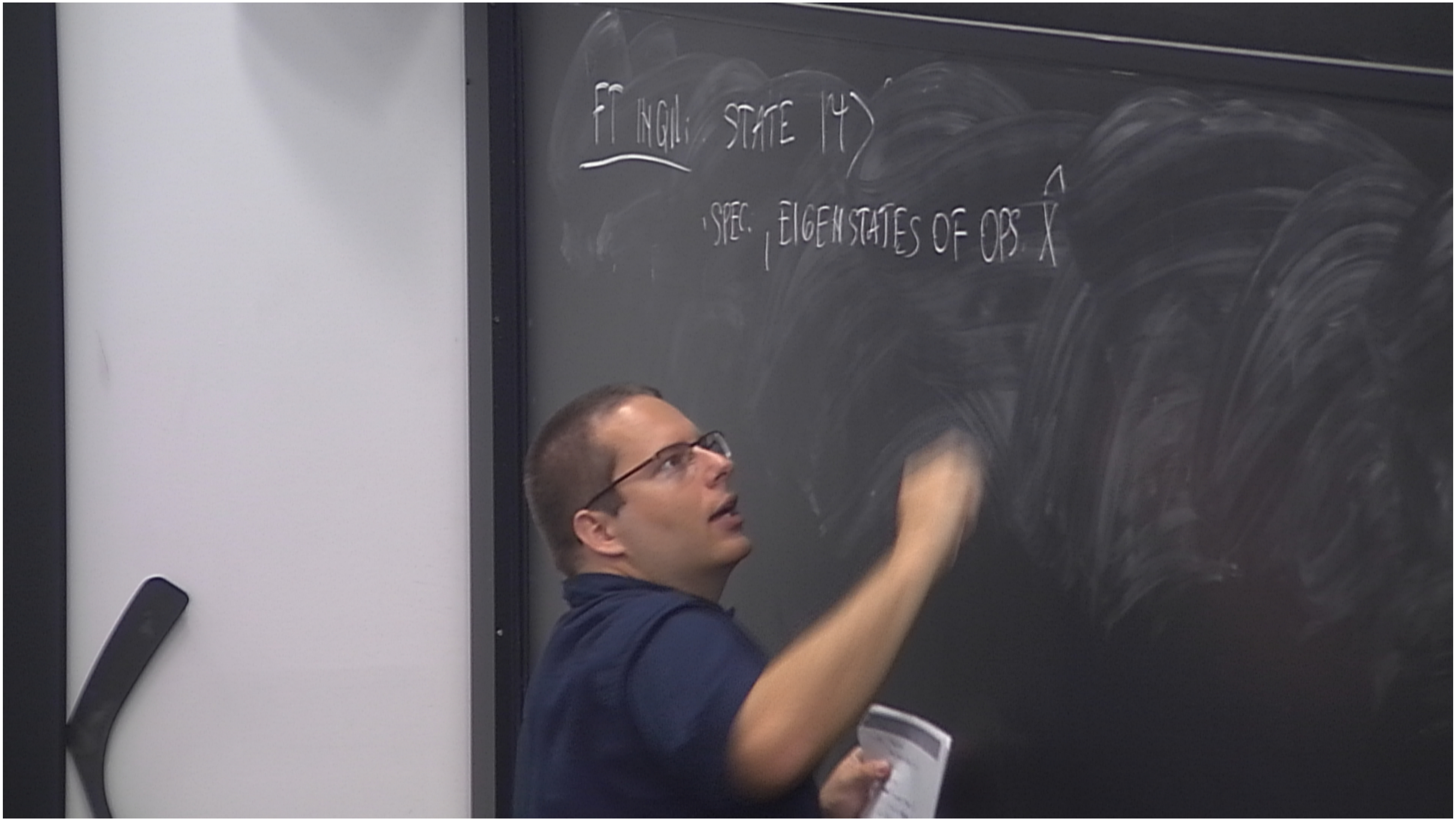

$u_0(y)\,dy + \int_0^t \int_{\mathbb{R}^d} \frac{e^{-\frac{|x-y|^2}{4(t-\tau)}}}{[2(t-\tau)]^{d/2}}$

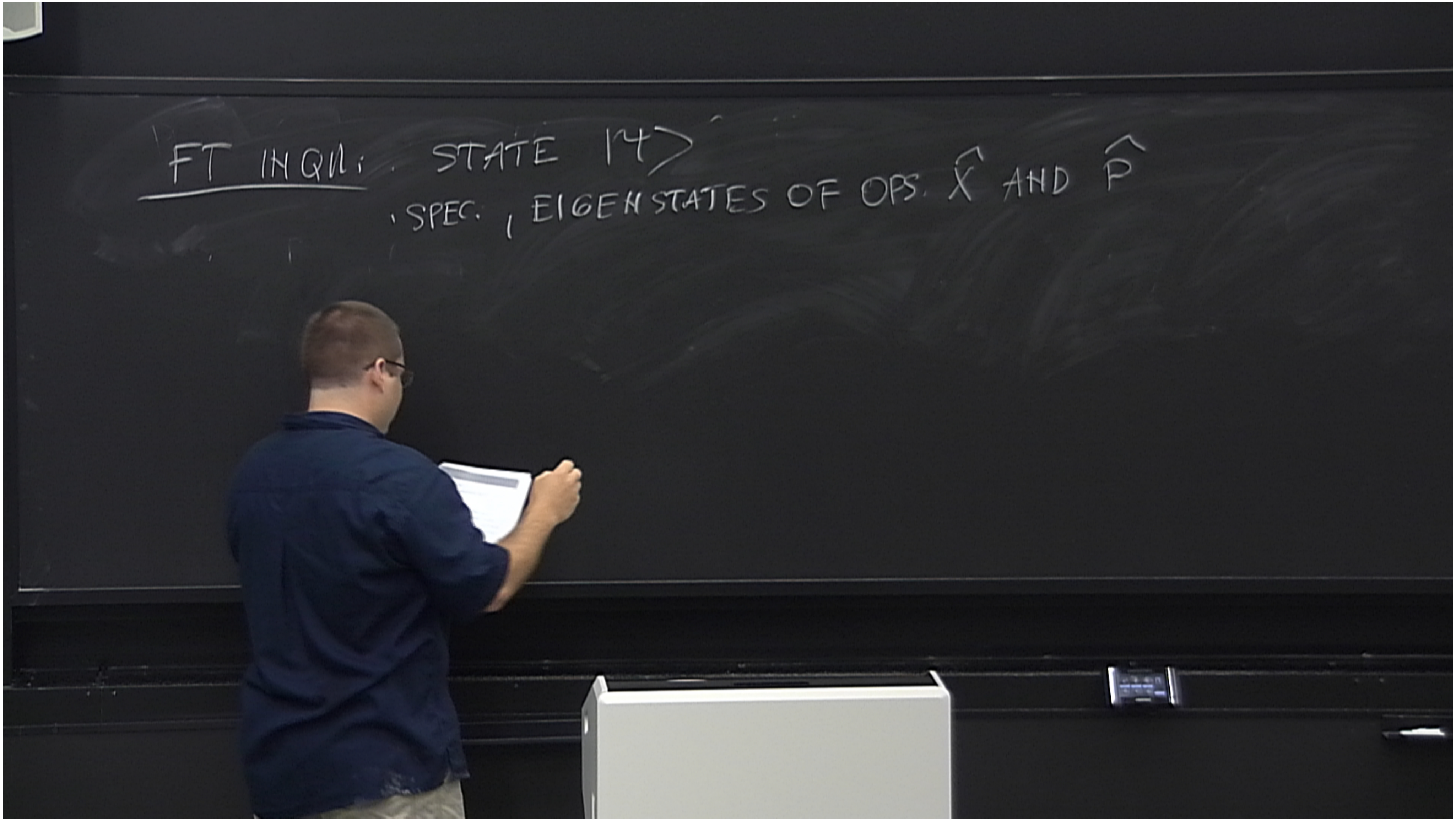

$u_0(y)\,dy + \int_0^t \int_{\mathbb{R}^d} \frac{e^{-\frac{|x-y|^2}{4(t-\tau)}}}{[2(t-\tau)]^{d/2}} f(\tau,$

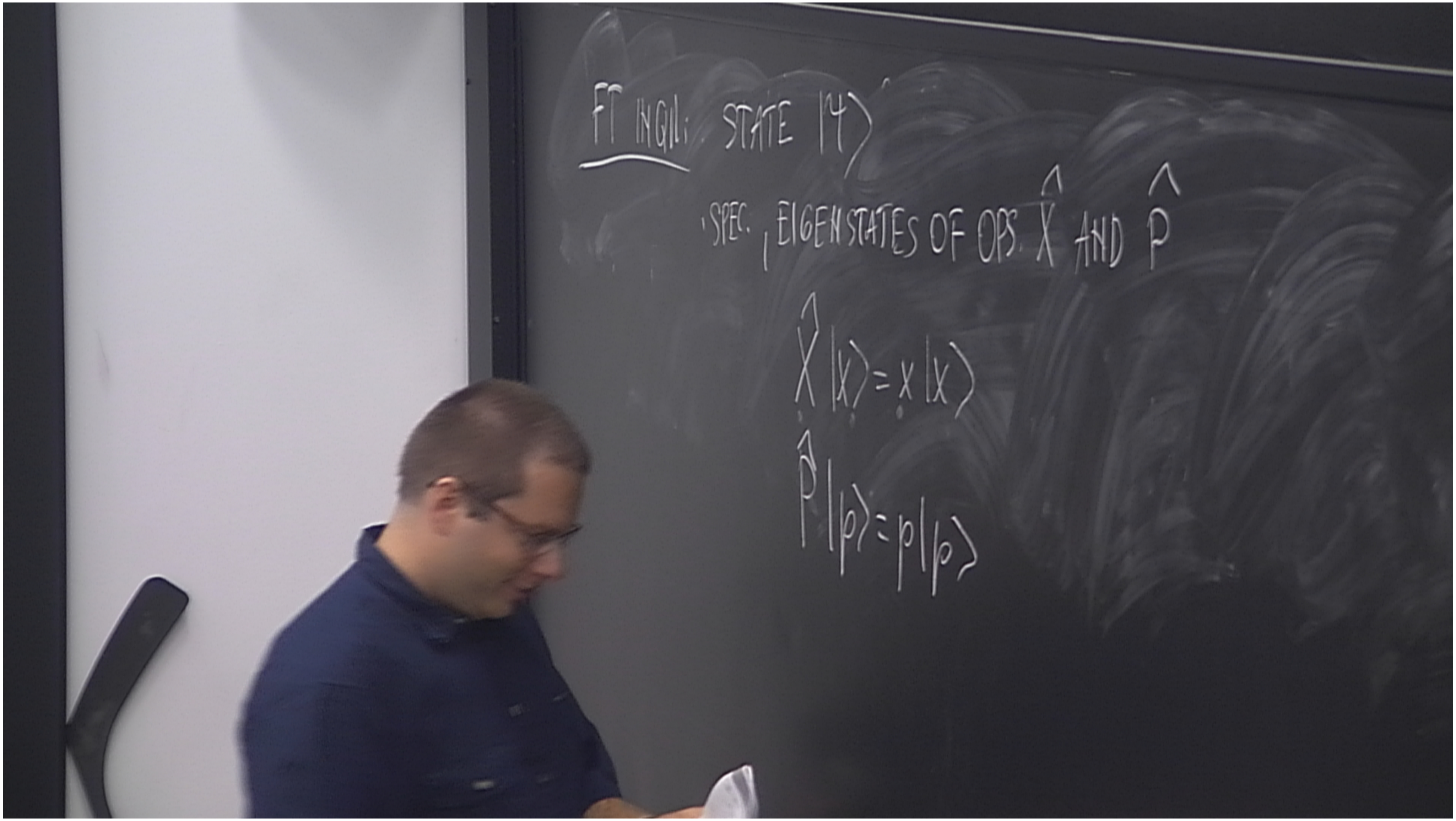



FT in QM:

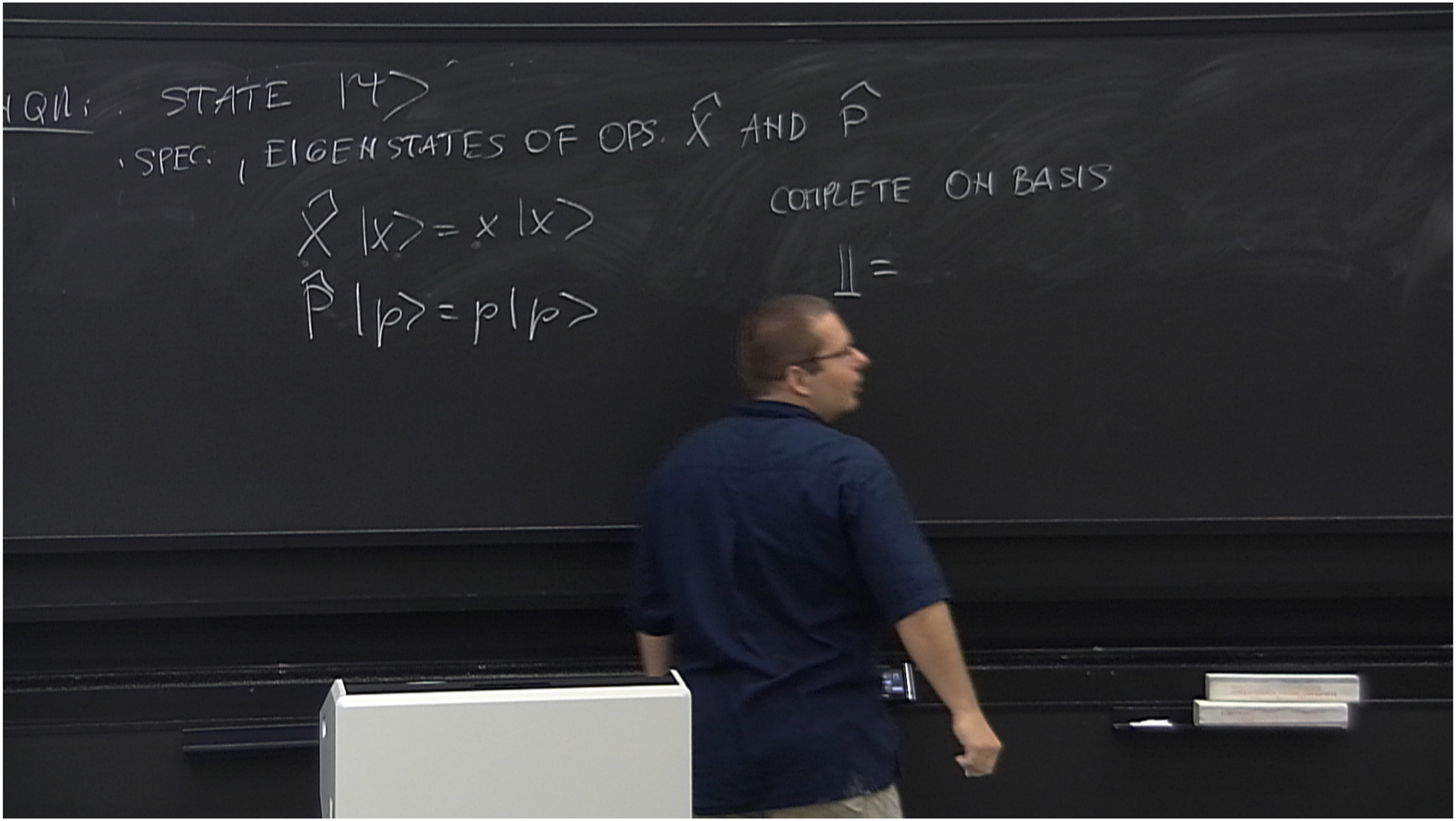

FT IN QM: STA

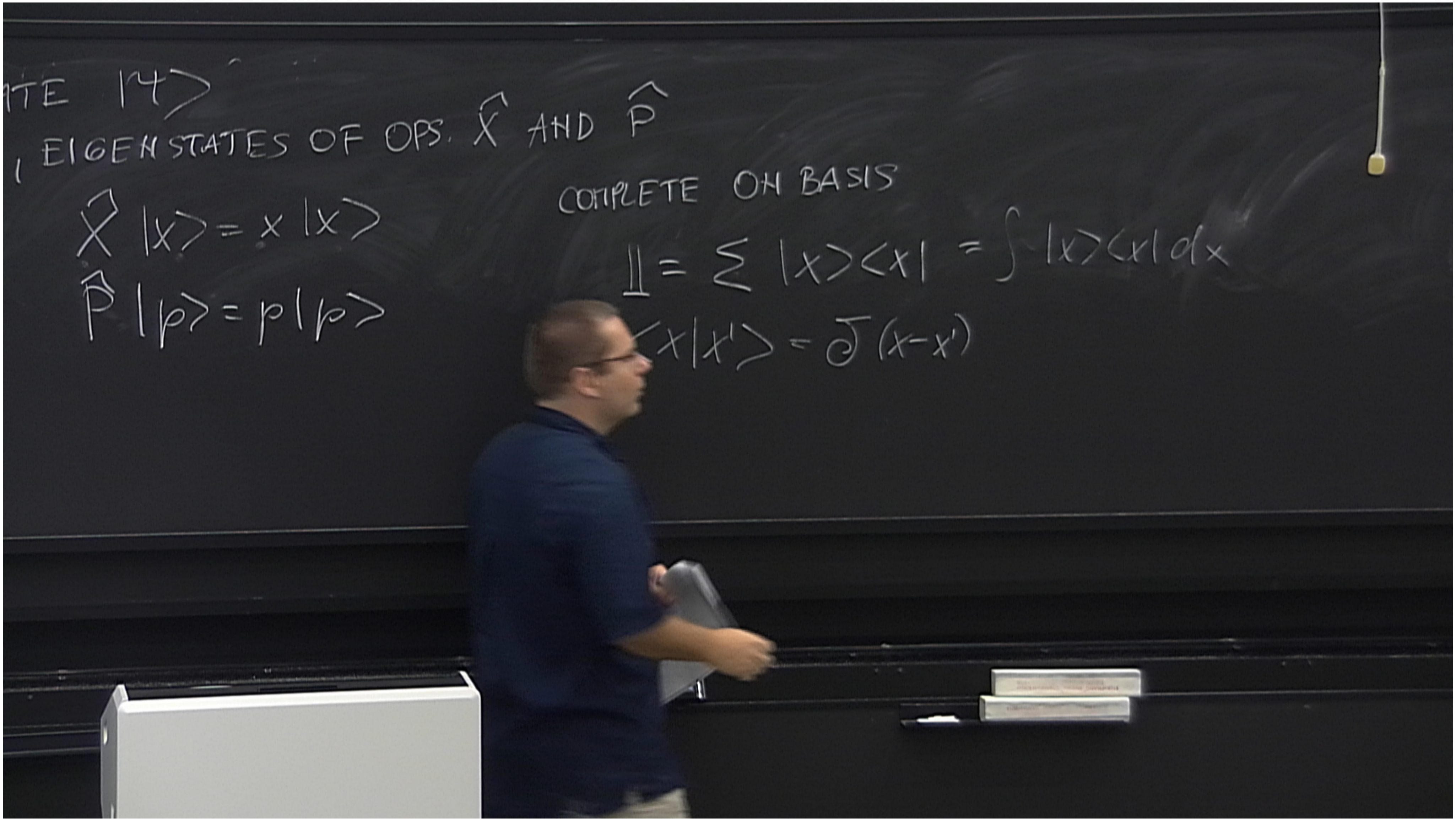

STATE |ψ⟩
EIGENSTATES OF OPS X̂


FT IN QM: STATE |ψ⟩
SPEC. EIGENSTATES OF OPS. X̂ AND P̂

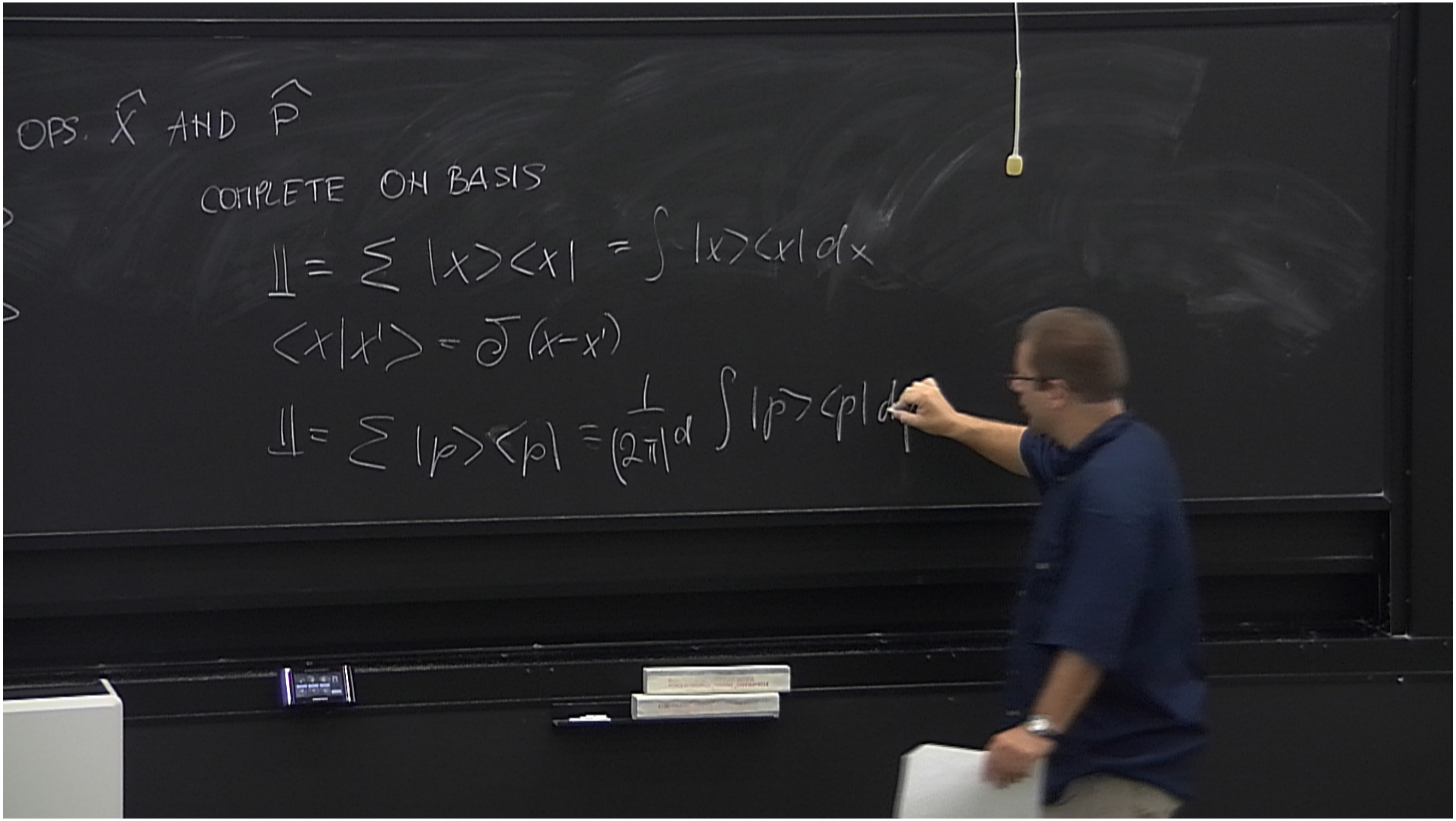


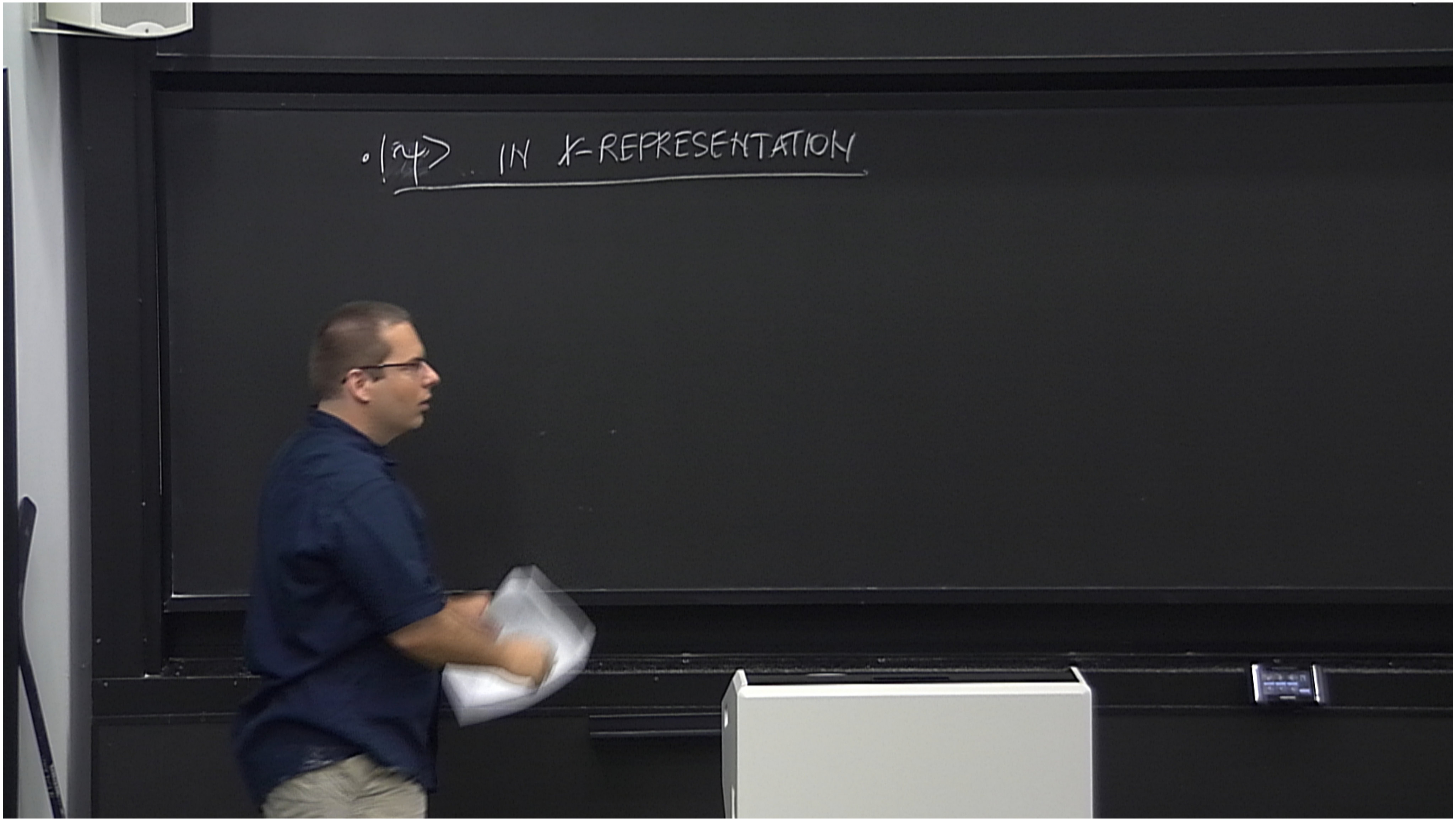
FT IN QM: STATE $|\psi\rangle$
SPEC. EIGENSTATES OF OPS. $\hat{X}$ AND $\hat{P}$
$\hat{X}|x\rangle = x|x\rangle$
$\hat{P}|p\rangle = p|p\rangle$

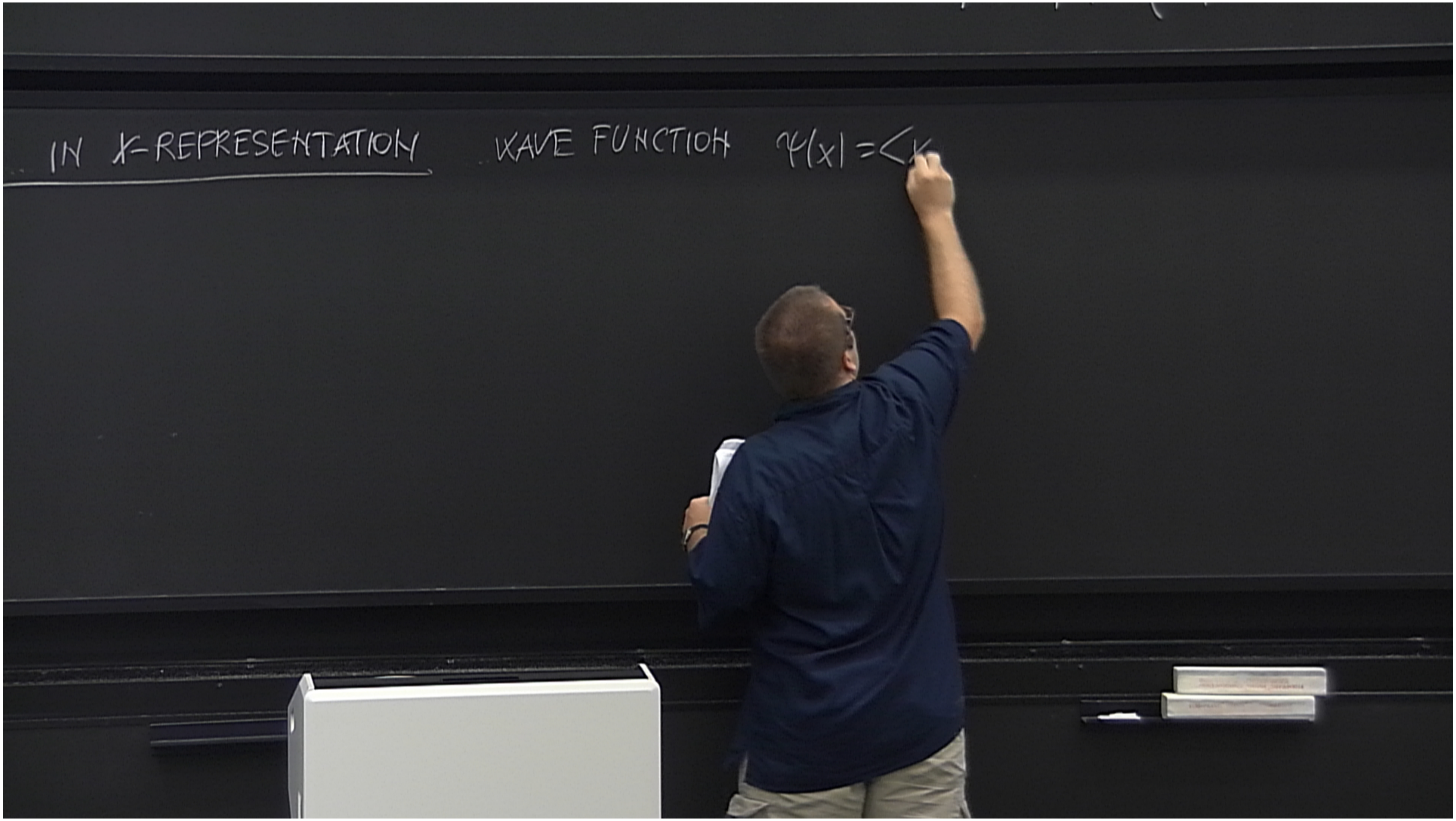

QM: . STATE |ψ>
· SPEC. EIGENSTATES OF OPS. X̂ AND P̂
COMPLETE ON BASIS
X̂ |x> = x |x>
P̂ |p> = p |p>
𝟙 =

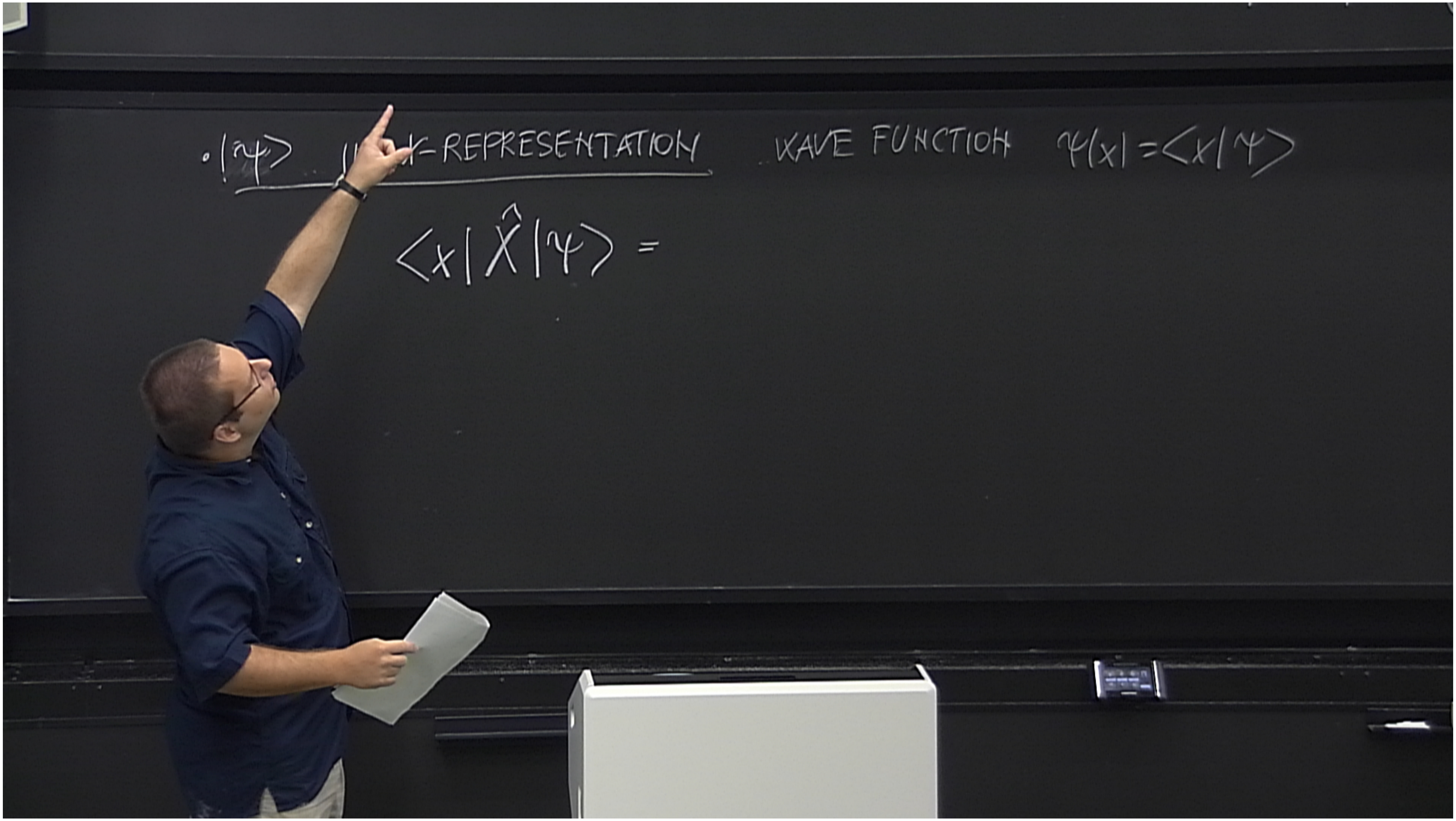


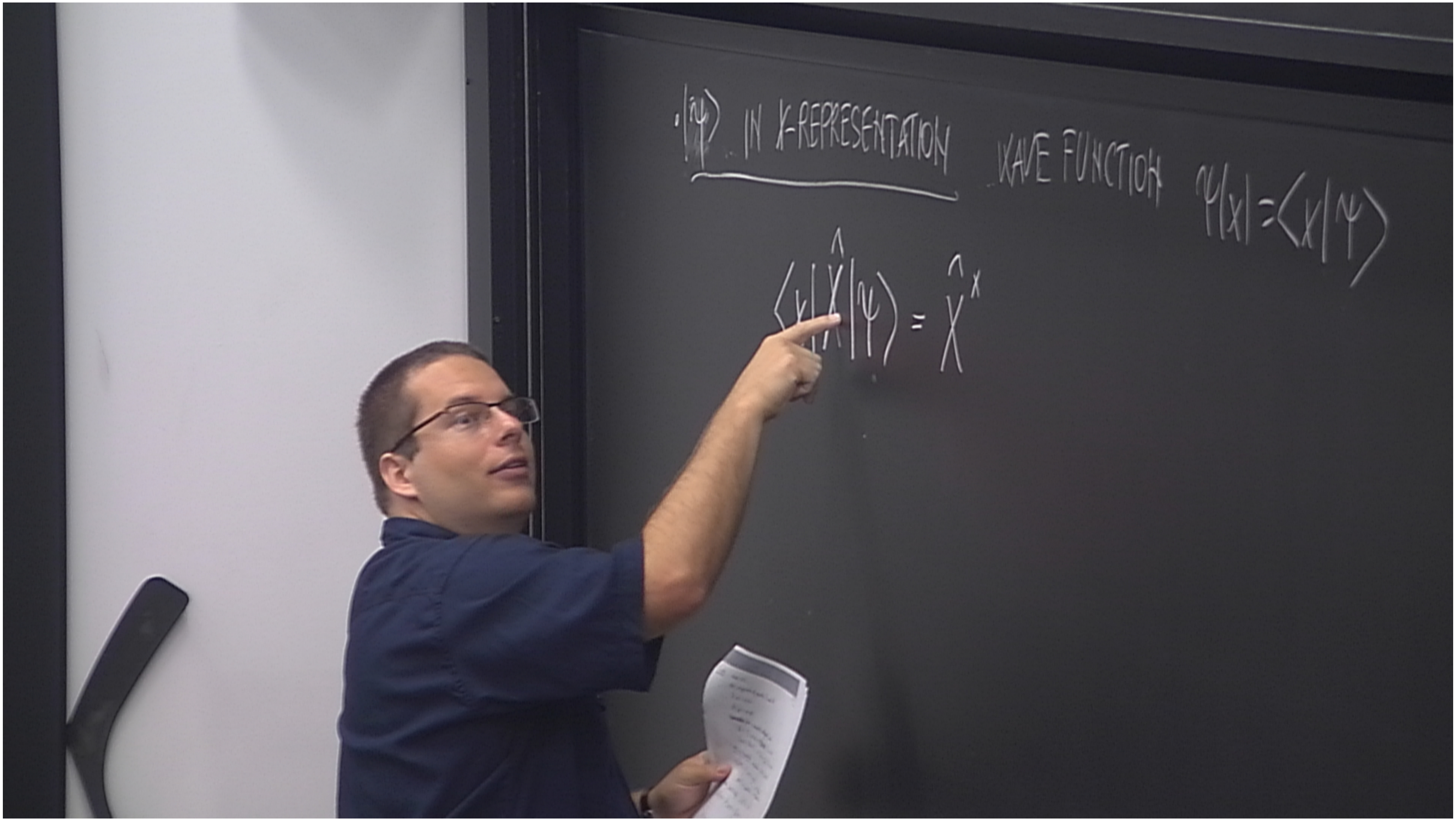
TE |ψ>
EIGENSTATES OF OPS. X̂ AND P̂
X̂ |x> = x |x>
P̂ |p> = p |p>
COMPLETE ON BASIS
𝟙 = Σ |x><x| = ∫ |x><x| dx
<x|x'> = δ(x−x')


OPS. X̂ AND P̂
COMPLETE ON BASIS
𝟙 = Σ |x><x| = ∫ |x><x| dx
<x|x'> = δ(x−x')
𝟙 = Σ |p><p| = 1/(2π)^d ∫ |p><p| d

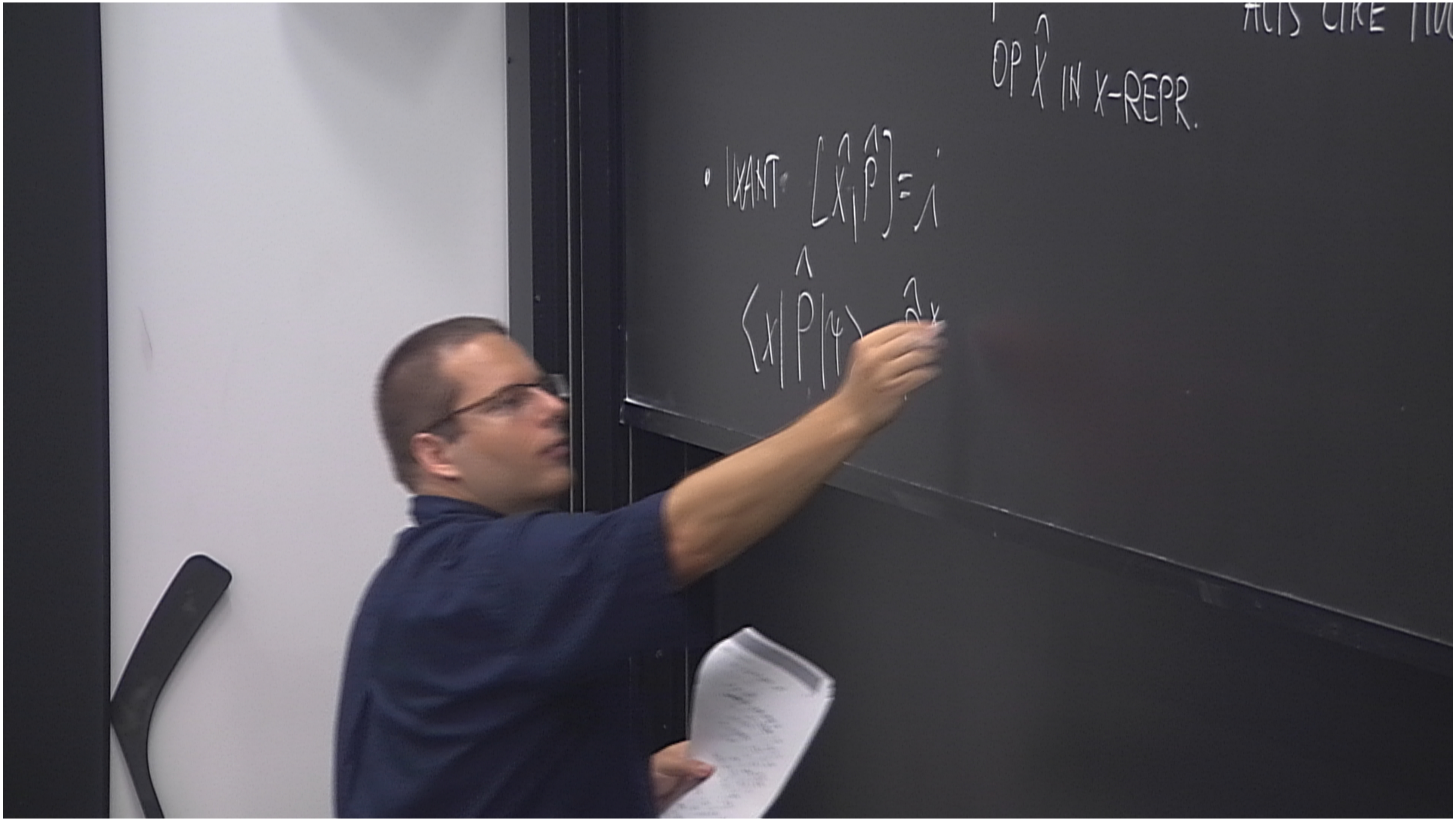

• |ψ⟩ IN X-REPRESENTATION

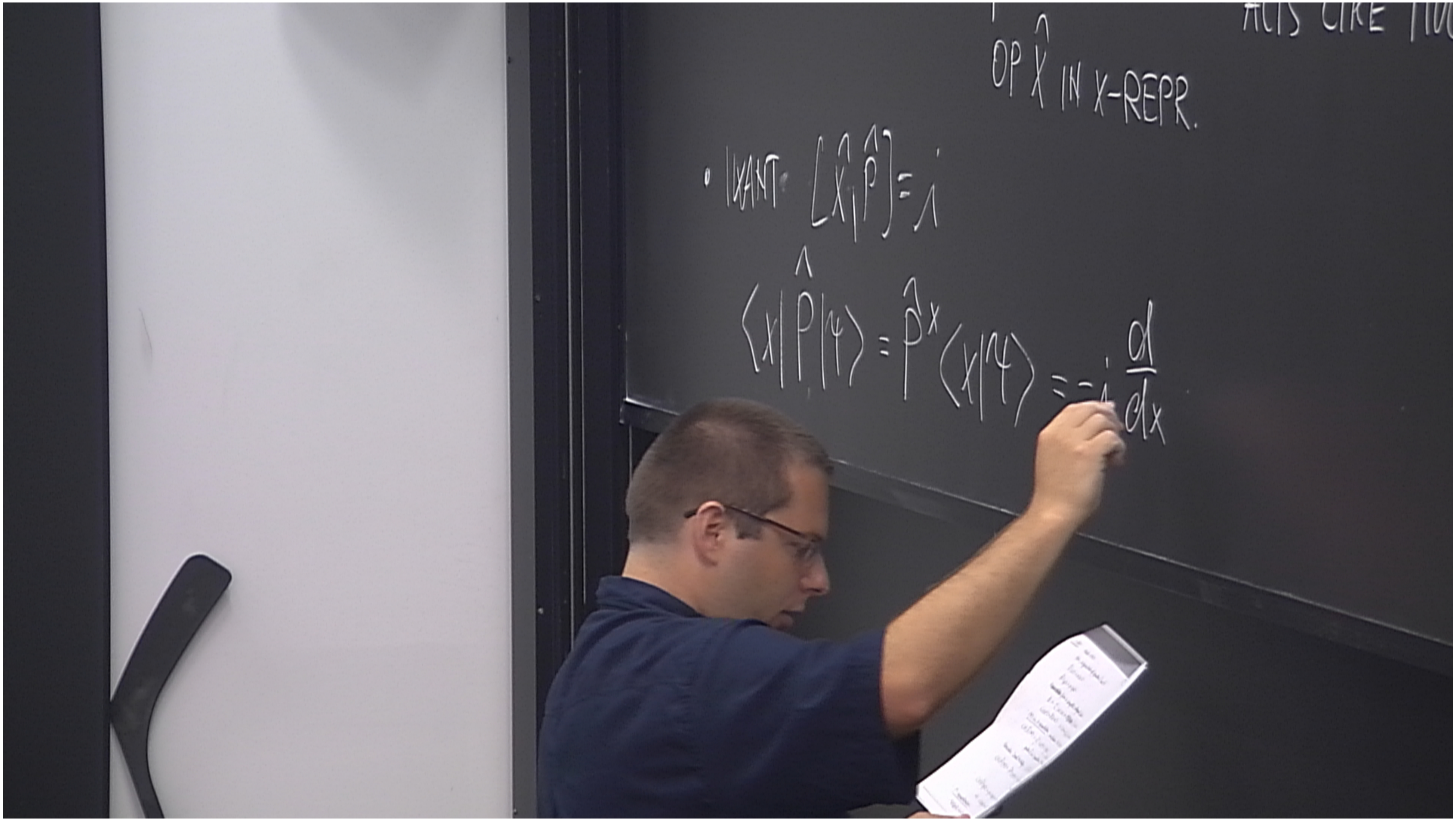

IN X-REPRESENTATION
WAVE FUNCTION $\psi(x) = \langle x$


$|\psi\rangle$
-REPRESENTATION
WAVE FUNCTION
$\psi(x) = \langle x|\psi\rangle$
$\langle x|\hat{X}|\psi\rangle =$

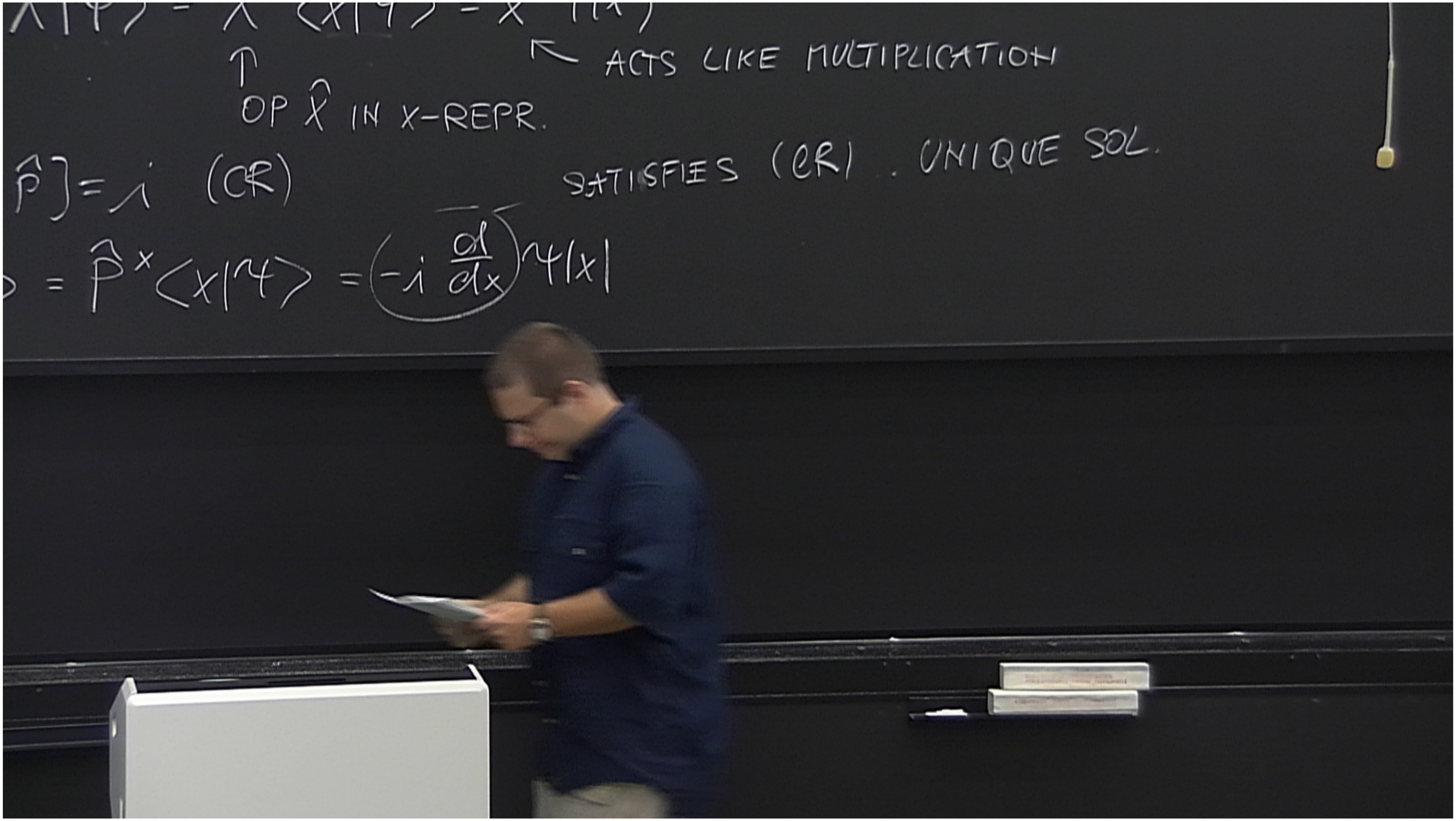


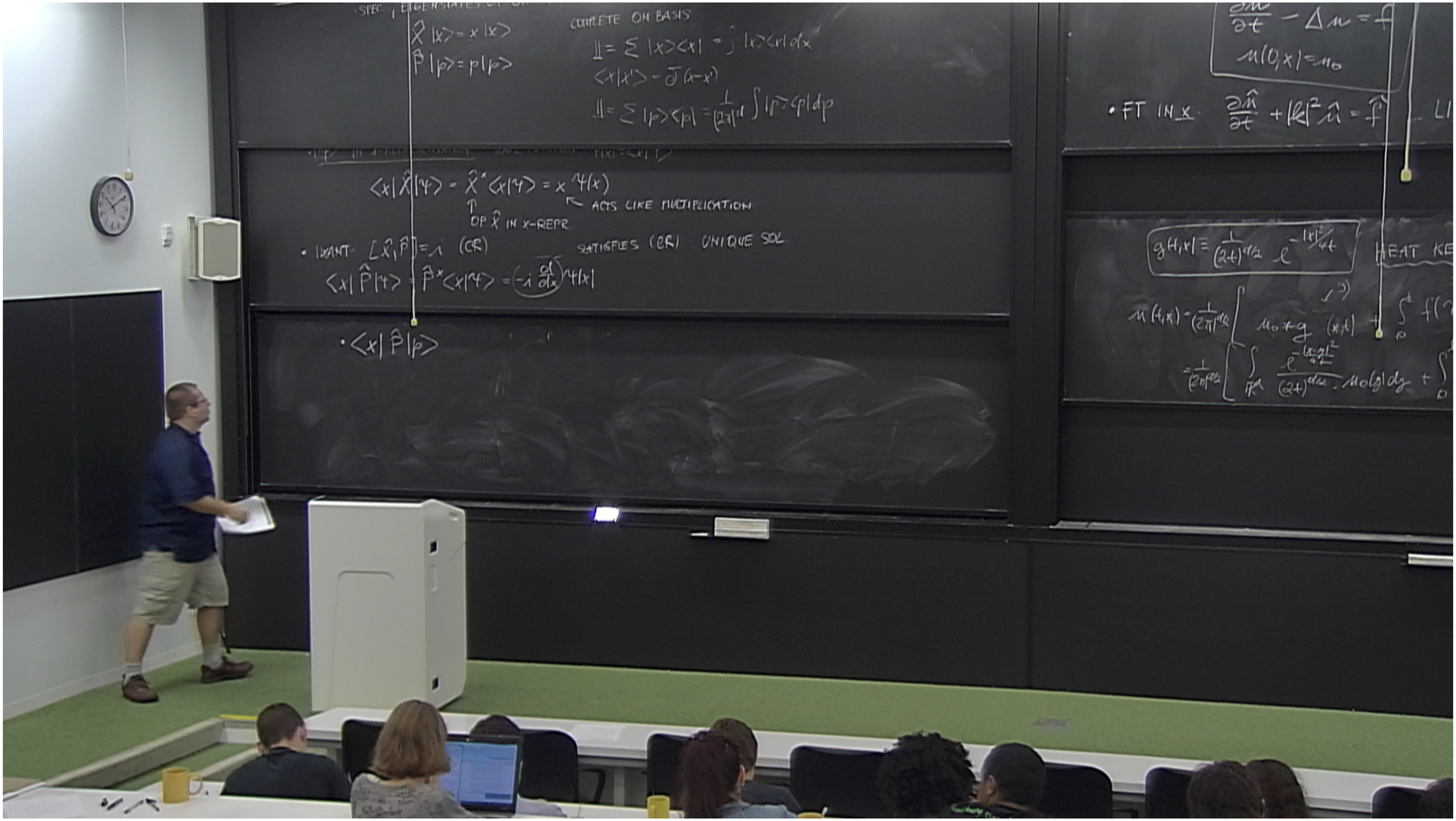
· |ψ⟩ IN X-REPRESENTATION
WAVE FUNCTION
ψ(x) = ⟨x|ψ⟩
⟨x|X̂|ψ⟩ = X̂^x

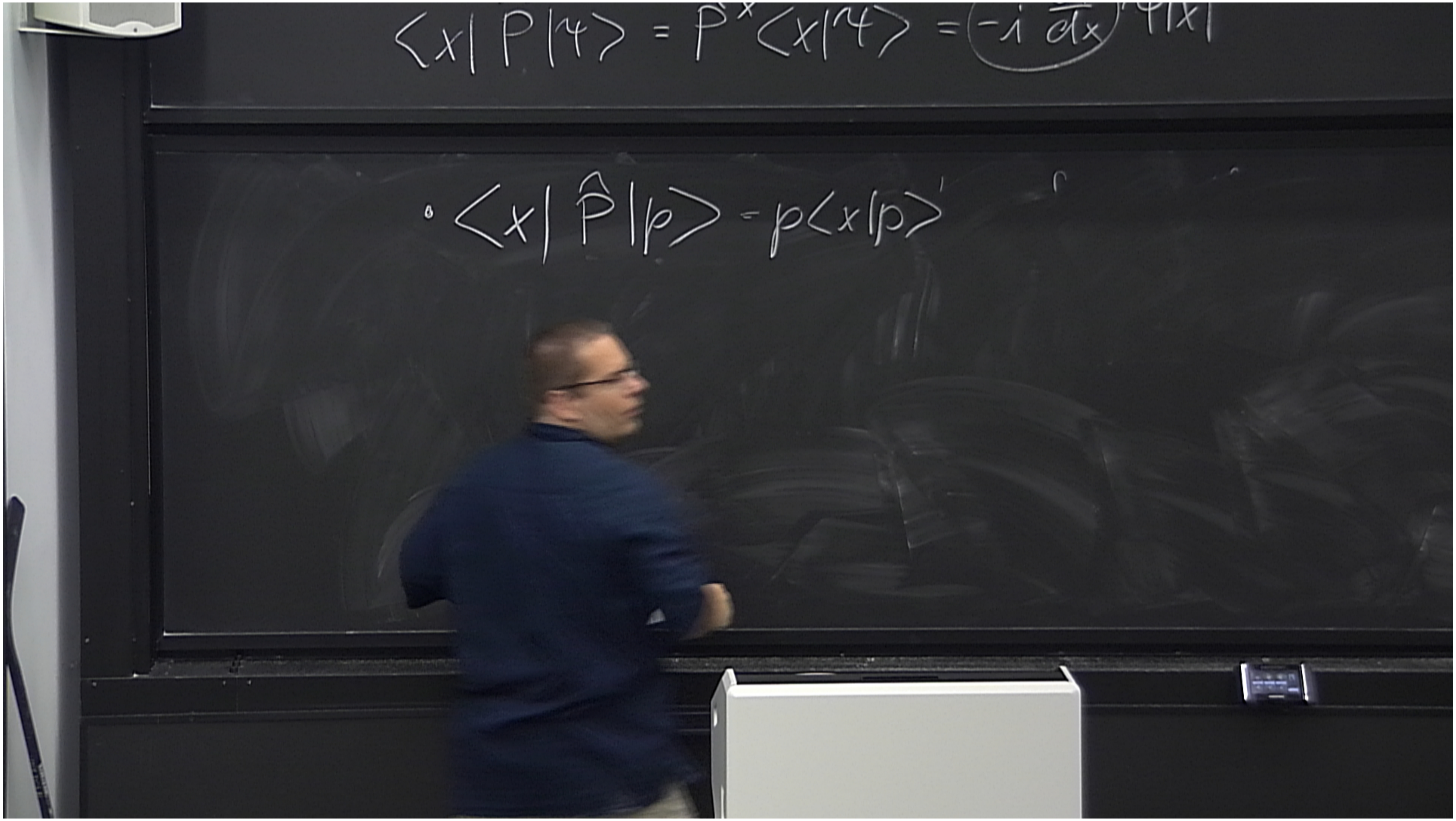

OP $\hat{X}$ IN X-REPR.
QUANT. $[\hat{X}, \hat{P}] = i$
$\langle x|\hat{P}|\psi\rangle$

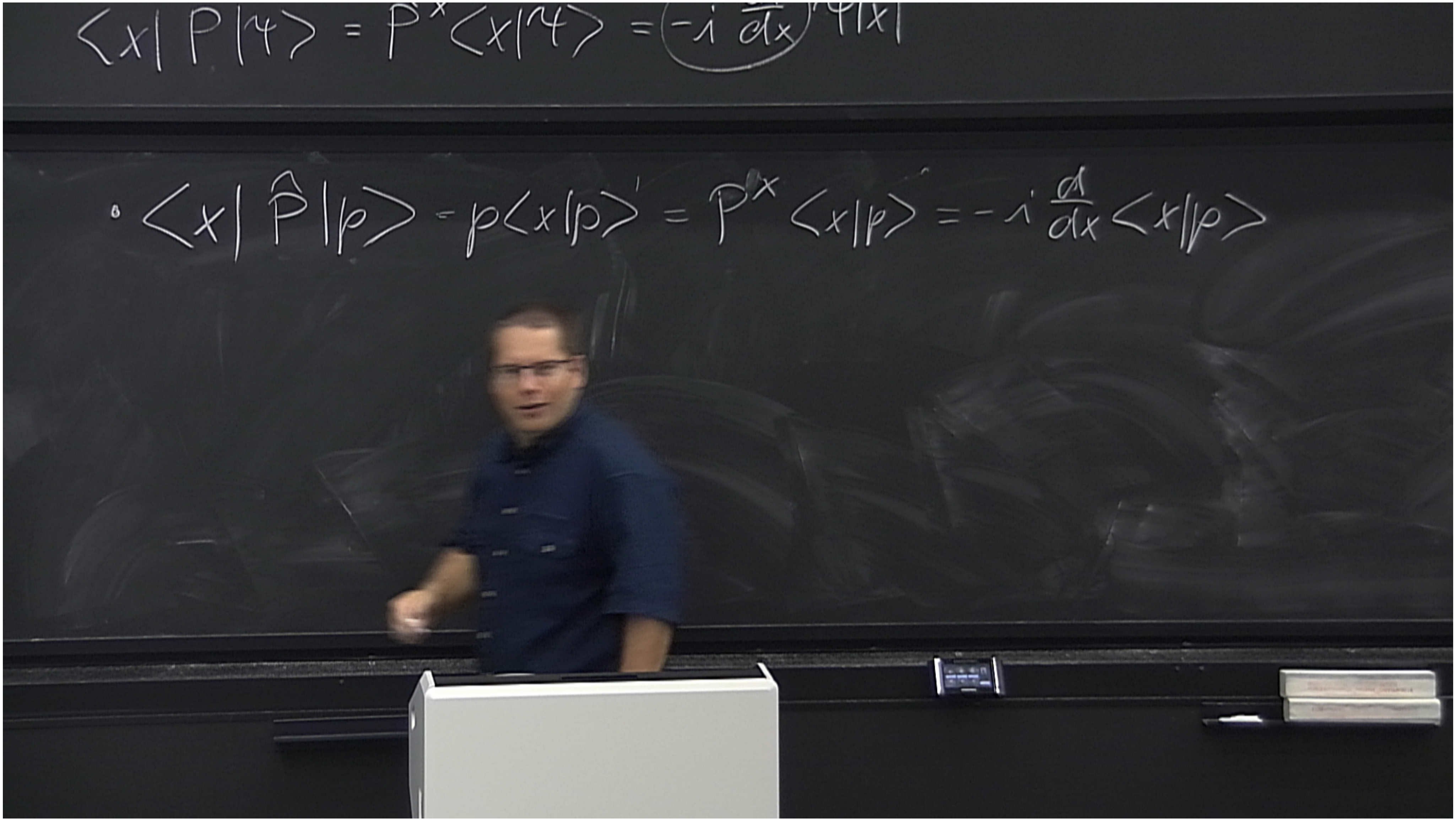


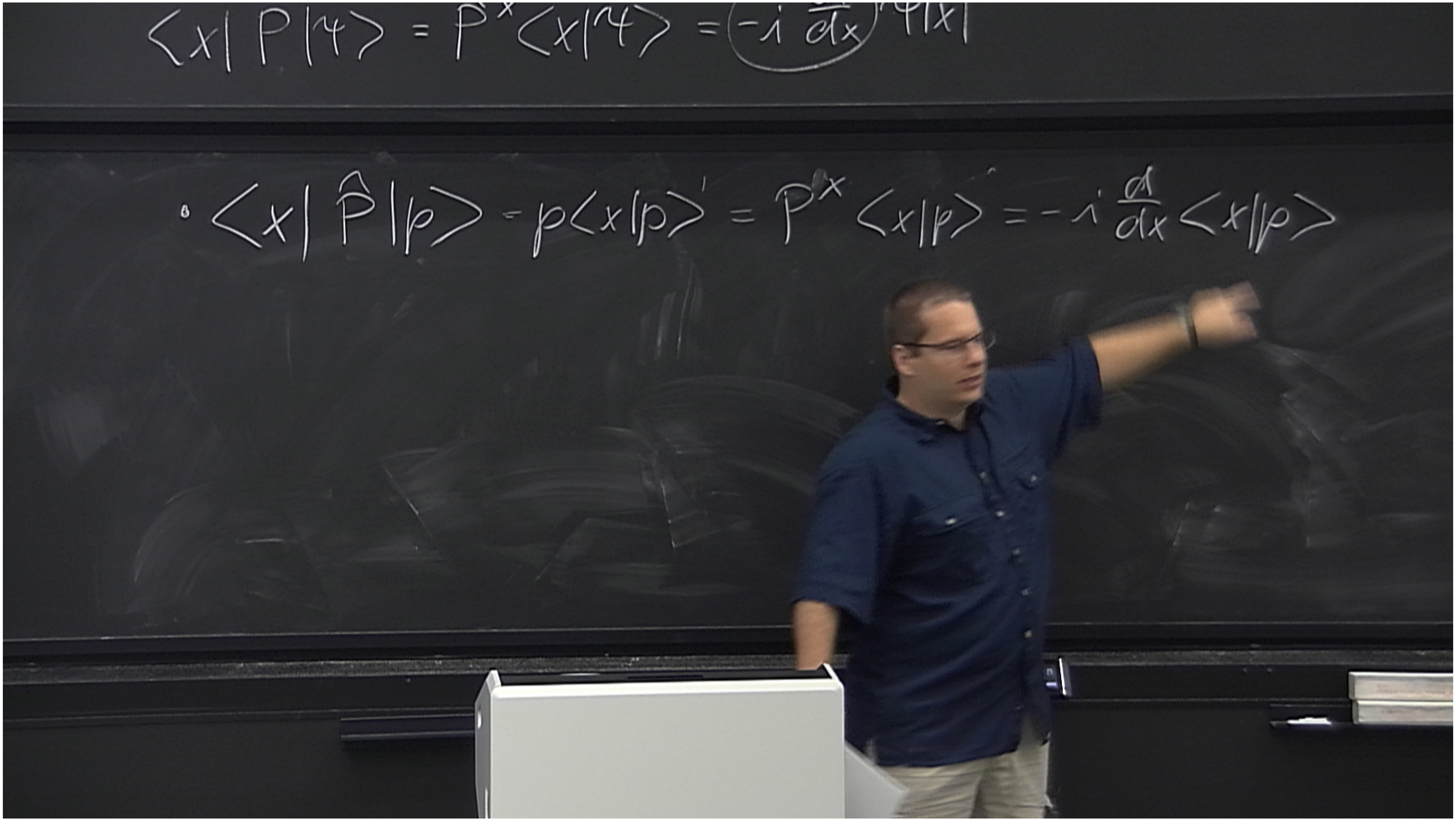
OP $\hat{X}$ IN X-REPR.
ACTS LIKE MU
• WANT $[\hat{X}, \hat{P}] = i$
$\langle x|\hat{P}|\psi\rangle = \hat{P}^x \langle x|\psi\rangle = -i \frac{d}{dx}$


ACTS LIKE MULTIPLICATION
OP x̂ IN X-REPR.
(CR)
SATISFIES (CR) . UNIQUE SOL.

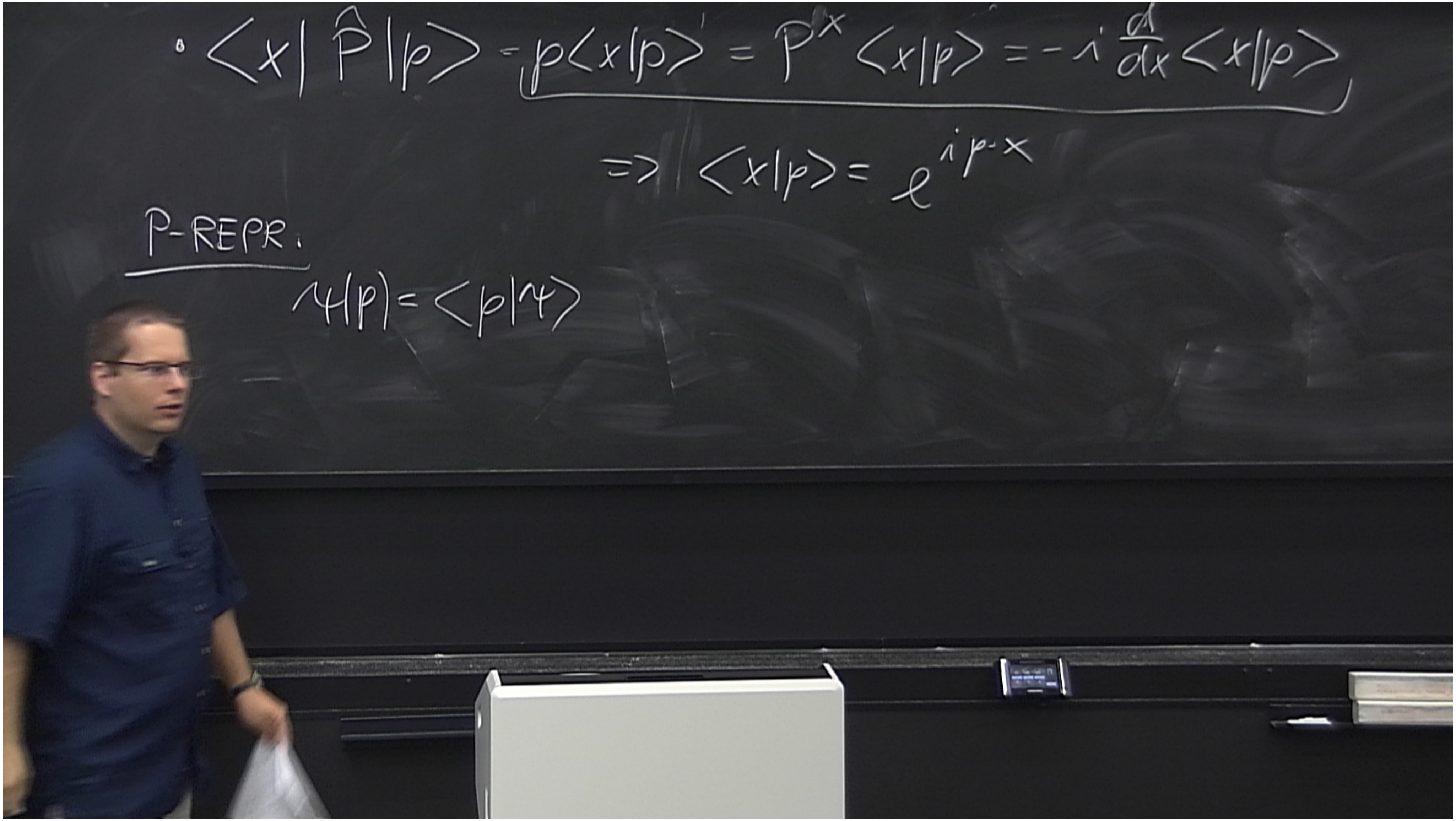

COMPLETE ON BASIS
ACTS LIKE MULTIPLICATION
OP X̂ IN X-REPR
SATISFIES (CR) UNIQUE SOL
• FT IN X
HEAT KE

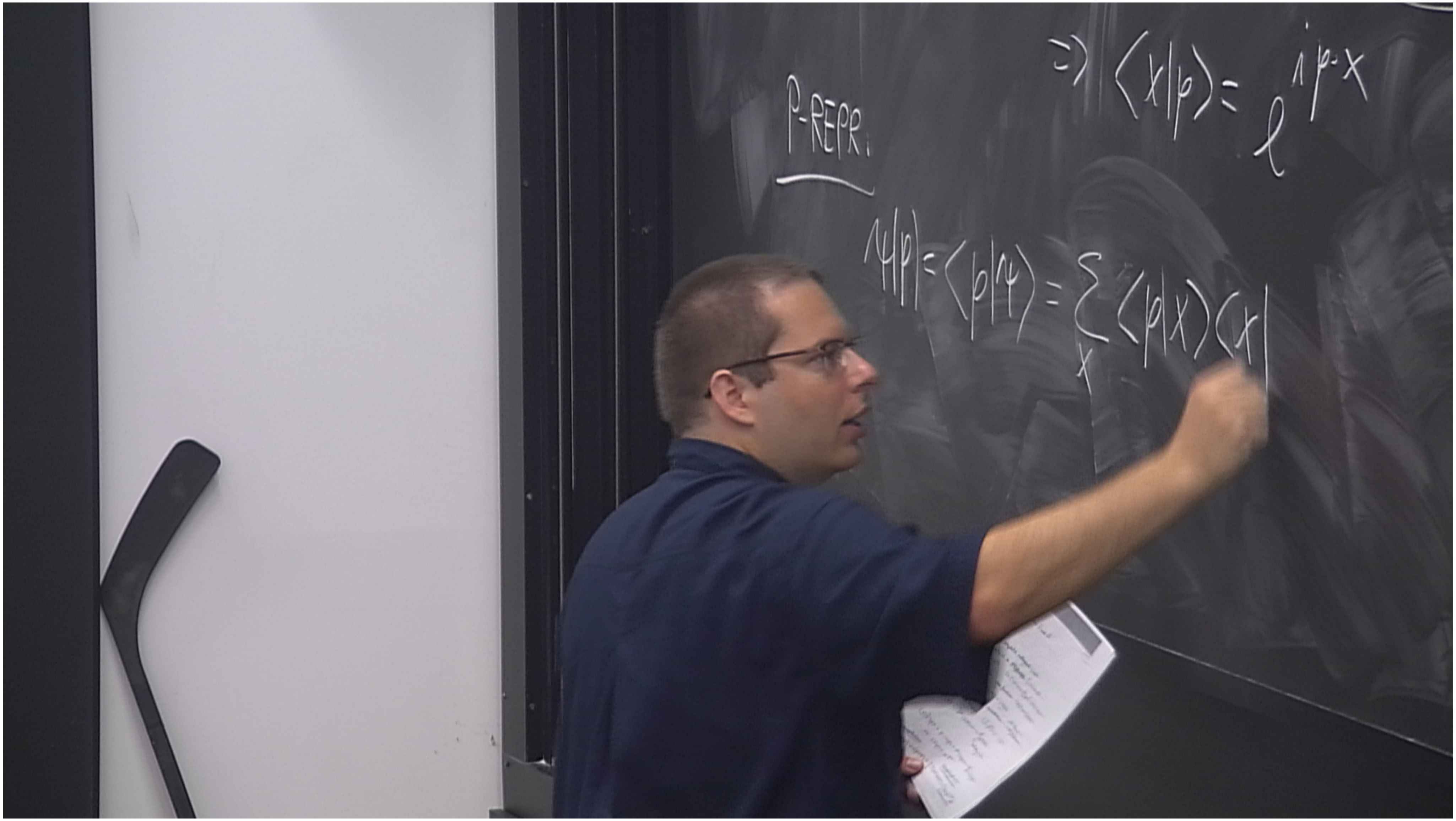

⟨x| P |ψ⟩ = P ⟨x|ψ⟩ = (-i d/dx) ψ(x)
⟨x| P̂ |p⟩ = p⟨x|p⟩

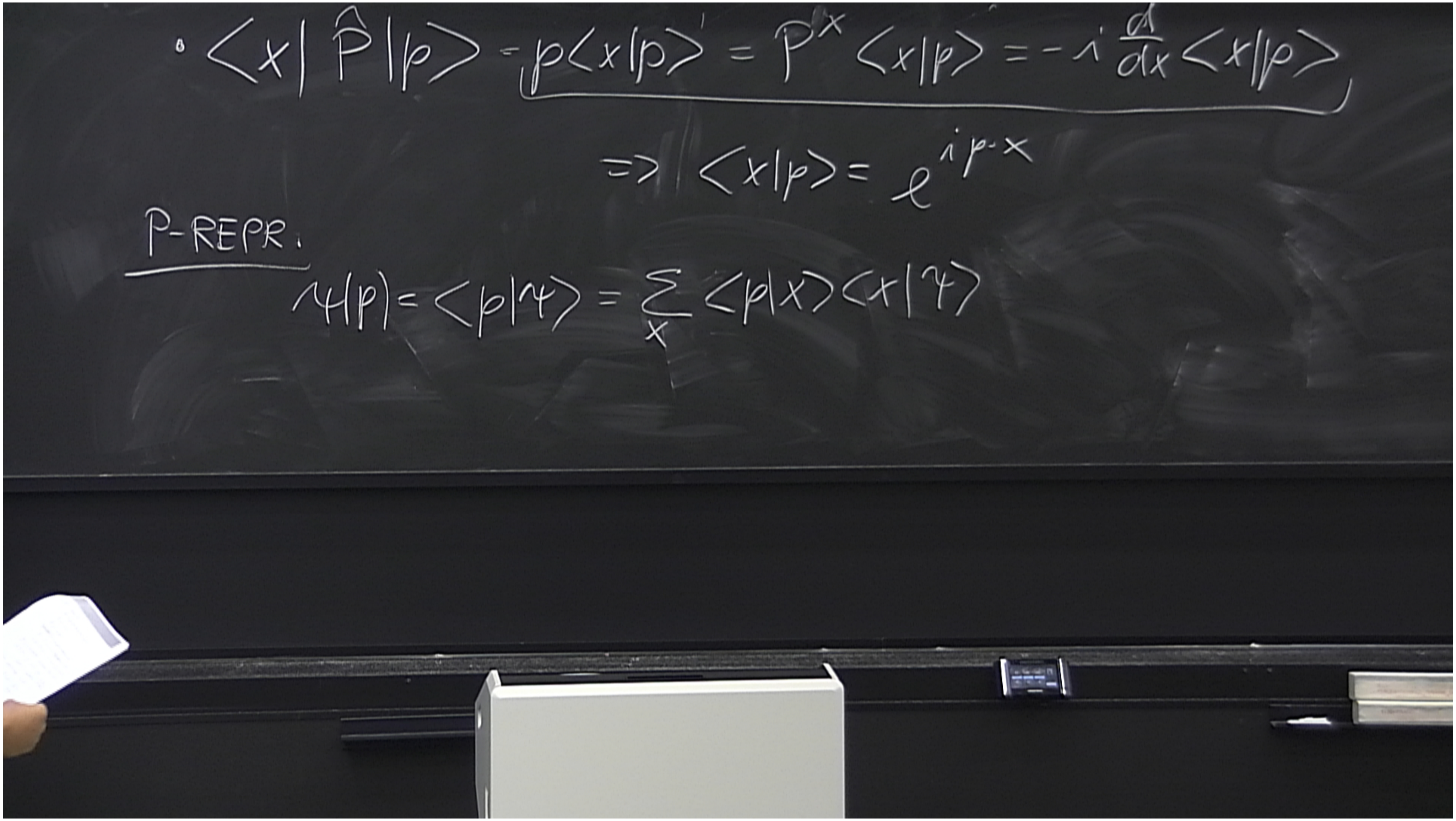

$\langle x| P |\psi\rangle = P^x \langle x|\psi\rangle = \left(-i \frac{d}{dx}\right)\psi(x)$
$\langle x| \hat{P} |p\rangle = p\langle x|p\rangle = P^x \langle x|p\rangle = -i \frac{d}{dx}\langle x|p\rangle$

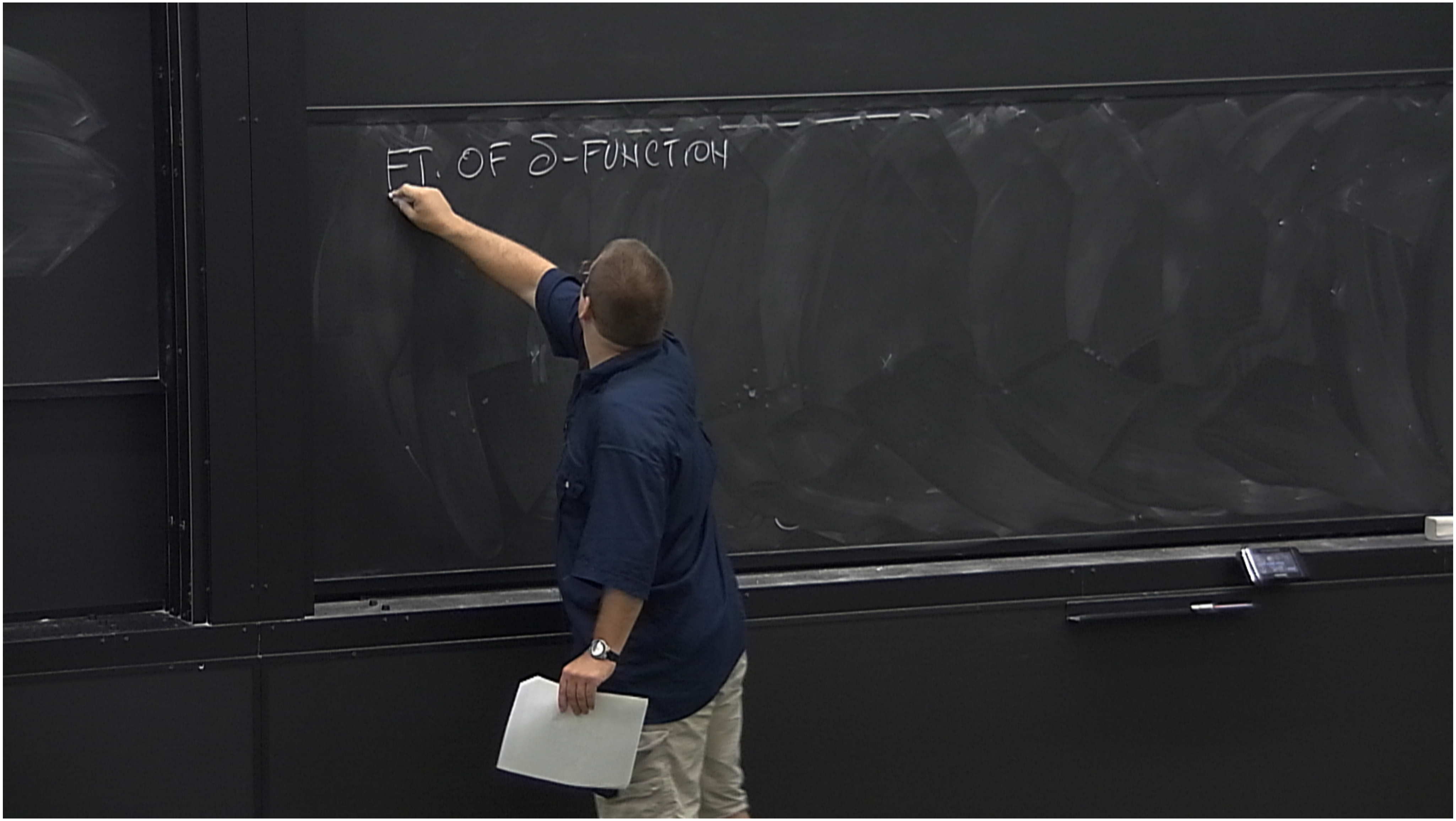


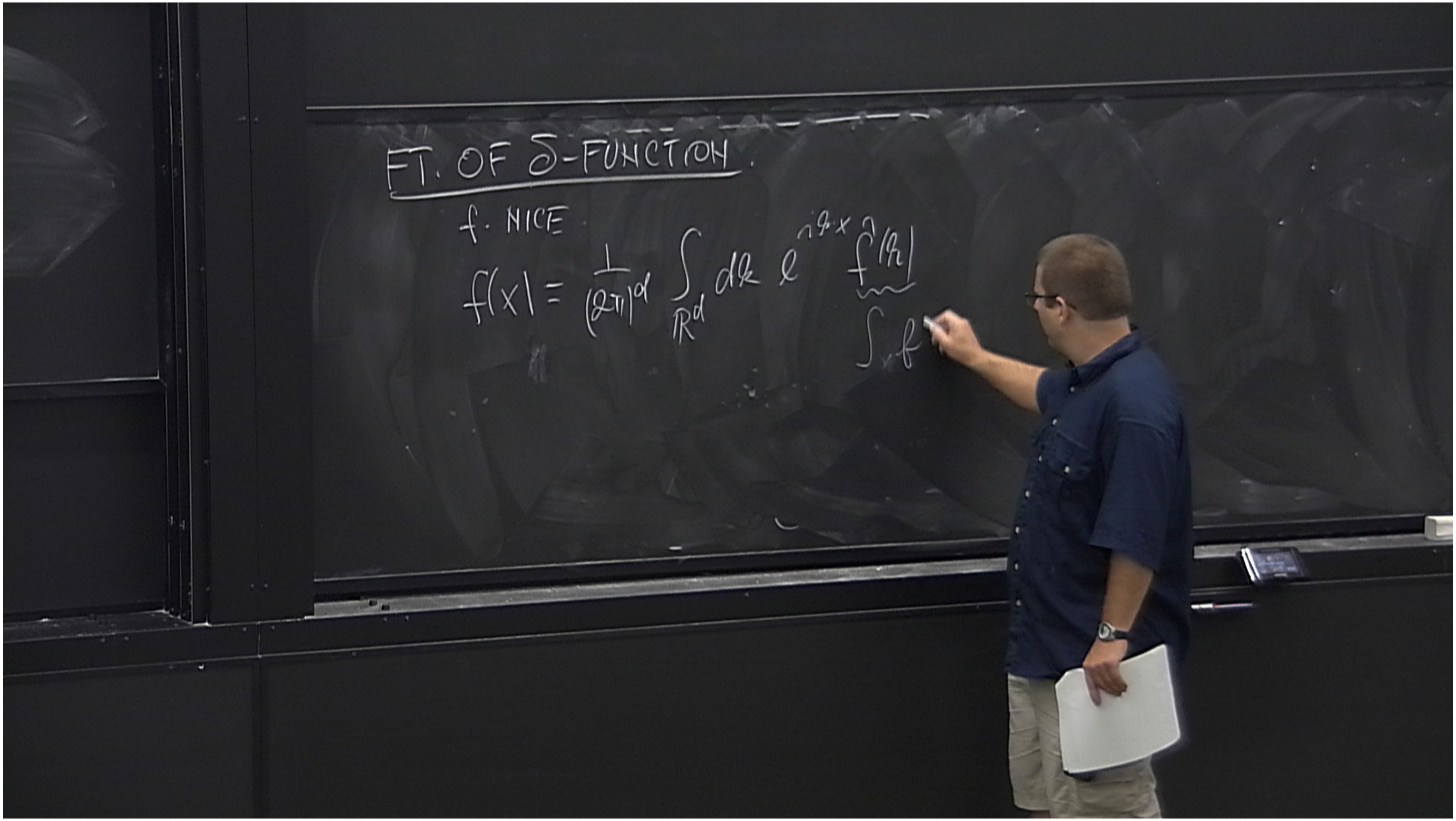
$\langle x| P |\psi\rangle = P^x \langle x|\psi\rangle = \left(-i \frac{d}{dx}\right)\psi(x)$
$\langle x| \hat{P} |p\rangle = p\langle x|p\rangle = P^x \langle x|p\rangle = -i \frac{d}{dx} \langle x|p\rangle$


⟨x| P̂ |p⟩ = p⟨x|p⟩ = P^x ⟨x|p⟩ = −i d/dx ⟨x|p⟩
⇒ ⟨x|p⟩ = e^{ip·x}
P-REPR.
ψ(p) = ⟨p|ψ⟩

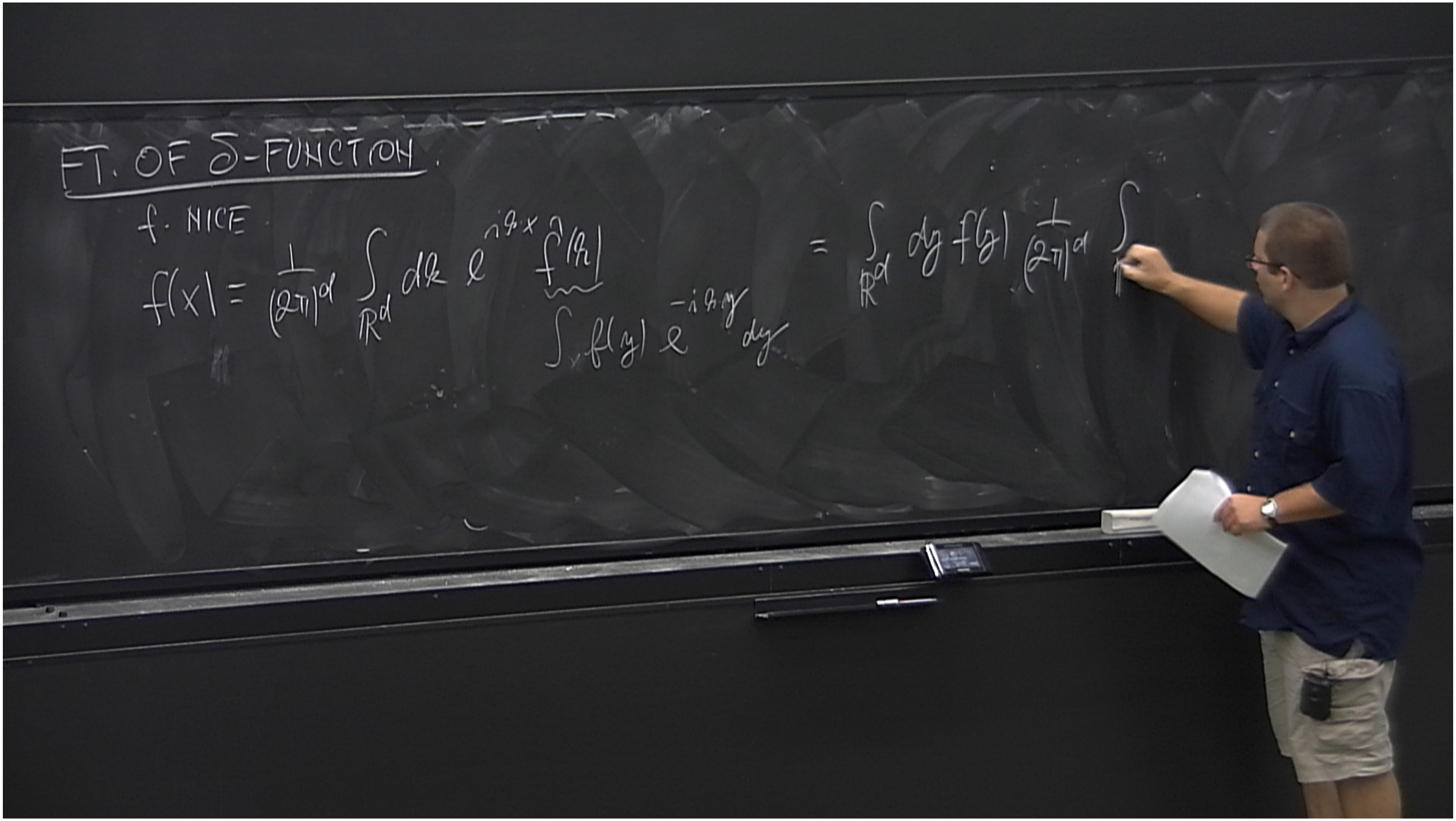

P-REPR.
⇒ ⟨x|p⟩ = e^{ip·x}
ψ(p) = ⟨p|ψ⟩ = Σ_x ⟨p|x⟩⟨x|


$\langle x | \hat{P} | p \rangle = p \langle x | p \rangle = P^x \langle x | p \rangle = -i \frac{d}{dx} \langle x | p \rangle$
$\Rightarrow \langle x | p \rangle = e^{i p \cdot x}$
P-REPR.
$\psi(p) = \langle p | \psi \rangle = \sum_x \langle p | x \rangle \langle x | \psi \rangle$


F.T. OF δ-FUNCTION



F.T. OF δ-FUNCTION
f. NICE
f(x) = 1/(2π)^d ∫_{R^d} dk e^{ik·x} f̂(k)
∫_x f

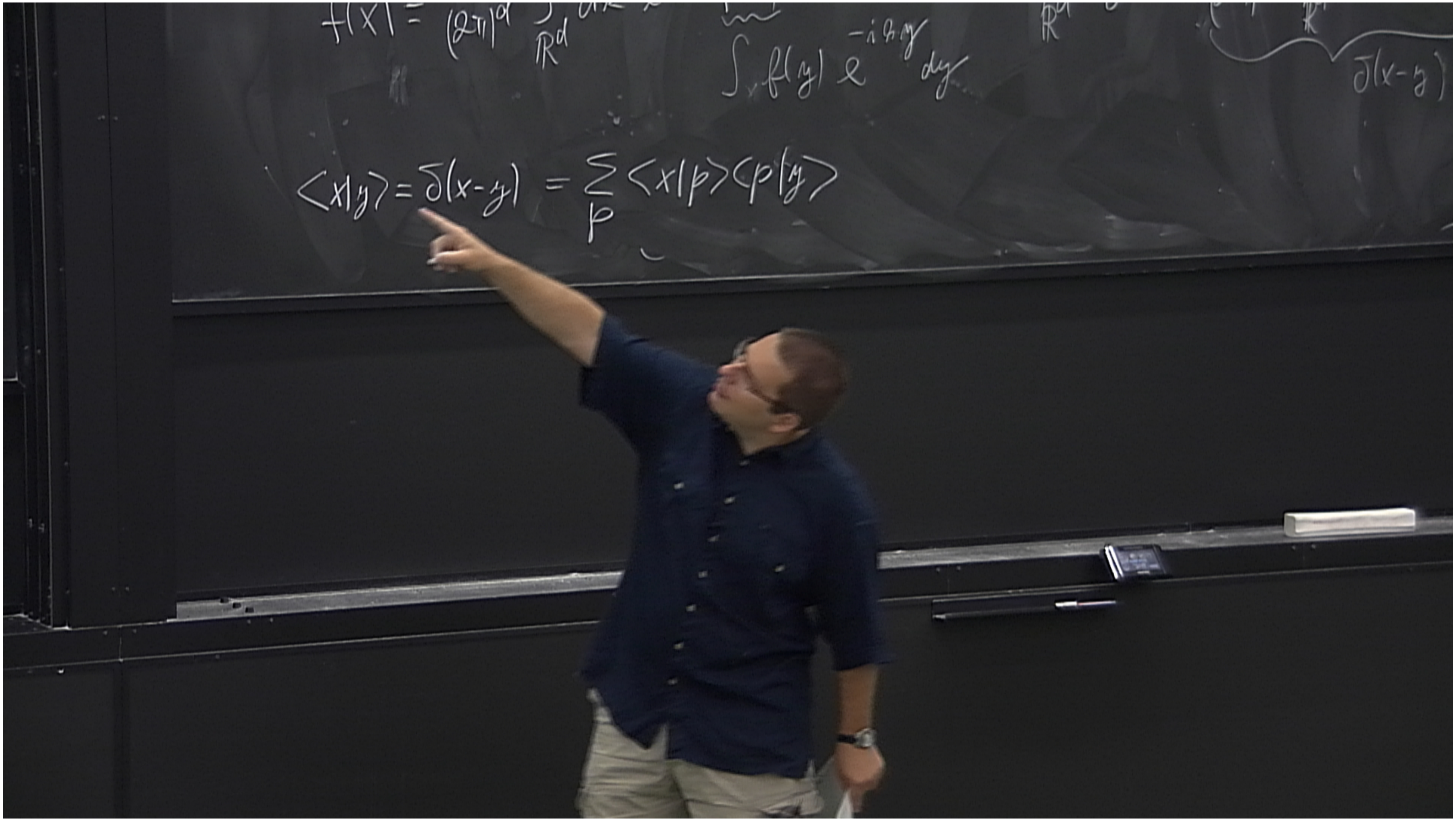

FT. OF δ-FUNCTION.
f · NICE

F.T. OF δ-FUNCTION

f. NICE

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dk \; e^{ik\cdot x} \hat{f}(k) = \int_{\mathbb{R}^d} dy \, f(y) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ik\cdot(x-y)} \, dk$$

$$\hat{f}(k) = \int_x f(y) \, e^{-ik\cdot y} \, dy$$


$\langle x|y\rangle = \delta(x-y) = \sum_p \langle x|p\rangle\langle p|y\rangle$
$\int_x f(y)\, e^{-i k \cdot y}\, dy$
$\delta(x-y)$