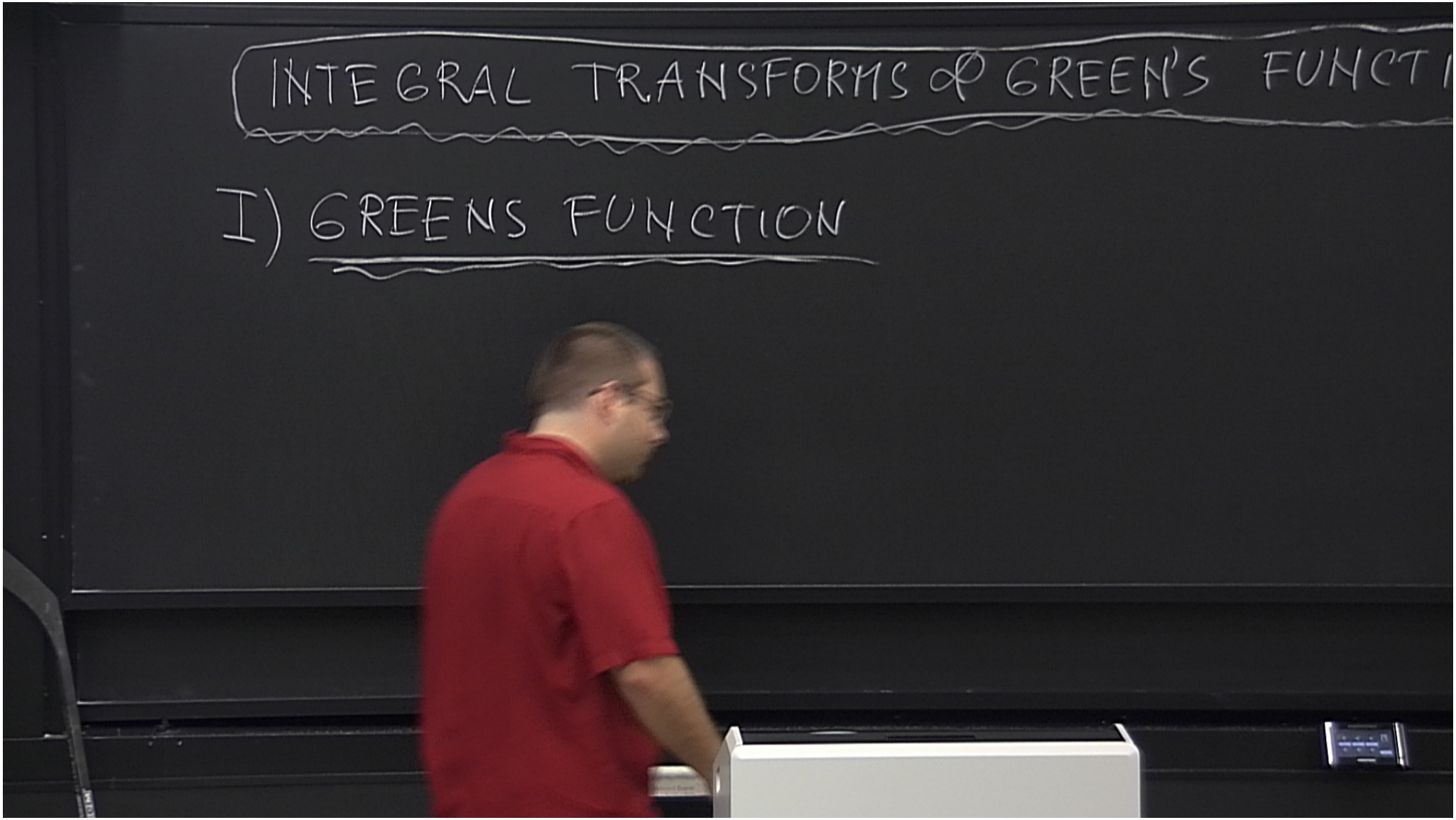
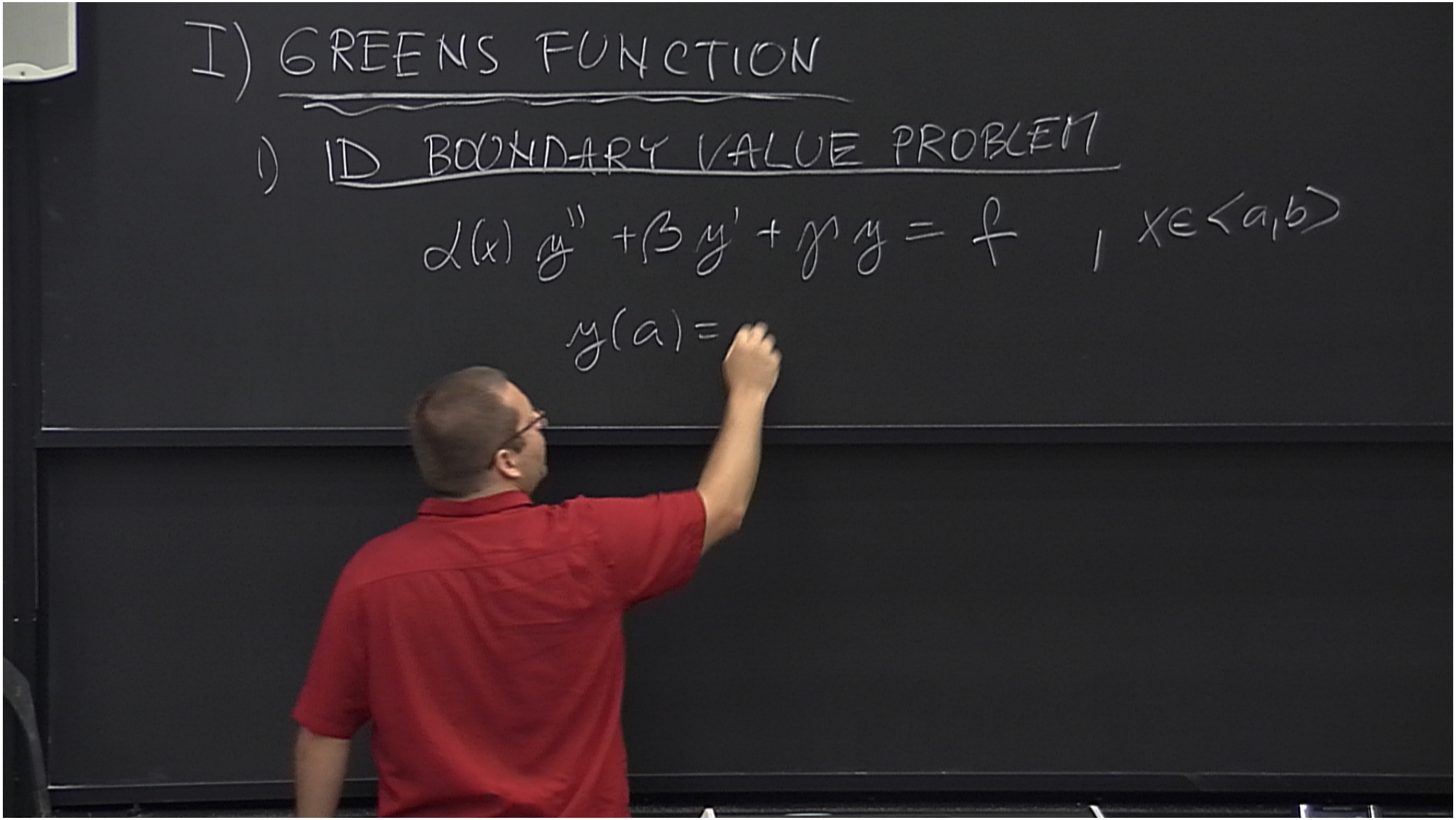
Title: 13/14 PSI - Integral Transformations & Green's Function - Lecture 1

Date: Aug 27, 2013 09:00 AM

URL: http://pirsa.org/13080028

Abstract:


INTEGRAL TRANSFORMS & GREEN'S FUNCT
I) GREENS FUNCTION

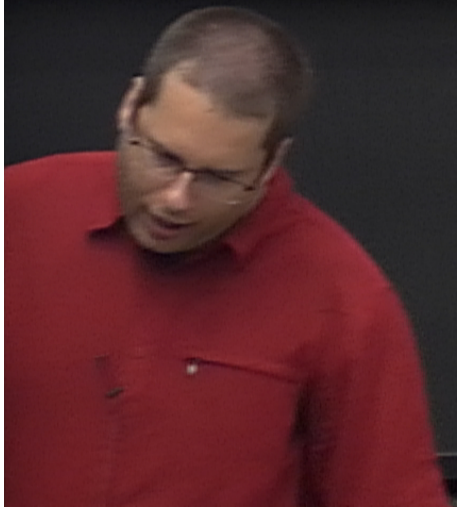

I) GREENS FUNCTION
1) 1D BOUNDARY VALUE PROBLEM
α(x) y'' + β y' + γ y = f , x ∈ ⟨a,b⟩
y(a) =

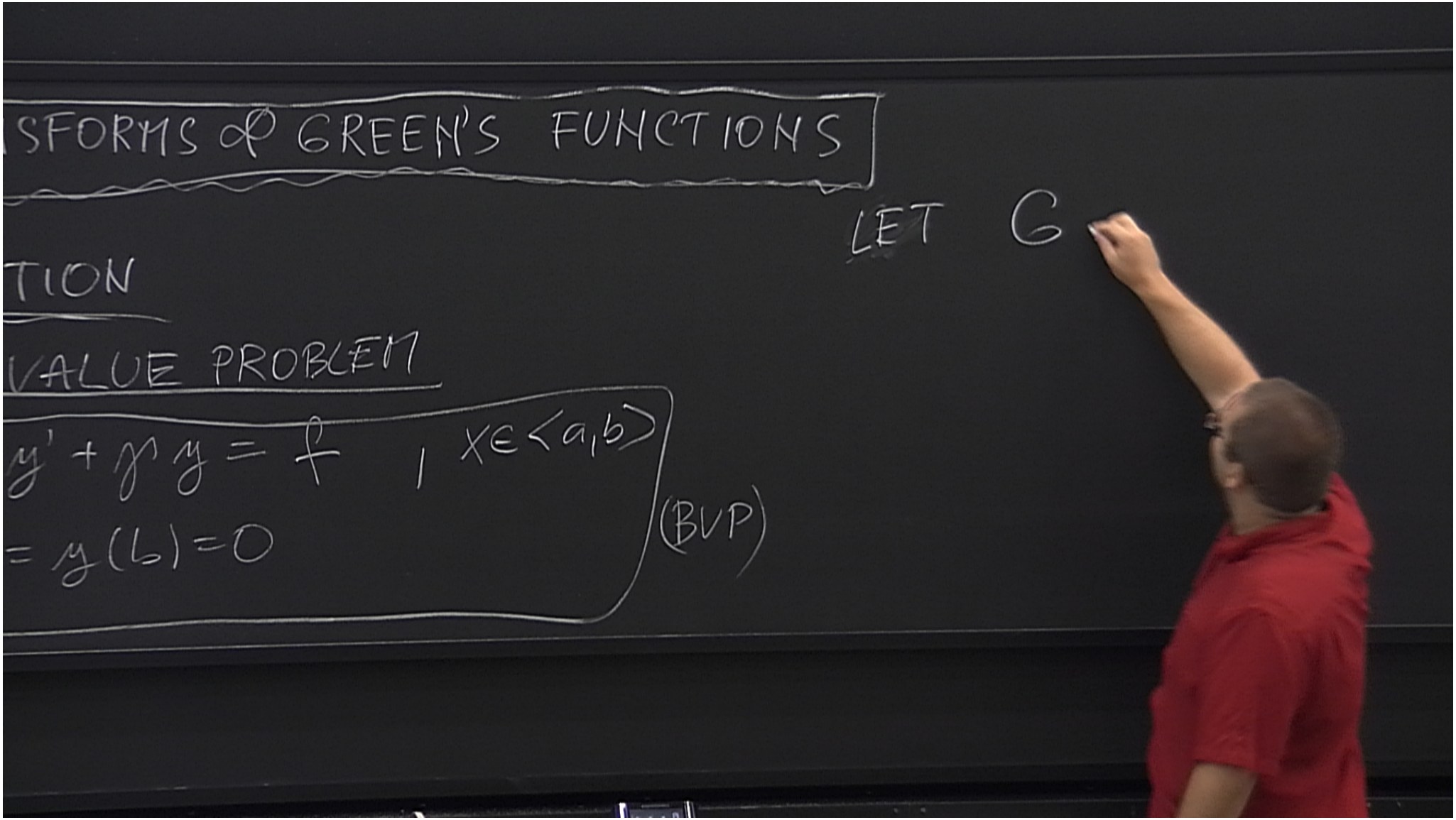
I) GREENS FUNCTION

1) 1D BOUNDARY VALUE PROBLEM

$$\alpha(x)\,y'' + \beta\,y' + \gamma\,y = f \quad , \; x \in \langle a,b \rangle$$

$$y(a) = y(b) = 0 \qquad \text{(BVP)}$$




SFORMS & GREEN'S FUNCTIONS
LET G
TION
VALUE PROBLEM
y' + γ y = f , x∈<a,b>
= y(b) = 0
(BVP)

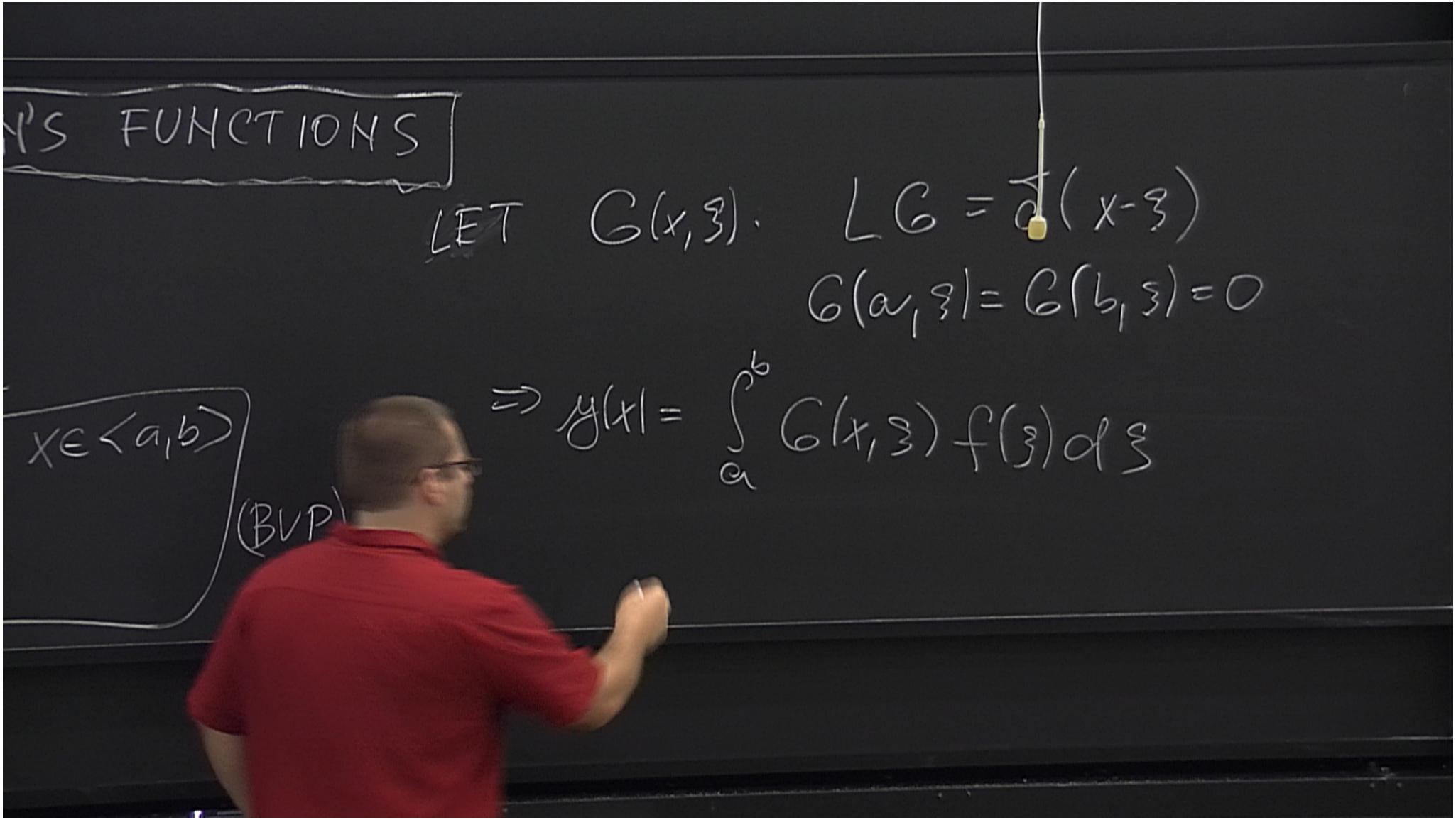
INTEGRAL TRANSFORMS & GREEN'S FUNCTIONS

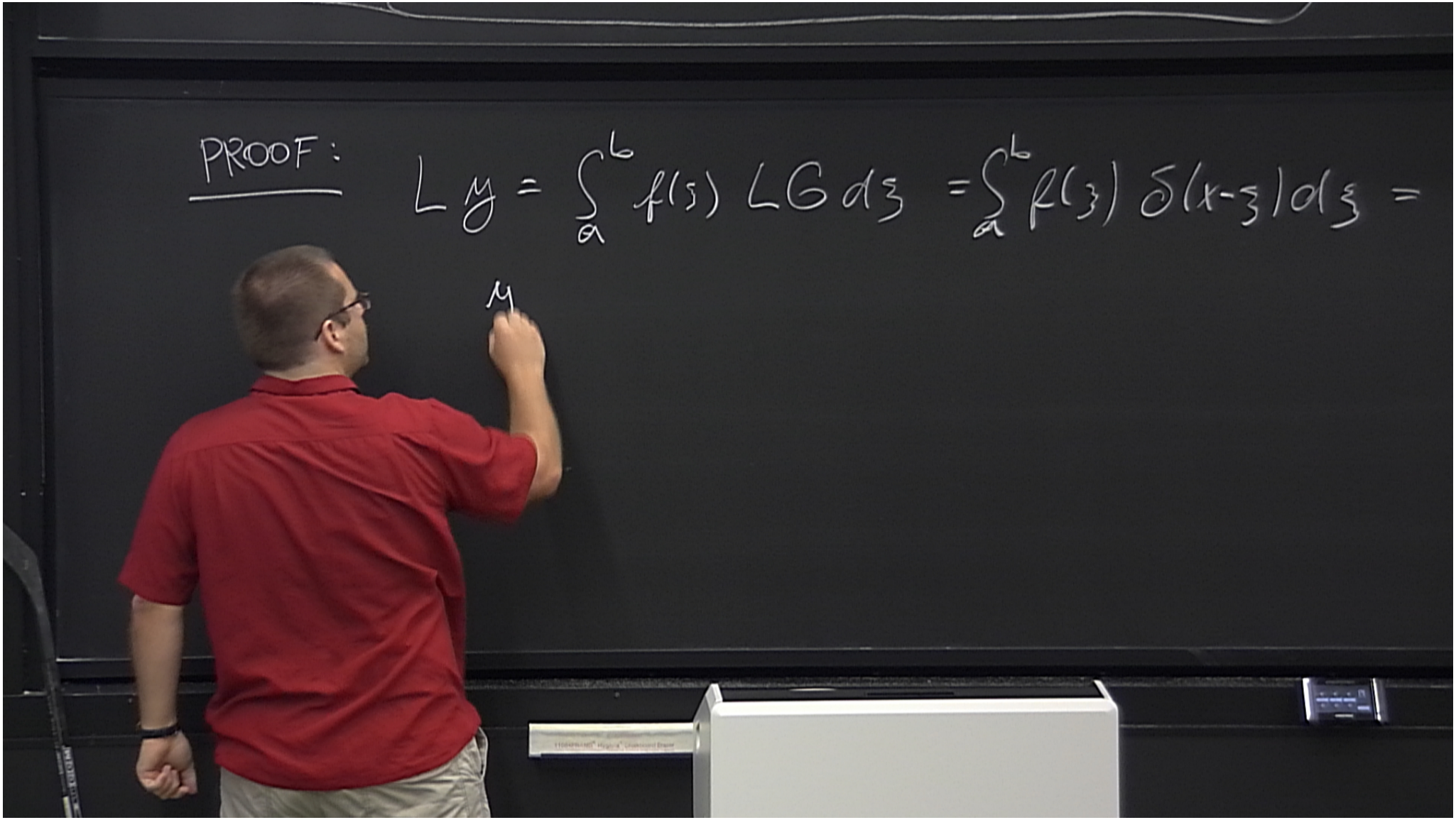
I) GREENS FUNCTION

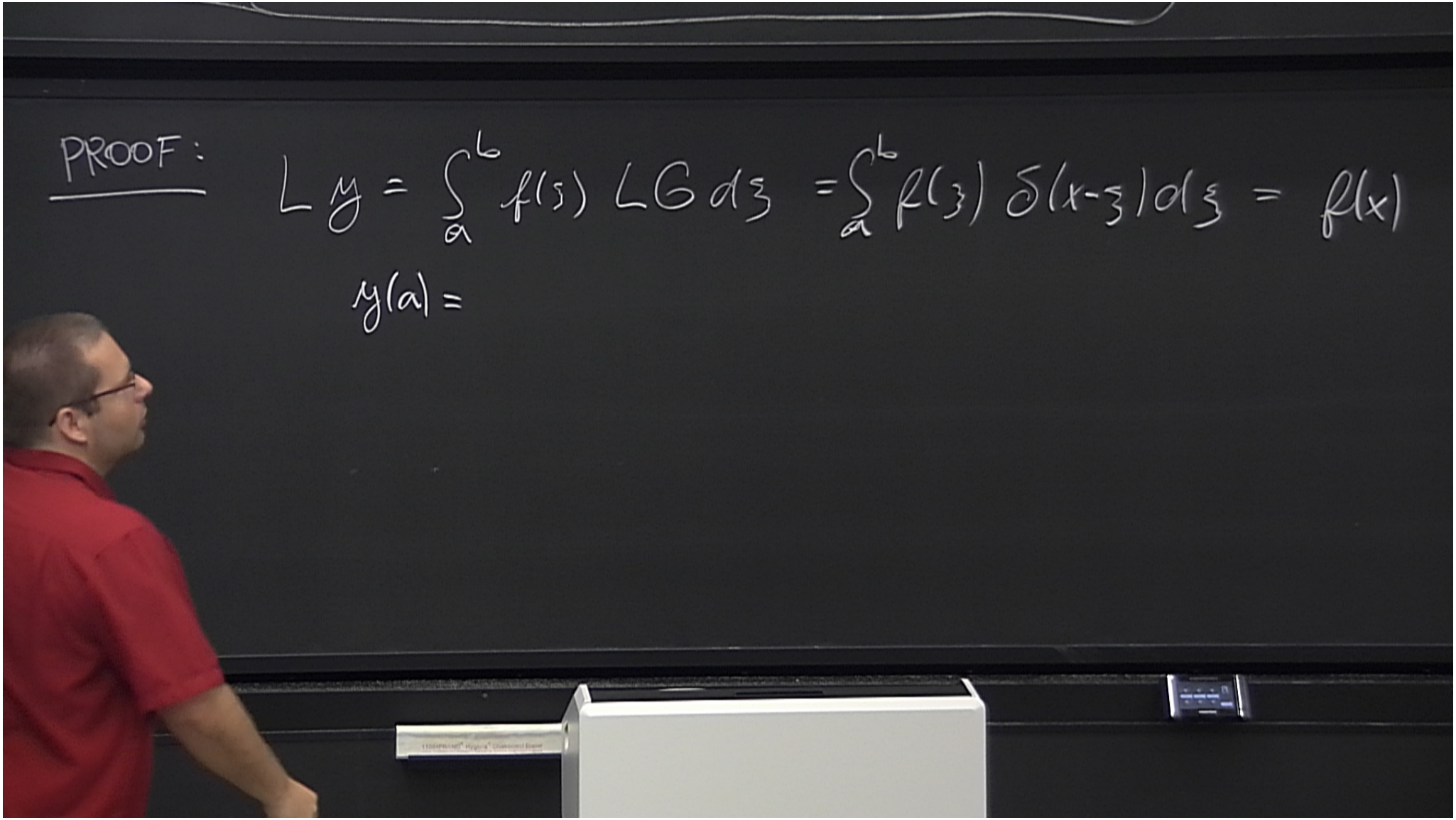
1) 1D BOUNDARY VALUE PROBLEM

$$Ly \equiv \alpha(x)\, y'' + \beta y' + \gamma y = f \quad , \; x \in \langle a, b \rangle$$

$$y(a) = y(b) = 0 \qquad \text{(BVP)}$$


'S FUNCTIONS
LET G(x,ξ). LG = δ(x-ξ)
G(a,ξ) = G(b,ξ) = 0
⇒ y(x) = ∫_a^b G(x,ξ) f(ξ) dξ
x∈<a,b>
(BVP)


PROOF:
$Ly = \int_a^b f(\xi)\, LG\, d\xi = \int_a^b f(\xi)\, \delta(x-\xi)\, d\xi =$
y


PROOF:
$Ly = \int_a^b f(\xi)\, LG\, d\xi = \int_a^b f(\xi)\, \delta(x-\xi)\, d\xi = f(x)$
$y(a) =$

PROOF:

$$L y = \int_a^b f(\xi)\, LG\, d\xi = \int_a^b f(\xi)\, \delta(x-\xi)\, d\xi = f(x)$$

$$y(a) = \int_a^b G(a,\xi)\, f(\xi)\, d\xi = 0 = y(b)$$

PROOF:

$$L y = \int_a^b f(\xi)\, LG\, d\xi = \int_a^b f(\xi)\, \delta(x-\xi)\, d\xi = f(x)$$

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PROOF:

$$Ly = \int_a^b f(\xi)\, LG\, d\xi = \int_a^b f(\xi)\, \delta(x-\xi)\, d\xi = f(x)$$

$$y(a) = \int_a^b G(a,\xi)\, f(\xi)\, d\xi = 0 = y(b)$$

HOW TO CONSTRUCT G ?

$$y = \int_a^b f(\xi)\, LG\, d\xi = \int_a^b f(\xi)\, \delta(x-\xi)\, d\xi = f(x)$$

$$y(a) = \int_a^b G(a,\xi)\, f(\xi)\, d\xi = 0 = y(b)$$

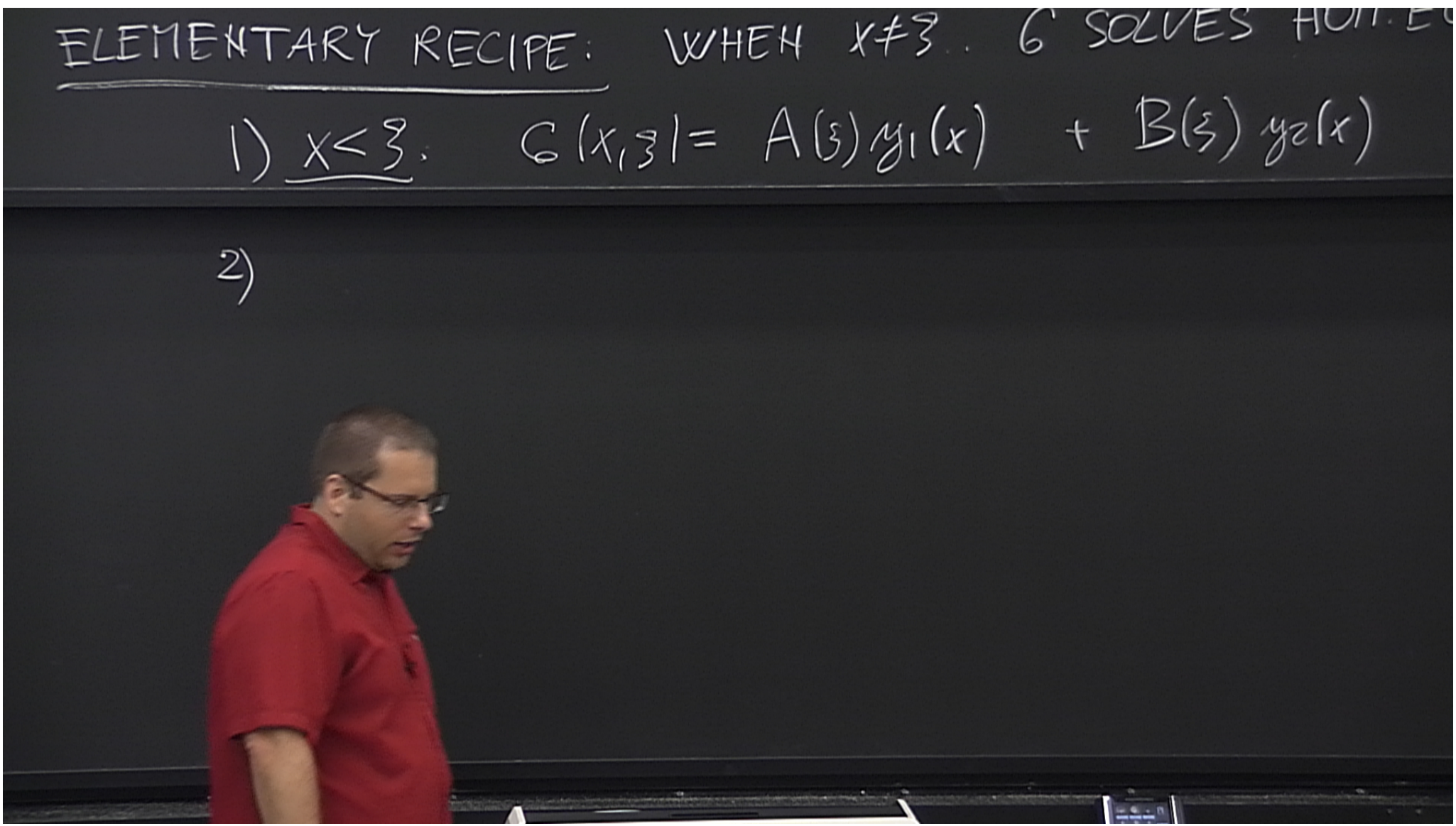
ONSTRUCT G ?

TARY RECIPE: WHEN $x \neq \xi$. G SOLVES HOM. EQ $LG = 0$

ELEMENTARY RECIPE: WHEN $x \neq \xi$. G SOLVES HOM. E

1) $x < \xi$. $G(x,\xi) = A(\xi)\, y_1(x) + B(\xi)\, y_2(x)$

2)



TARY RECIPE: WHEN $x \neq \xi$. G SOLVES HOM. EQ $LG = 0$

) $x < \xi$. $G(x,\xi) = A(\xi)\, y_1(x) + B(\xi)\, y_2(x)$

$x > \xi$: $G(x,\xi) = C(\xi)\, y_1(x) + D(\xi)\, y_2(x)$

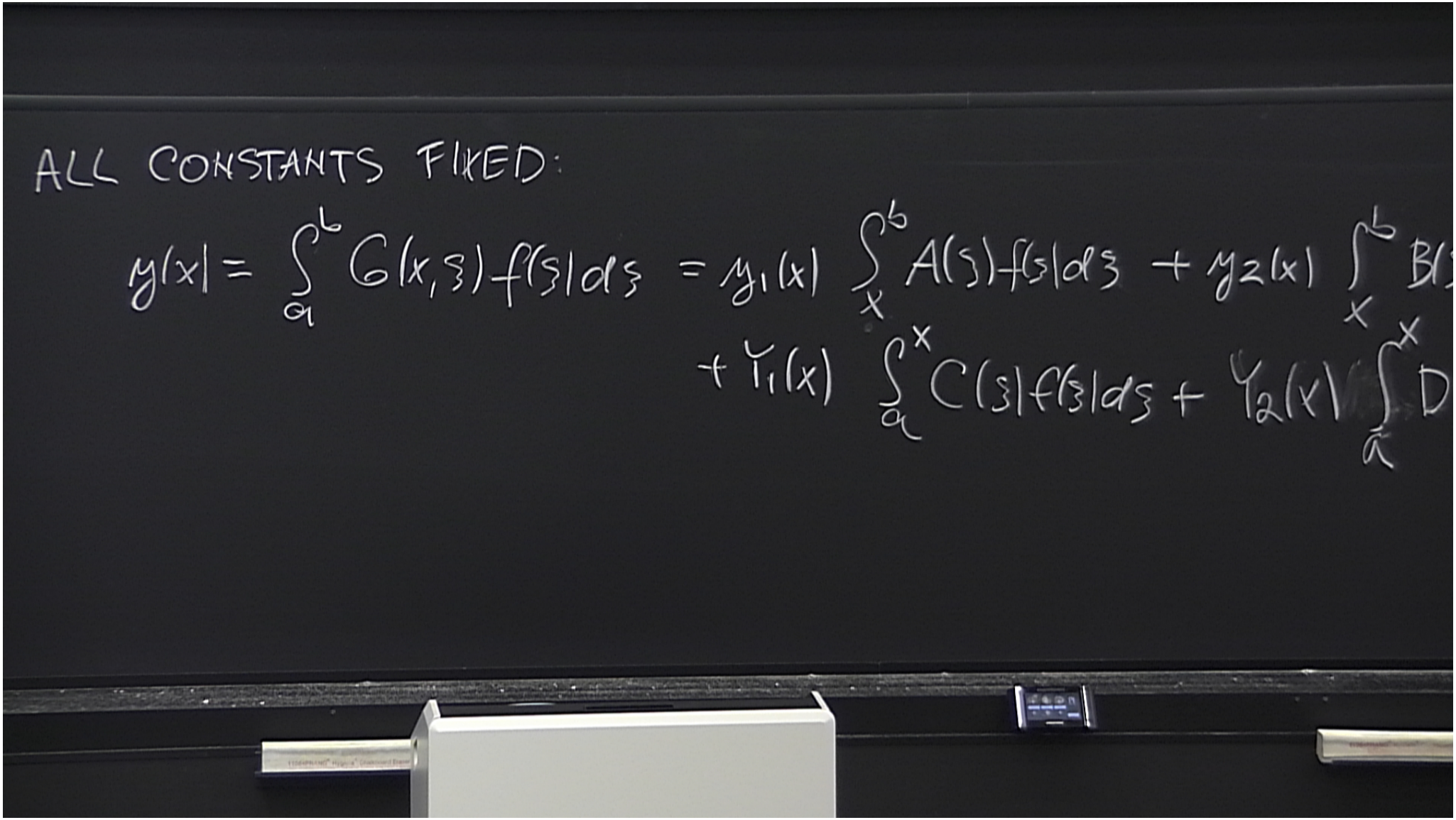
$G(a,\xi) = 0 = A\, y_1(a) + B\, y_2(a) = 0$ ... SOLVE FOR A

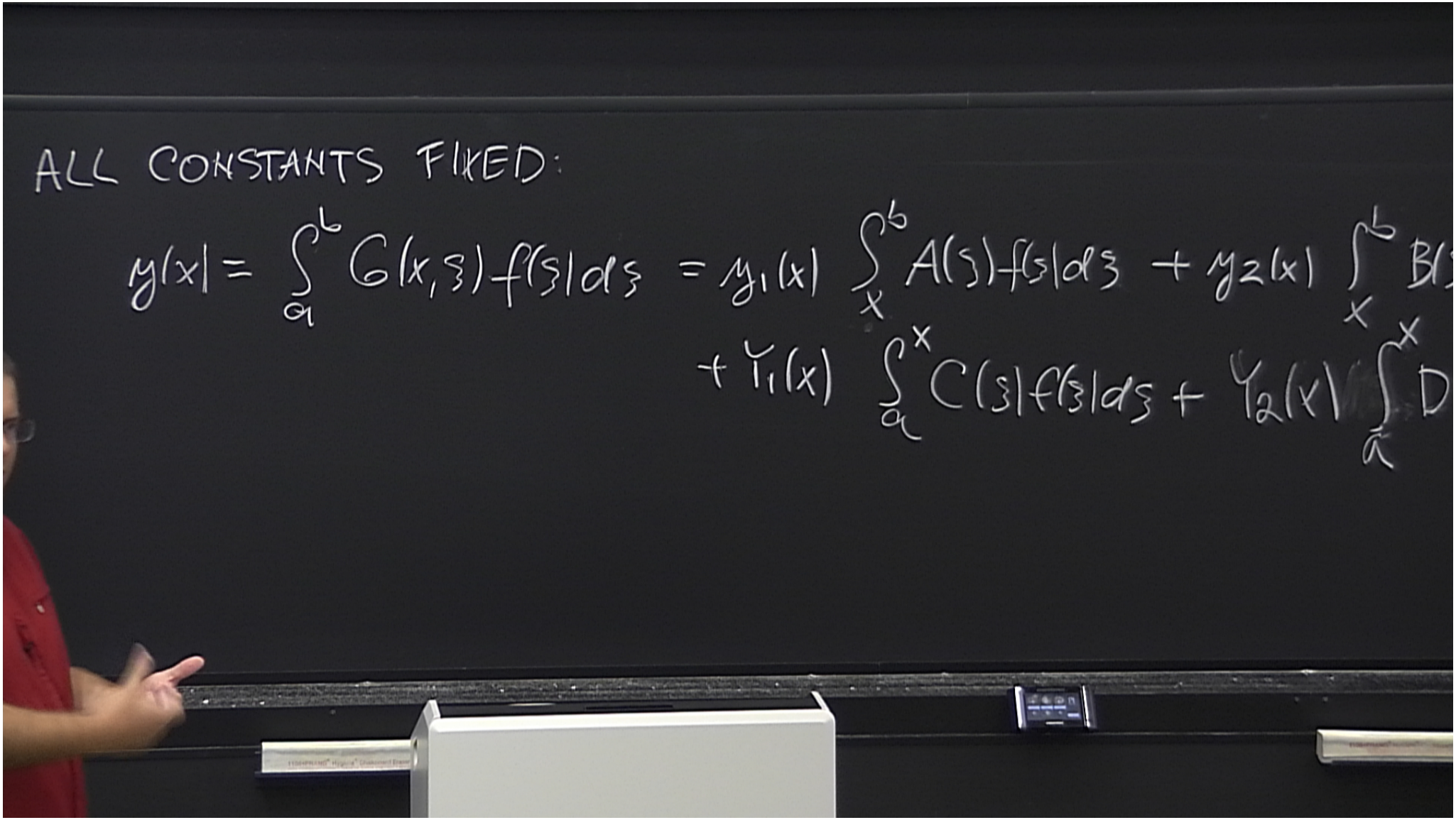
) $G(b,\xi) = 0 = C\, y_1(b) + D\, y_2(b) = 0$ ... C

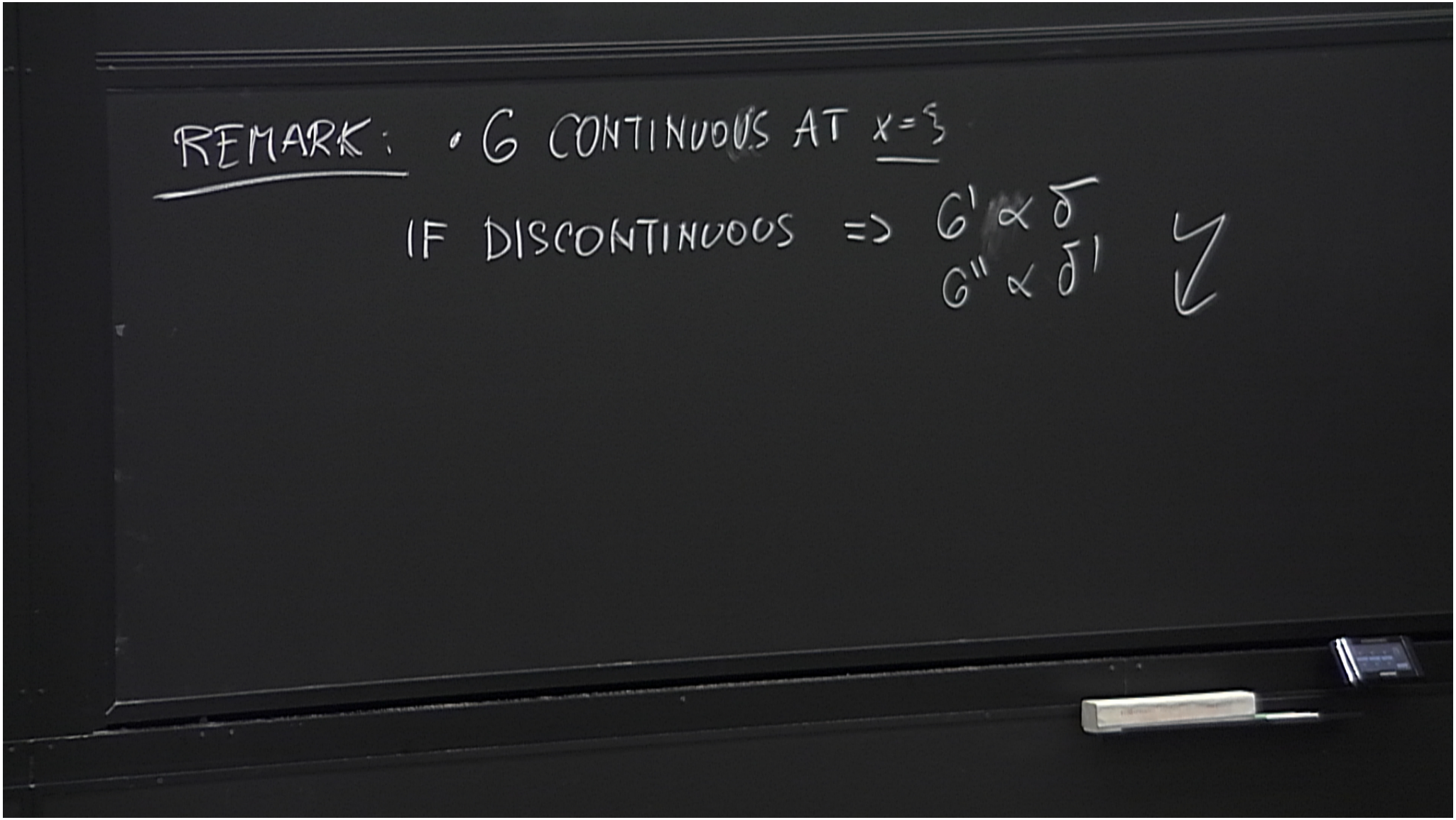
) G CONTINUOUS AT $x = \xi$ $A\, y_1(\xi) + B\, y_2(\xi) = C\, y_1(\xi) + D\, y_2(\xi)$. ELIMINATE . B

ALL CONSTANTS FIXED:

$$y(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi = y_1(x) \int_x^b A(\xi) f(\xi)\, d\xi + y_2(x) \int_x^b B(\xi$$


ALL CONSTANTS FIXED:
$y(x) = \int_a^b G(x,s) f(s)\,ds = y_1(x) \int_x^b A(s) f(s)\,ds + y_2(x) \int_x^b B($
$+ y_1(x) \int_a^x C(s) f(s)\,ds + y_2(x) \int_a^x D$


ALL CONSTANTS FIXED:
$y(x) = \int_a^b G(x,s) f(s)\,ds = y_1(x) \int_x^b A(s) f(s)\,ds + y_2(x) \int_x^b B($
$+ y_1(x) \int_a^x C(s) f(s)\,ds + y_2(x) \int_a^x D$


REMARK: • G CONTINUOUS AT x=ξ
IF DISCONTINUOUS => G' ∝ δ
G'' ∝ δ'

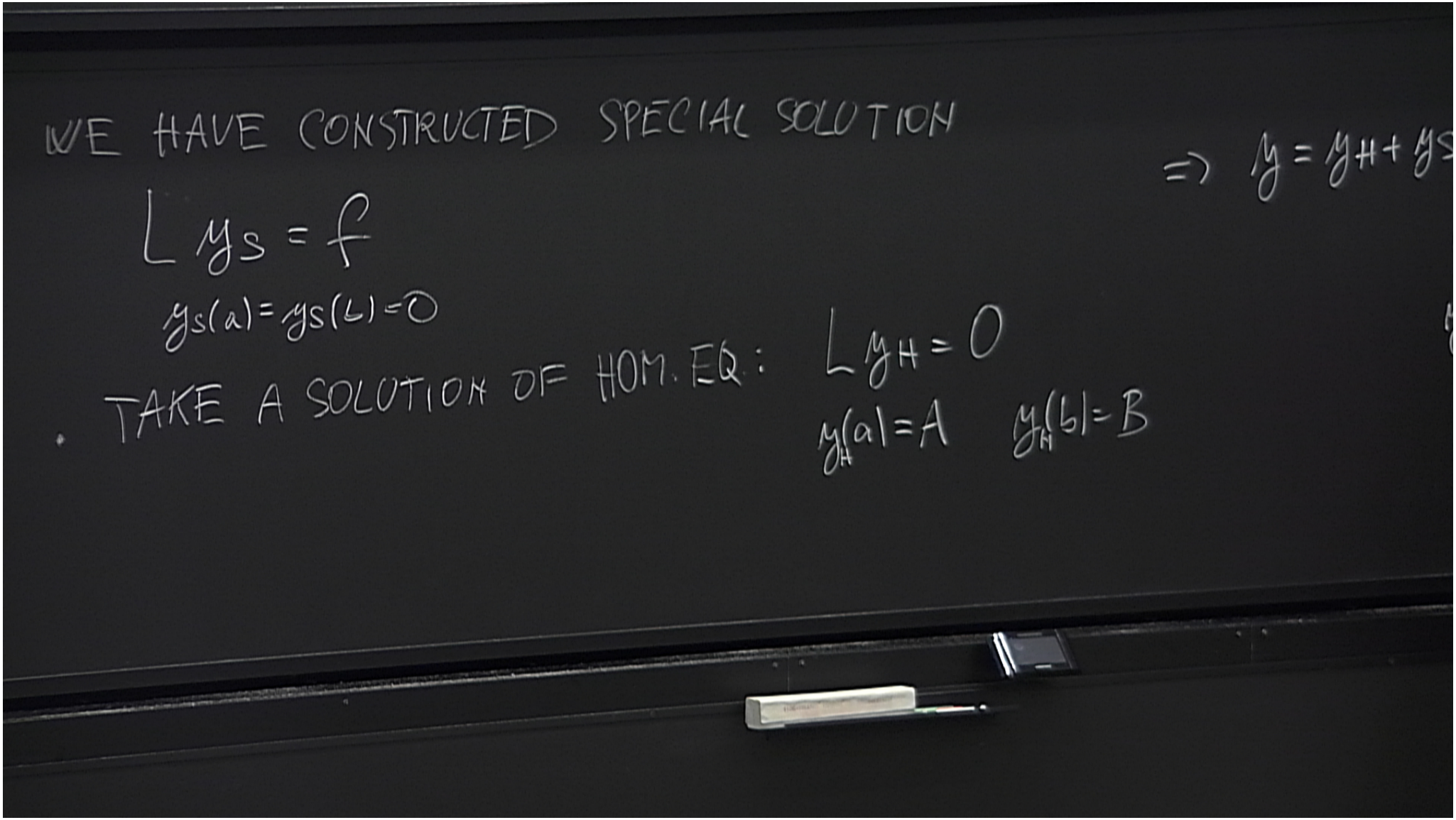
WE HAVE CONSTRUCTED SPECIAL SOLUTION

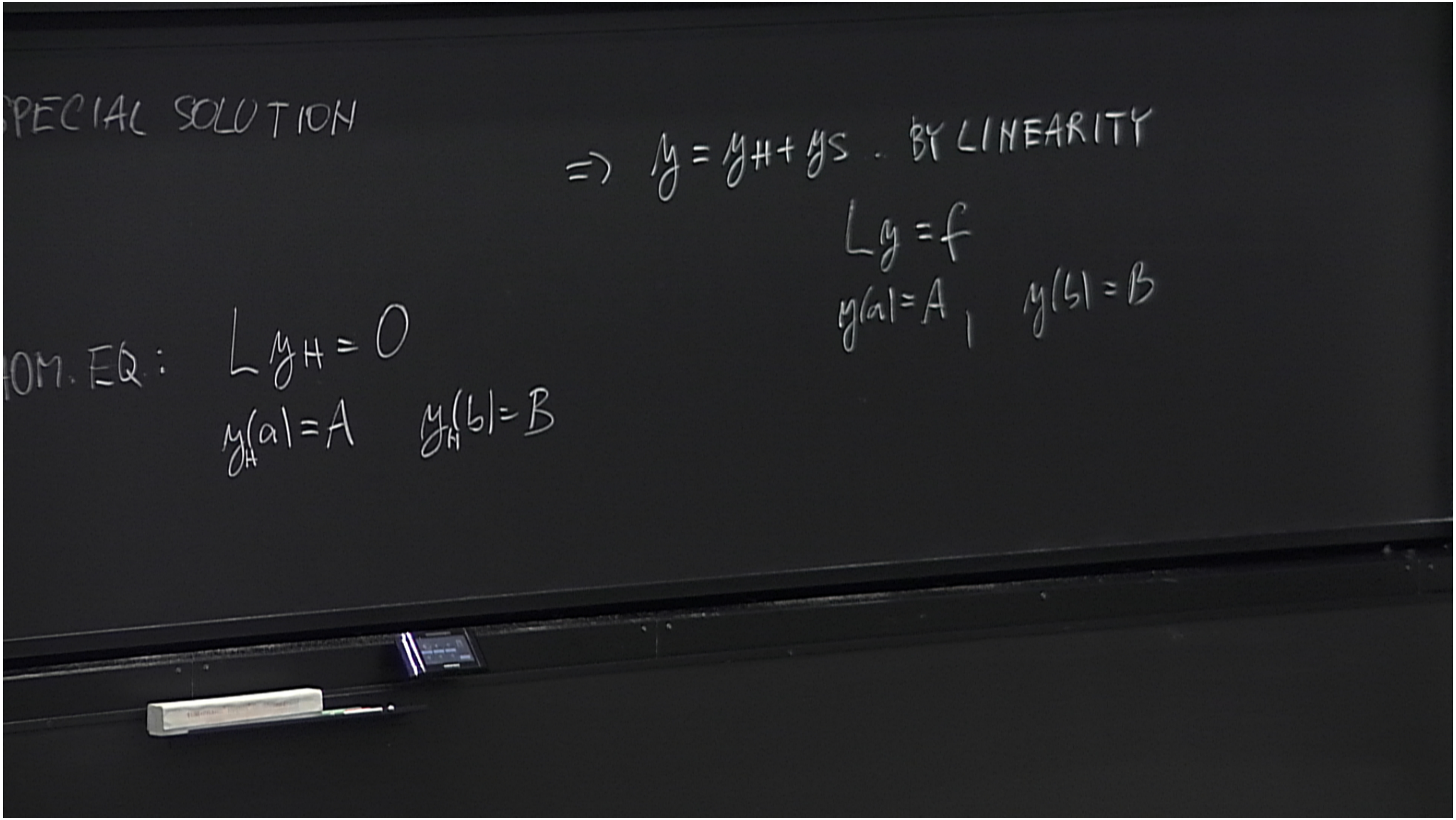
$$L y_S = f$$

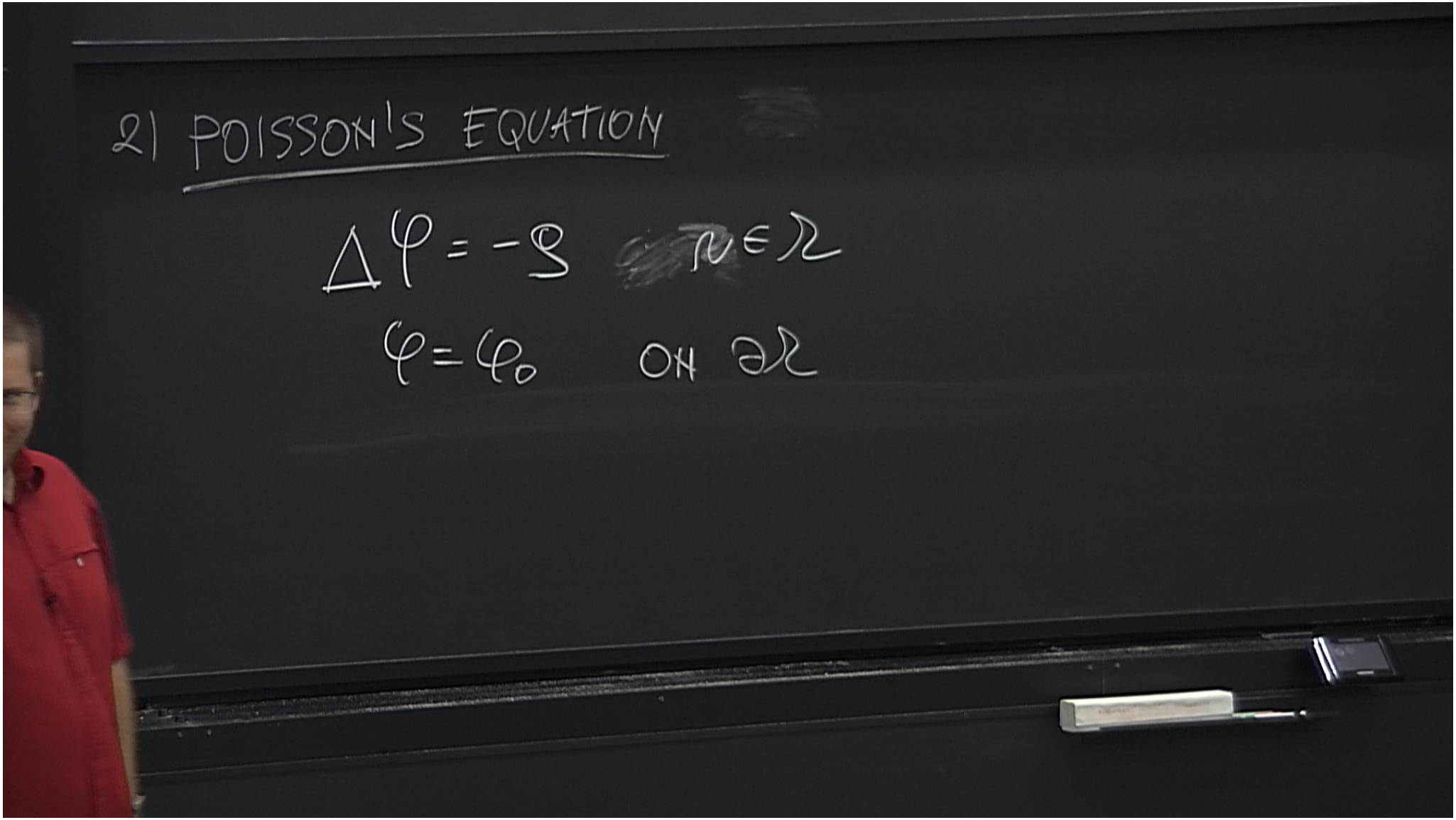
$$y_S(a) = y_S(L) = 0$$

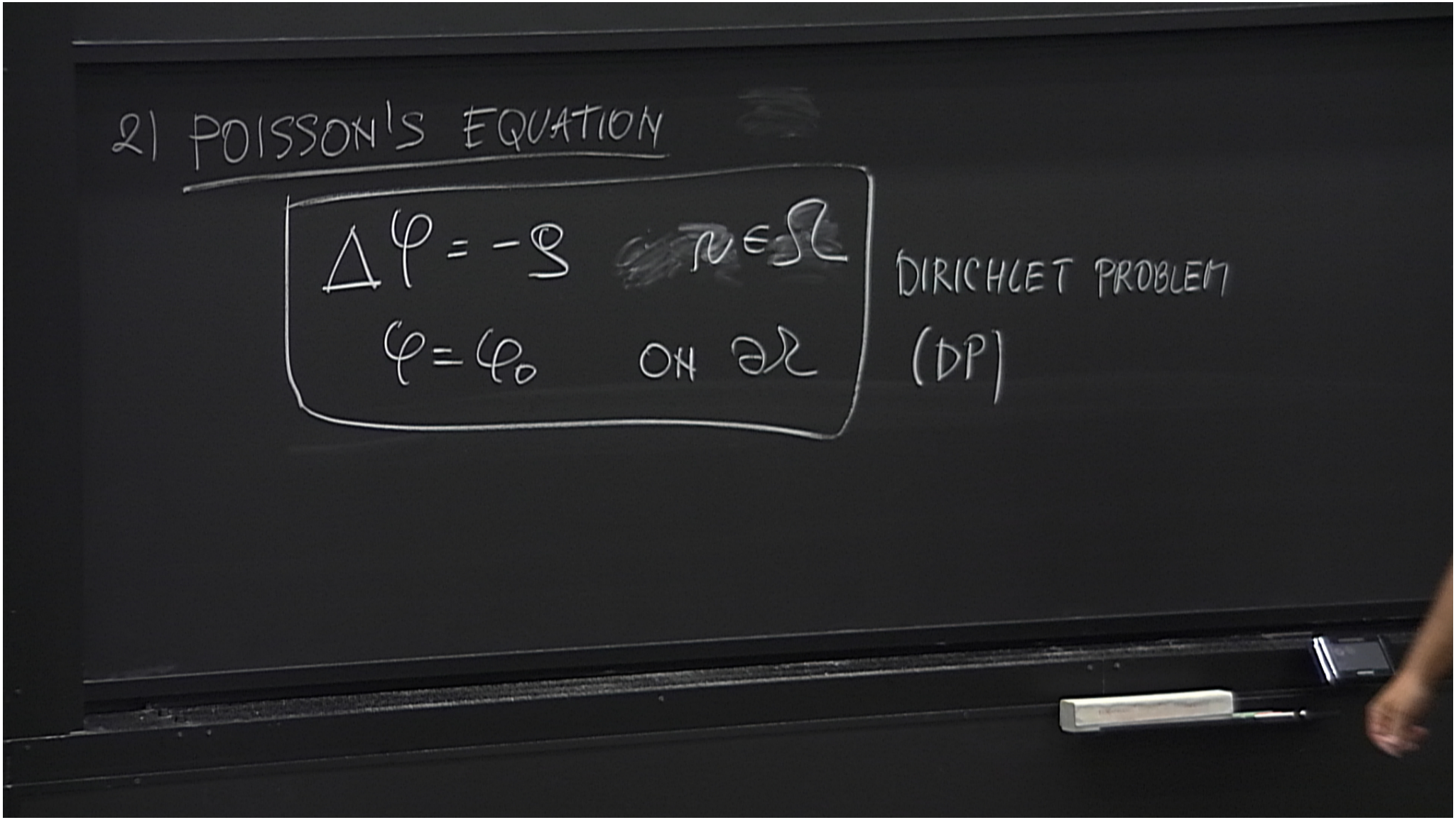
. TAKE A SOLUTION OF HOM. EQ: $L y_H = 0$

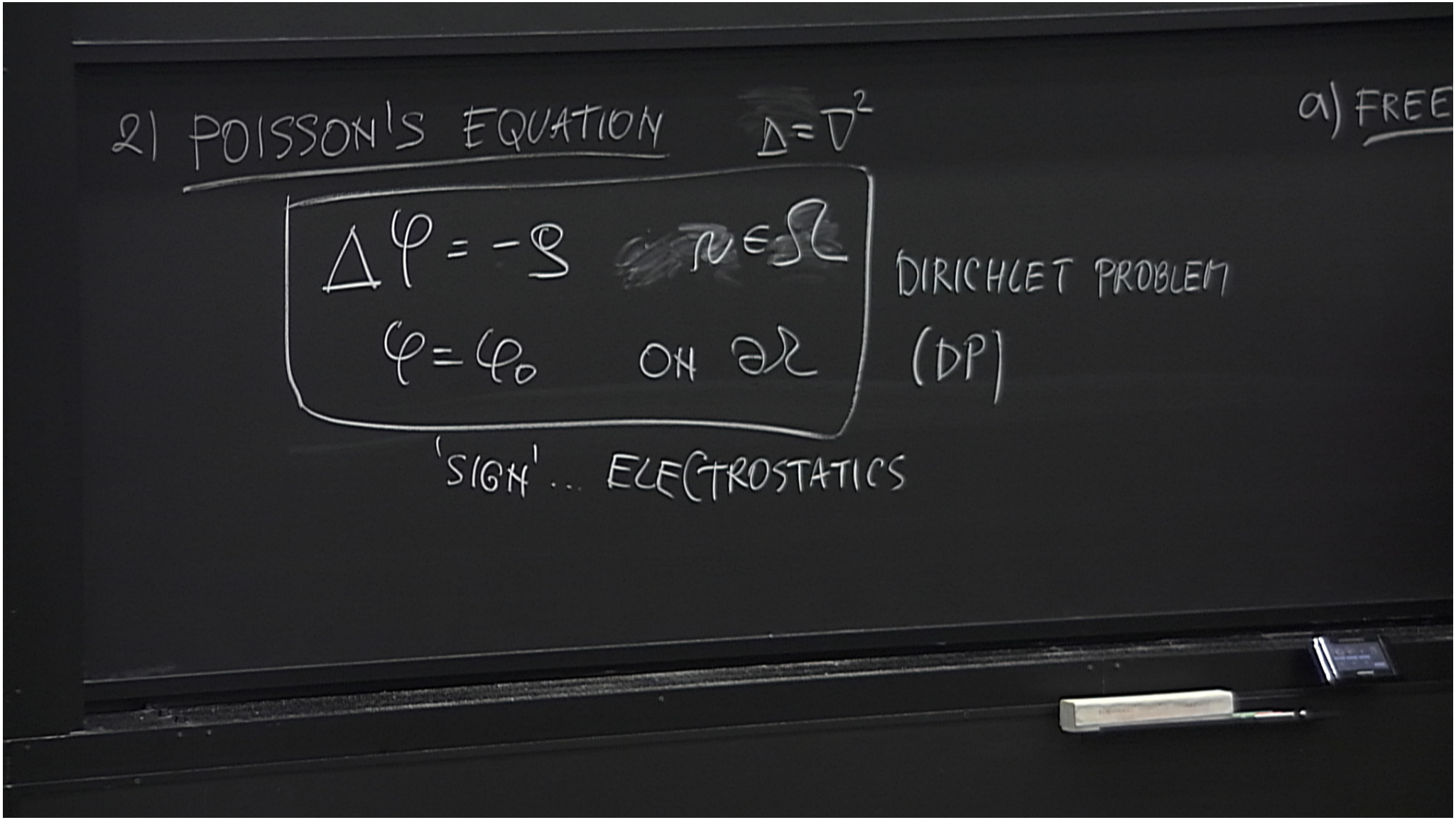
$$y_H(a) = A \qquad y_H(b) = B$$


WE HAVE CONSTRUCTED SPECIAL SOLUTION
$\Rightarrow y = y_H + y_S$
$L y_S = f$
$y_S(a) = y_S(b) = 0$
TAKE A SOLUTION OF HOM. EQ: $L y_H = 0$
$y_H(a) = A \quad y_H(b) = B$


SPECIAL SOLUTION
$\Rightarrow y = y_H + y_S$. BY LINEARITY
$Ly = f$
$y(a) = A$, $y(b) = B$
HOM. EQ: $Ly_H = 0$
$y_H(a) = A$ $y_H(b) = B$


21 POISSON'S EQUATION
$\Delta\varphi = -\rho$ $x \in \Omega$
$\varphi = \varphi_0$ on $\partial\Omega$

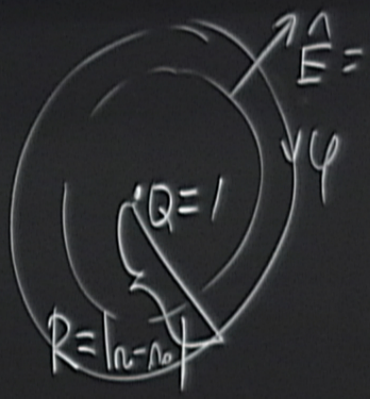

21 POISSON'S EQUATION
$\Delta\varphi = -\rho$ $x \in \Omega$
$\varphi = \varphi_0$ on $\partial\Omega$
DIRICHLET PROBLEM
(DP)

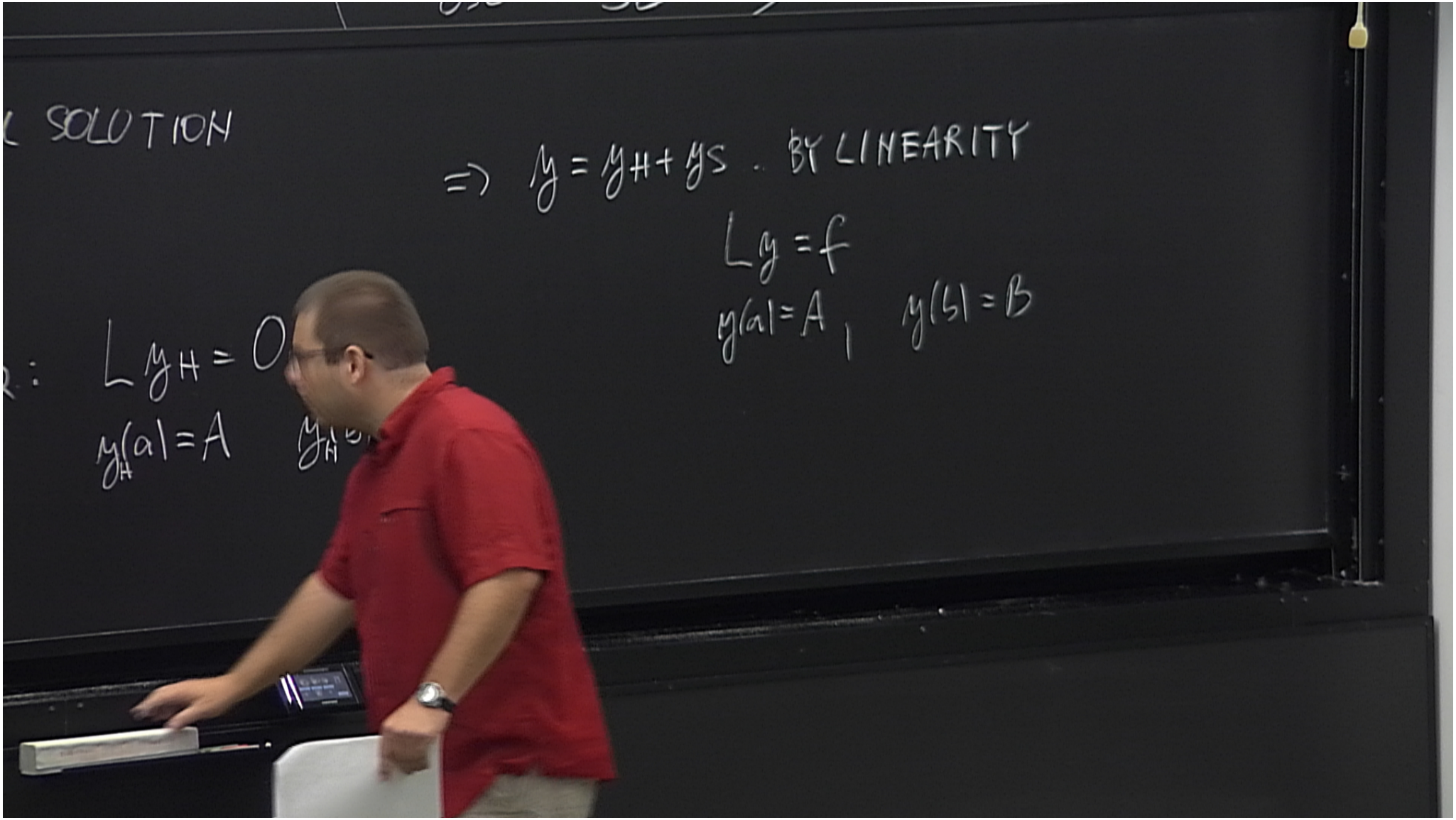

2) POISSON'S EQUATION $\Delta = \nabla^2$
a) FREE
$\Delta\varphi = -\rho$ $x \in \Omega$
$\varphi = \varphi_0$ ON $\partial\Omega$
DIRICHLET PROBLEM
(DP)
'SIGN' .. ELECTROSTATICS

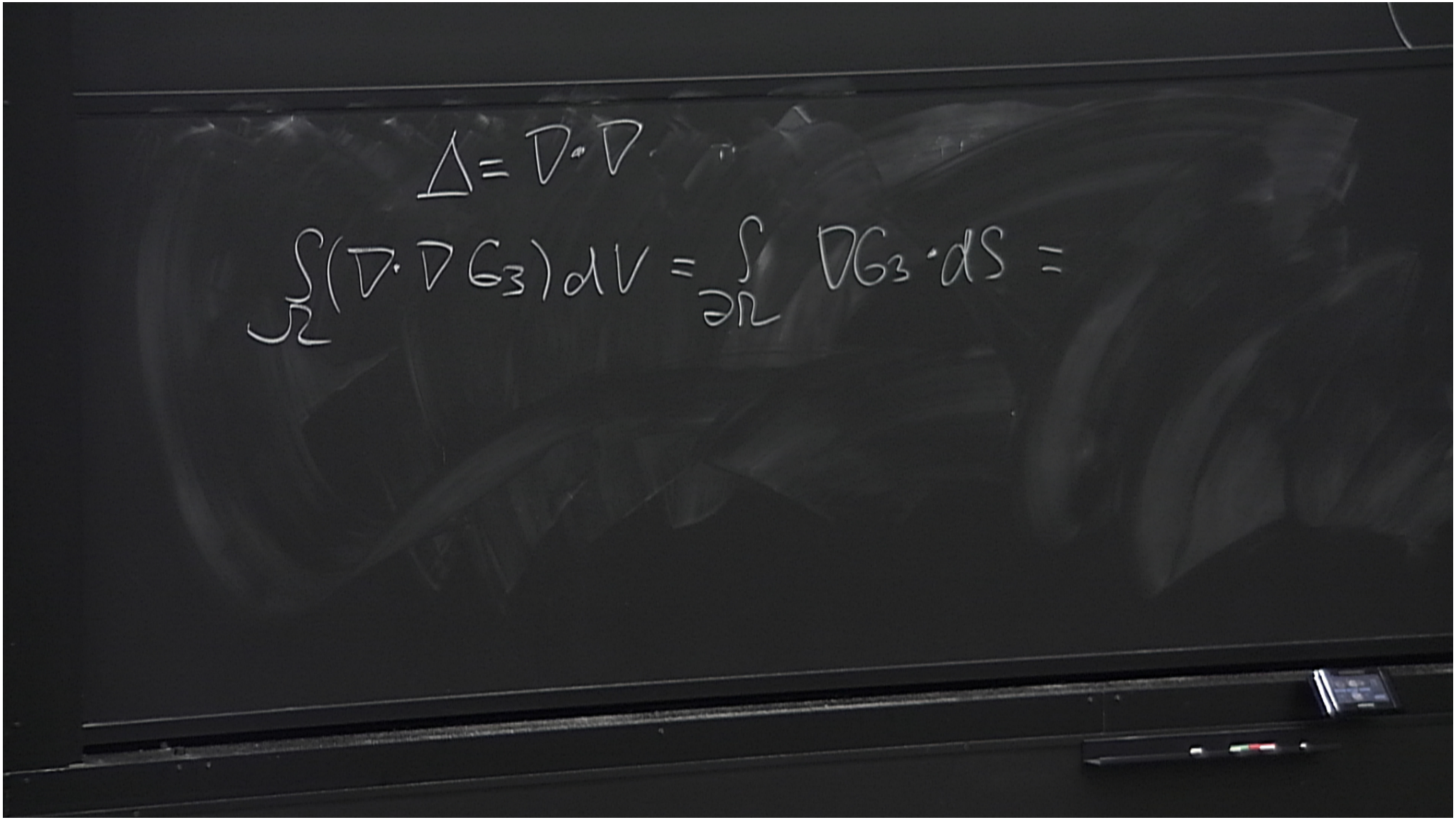
a) FREE SPACE GREEN'S FUNCTION

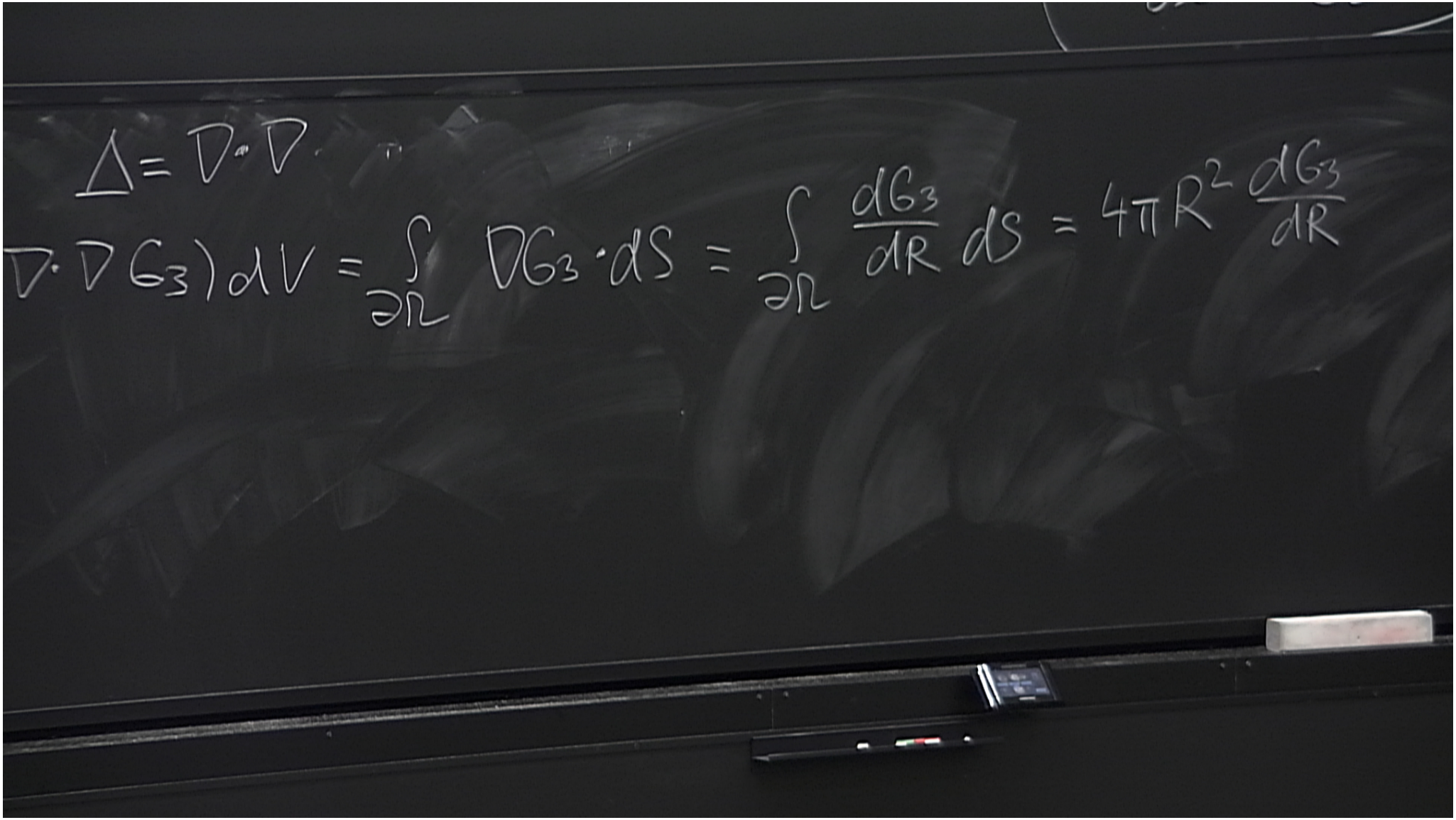
$$\Delta G_3 = \delta(r - r_0)$$

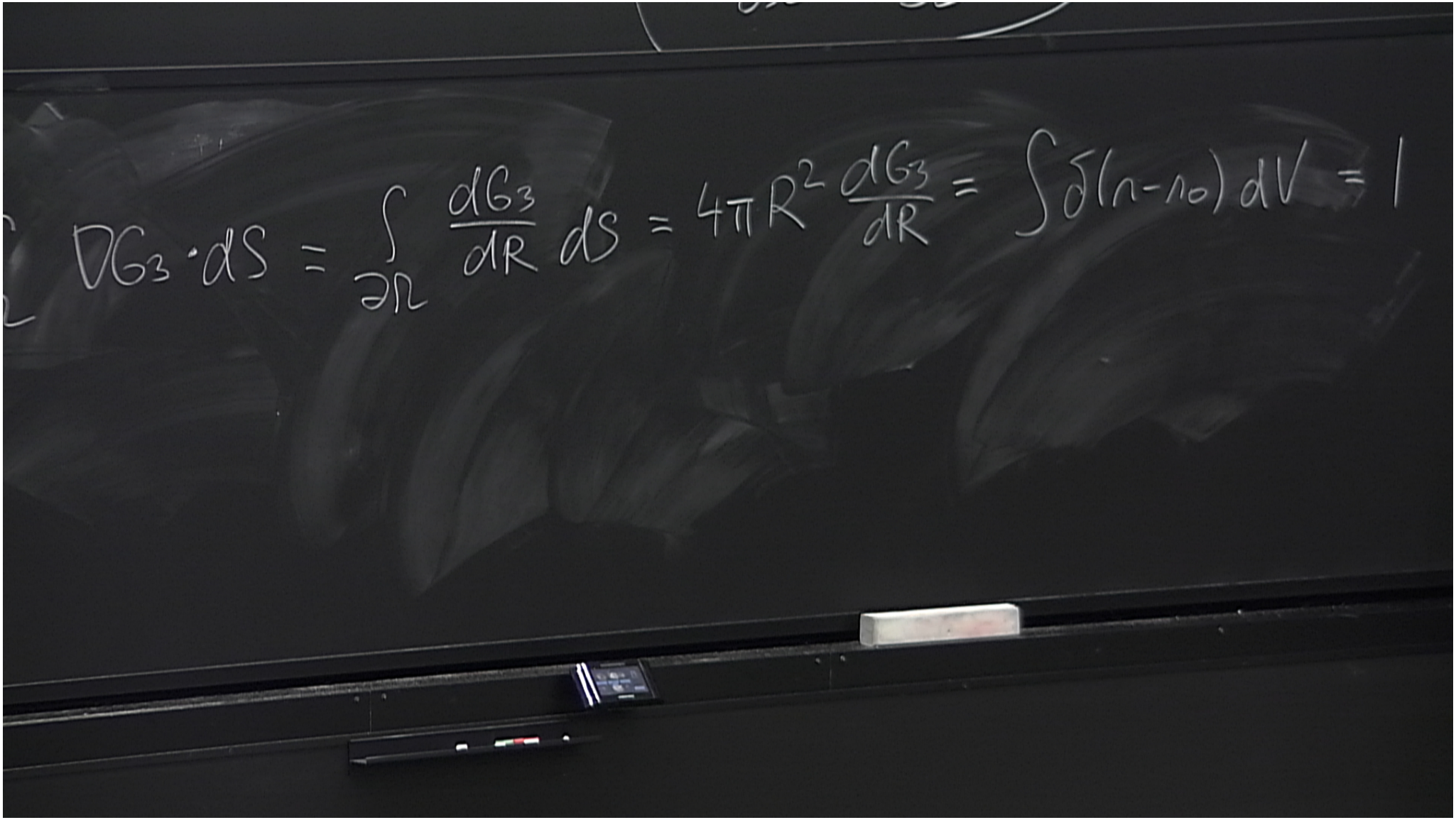
T PROBLEM




SOLUTION
$\Rightarrow y = y_H + y_S$. BY LINEARITY
$Ly = f$
$y(a) = A$, $y(b) = B$
$Ly_H = 0$
$y_H(a) = A$

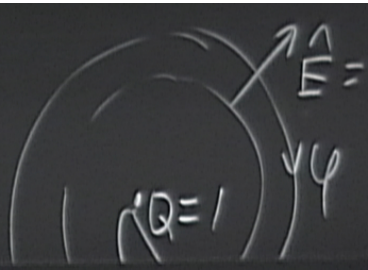

$\Delta = D \cdot D$
$\int_\Omega (D \cdot D G_3)\, dV = \int_{\partial\Omega} DG_3 \cdot dS =$


$\Delta = D \cdot D$
$D \cdot D G_3) dV = \int_{\partial\Omega} DG_3 \cdot dS = \int_{\partial\Omega} \frac{dG_3}{dR} dS = 4\pi R^2 \frac{dG_3}{dR}$


$\int_{\partial\Omega} DG_3 \cdot dS = \int_{\partial\Omega} \frac{dG_3}{dR} dS = 4\pi R^2 \frac{dG_3}{dR} = \int \delta(r - r_0)\, dV = 1$

a) FREE SPACE GREEN'S FUNCTION

$$\Lambda G_3 = \delta(r - r_0)$$

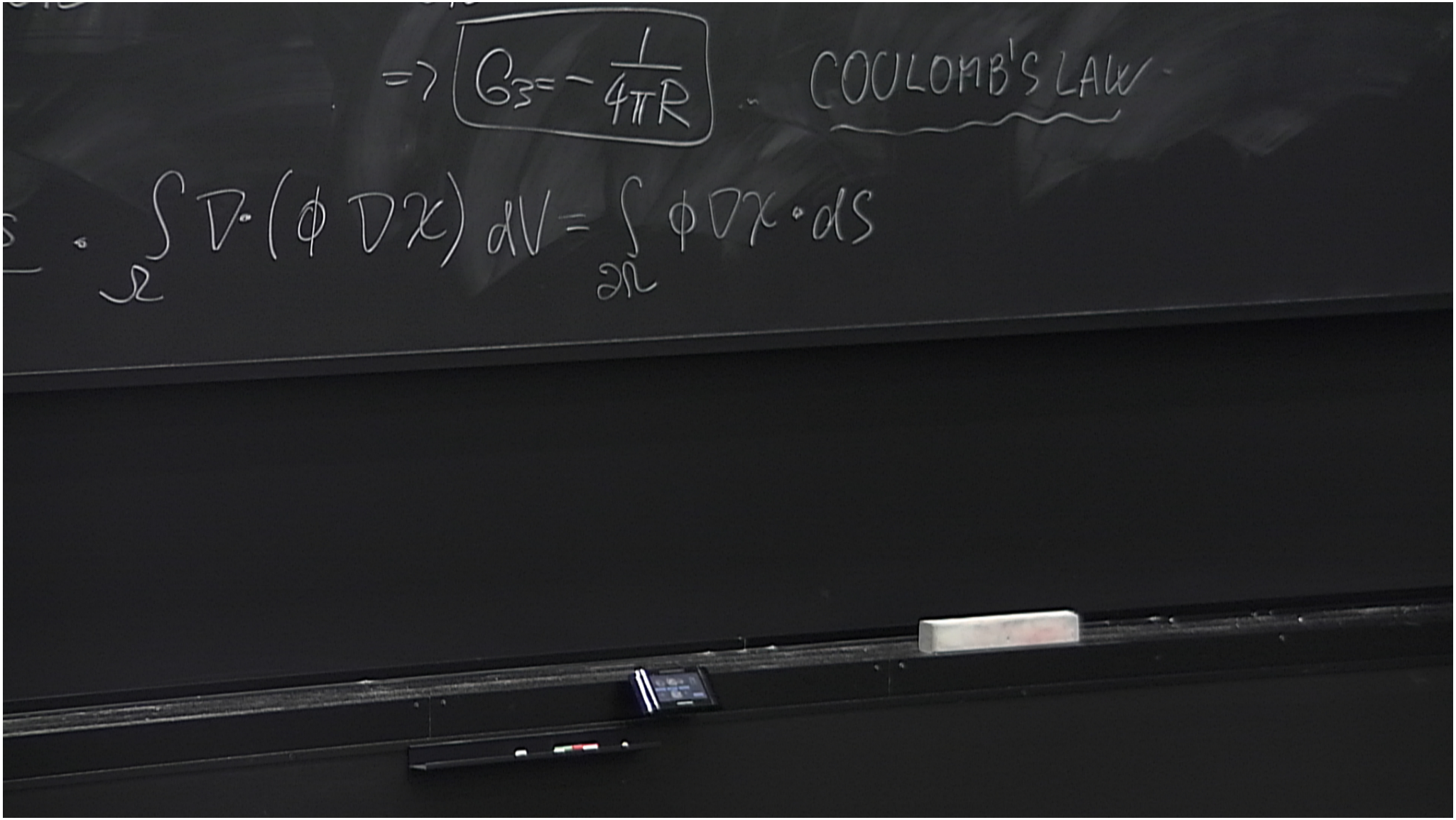


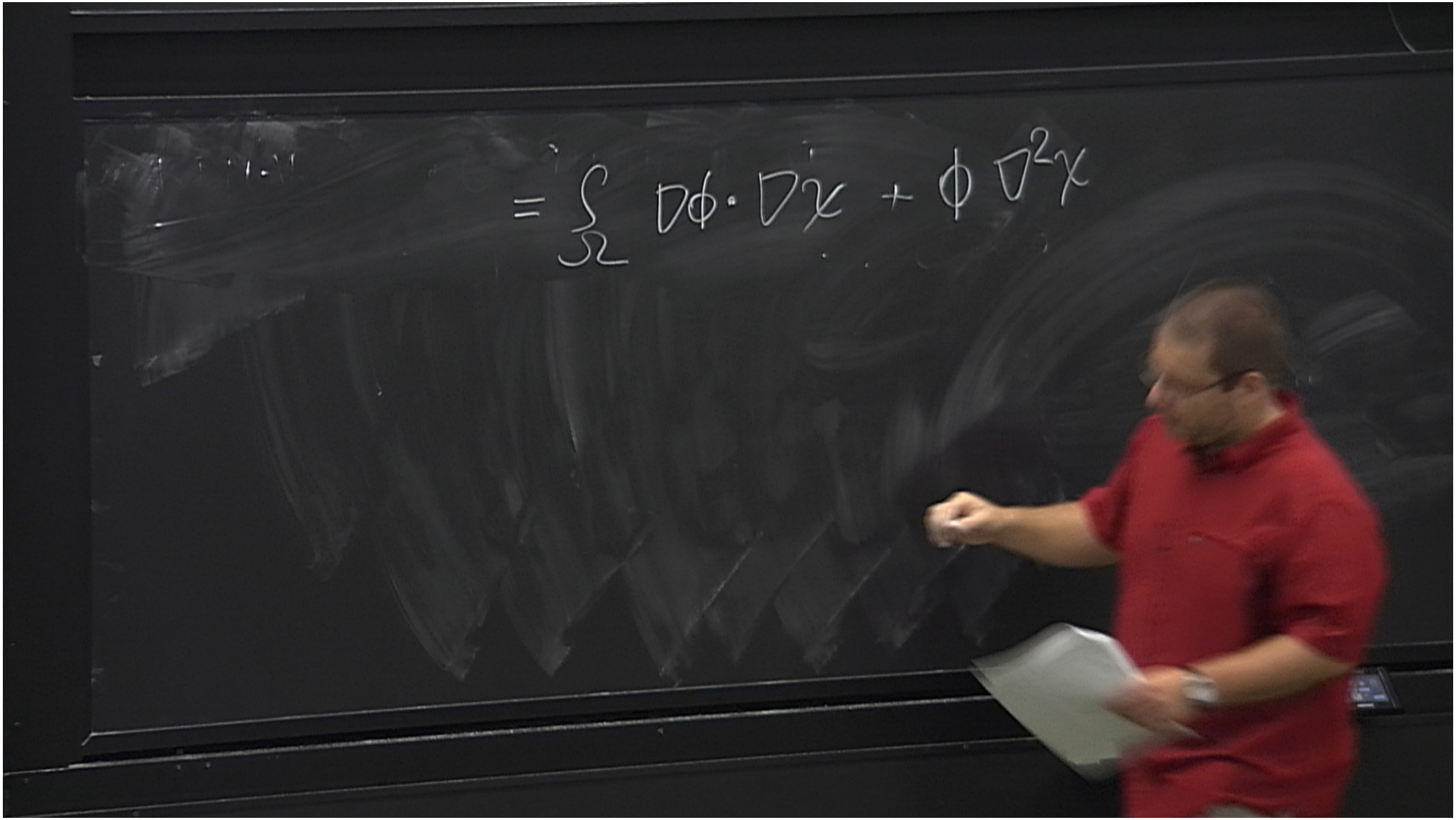
$$\int_{\partial\Omega} \frac{dG_3}{dR}\, dS = 4\pi R^2 \frac{dG_3}{dR} = \int \delta(r - r_0)\, dV = 1$$

TOTAL CHARGE INSIDE

$$\Rightarrow \boxed{G_3 = -\frac{1}{4\pi R}}$$

COULOMB'S LAW


$\Rightarrow \boxed{G_3 = -\frac{1}{4\pi R}}$ .. COULOMB'S LAW
$\int_{\Omega} \nabla \cdot (\phi \nabla \chi)\, dV = \int_{\partial\Omega} \phi \nabla \chi \cdot dS$


$= \int_{S^2} D\phi \cdot D\chi + \phi D^2\chi$

$$= \int_\Omega (\nabla\phi \cdot \nabla\chi + \phi \nabla^2\chi)\, dV = \int_{\partial\Omega} \phi \frac{\partial\chi}{\partial n}\, dS$$

1ST GREEN'S

$$= \int_{\Omega} (\nabla\phi \cdot \nabla\chi + \phi \nabla^2\chi)\, dV = \int_{\partial\Omega} \phi \frac{\partial\chi}{\partial n}\, dS$$ (1) 1ST G

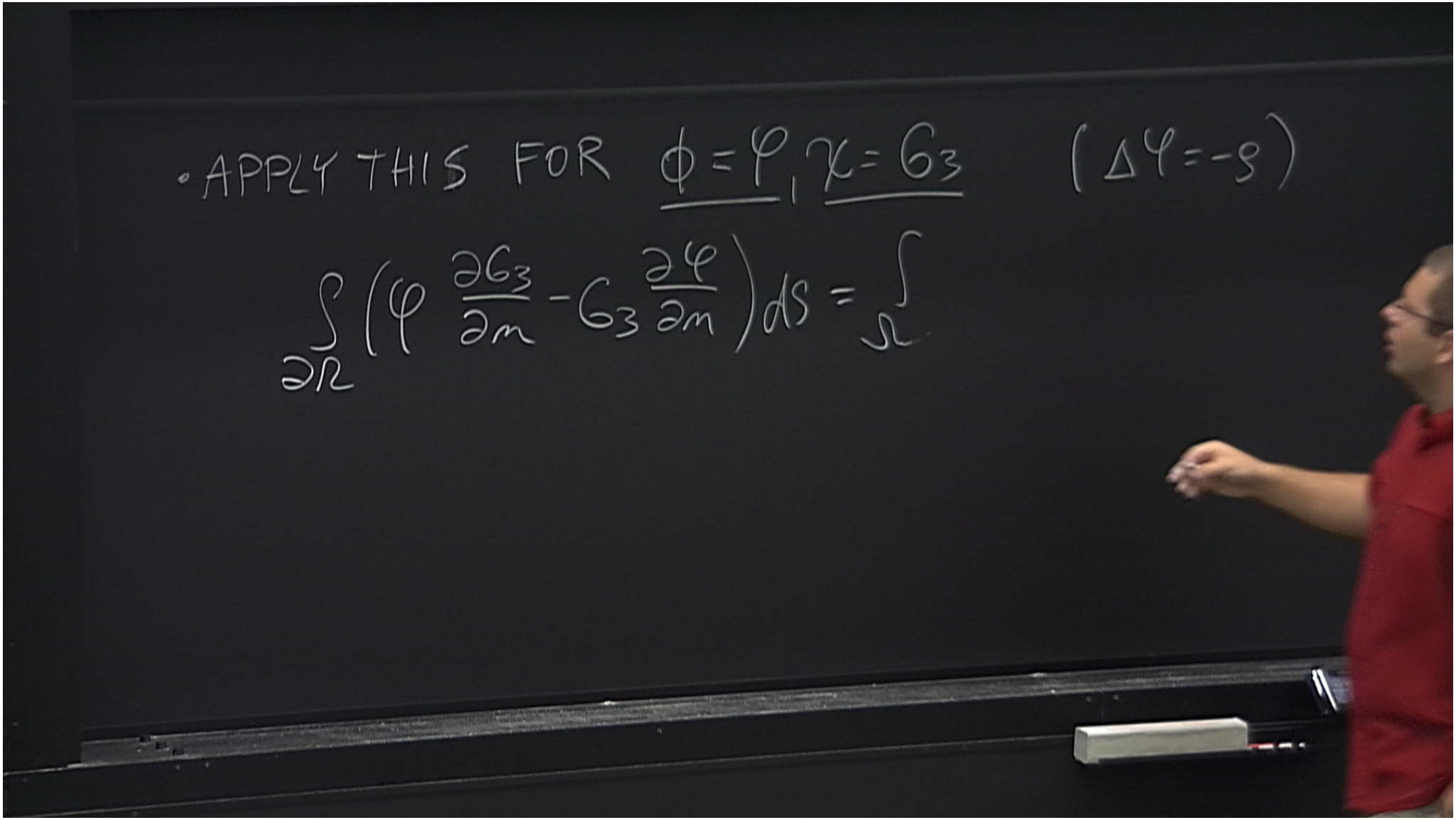
$\phi \leftrightarrow \chi$ (2) ... SUBSTRACT (1) AND (2)

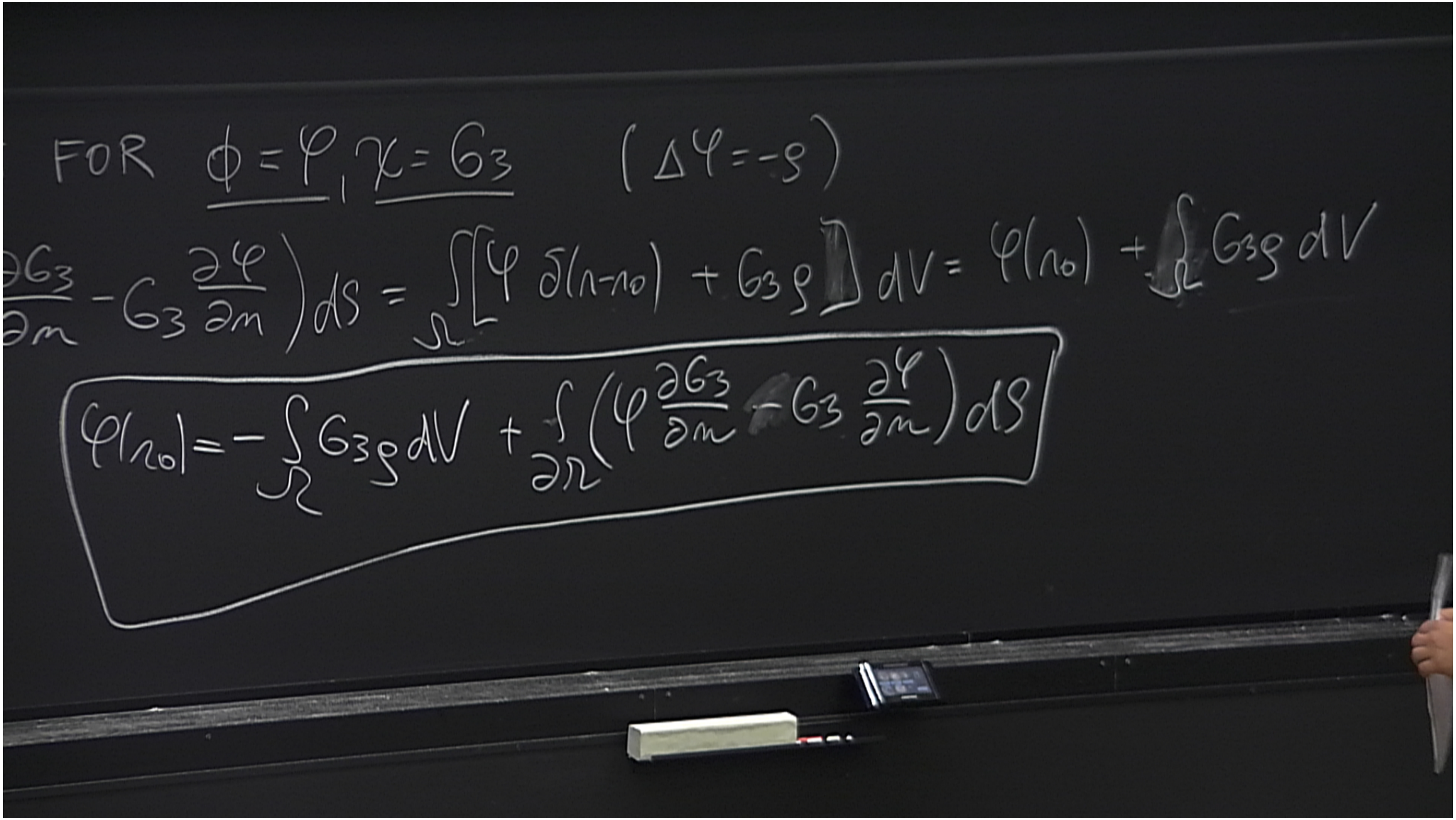
$$\int_{\partial\Omega} \left(\phi \frac{\partial\chi}{\partial n} - \chi \frac{\partial\phi}{\partial n}\right) dS =$$

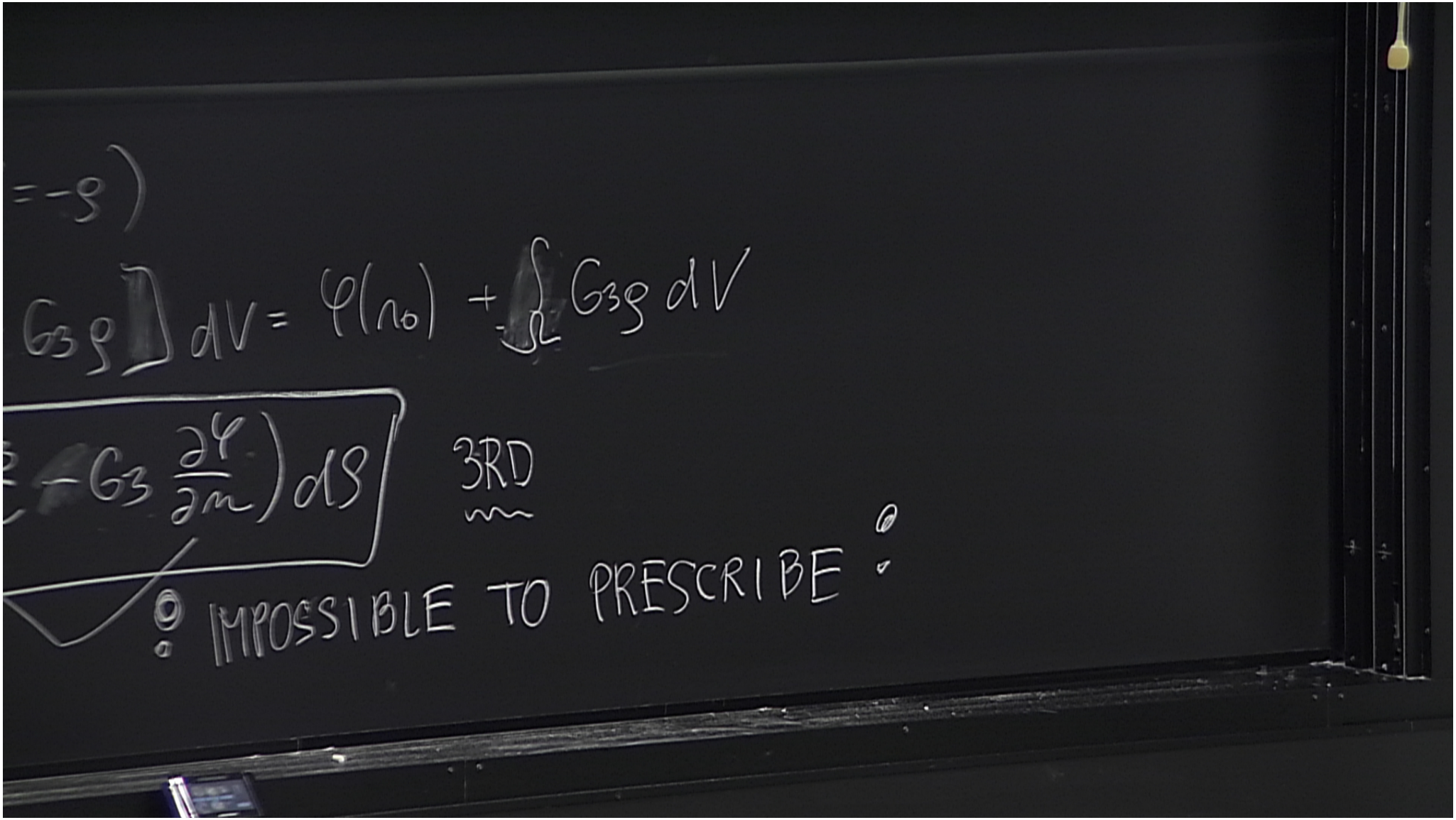
$$= \underset{(1)}{\int_\Omega (\nabla\phi \cdot \nabla\chi + \phi \nabla^2\chi)\, dV = \int_{\partial\Omega} \phi \frac{\partial\chi}{\partial n}\, dS}$$ 1ST G

$\phi \leftrightarrow \chi$ ... SUBSTRACT (1) AND (2)

$$\int_{\partial\Omega} \left(\phi \frac{\partial\chi}{\partial n} - \chi \frac{\partial\phi}{\partial n}\right) dS = \int_\Omega (\phi \Delta\chi - \chi \Delta\phi)$$


• APPLY THIS FOR $\phi = \varphi, \chi = G_3$ $(\Delta\varphi = -\delta)$
$\int_{\partial\Omega} \left(\varphi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \varphi}{\partial n}\right) dS = \int_{\Omega}$


FOR $\phi = \varphi, \chi = G_3$ $(\Delta\varphi = -\rho)$
$\left(\frac{\partial G_3}{\partial n} - G_3 \frac{\partial \varphi}{\partial n}\right) dS = \int_\Omega \left[\varphi\, \delta(r - r_0) + G_3 \rho\right] dV = \varphi(r_0) + \int_\Omega G_3 \rho\, dV$
$\varphi(r_0) = -\int_\Omega G_3 \rho\, dV + \int_{\partial\Omega} \left(\varphi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \varphi}{\partial n}\right) dS$


=-g)
G3g] dV = φ(n0) + ∫ G3g dV
-G3 ∂φ/∂n) dS
3RD
IMPOSSIBLE TO PRESCRIBE!

ELEMENTARY RECIPE: WHEN $x \neq 3$. G SOLVES

c) DIRECHLET GREEN'S FUNCTION ON Ω

1) $\Delta G(x, x_0) = 0$ EXCEPT x_0

2) $G(x, x_0)|_{\partial\Omega} = 0$

ELEMENTARY RECIPE: WHEN $x \neq 3$. G SOLVES

c) DIRECHLET GREEN'S FUNCTION ON Ω

1) $\Delta G(x, x_0) = 0$ EXCEPT x_0

2) $G(x, x_0)|_{\partial\Omega} = 0$

ELEMENTARY RECIPE: WHEN $\chi \neq 3$. G SOLVES

c) DIRECHLET GREEN'S FUNCTION ON Ω

1) $\Delta G(x, x_0) = 0$ EXCEPT x_0

2) $G(x, x_0)|_{\partial\Omega} = 0$

3) $G = G_3 + H$, H ... IS FINITE AT x_0 & $\Delta H = 0$ ON Ω

$1) + 3) \Rightarrow \Delta G = \delta(x - x_0)$ ON Ω

- 2ND → 3RD $\underline{X = G}$

$$\Rightarrow \boxed{\varphi(x_0) = -\int_\Omega G g + \int_{\partial\Omega} \varphi \frac{\partial G}{\partial n} dS}$$

SOLUTION TO (DP)

AT x_0

ON $\partial\Omega$

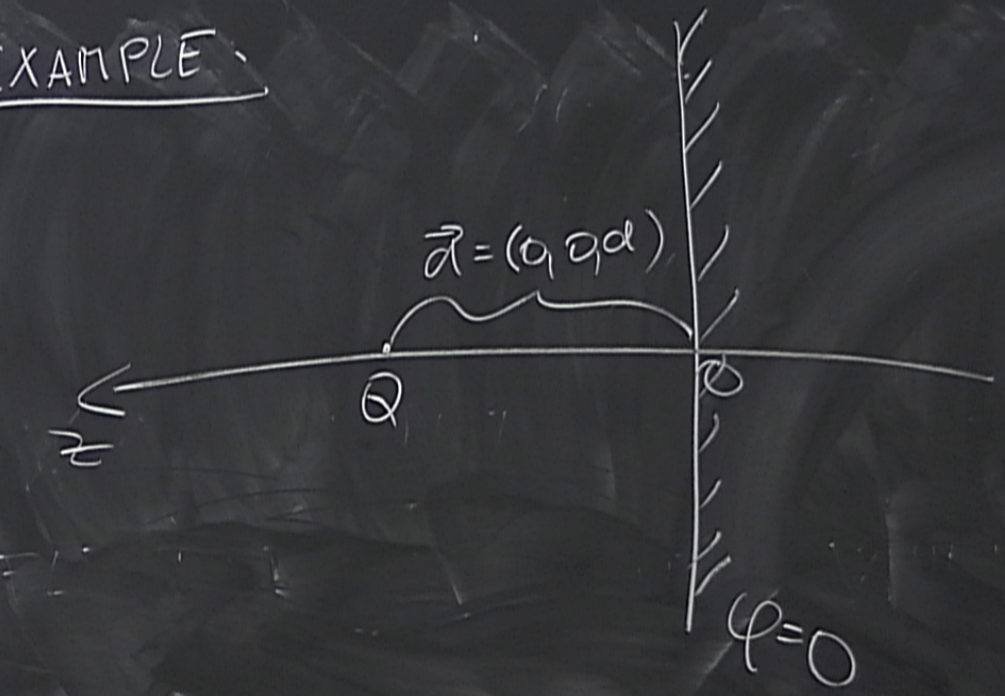
$f(\xi) f(\xi) d\xi + y_2(x) \int_x^{} B(\xi) f(\xi) d\xi +$

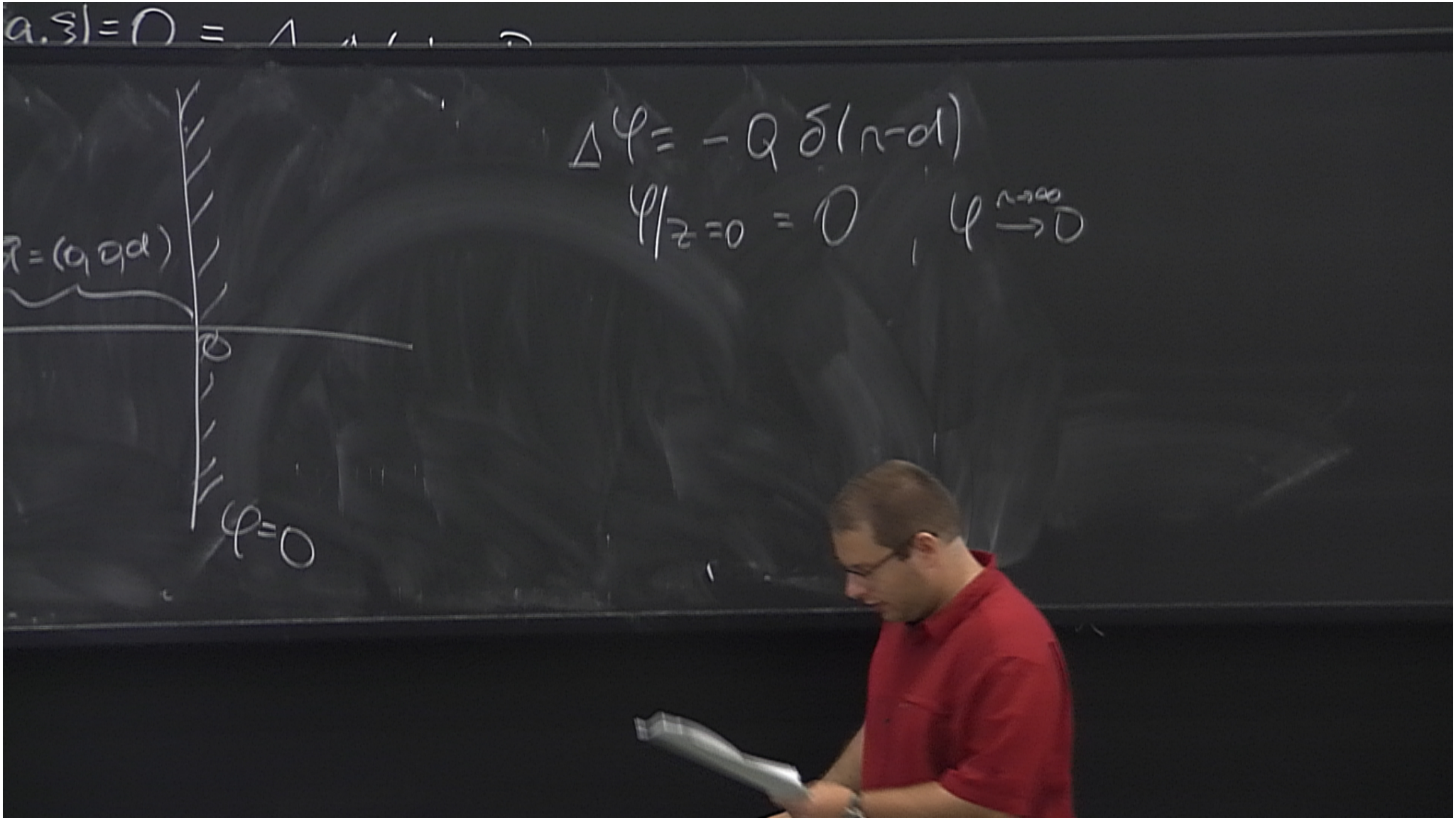
3) $G(a,\xi) = 0 = A\, y_1(a) + B\, y_2(a) = 0$ ...

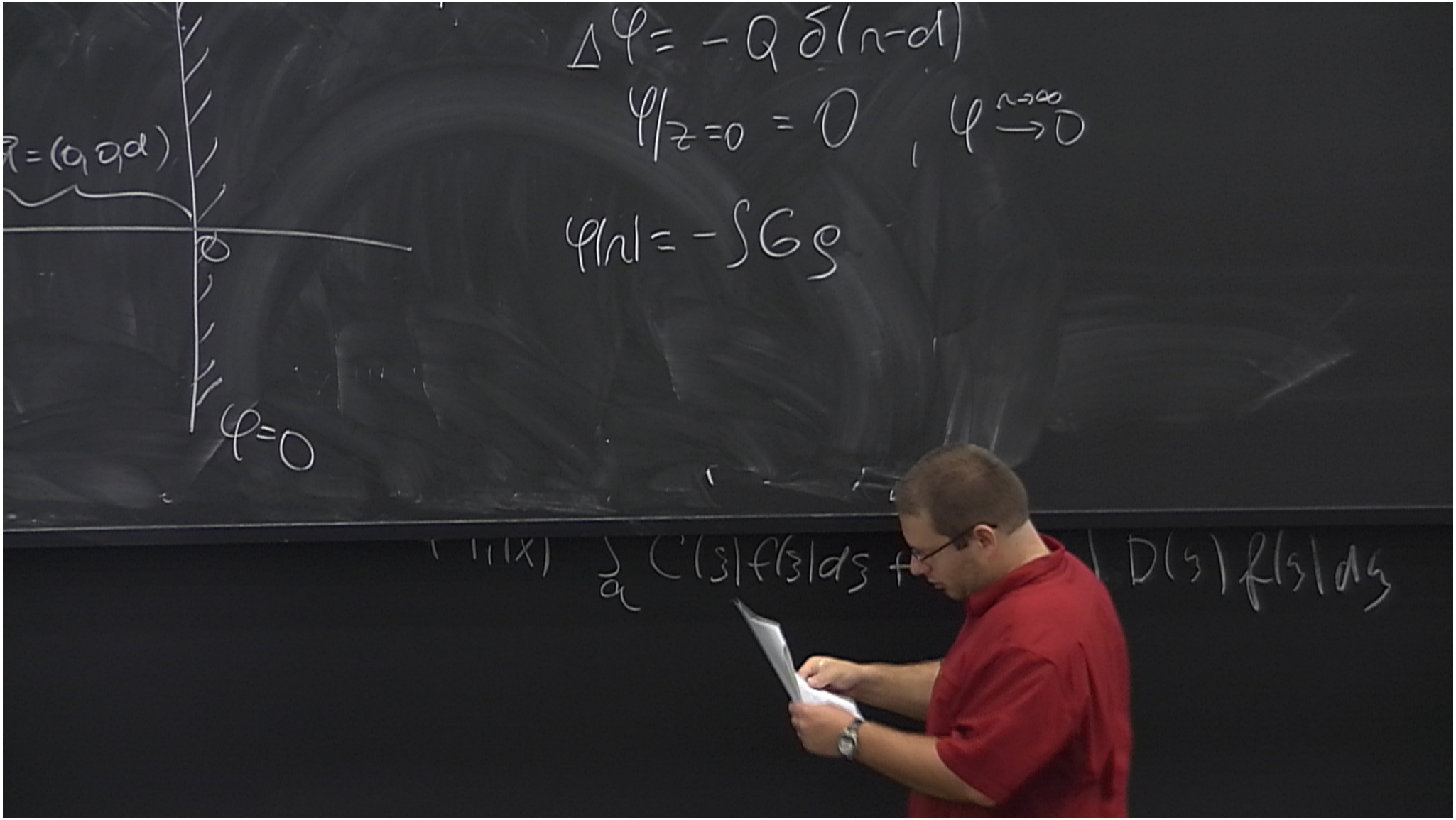
4) $G(b,\xi) = 0 = C$ ...

EXAMPLE:

$\Delta\varphi = -Q\,\delta($

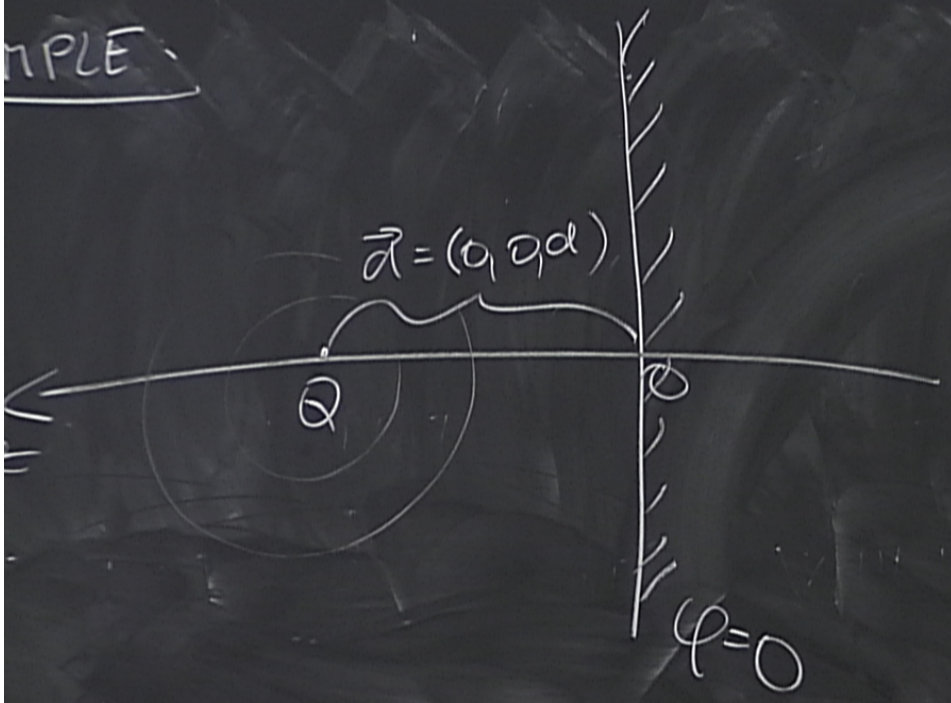



$\Delta\varphi = -Q\,\delta(r-a)$
$\varphi|_{z=0} = 0$, $\varphi \xrightarrow{r\to\infty} 0$
$\vec{r} = (0,0,a)$
$\varphi = 0$


$\Delta\varphi = -Q\,\delta(r-d)$
$\varphi|_{z=0} = 0$, $\varphi \overset{r\to\infty}{\to} 0$
$\varphi(r) = -\int G\,\varrho$
$R = (0,0,d)$
$\varphi = 0$

2) $x > 3$: $G(x,\xi) = C(\xi)\,\psi_1(x) + D(\xi)\,\psi_2(x)$

MPLE:



$$\Delta\varphi = -Q\,\delta(r-a)$$

$$\varphi|_{z=0} = 0, \quad \varphi \xrightarrow{r\to\infty} 0$$

$$\varphi(r) = -\int G\rho = QG(a,r)$$

NEED DIR. GREENS FUNCTION

$$\psi_1(x) \int_a^{\infty} C(\xi) f(\xi)\,d\xi + \psi_2(x) \int_{\vec{a}} D($$

$\varphi|_{z=0} = 0 \quad , \quad \varphi \xrightarrow{r\to\infty} 0$

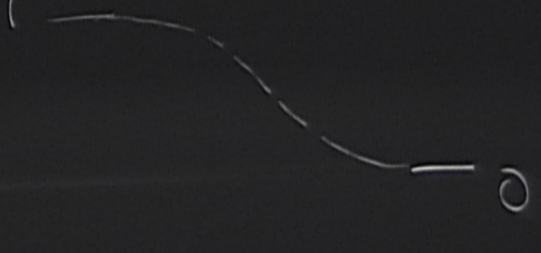
$\varphi(r) = -\int G g = \underline{Q G(d, r)}$

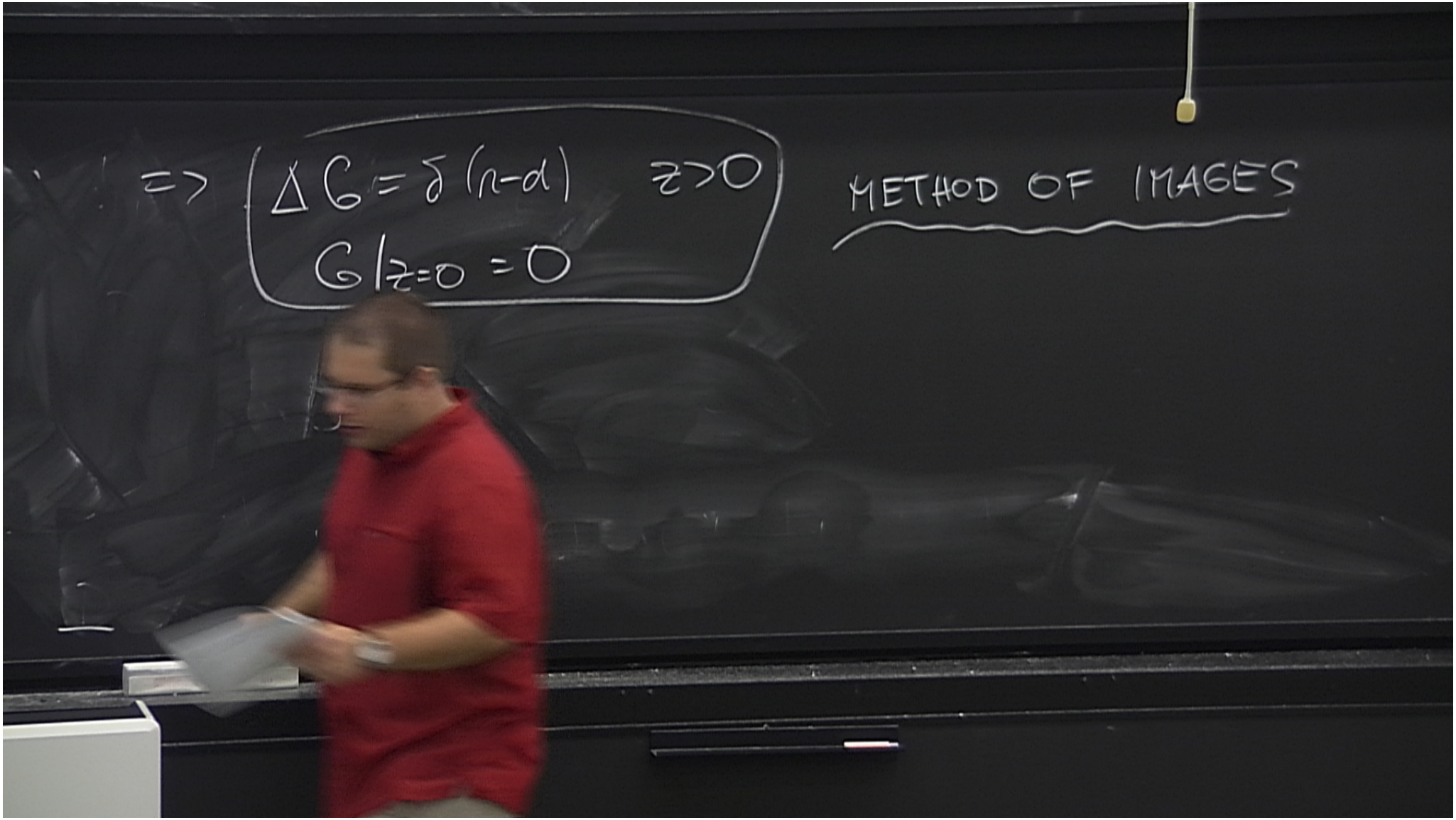
ED DIR. GREENS FUNCTION

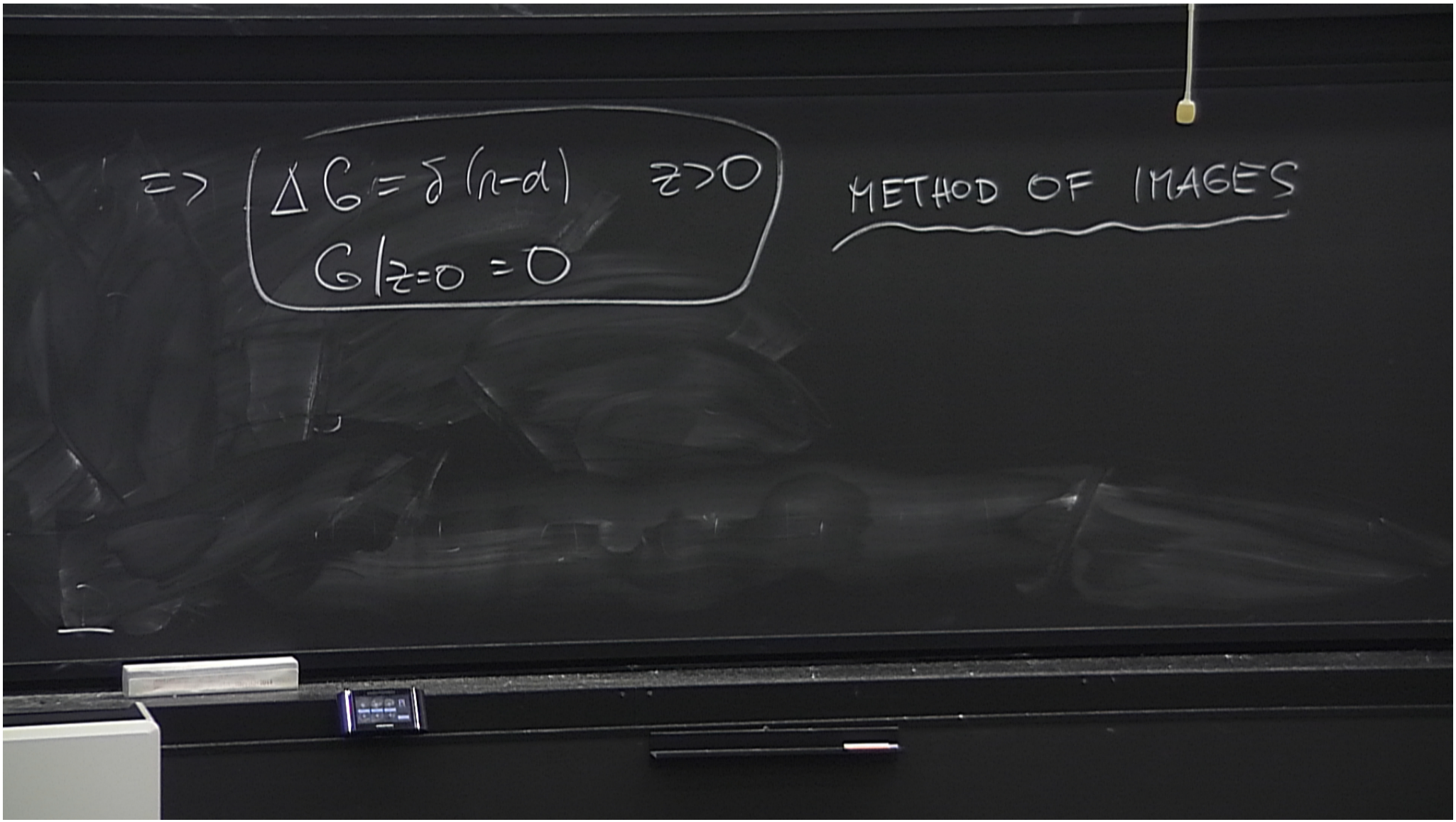
$$G = -\frac{1}{4\pi}\frac{1}{|r-d|} + \frac{1}{4\pi}\frac{1}{|r+d|} = -\frac{1}{4\pi}\left(\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right.$$

$$\int_x A(s) f(s)\,ds + y_2(x) \int_x B(s) f(s)\,ds +$$

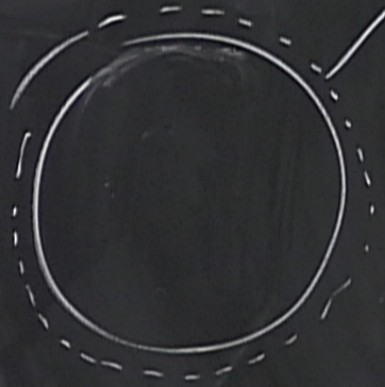
$$\int_a^x C(\quad)\,ds + y_2(x) \int_a^x D(s) f(s)\,ds$$



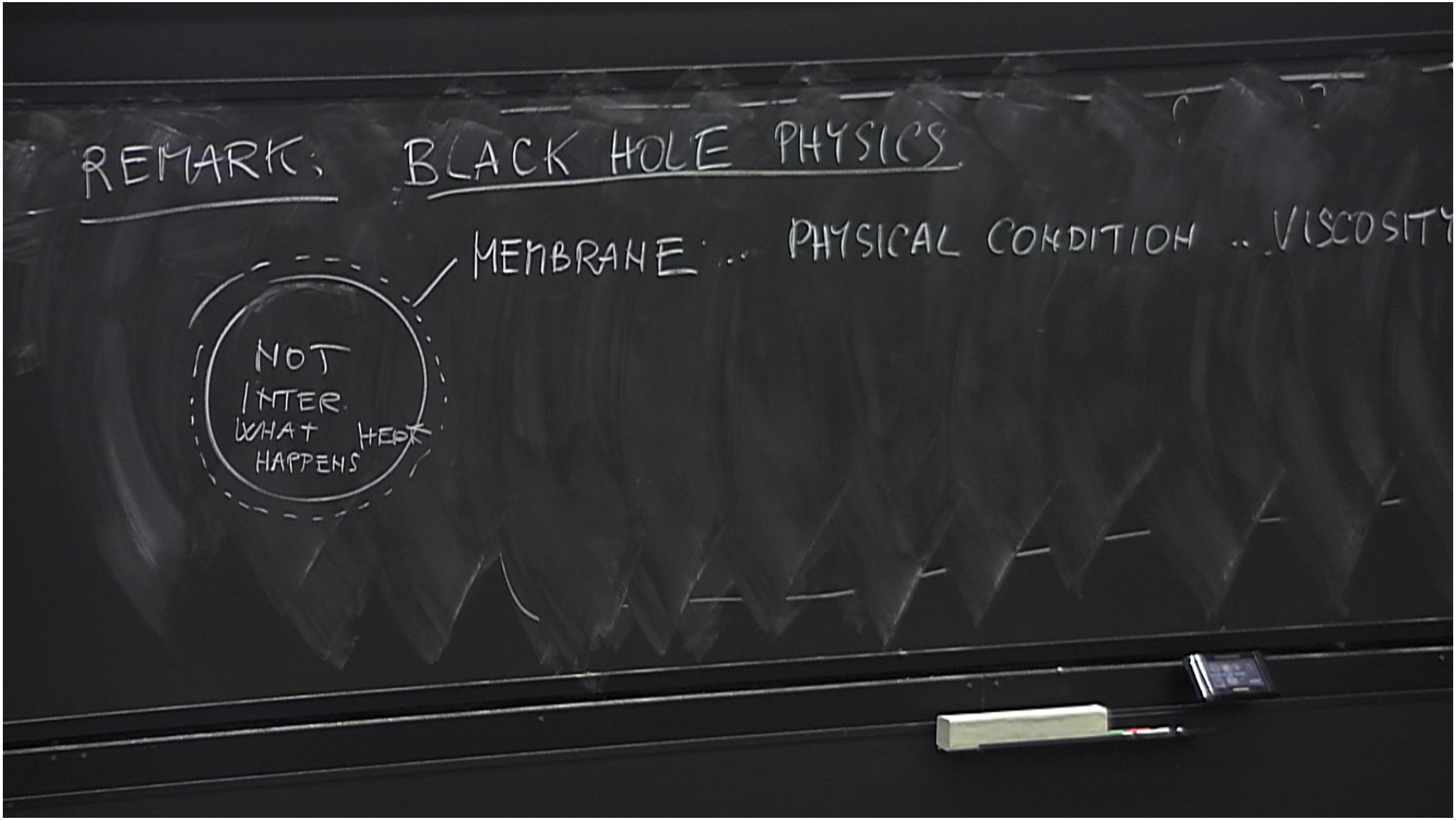

=>
$\Delta G = \delta(r-d)$ $z>0$
$G|_{z=0} = 0$
METHOD OF IMAGES

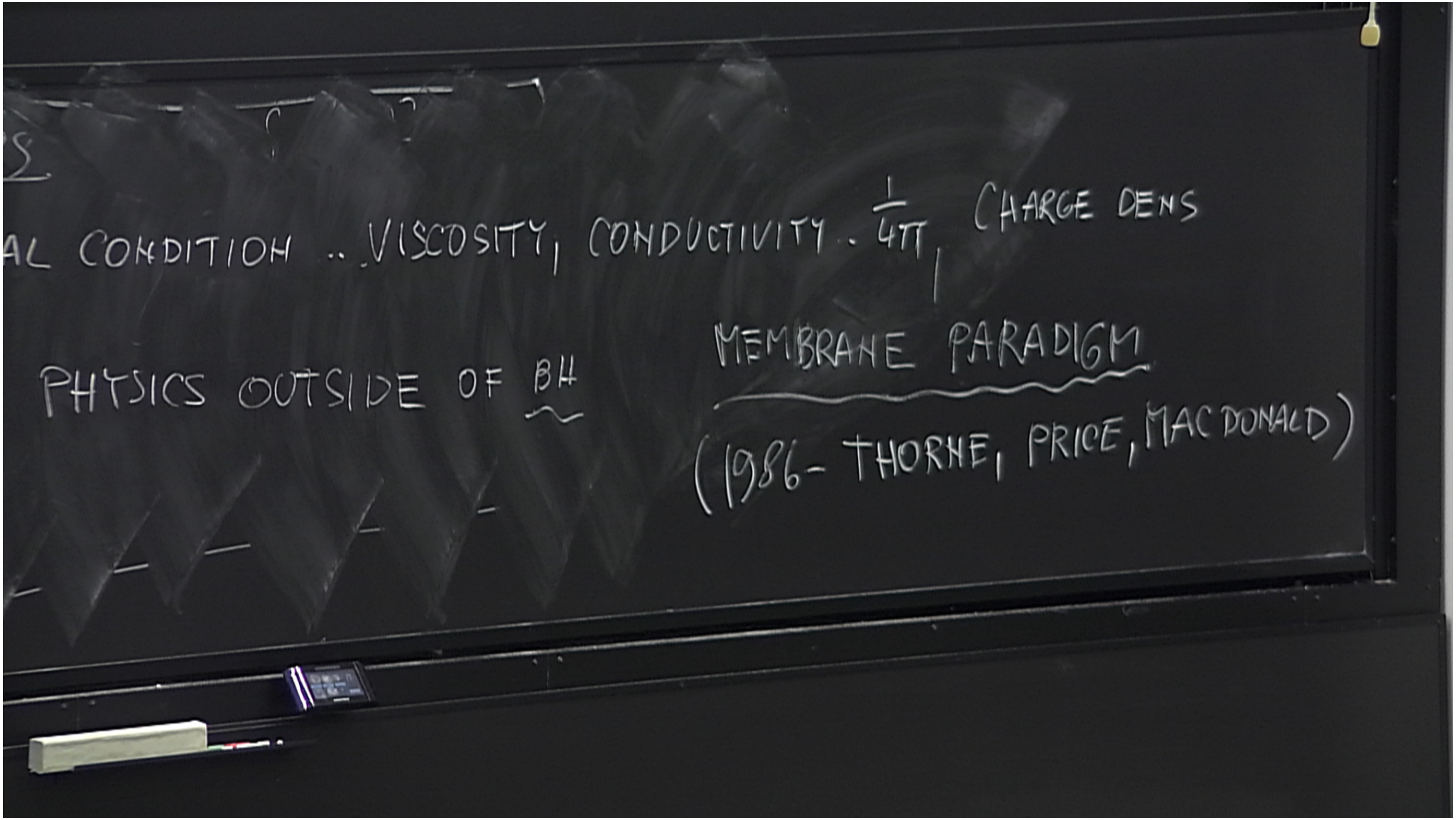

=>
$\Delta G = \delta(r-d)$ $z>0$
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METHOD OF IMAGES

REMARK: BLACK HOLE PHYSICS



MEMBRANE ... PHYSICAL CONDITION .. VISCO


REMARK: BLACK HOLE PHYSICS
MEMBRANE .. PHYSICAL CONDITION .. VISCOSITY
NOT
INTER
WHAT
HAPPENS
HERE


AL CONDITION .. VISCOSITY, CONDUCTIVITY . $\frac{1}{4\pi}$, CHARGE DENS
PHYSICS OUTSIDE OF BH
MEMBRANE PARADIGM
(1986 - THORNE, PRICE, MACDONALD)