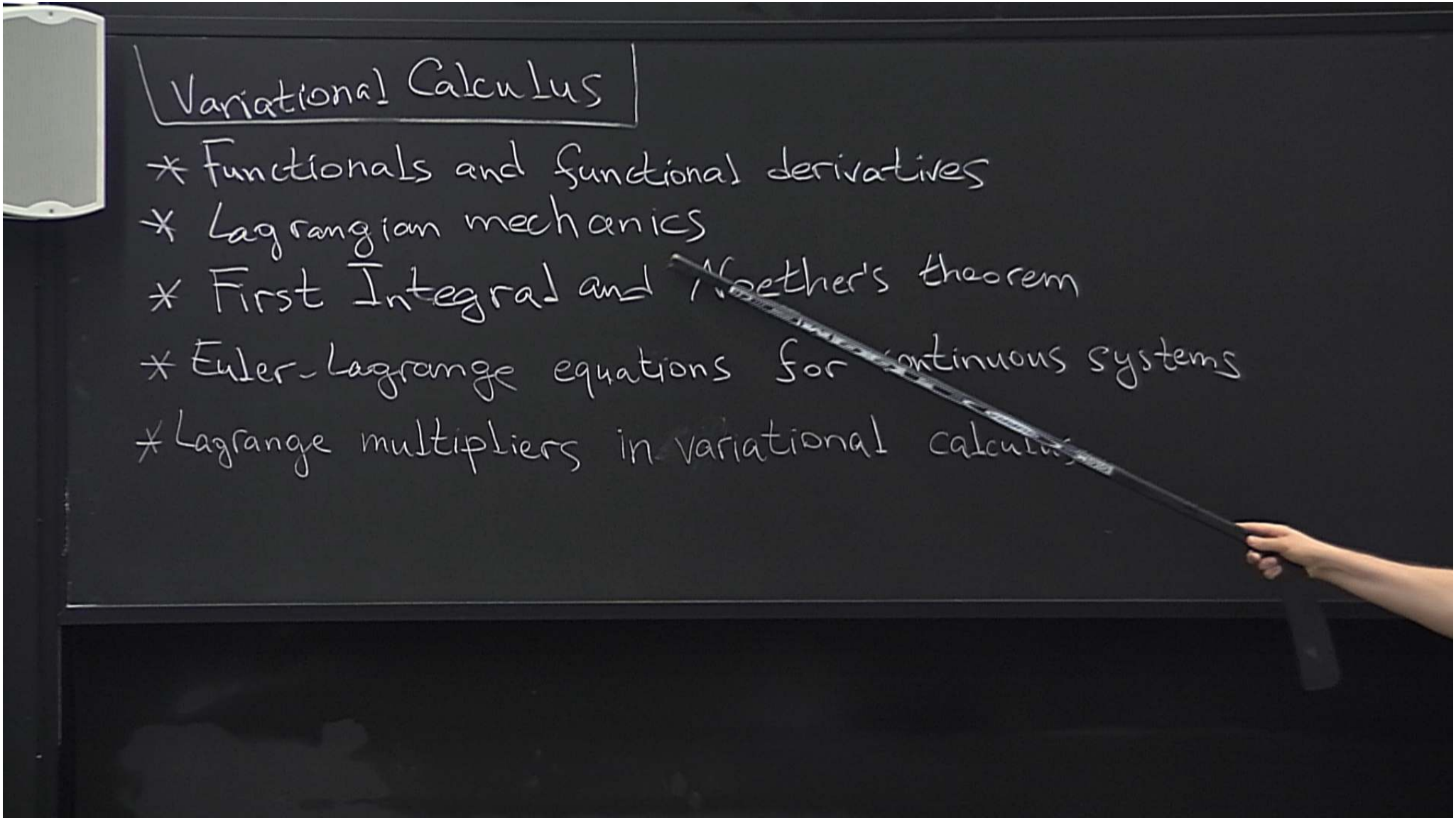
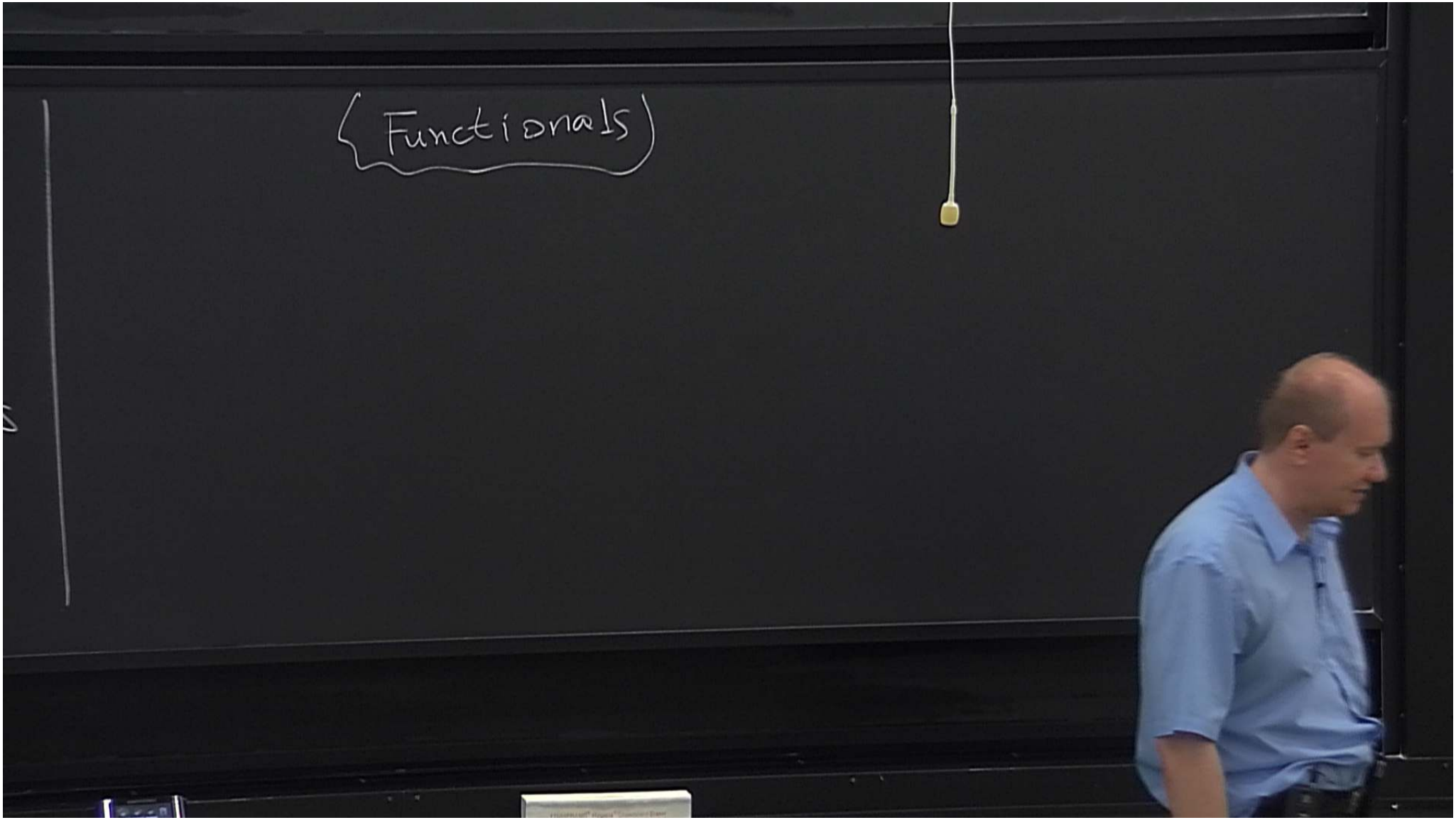
Title: 13/14 PSI - Variational Calculus & Guassian Integrals - Lecture 3

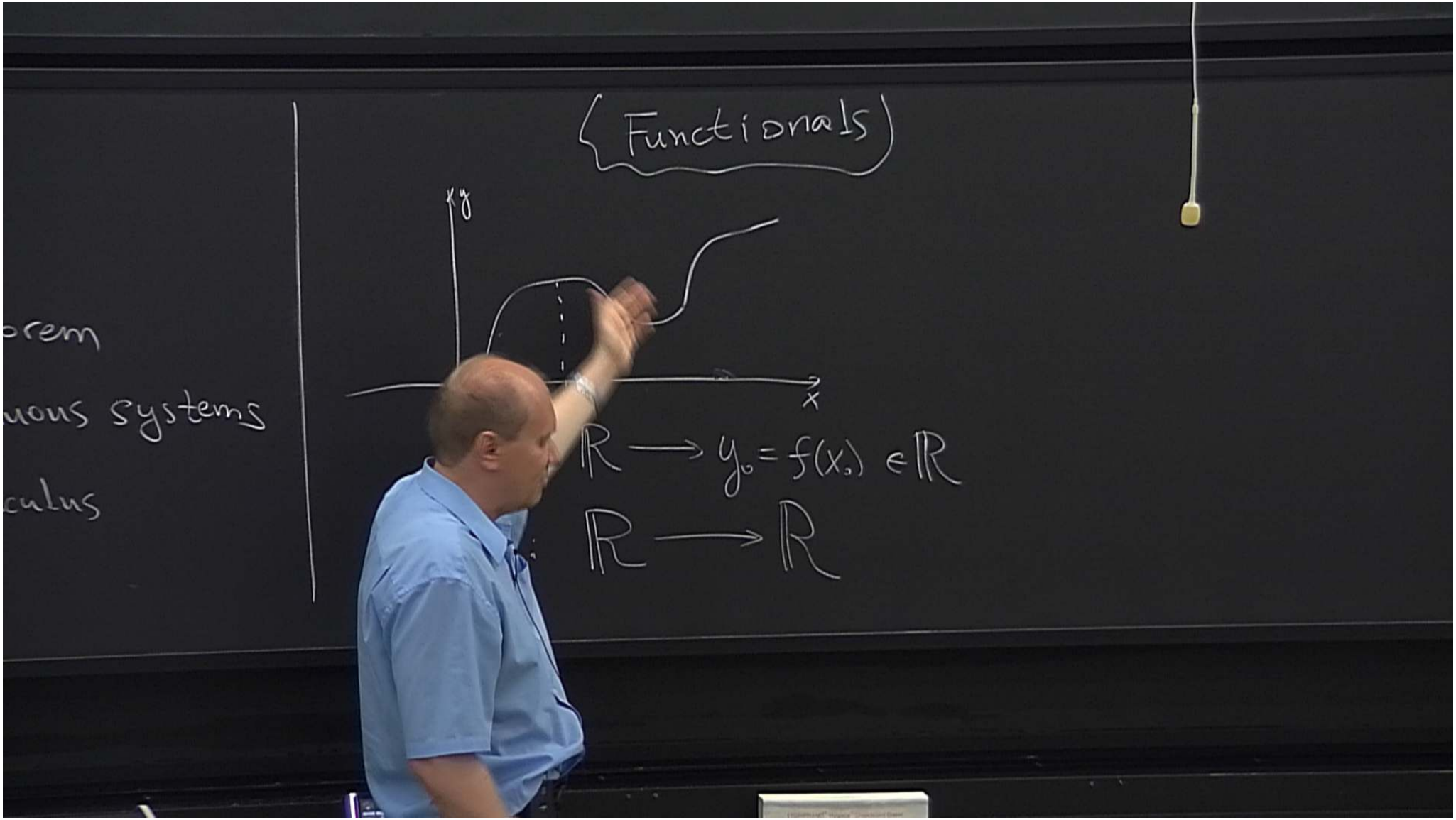
Date: Aug 23, 2013 10:30 AM

URL: http://pirsa.org/13080024

Abstract:


Variational Calculus
* Functionals and functional derivatives
* Lagrangian mechanics
* First Integral and Noether's theorem
* Euler-Lagrange equations for continuous systems
* Lagrange multipliers in variational calculus


Functionals


Functionals
y
x
$\mathbb{R} \longrightarrow y_0 = f(x_0) \in \mathbb{R}$
$\mathbb{R} \longrightarrow \mathbb{R}$
orem
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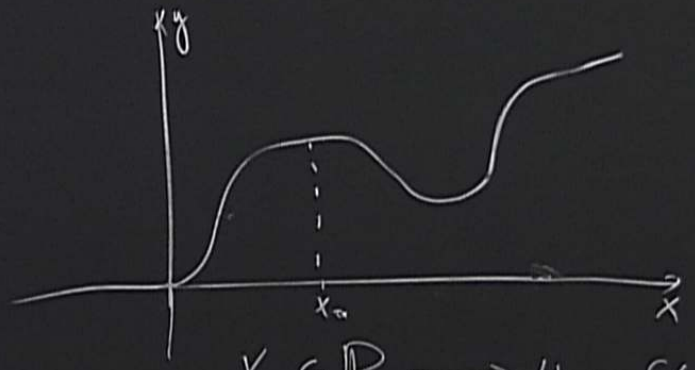
orem

uous systems

culus

Functionals



$x_0 \in \mathbb{R} \longrightarrow y_0 = f(x_0) \in \mathbb{R}$

$f: \mathbb{R} \longrightarrow \mathbb{R}$

orem

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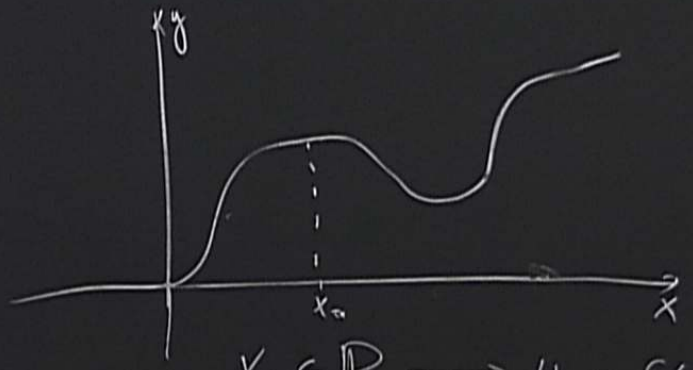
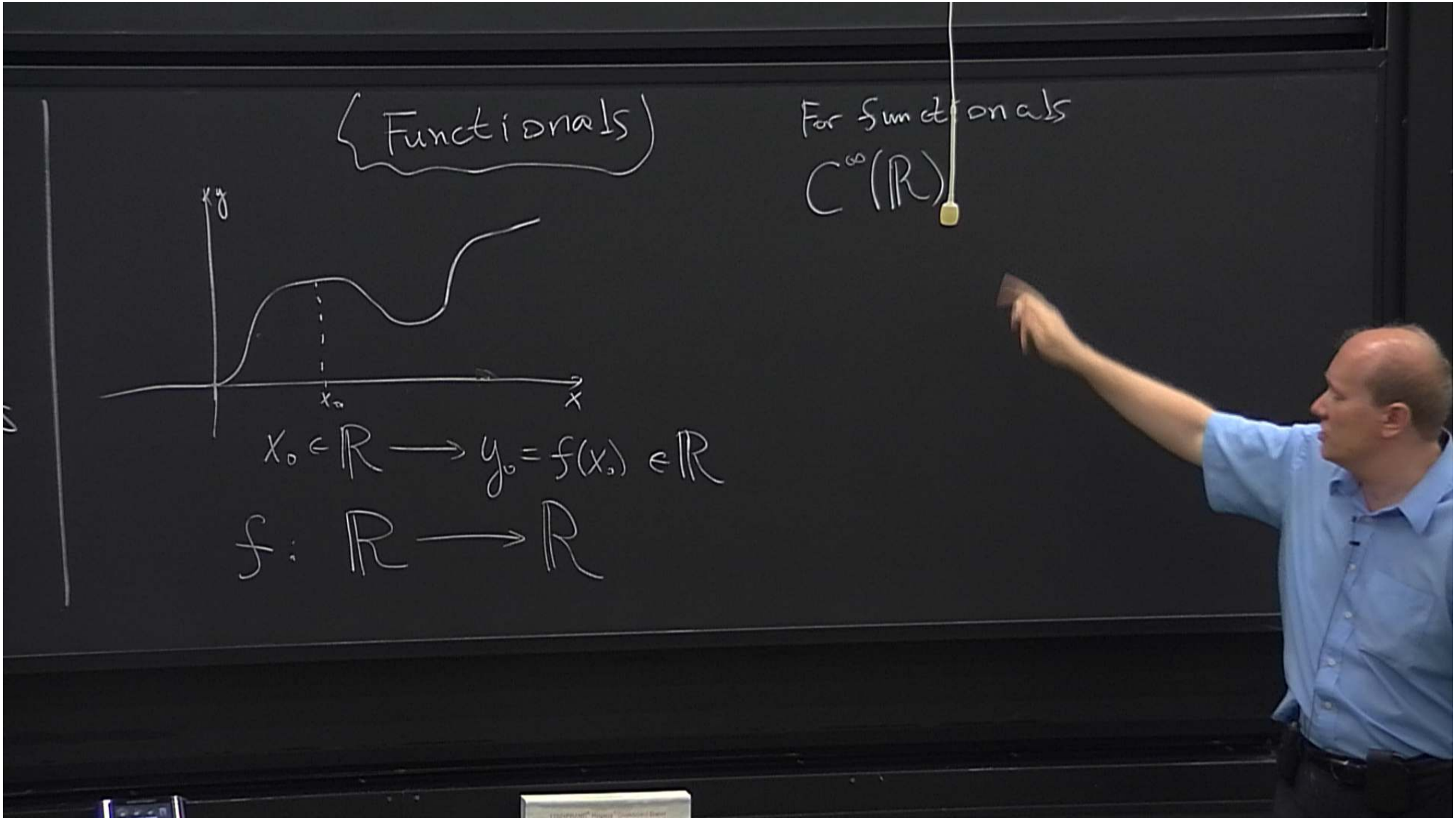
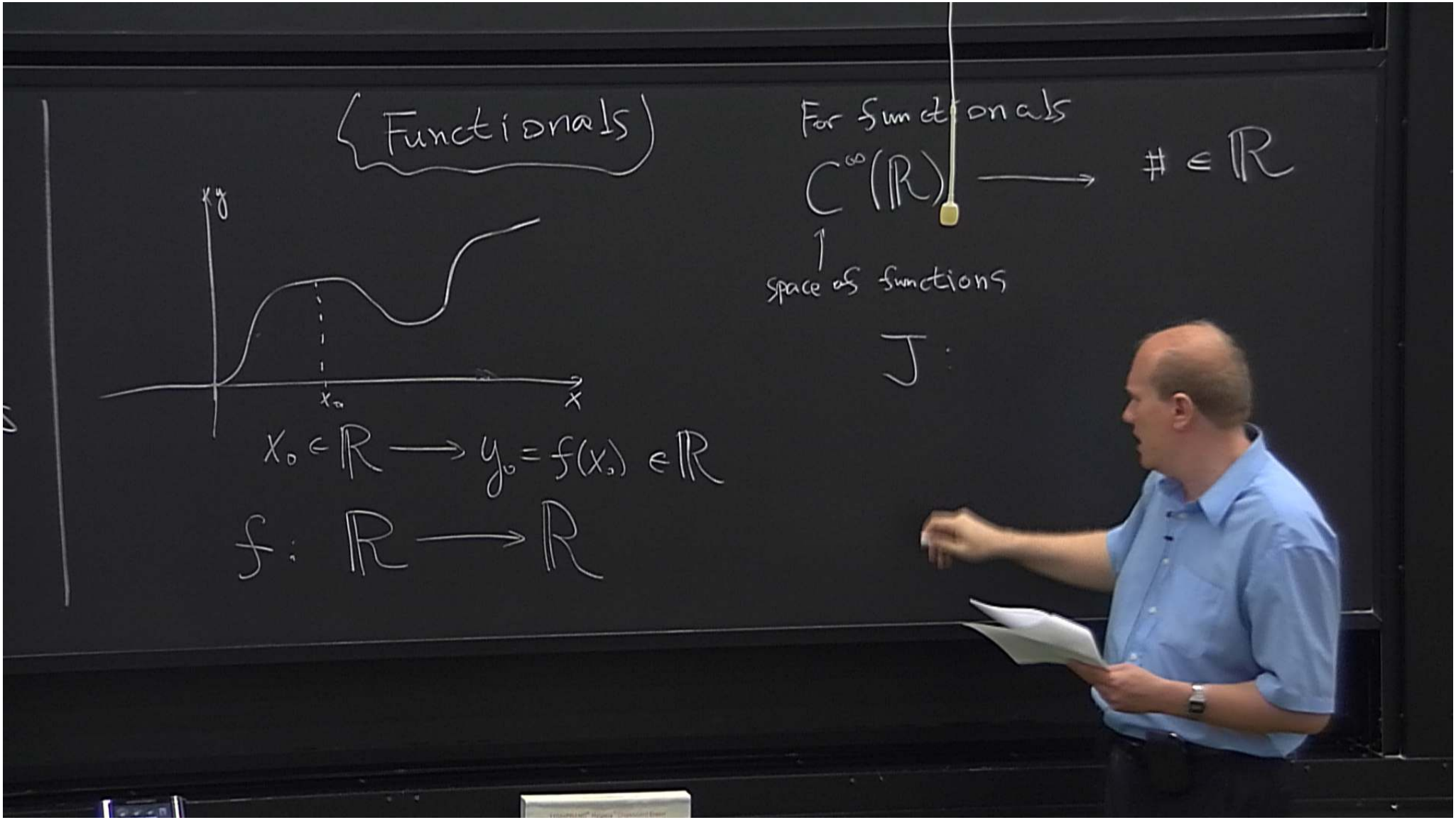
Functionals



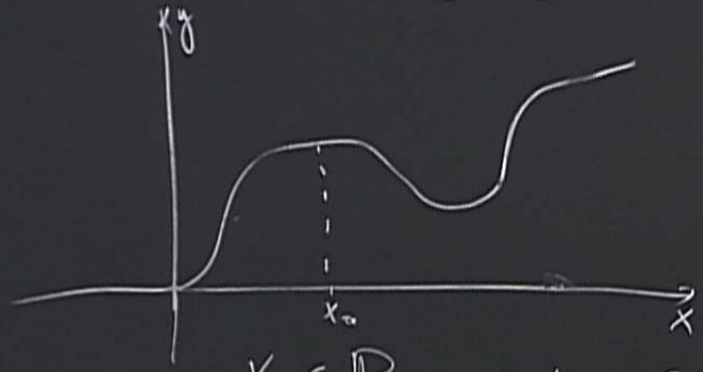
$x_0 \in \mathbb{R} \longrightarrow y_0 = f(x_0) \in \mathbb{R}$

$f : \mathbb{R} \longrightarrow \mathbb{R}$


Functionals
For functionals
$C^\infty(\mathbb{R})$
y
x_0
x
$x_0 \in \mathbb{R} \longrightarrow y_0 = f(x_0) \in \mathbb{R}$
$f: \mathbb{R} \longrightarrow \mathbb{R}$


Functionals
For functionals
$C^\infty(\mathbb{R})$
$\# \in \mathbb{R}$
space of functions
J :
y
x
x_0
$x_0 \in \mathbb{R} \longrightarrow y_0 = f(x_0) \in \mathbb{R}$
$f : \mathbb{R} \longrightarrow \mathbb{R}$

Functionals



$x_0 \in \mathbb{R} \longrightarrow y_0 = f(x_0) \in \mathbb{R}$

$f : \mathbb{R} \longrightarrow \mathbb{R}$

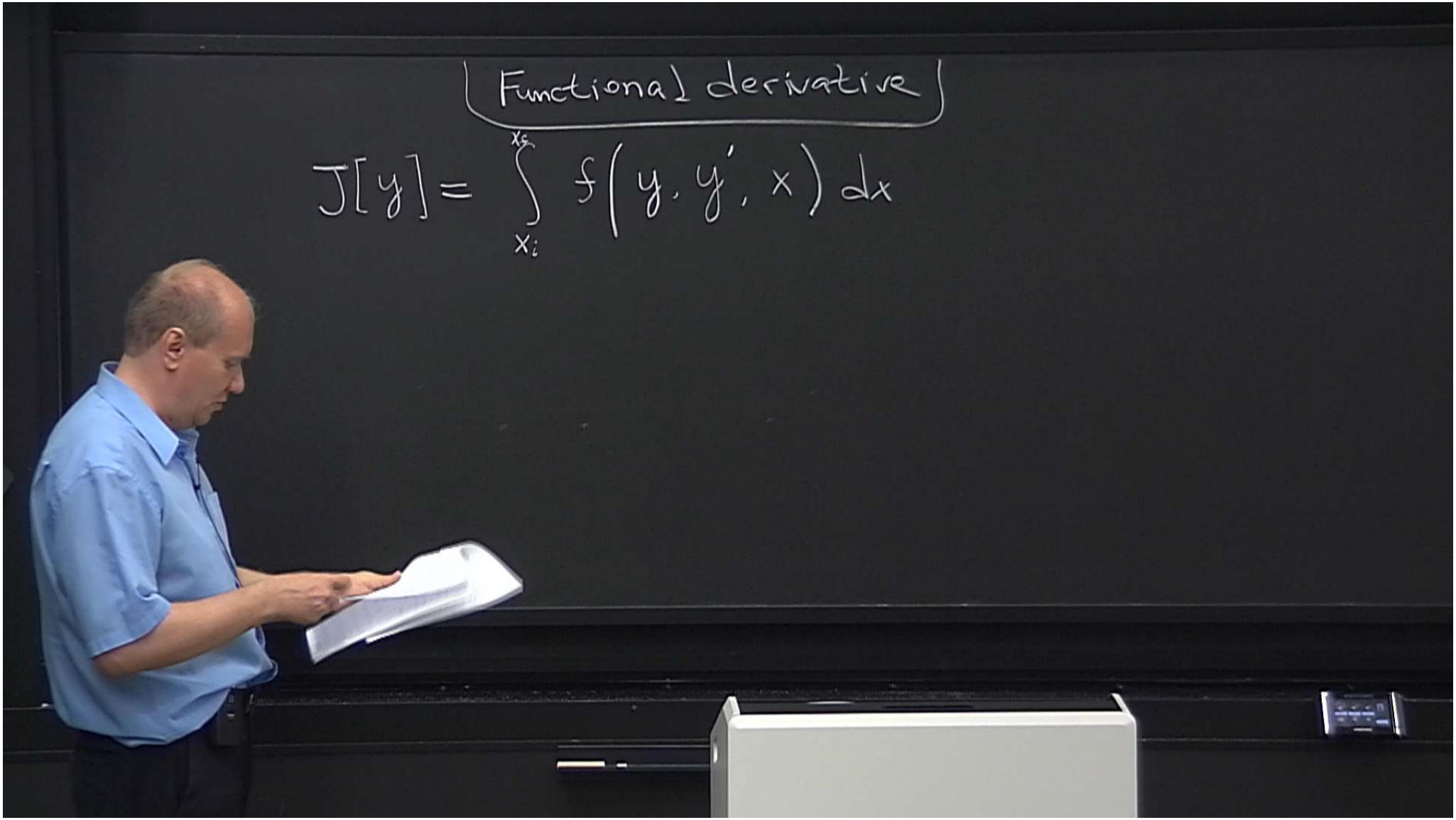
For functionals

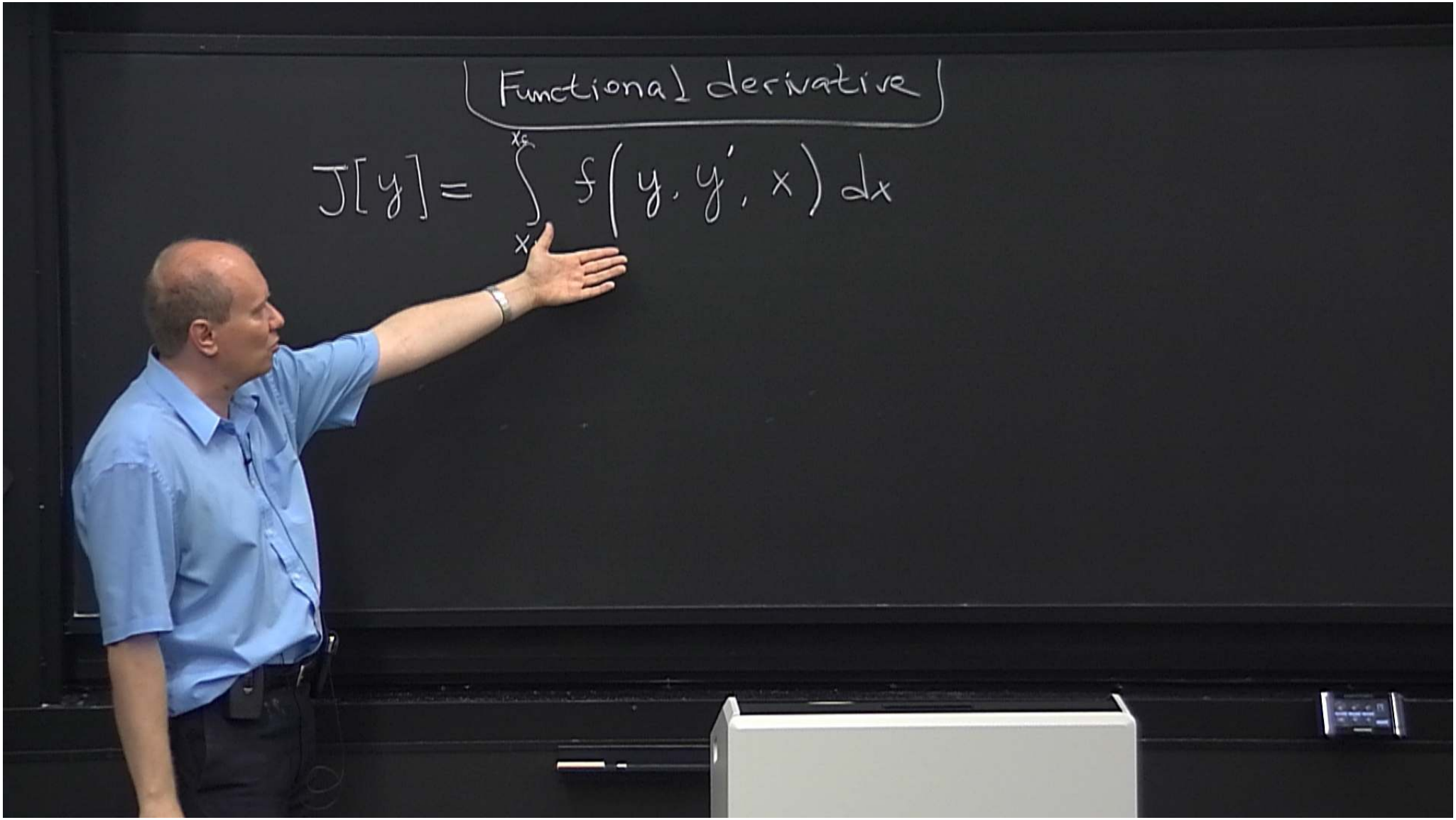
$C^\infty(\mathbb{R}) \longrightarrow \# \in \mathbb{R}$

↑ space of functions

$J : C^\infty(\mathbb{R}) \longrightarrow \mathbb{R}$

↑ functional


Functional derivative
$J[y] = \int_{x_i}^{x_f} f(y, y', x)\, dx$


Functional derivative
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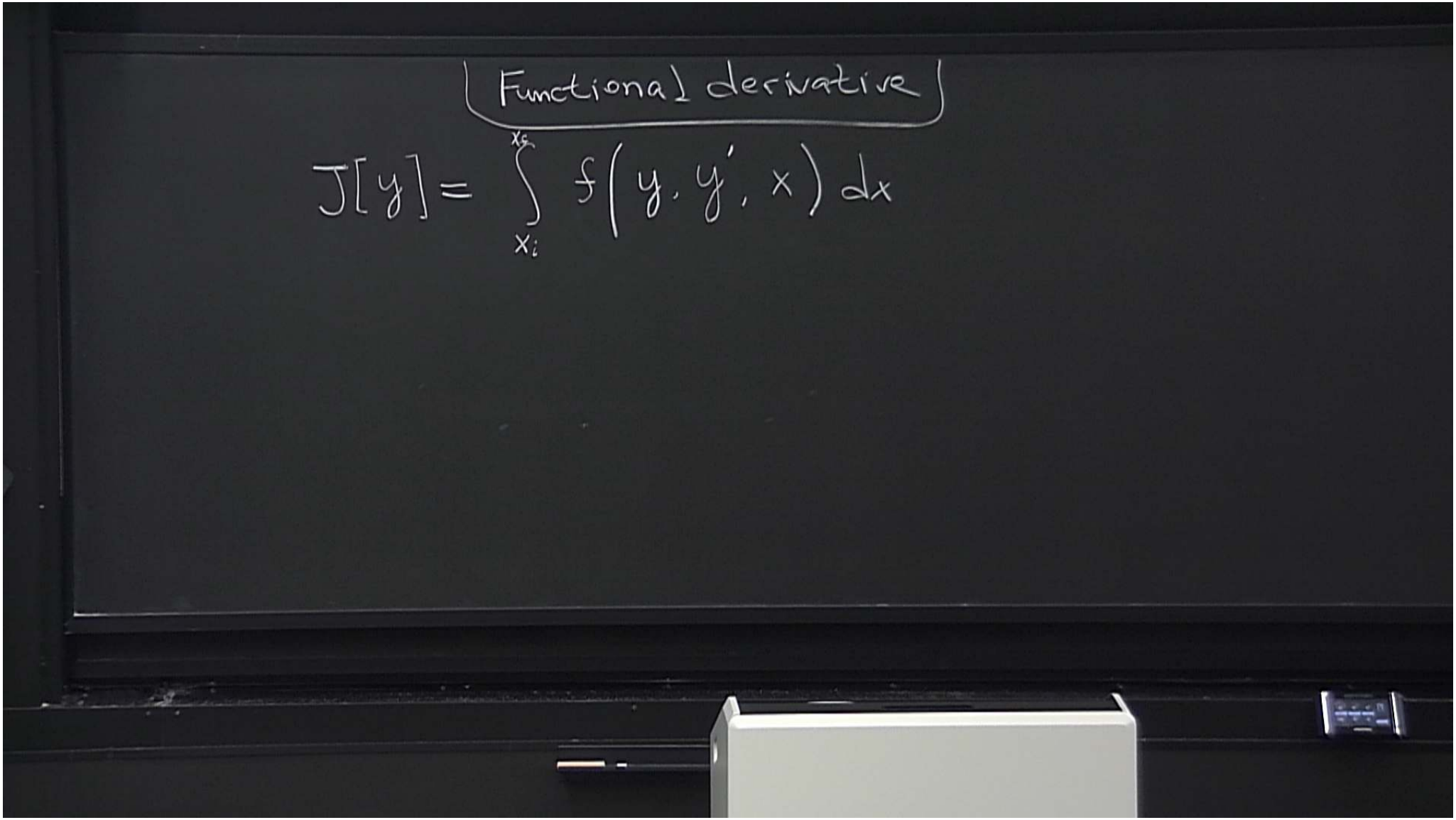
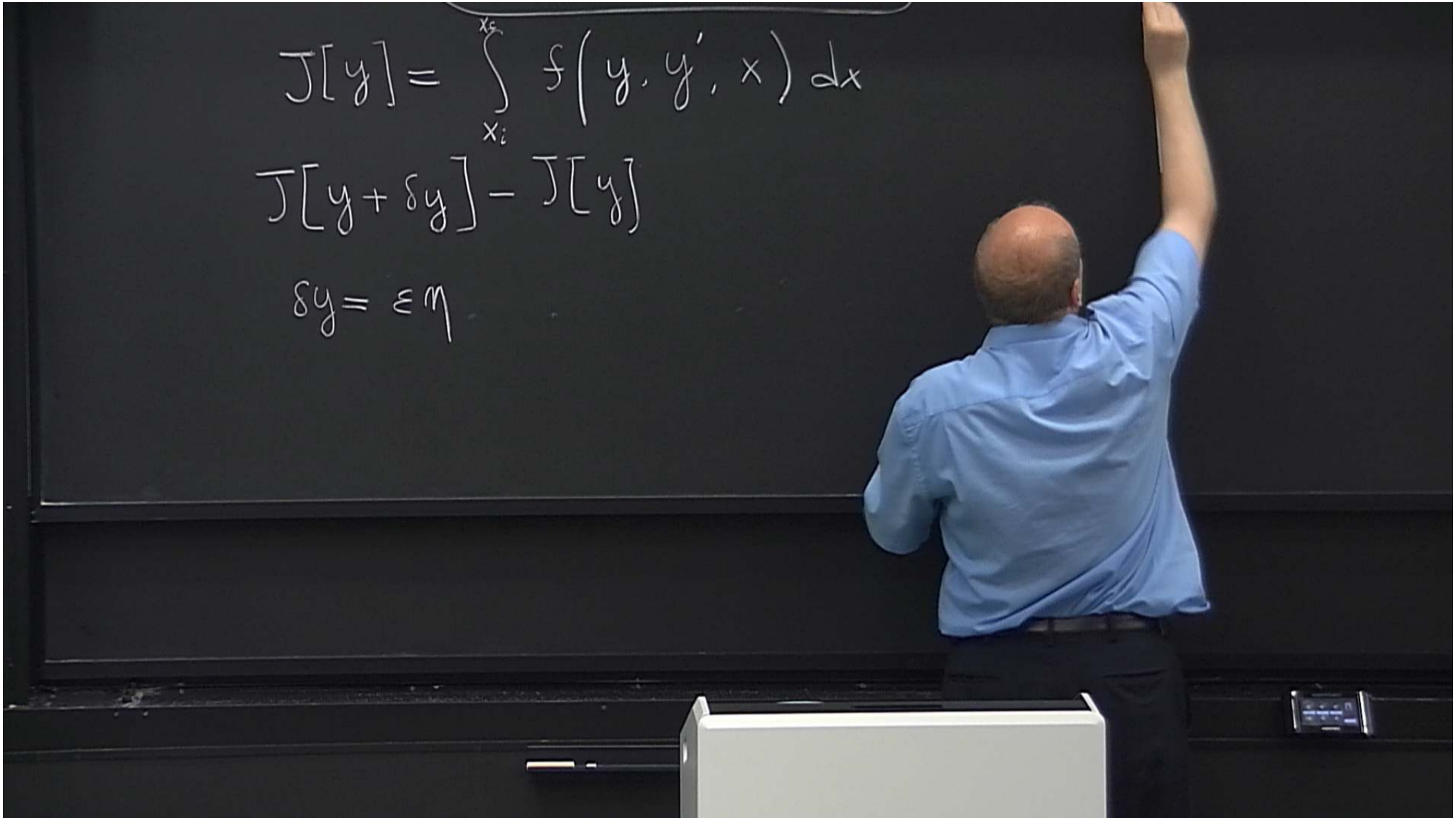
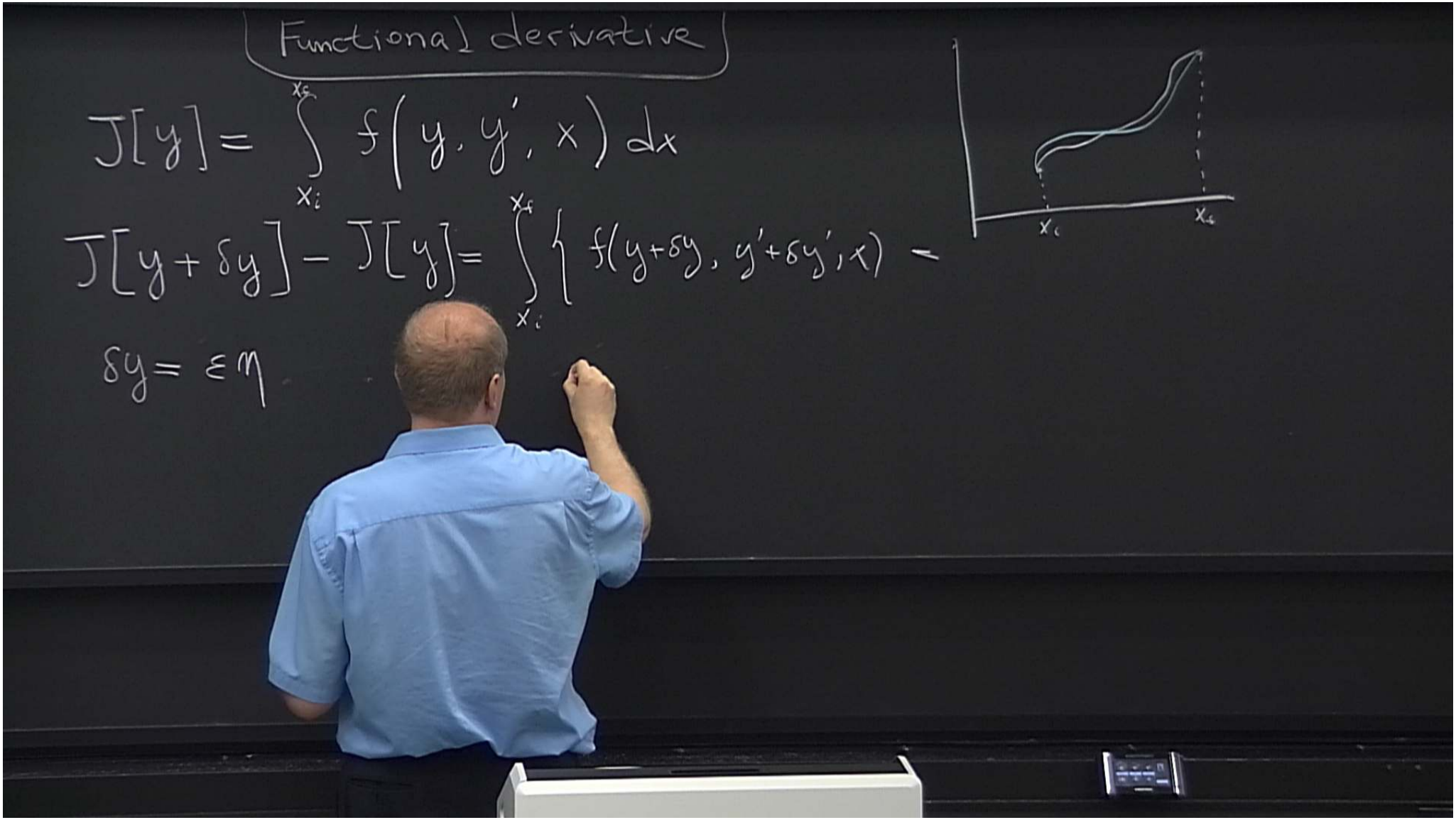
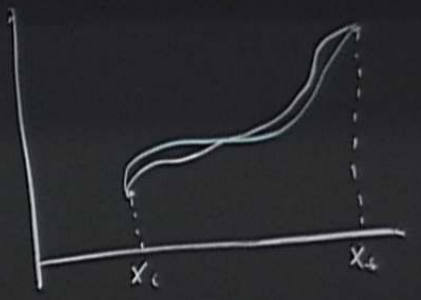

Functional derivative
$J[y] = \int_{x_i}^{x_f} f(y, y', x)\, dx$


$J[y] = \int_{x_i}^{x_f} f(y, y', x)\, dx$
$J[y+\delta y] - J[y]$
$\delta y = \varepsilon \eta$


Functional derivative
$J[y] = \int_{x_i}^{x_f} f(y, y', x)\, dx$
$J[y+\delta y] - J[y] = \int_{x_i}^{x_f} \{ f(y+\delta y, y'+\delta y', x) -$
$\delta y = \varepsilon \eta$
x_i
x_f

Functional derivative

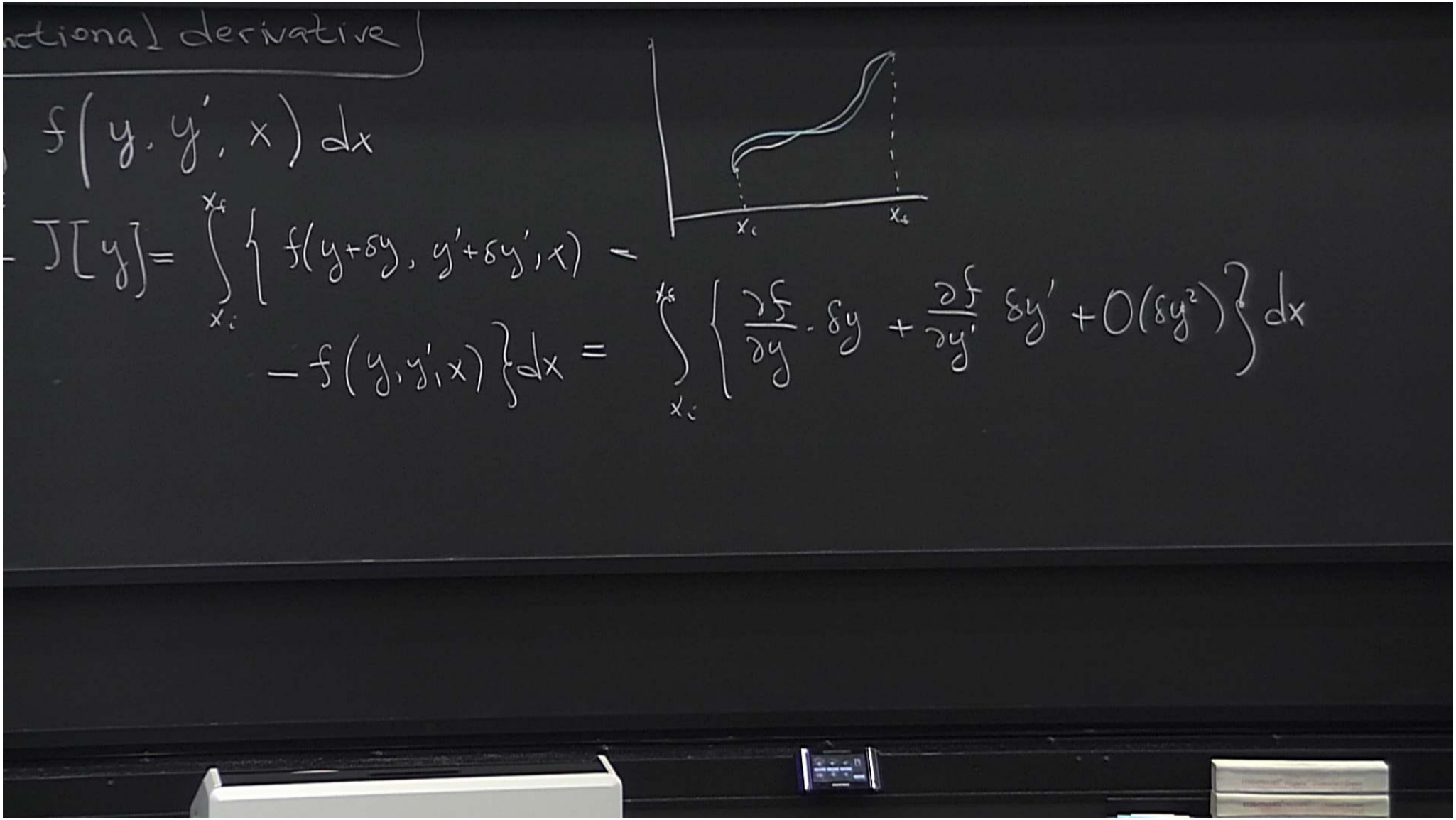
$$J[y] = \int_{x_i}^{x_f} f(y, y', x)\, dx$$

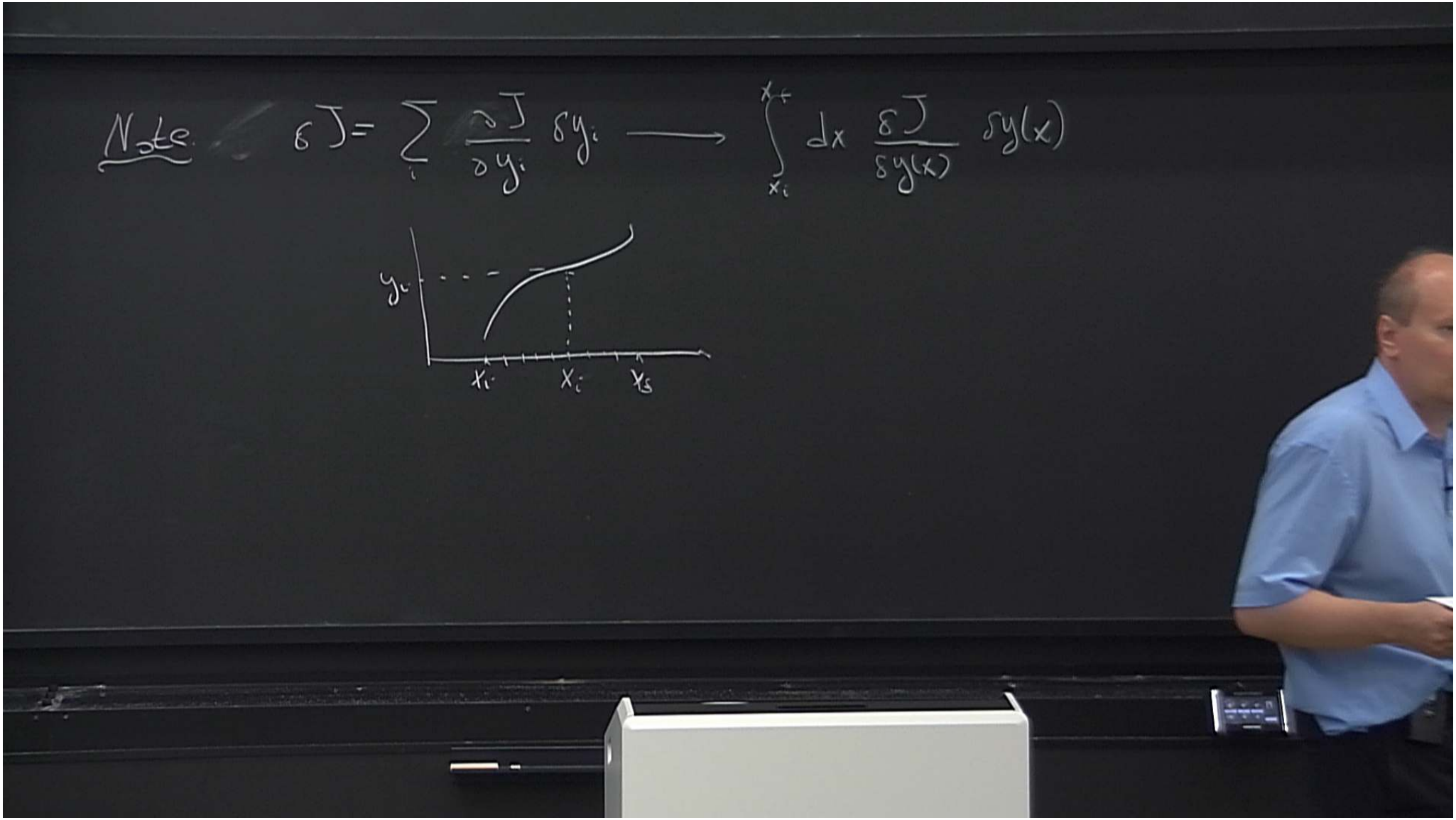


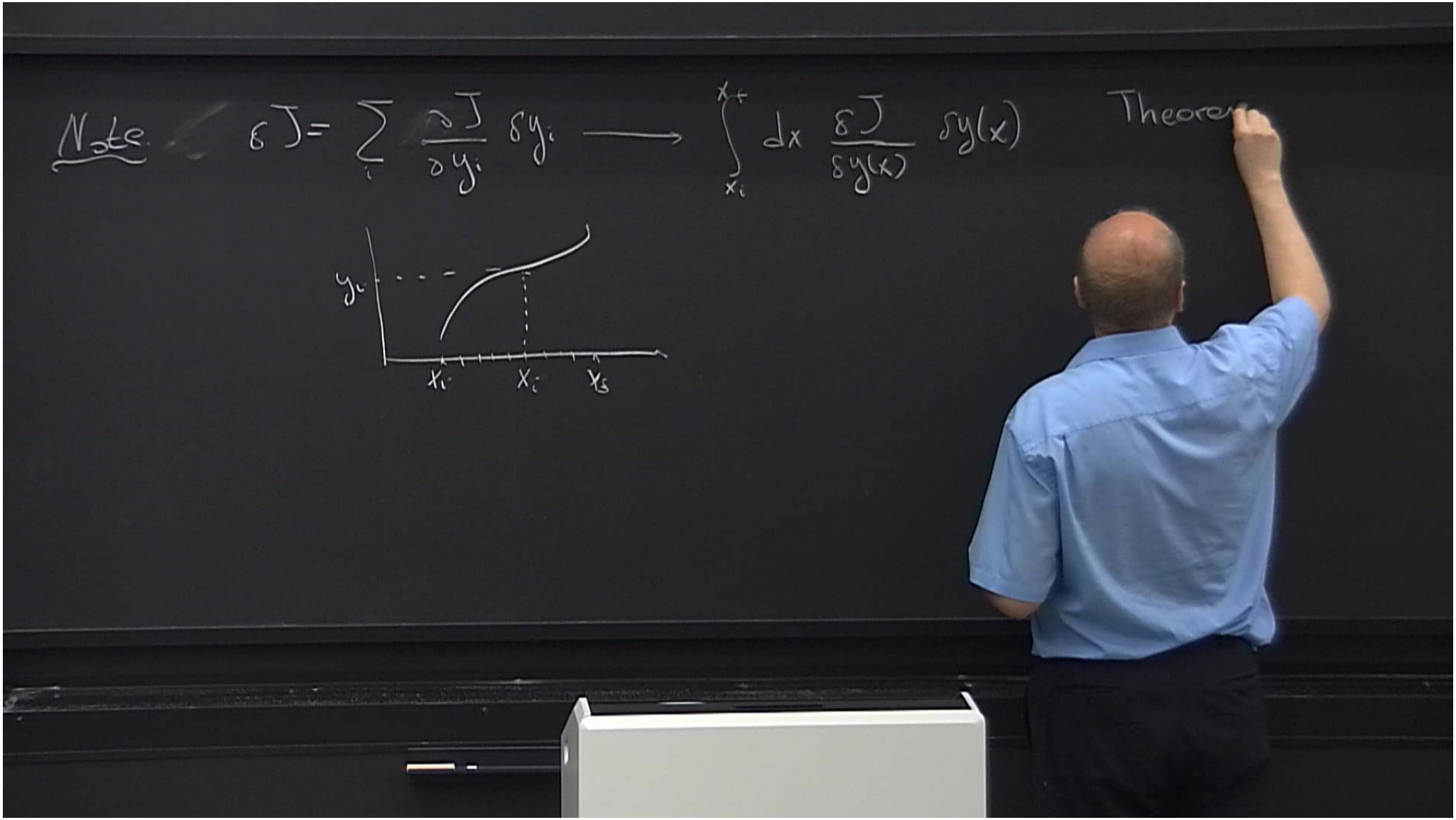
$$J[y+\delta y] - J[y] = \int_{x_i}^{x_f} \{ f(y+\delta y, y'+\delta y', x) - f(y, y', x) \}\, dx =$$

$$= \int_{x_i}^{x_f} \left\{ \frac{\partial f}{\partial y} \cdot \delta y + \frac{\partial f}{\partial y'} \delta y' \right.$$

$$\delta y = \varepsilon \eta$$

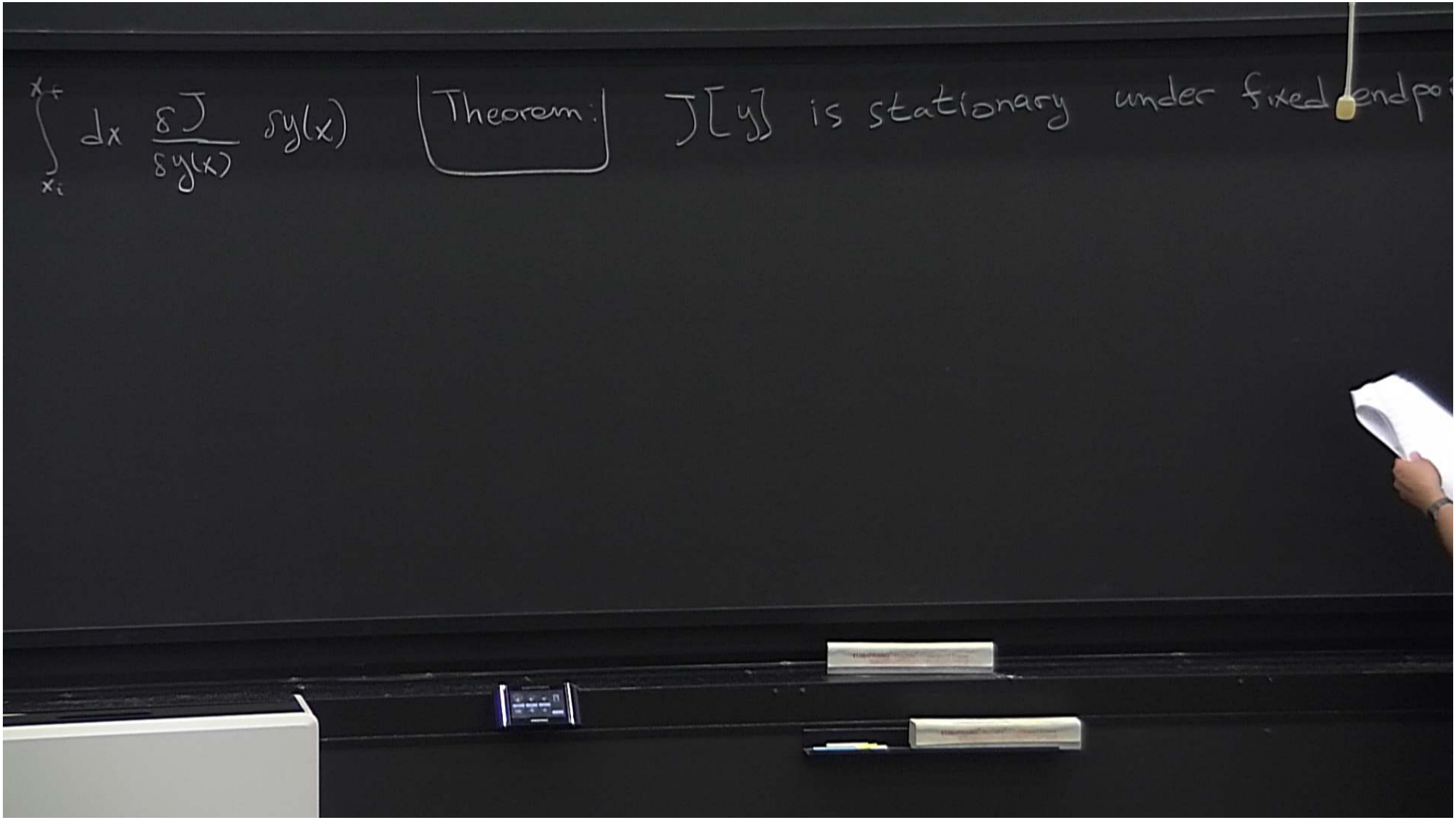

ctional derivative
$f(y, y', x)\,dx$
$J[y] = \int_{x_i}^{x_f} \{ f(y+\delta y, y'+\delta y', x) -$
$-f(y, y', x)\}dx = \int_{x_i}^{x_f} \left\{ \frac{\partial f}{\partial y}\cdot \delta y + \frac{\partial f}{\partial y'}\delta y' + O(\delta y^2) \right\} dx$
x_i
x_f

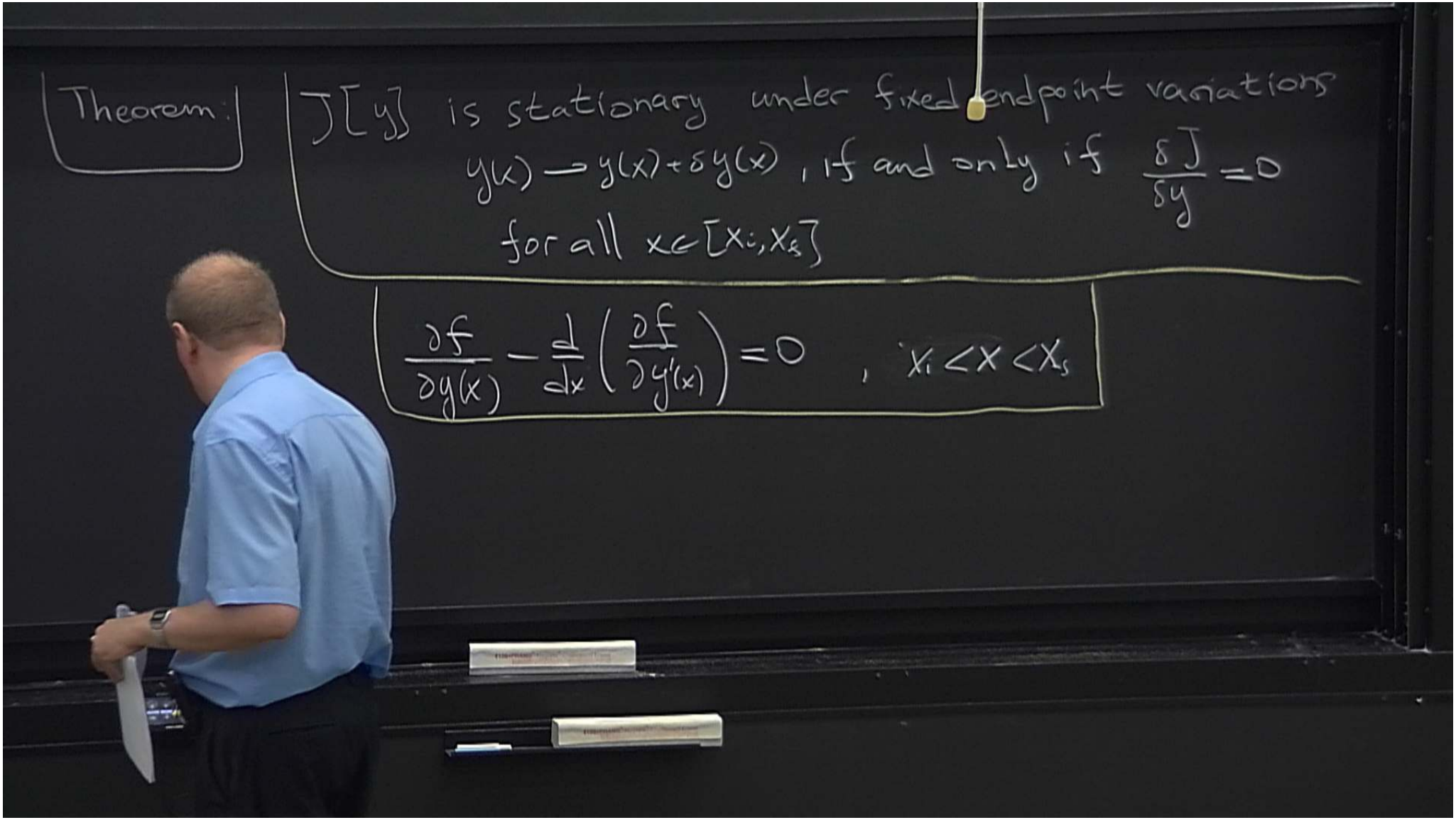

Note: $\delta J = \sum_i \frac{\partial J}{\partial y_i} \delta y_i \longrightarrow \int_{x_i}^{x_f} dx \frac{\delta J}{\delta y(x)} \delta y(x)$
y_i
x_i
x_i
x_f

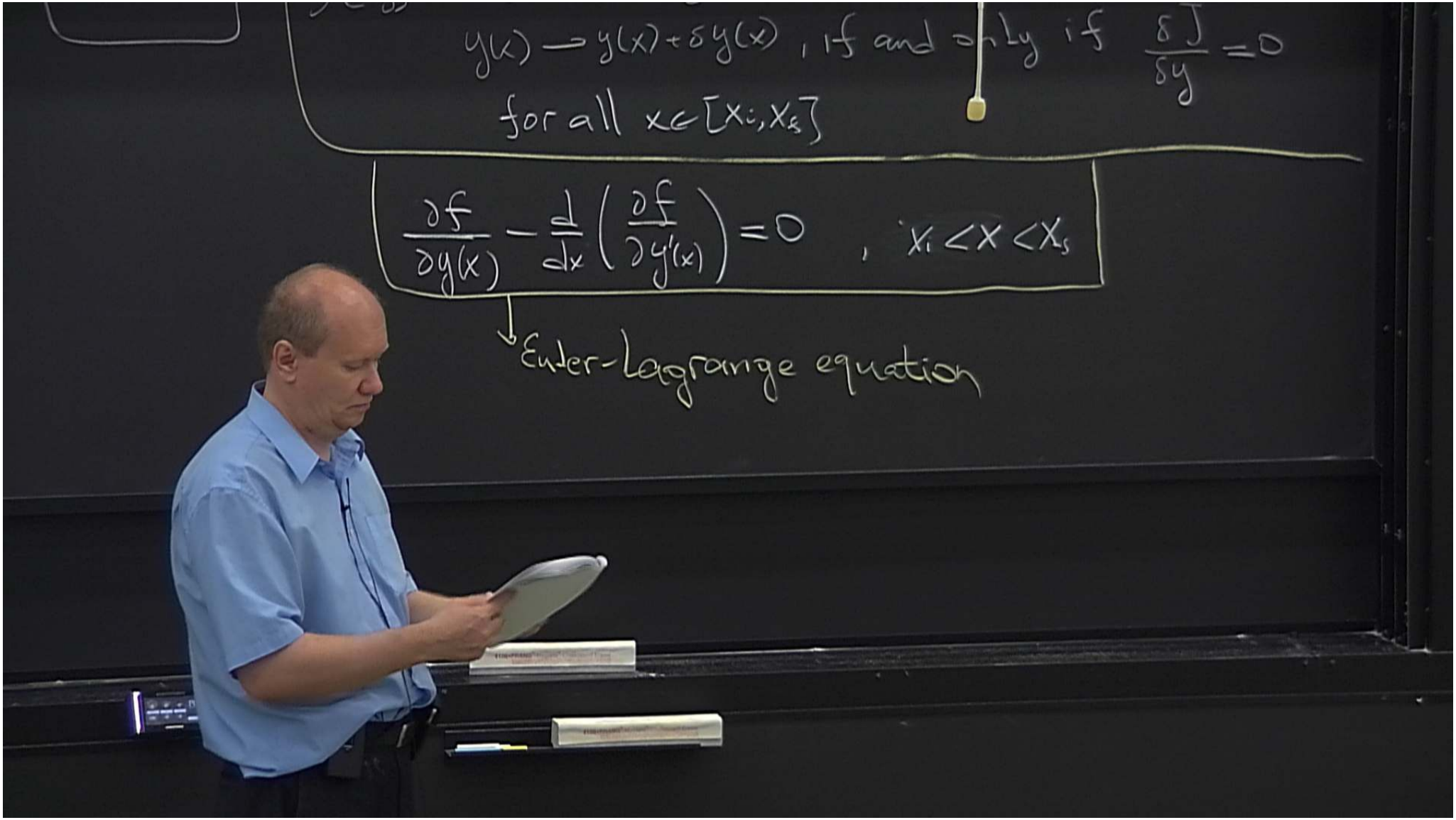

Note
$\delta J = \sum_i \frac{\partial J}{\partial y_i} \delta y_i \longrightarrow \int_{x_i}^{x_f} dx \frac{\delta J}{\delta y(x)} \delta y(x)$
Theore
y_i
x_i
x_i
x_s

$$\int_{x_i}^{x_f} dx \, \frac{\delta J}{\delta y(x)} \, \delta y(x)$$

Theorem: $J[y]$ is stationary under fixed endpo




Theorem:
J[y] is stationary under fixed endpoint variations
$y(x) \to y(x) + \delta y(x)$, if and only if $\frac{\delta J}{\delta y} = 0$
for all $x \in [x_i, x_s]$
$\frac{\partial f}{\partial y(x)} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'(x)}\right) = 0$, $x_i < x < x_s$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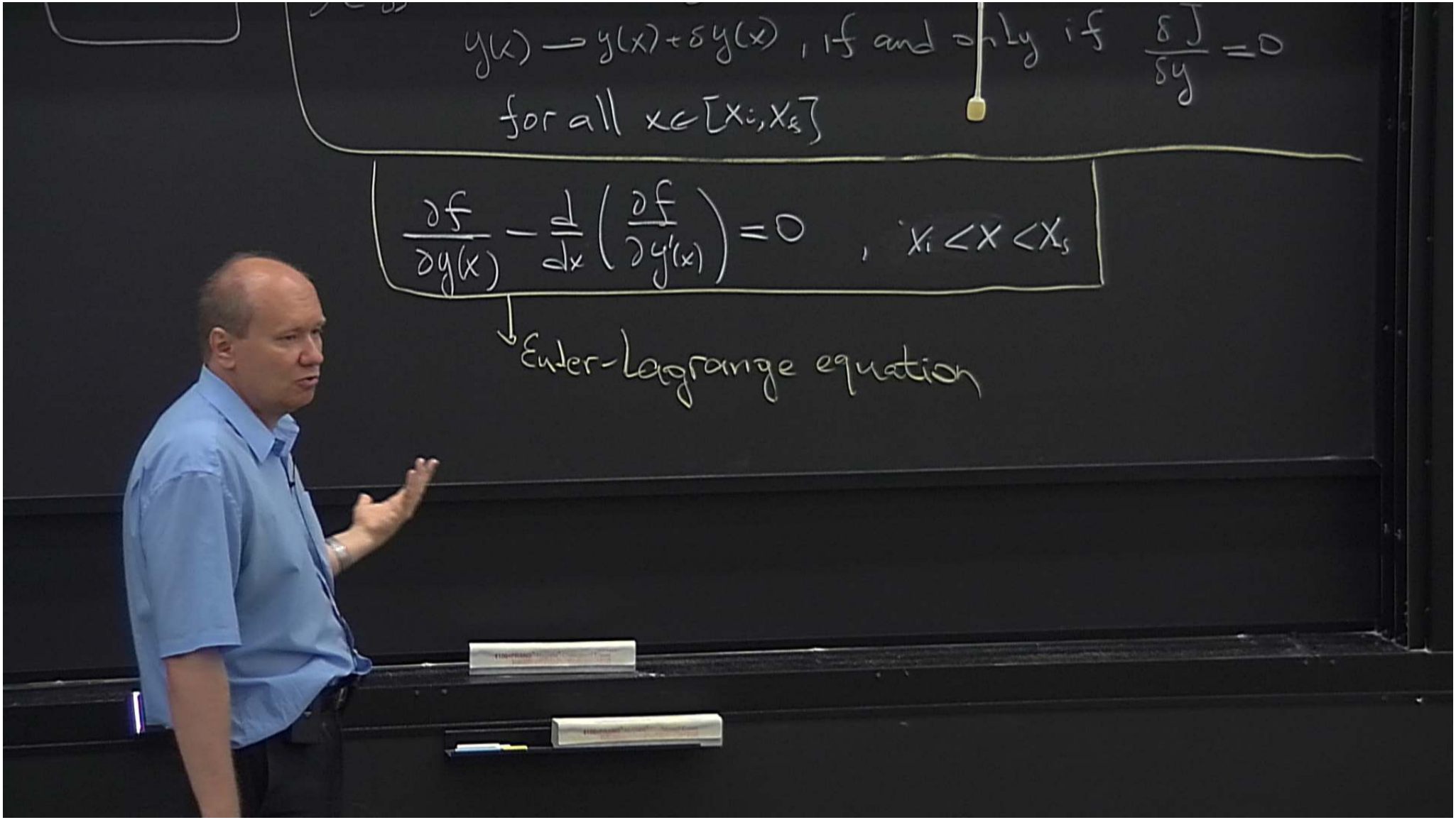
Euler-Lagrange equation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Euler-Lagrange equation

$y(x) \to y(x) + \delta y(x)$, if and only if $\frac{\delta J}{\delta y} = 0$

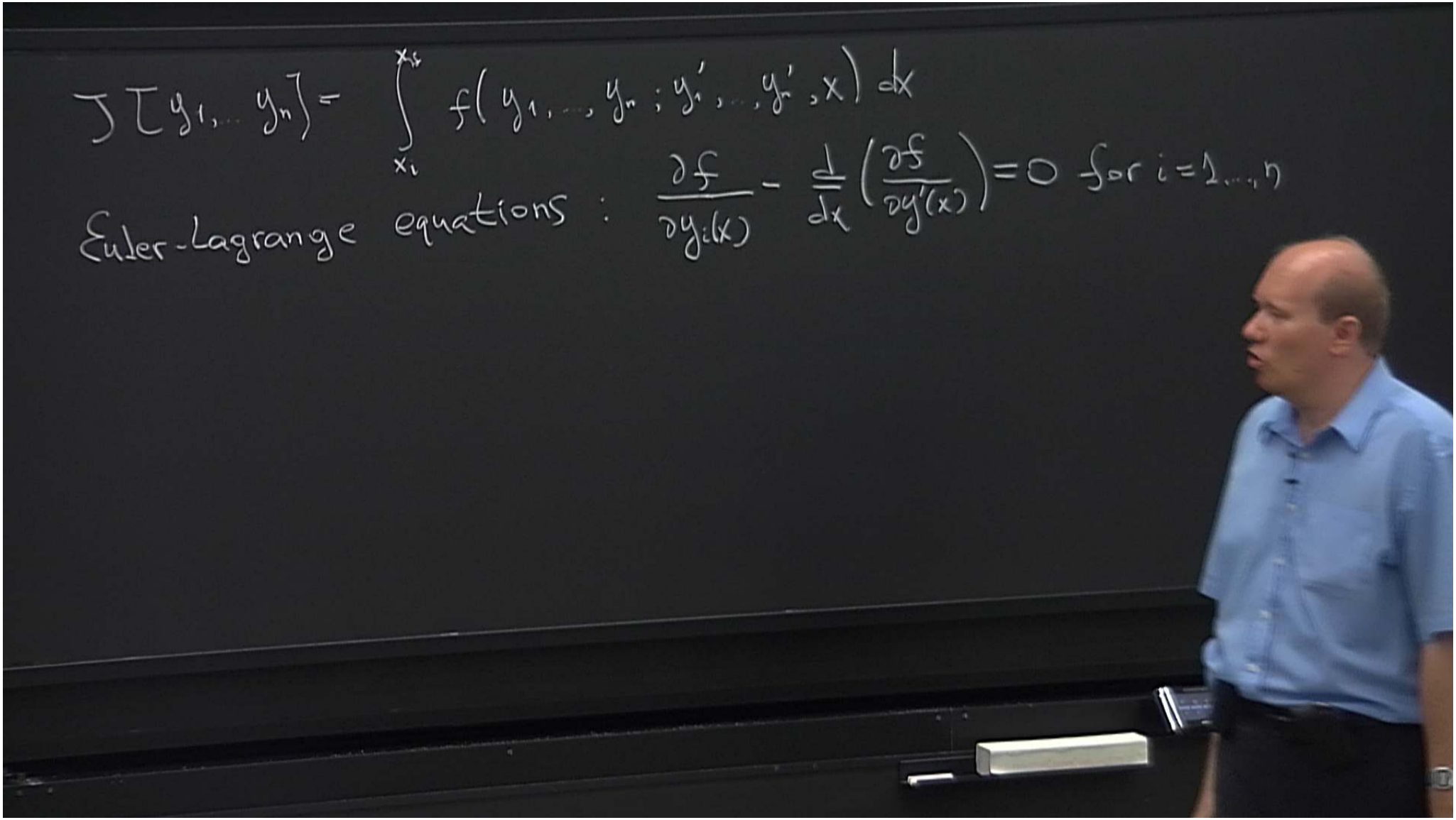
for all $x \in [x_i, x_s]$

$$\frac{\partial f}{\partial y(x)} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'(x)}\right) = 0 \quad , \quad x_i < x < x_s$$

Euler-Lagrange equation

$$J[y_1, \dots y_n] = \int_{x_i}^{x_f} f(y_1, \dots, y_n ; y_1', \dots, y_n', x)\, dx$$

Euler-Lagrange equations : $\frac{\partial f}{\partial y_i(x)} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'(x)}\right) = 0$ for $i = 1 \dots n$



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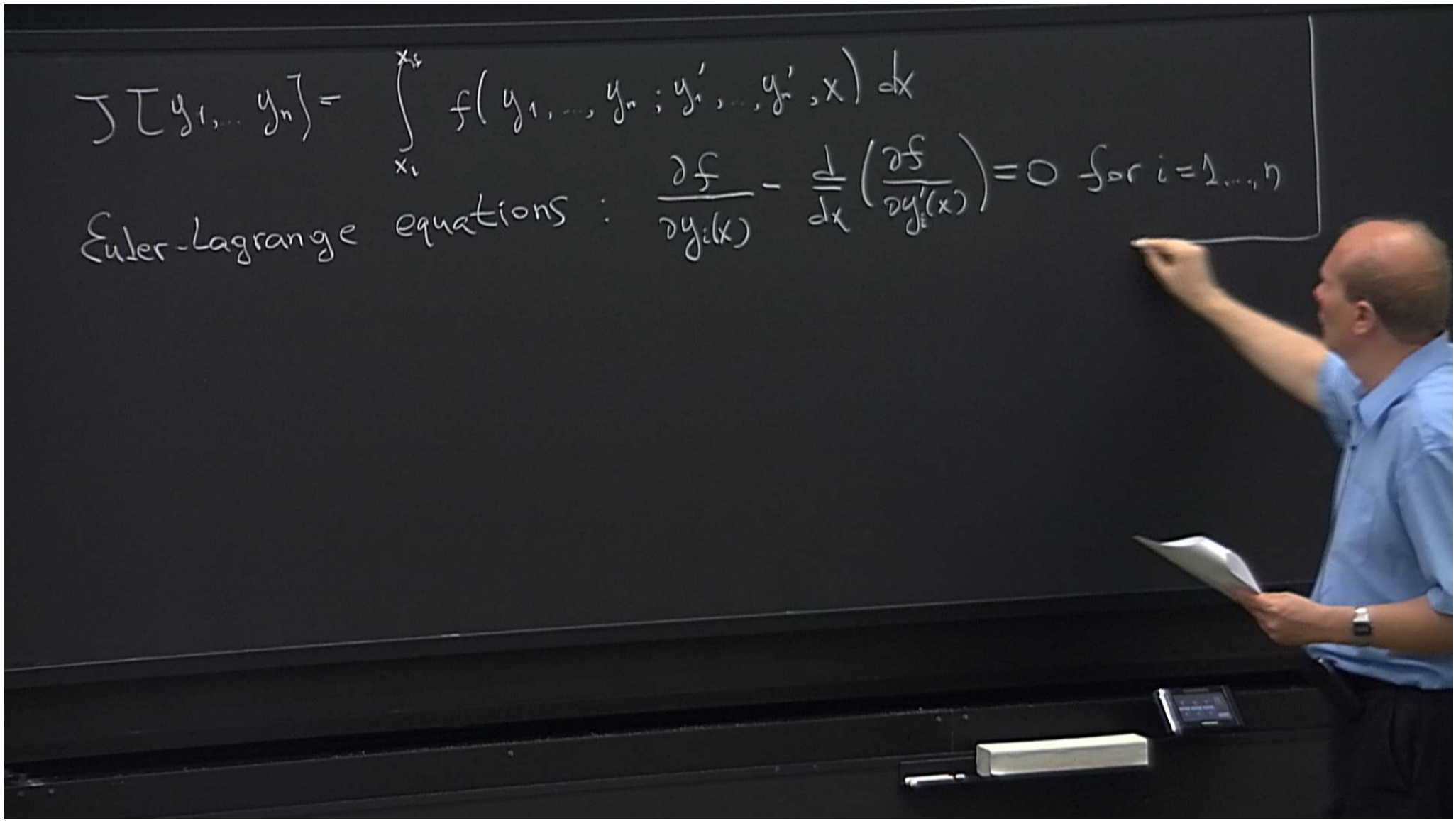
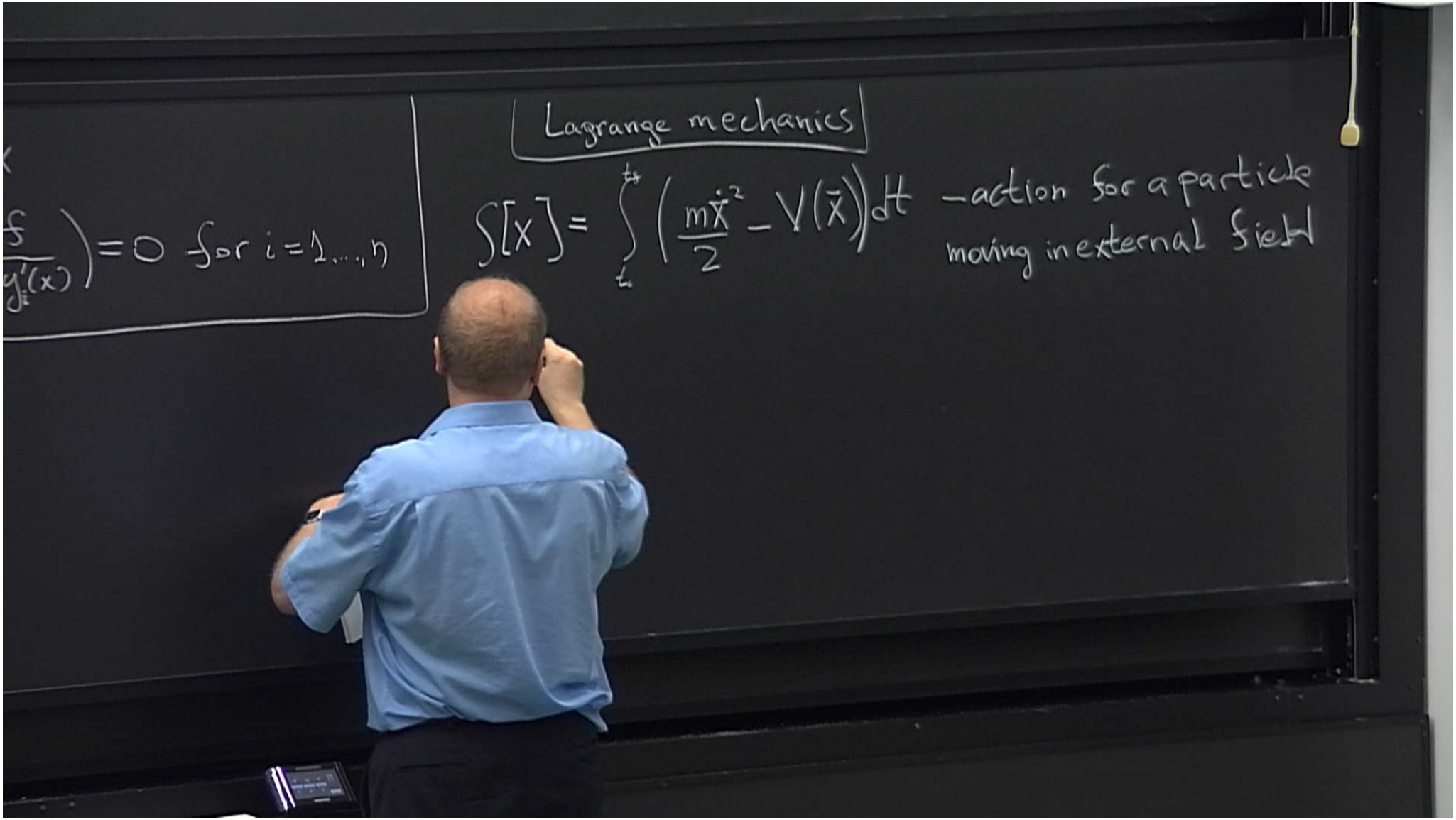
Euler-Lagrange equations : $$\frac{\partial f}{\partial y_i(x)} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'(x)}\right) = 0 \quad \text{for } i = 1 \dots n$$




$\frac{\delta f}{\delta y_i'(x)}\Big) = 0$ for $i = 1, \ldots, n$
Lagrange mechanics
$S[x] = \int_{t_-}^{t_+} \left(\frac{m\dot{\vec{x}}^2}{2} - V(\vec{x}) \right) dt$ – action for a particle moving in external field

$\frac{\delta f}{\delta y_i'(x)}\Big) = 0$ for $i = 1, \dots, n$

Lagrange mechanics

$$S[x] = \int_{t_-}^{t_+} \underbrace{\left(\frac{m\dot{\vec{x}}^2}{2} - V(\vec{x})\right)}_{L} dt$$

– action for a particle moving in external field

$\vec{x} = (x_1, x_2, x_3)$

Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0, \quad i = 1, 2, 3$$

$$J[y_1, \dots y_n] = \int_{x_i}^{x_f} f(y_1, \dots, y_n ; y_1', \dots, y_n', x)\, dx$$

Euler-Lagrange equations : $\frac{\partial f}{\partial y_i(x)} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'(x)}\right) = 0$ for $i = 1 \dots n$

L – Lagrangian

$$S[q_i] = \int_{t_i}^{t_f} L(q_i, \dot{q}_i, t)\, dt$$

Lagrangian

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

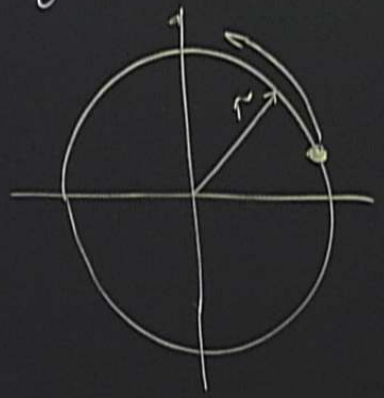
q_i - generalized coordinates

$$S[q_i] = \int_{t_i}^{t_f} L(q_i, \dot{q}_i, t)\, dt$$ Lagrangian

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

q_i - generalized coordinates

Ex:



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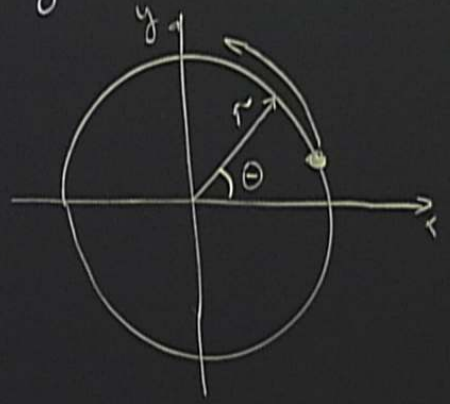
q_i - generalized coordinates

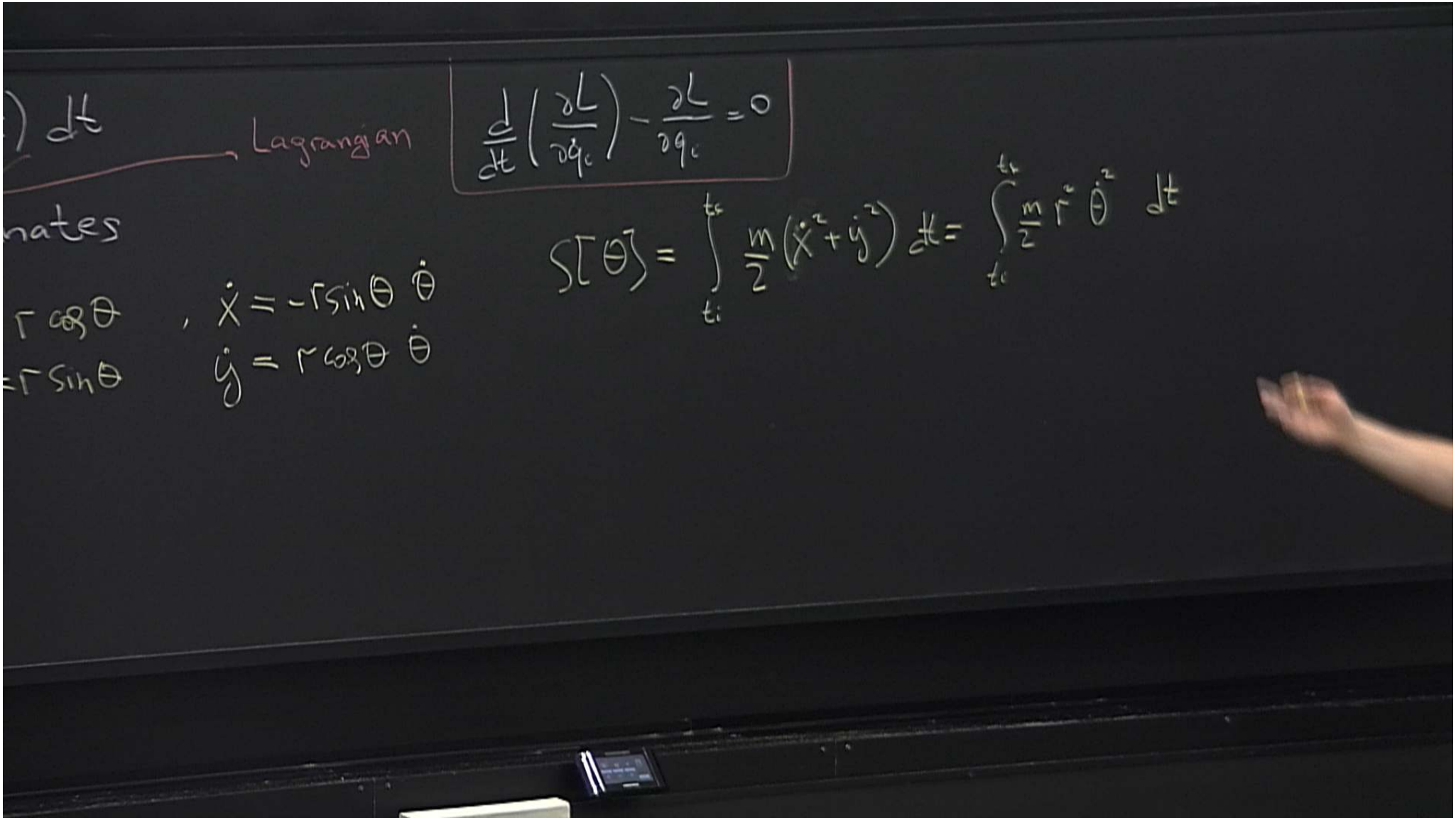
Ex:

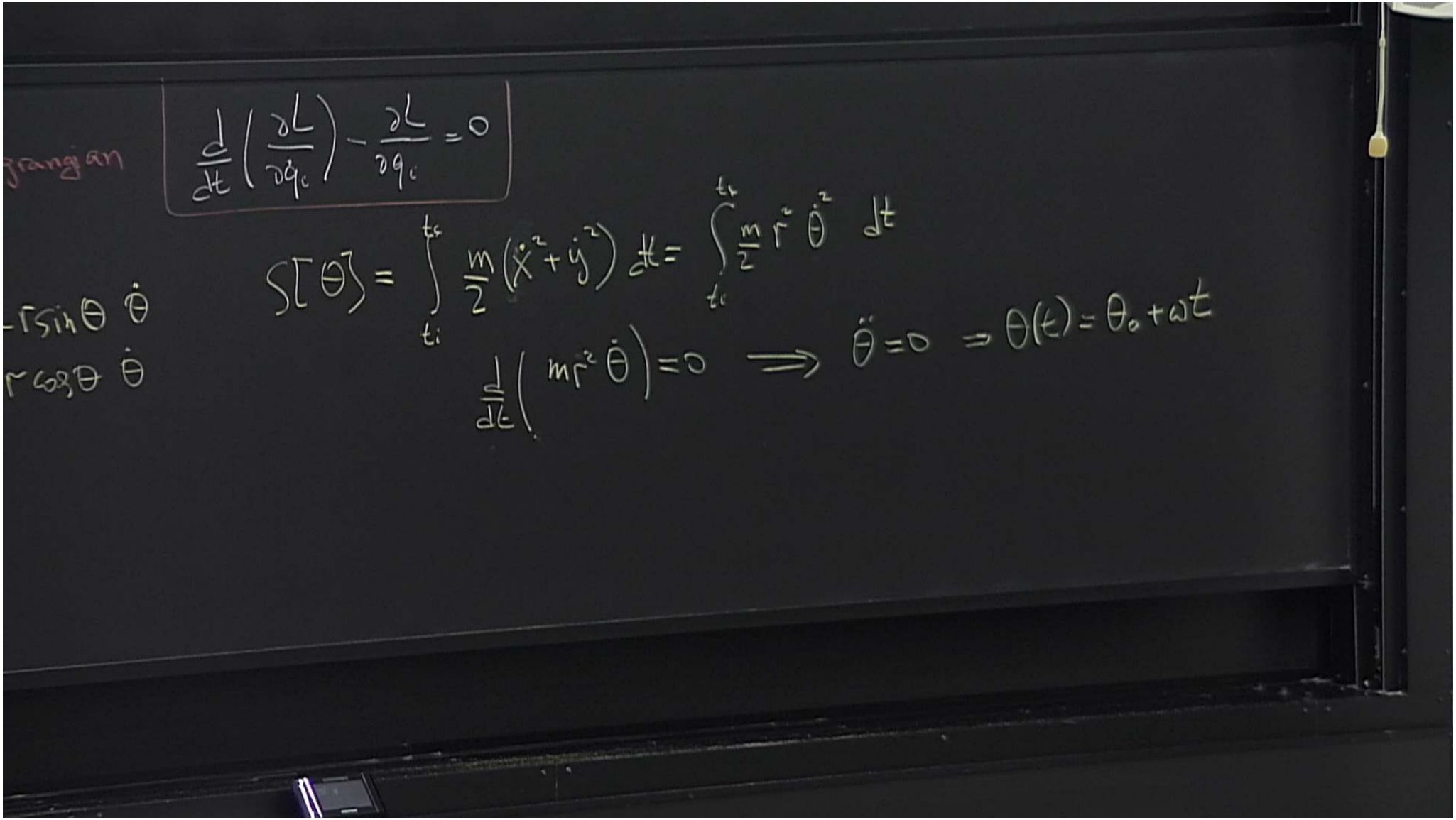


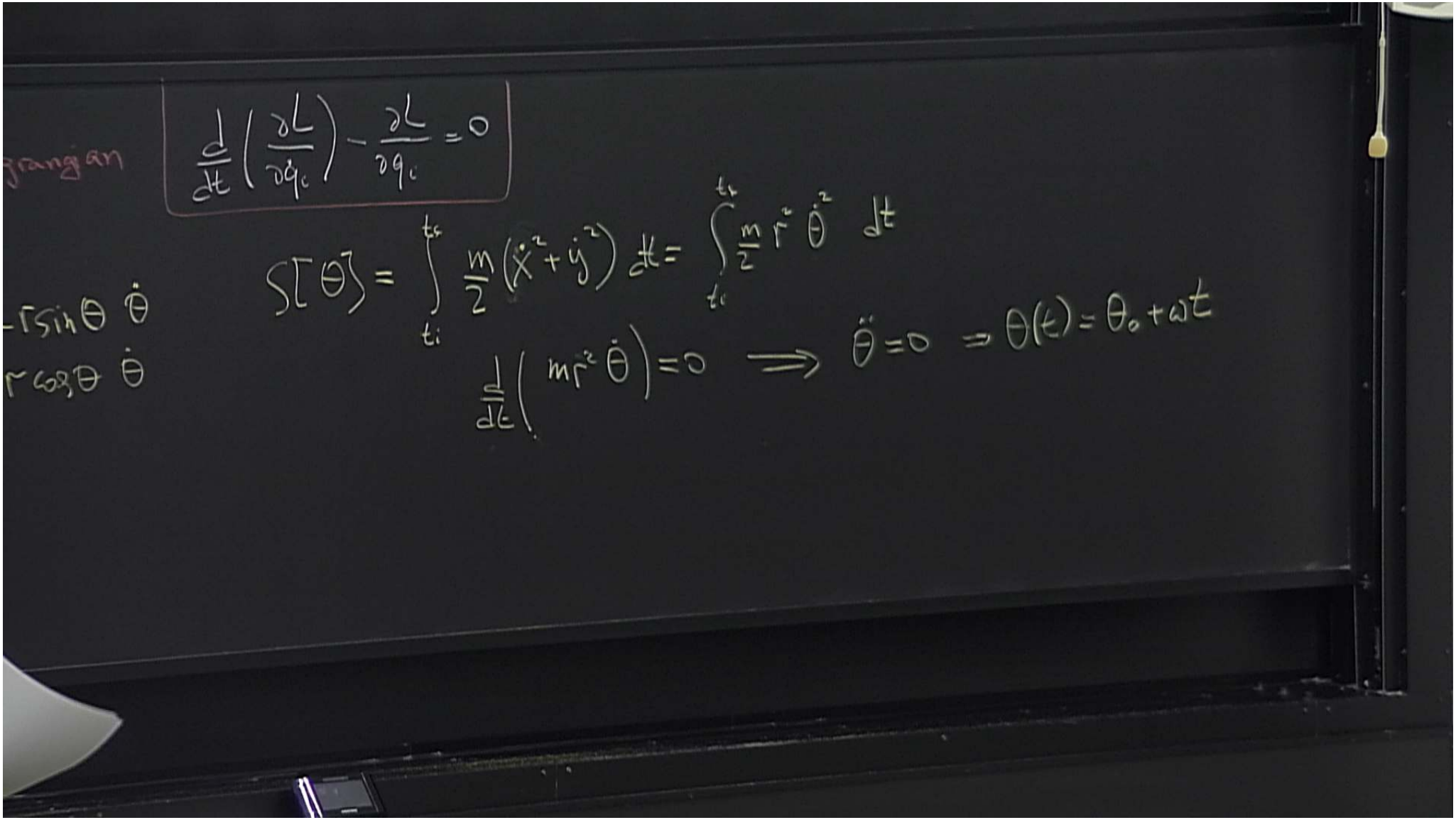
$x = r\cos\theta$, $\dot{x} = -r\sin\theta\,\dot{\theta}$

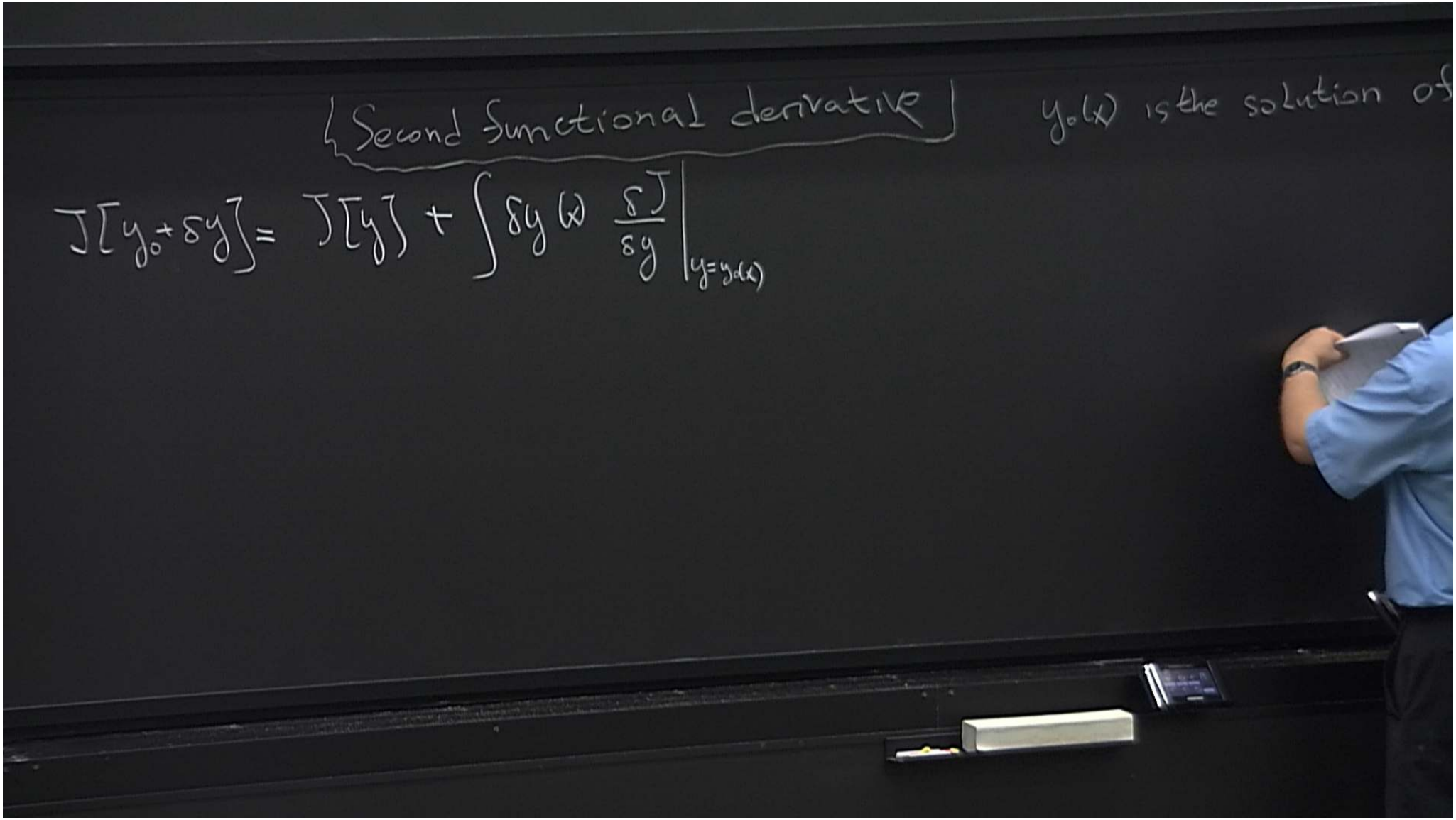
$y = r\sin\theta$ $\dot{y} = r\cos\theta\,\dot{\theta}$

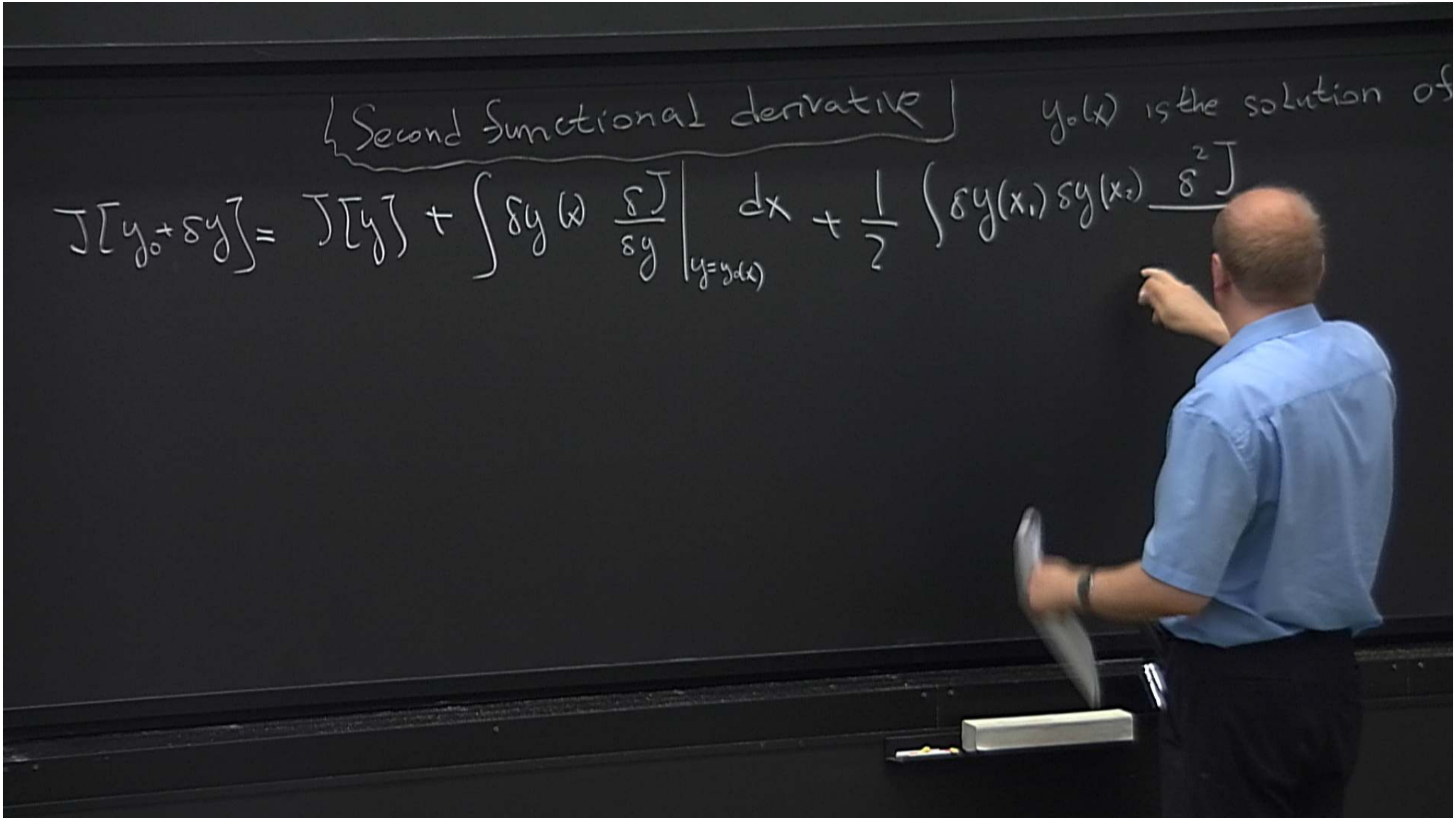
$S[\theta] = \int_{t_i}$

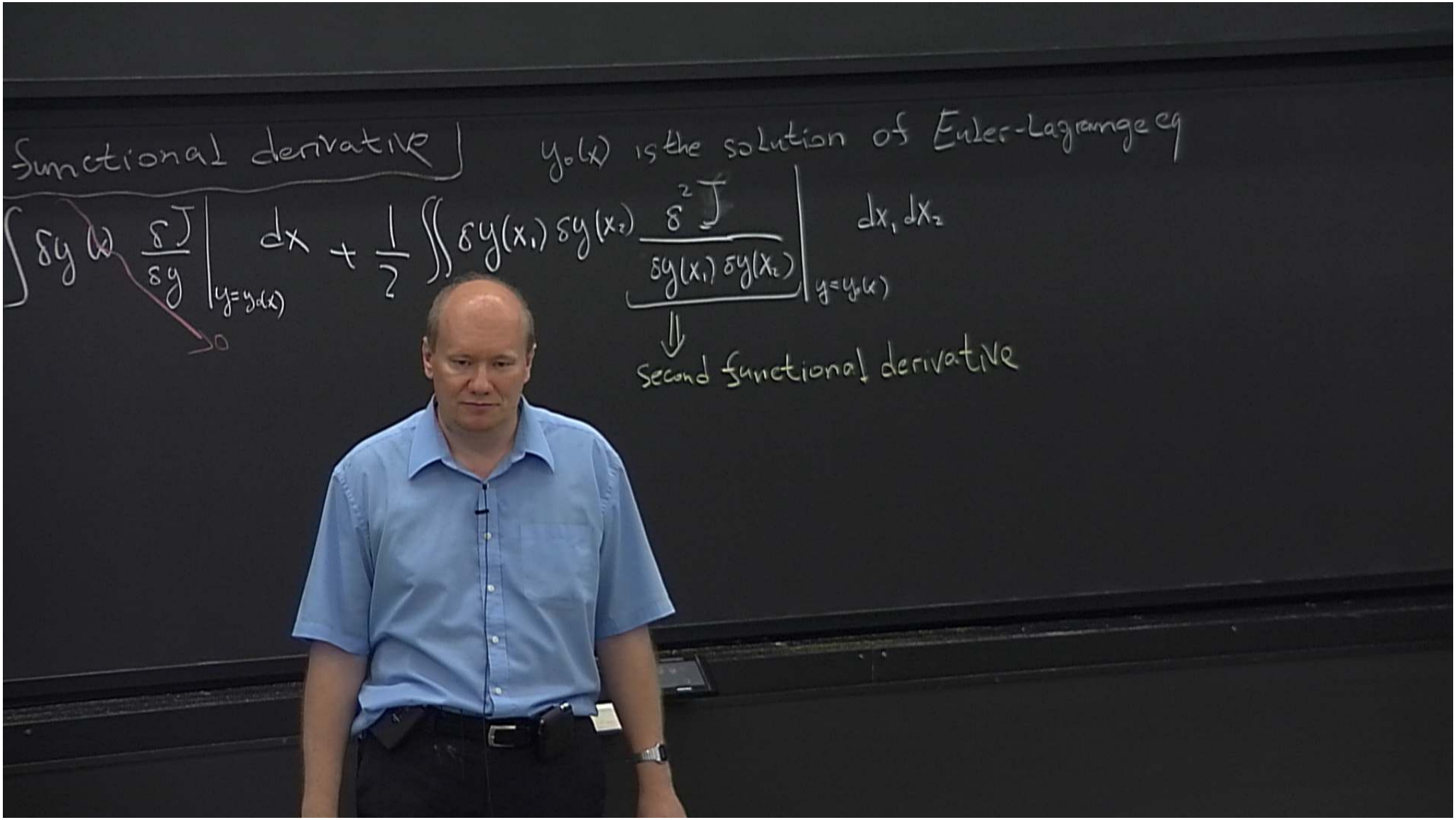

dt
Lagrangian
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$
nates
$r\cos\theta$
$= r\sin\theta$
$\dot{x} = -r\sin\theta\,\dot{\theta}$
$\dot{y} = r\cos\theta\,\dot{\theta}$
$S[\theta] = \int_{t_i}^{t_f} \frac{m}{2}(\dot{x}^2 + \dot{y}^2)\,dt = \int_{t_i}^{t_f} \frac{m}{2} r^2 \dot{\theta}^2\,dt$

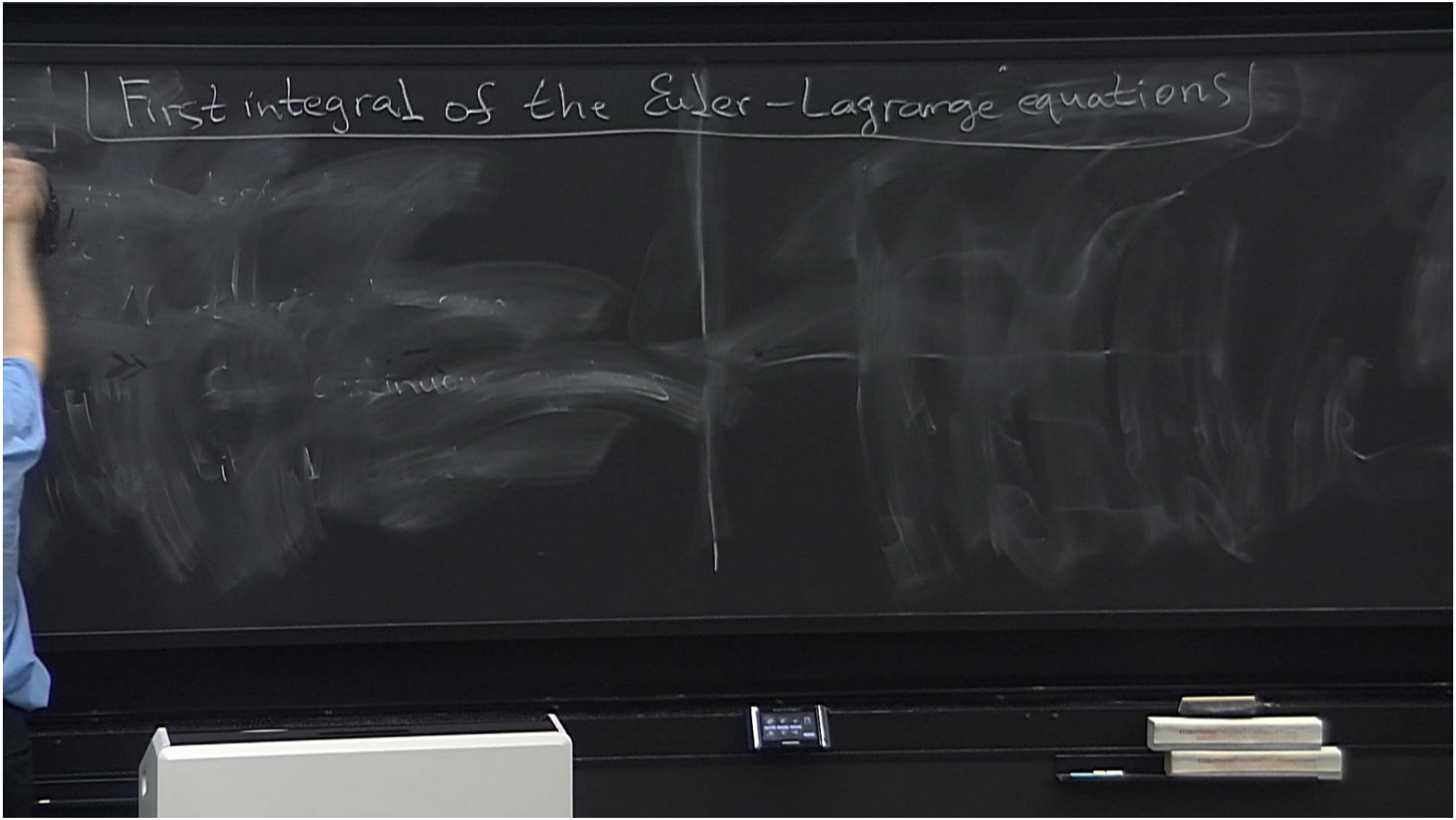

grangian
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$
$-r\sin\theta\,\dot{\theta}$
$r\cos\theta\,\dot{\theta}$
$S[\theta] = \int_{t_i}^{t_f} \frac{m}{2}(\dot{x}^2 + \dot{y}^2)\,dt = \int_{t_i}^{t_f} \frac{m}{2} r^2 \dot{\theta}^2\,dt$
$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \Longrightarrow \ddot{\theta} = 0 \Rightarrow \theta(t) = \theta_0 + \omega t$

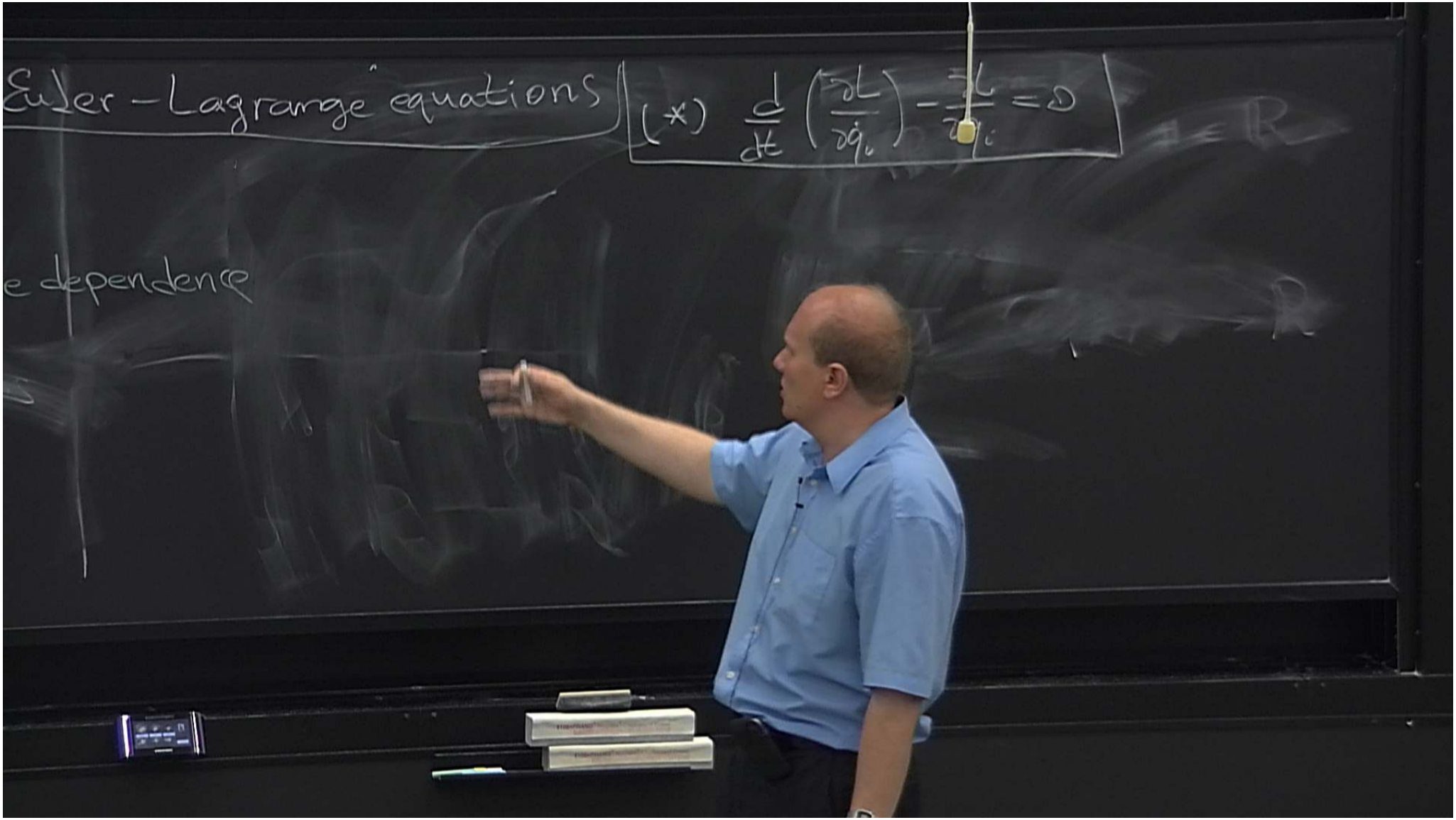

grangian
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$
$-r\sin\theta\,\dot{\theta}$
$r\cos\theta\,\dot{\theta}$
$S[\theta] = \int_{t_i}^{t_f} \frac{m}{2}(\dot{x}^2 + \dot{y}^2)\,dt = \int_{t_i}^{t_f} \frac{m}{2} r^2 \dot{\theta}^2\,dt$
$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \Longrightarrow \ddot{\theta} = 0 \Rightarrow \theta(t) = \theta_0 + \omega t$

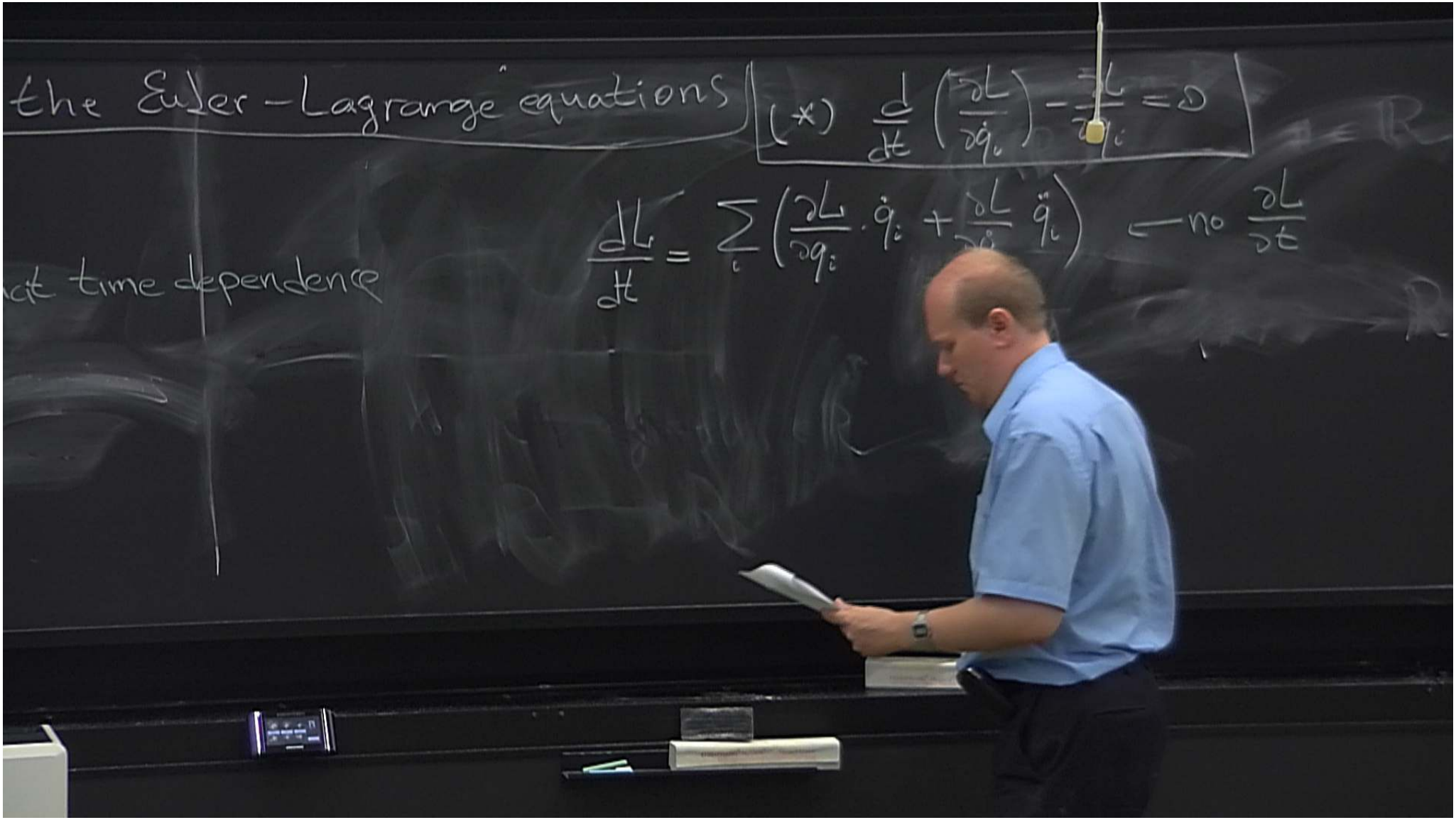

Second functional derivative
$y_0(x)$ is the solution of
$J[y_0+\delta y] = J[y] + \int \delta y(x) \left.\frac{\delta J}{\delta y}\right|_{y=y_0(x)}$

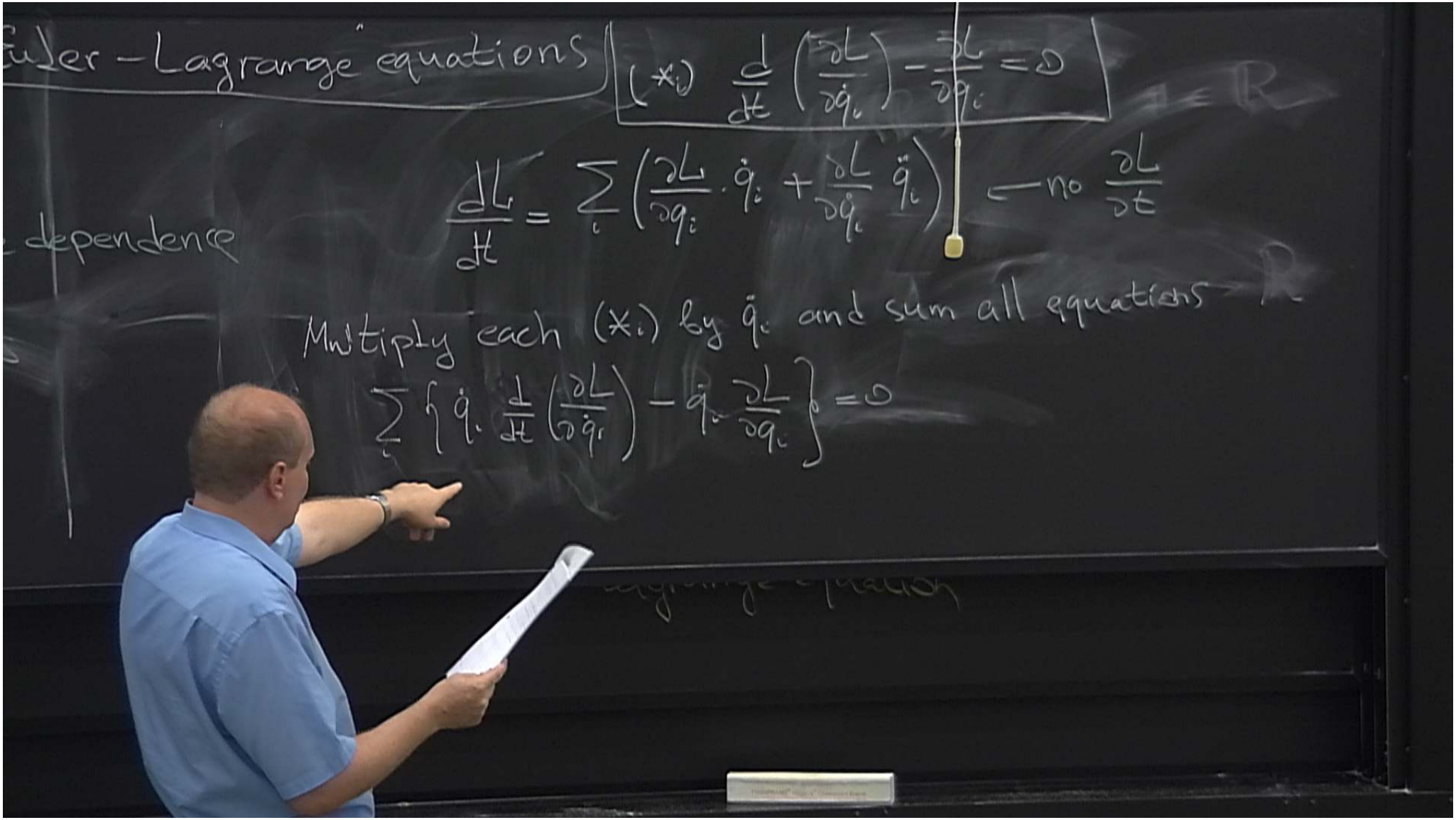

{Second functional derivative}
$y_0(x)$ is the solution of
$J[y_0+\delta y] = J[y] + \int \delta y(x) \left.\frac{\delta J}{\delta y}\right|_{y=y_0(x)} dx + \frac{1}{2} \int \delta y(x_1)\, \delta y(x_2)$

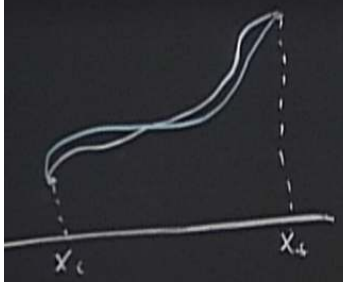

functional derivative
$y_0(x)$ is the solution of Euler-Lagrange eq
$\int \delta y(x) \left.\frac{\delta J}{\delta y}\right|_{y=y_0(x)} dx + \frac{1}{2} \iint \delta y(x_1)\, \delta y(x_2) \left.\frac{\delta^2 J}{\delta y(x_1)\, \delta y(x_2)}\right|_{y=y_0(x)} dx_1\, dx_2$
$\to 0$
$\Downarrow$
second functional derivative


First integral of the Euler-Lagrange equations


Euler-Lagrange equations
(*) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$
e dependence


the Euler-Lagrange equations
(*) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$
$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i}\cdot \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right)$ ← no $\frac{\partial L}{\partial t}$
ct time dependence


Euler-Lagrange equations
$(*_i)\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$
dependence
$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right)$ ← no $\frac{\partial L}{\partial t}$
Multiply each $(*_i)$ by $\dot{q}_i$ and sum all equations
$\sum_i \left\{\dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \dot{q}_i \frac{\partial L}{\partial q_i}\right\} = 0$

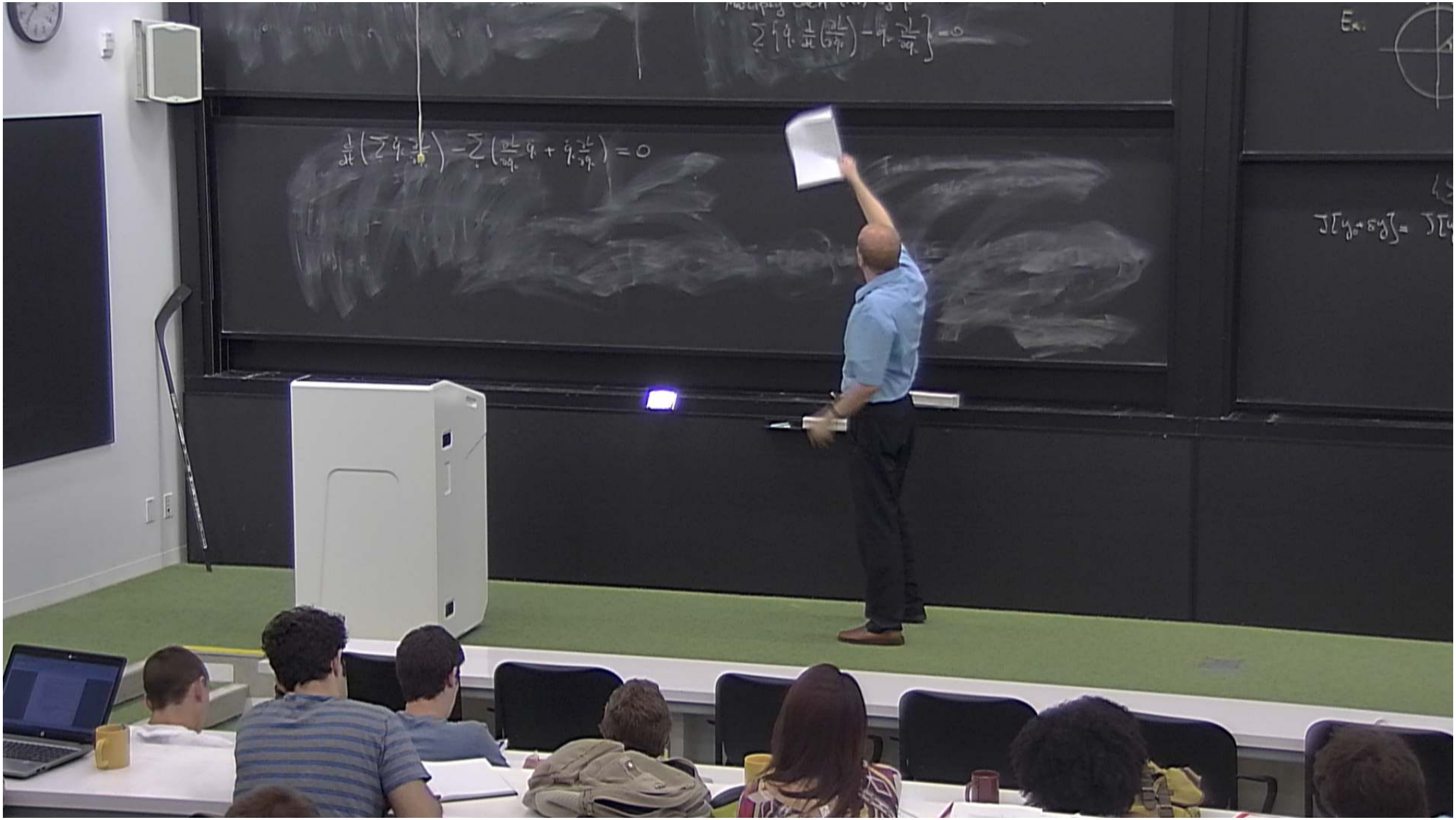


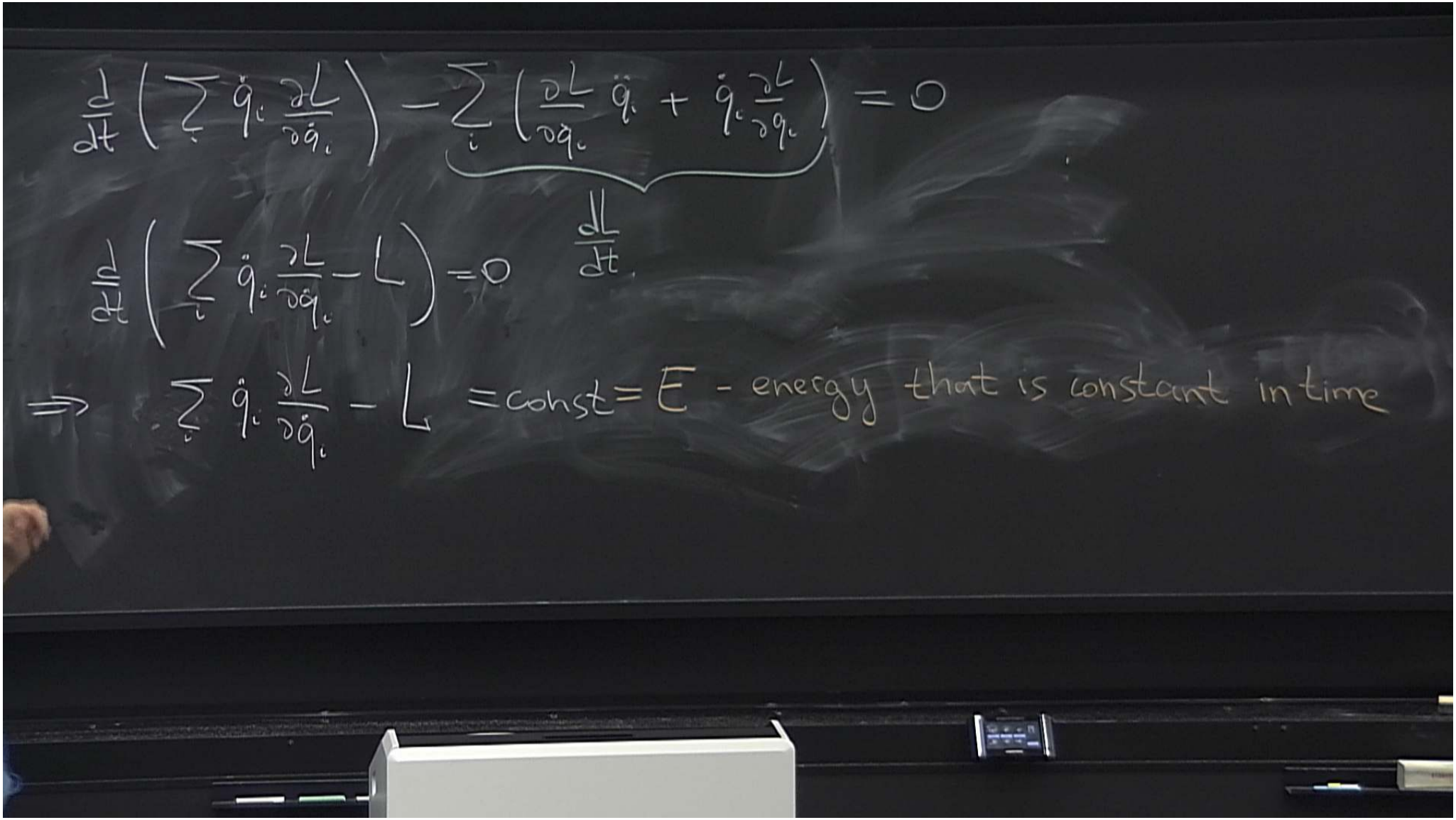
Fixed boundary conditions

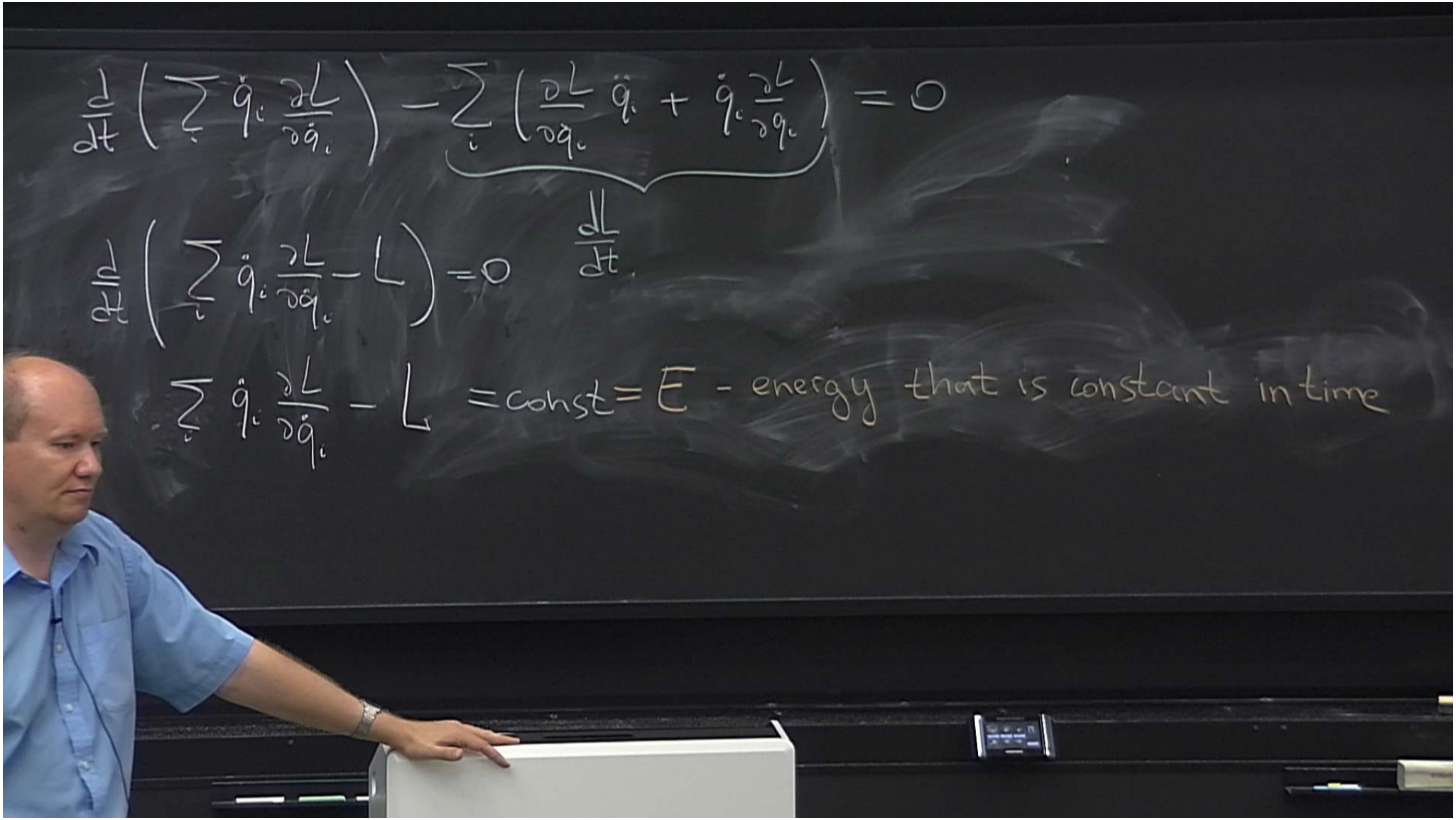
$$\delta y(x_1) = \delta y(x_2) = 0$$

$$\left\{ \frac{\partial f}{\partial y} \cdot \delta y + \frac{\partial f}{\partial y'} \delta y' + O(\delta y^2) \right\} dx = \int_{x_1}^{x_2} \left\{ \underbrace{\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right]}_{\frac{\delta J}{\delta y}} \delta y + O(\delta y^2) \right\} dx$$

$\frac{\delta J}{\delta y}$ - first functional derivative




$\frac{d}{dt}\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - \sum_i \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \dot{q}_i \frac{\partial L}{\partial q_i}\right) = 0$
$\frac{dL}{dt}$
$\frac{d}{dt}\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) = 0$
$\Rightarrow \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \text{const} = E$ - energy that is constant in time


$\frac{d}{dt}\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - \sum_i \left(\frac{\partial L}{\partial q_i}\dot{q}_i + \dot{q}_i \frac{\partial L}{\partial q_i}\right) = 0$
$\frac{dL}{dt}$
$\frac{d}{dt}\left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) = 0$
$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \text{const} = E$ - energy that is constant in time