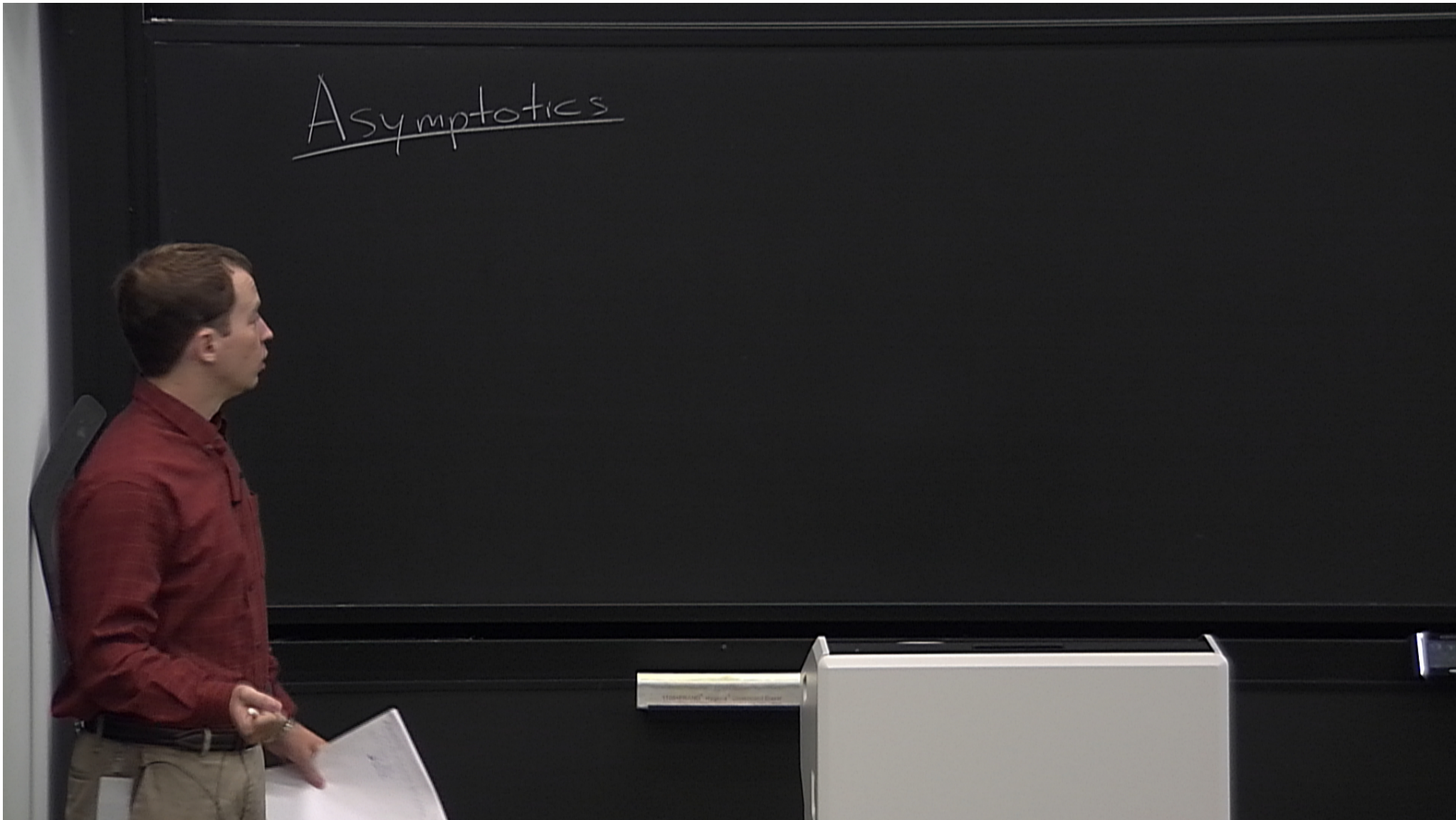


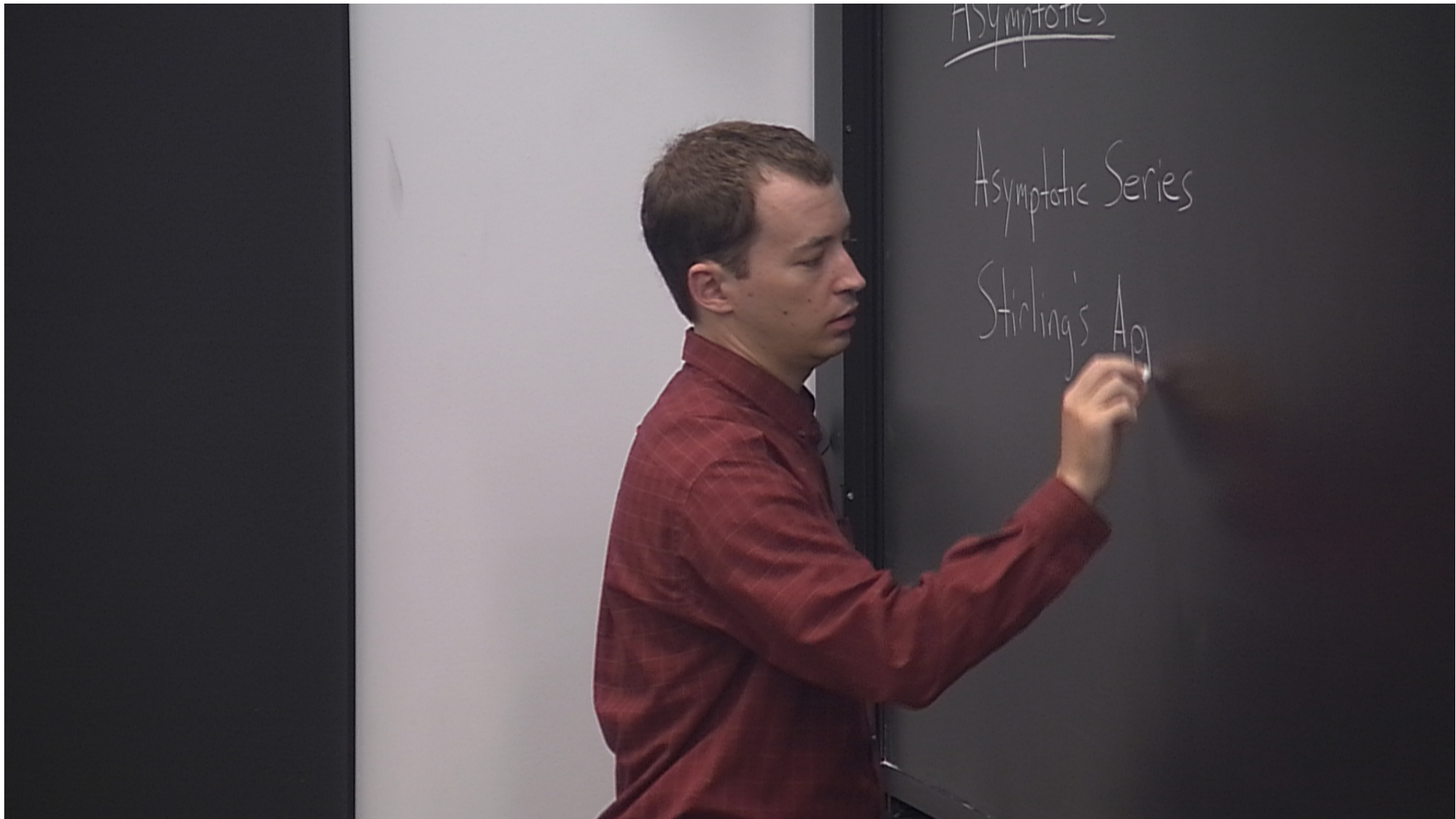
Title: 13/14 PSI - Distributions & Special Functions - Lecture 4

Date: Aug 26, 2013 09:00 AM

URL: <http://pirsa.org/13080021>

Abstract:





Asymptotics

Asymptotic Series

Stirling's App

Asymptotics

Asymptotic Series

Stirling's Approximation



Asymptotics

Asymptotic Series

Stirling's Approximation

Saddle-point Method

$$Z(\lambda) = \int_{-\infty}^{\infty} e^{-x^2 - \lambda x^4} dx$$

series
approximation
Method

$$Z(\lambda) = \int_{-\infty}^{\infty} e^{-x^2 - \lambda x^4} dx$$

Asymptotics

Asymptotic Series

Steepest Descent

$$Z(\lambda) = \int_{-\infty}^{\infty} e^{-x^2 - \lambda x^4} dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2}$$

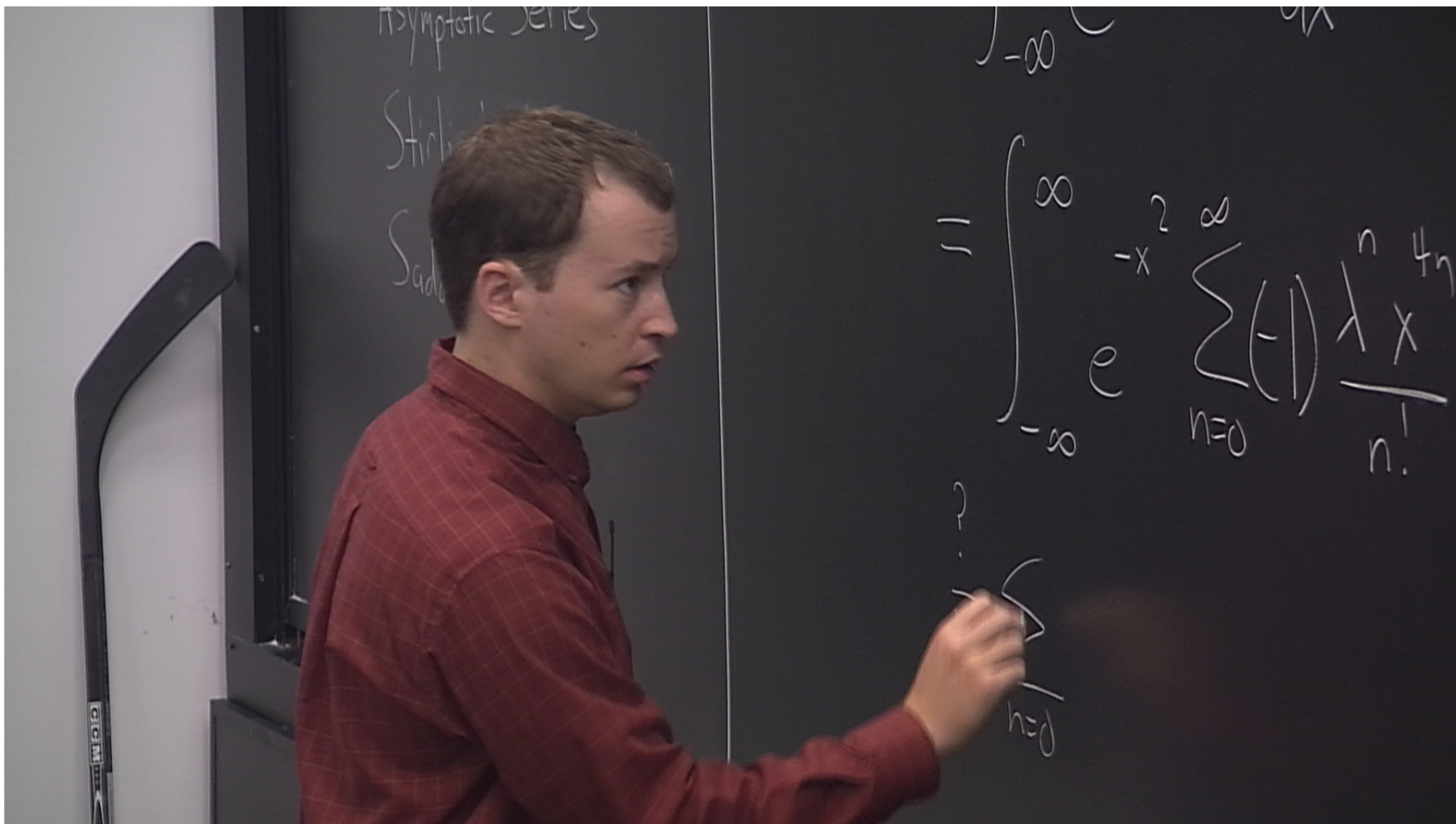
Asymptotics

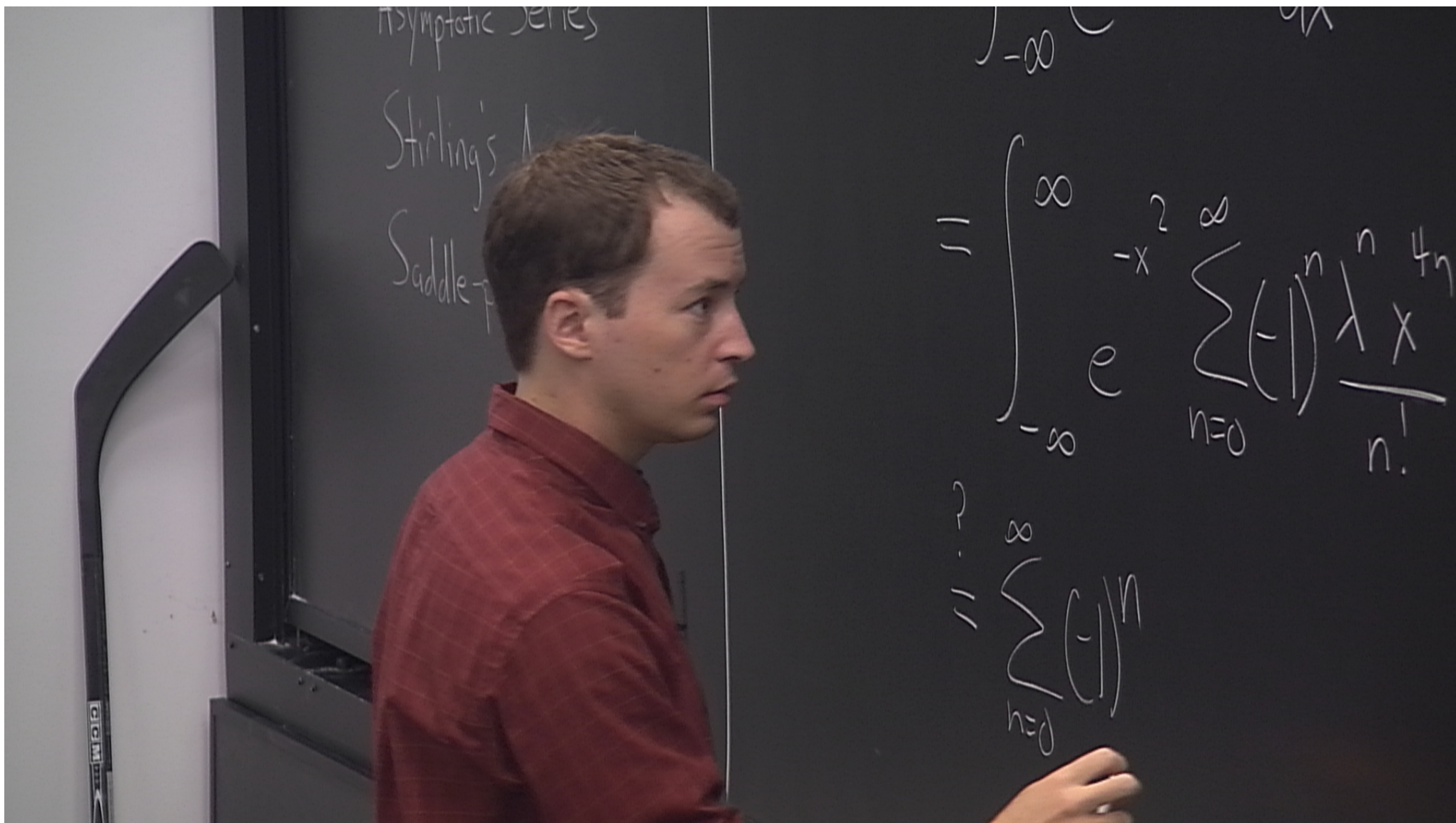
Asymptotic Series

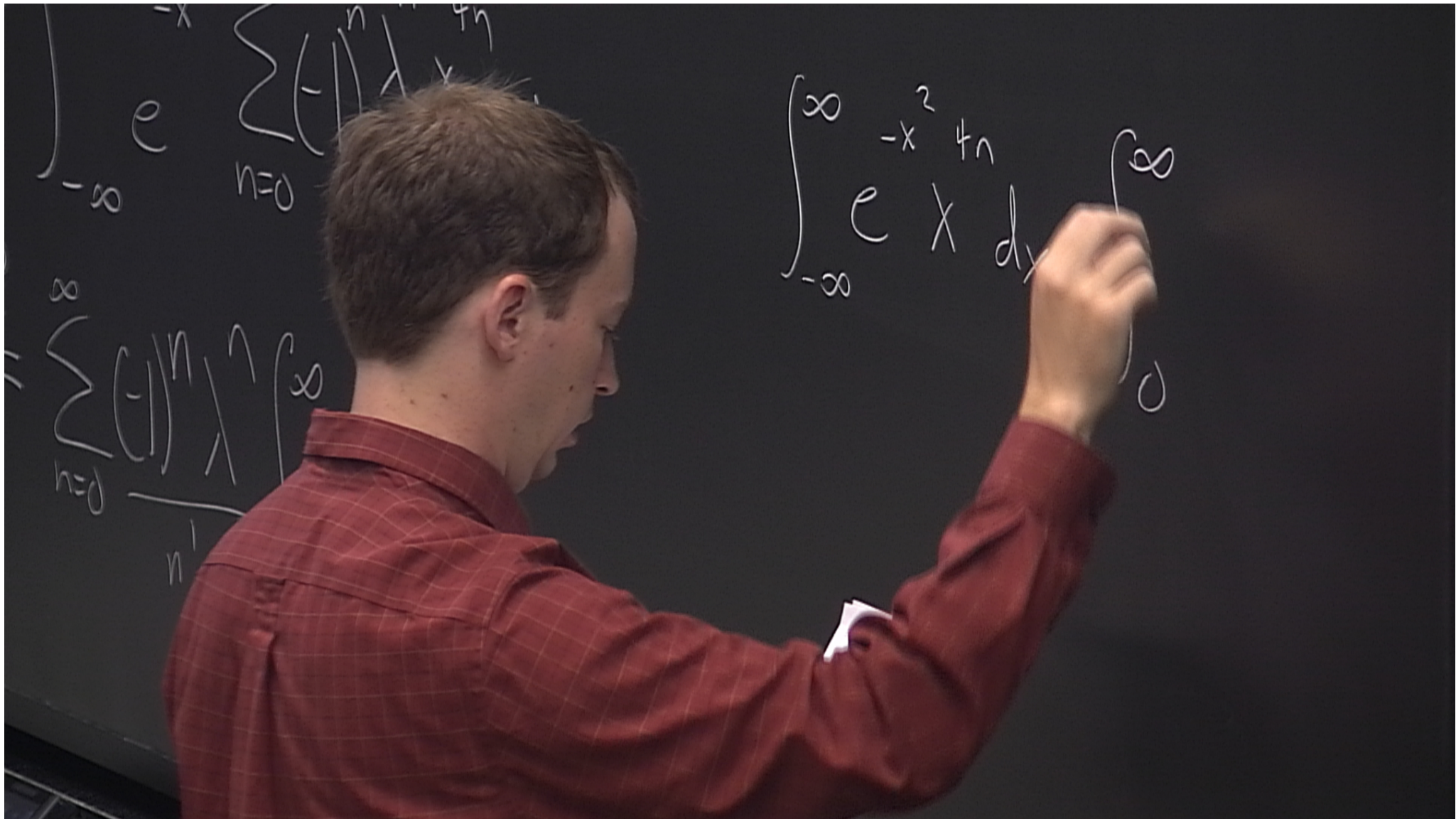
Steepest Descent
Approximation

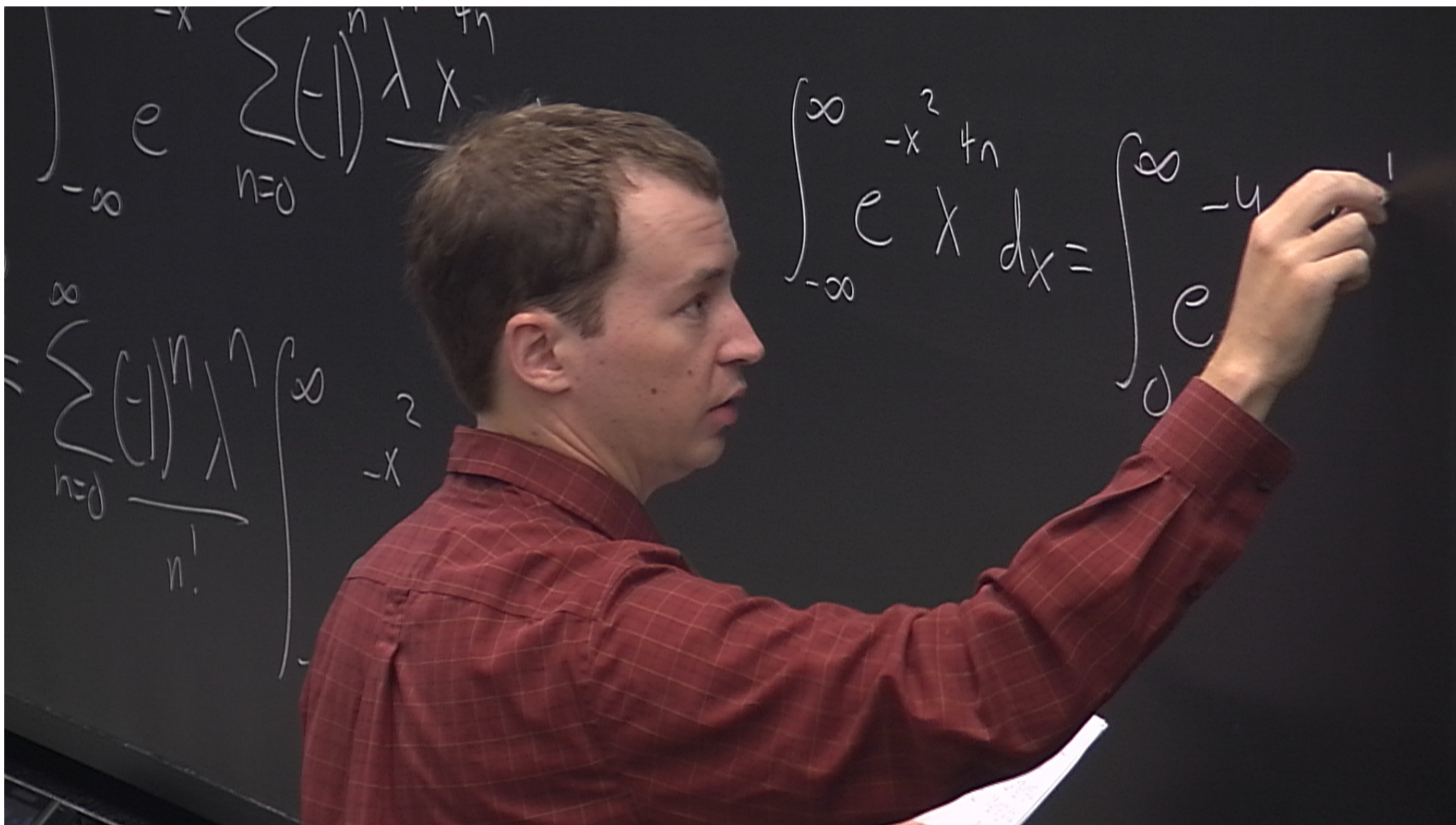
$$Z(\lambda) = \int_{-\infty}^{\infty} e^{-x^2 - \lambda x^4} dx$$

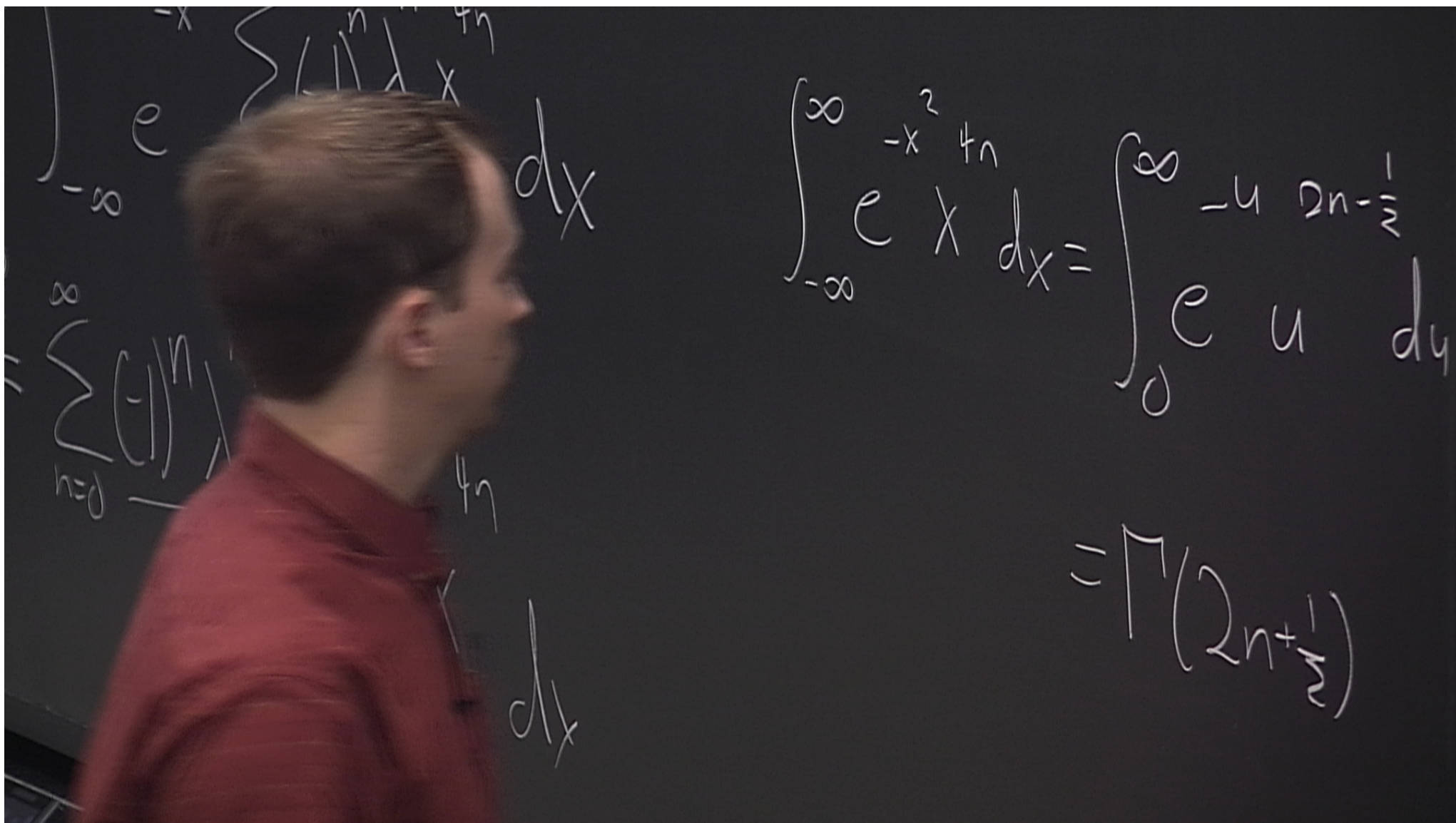
$$\int_{-\infty}^{\infty} e^{-x^2} dx$$











$$Z(\lambda) = \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n}{n!} \Gamma(2n + \frac{1}{2})$$

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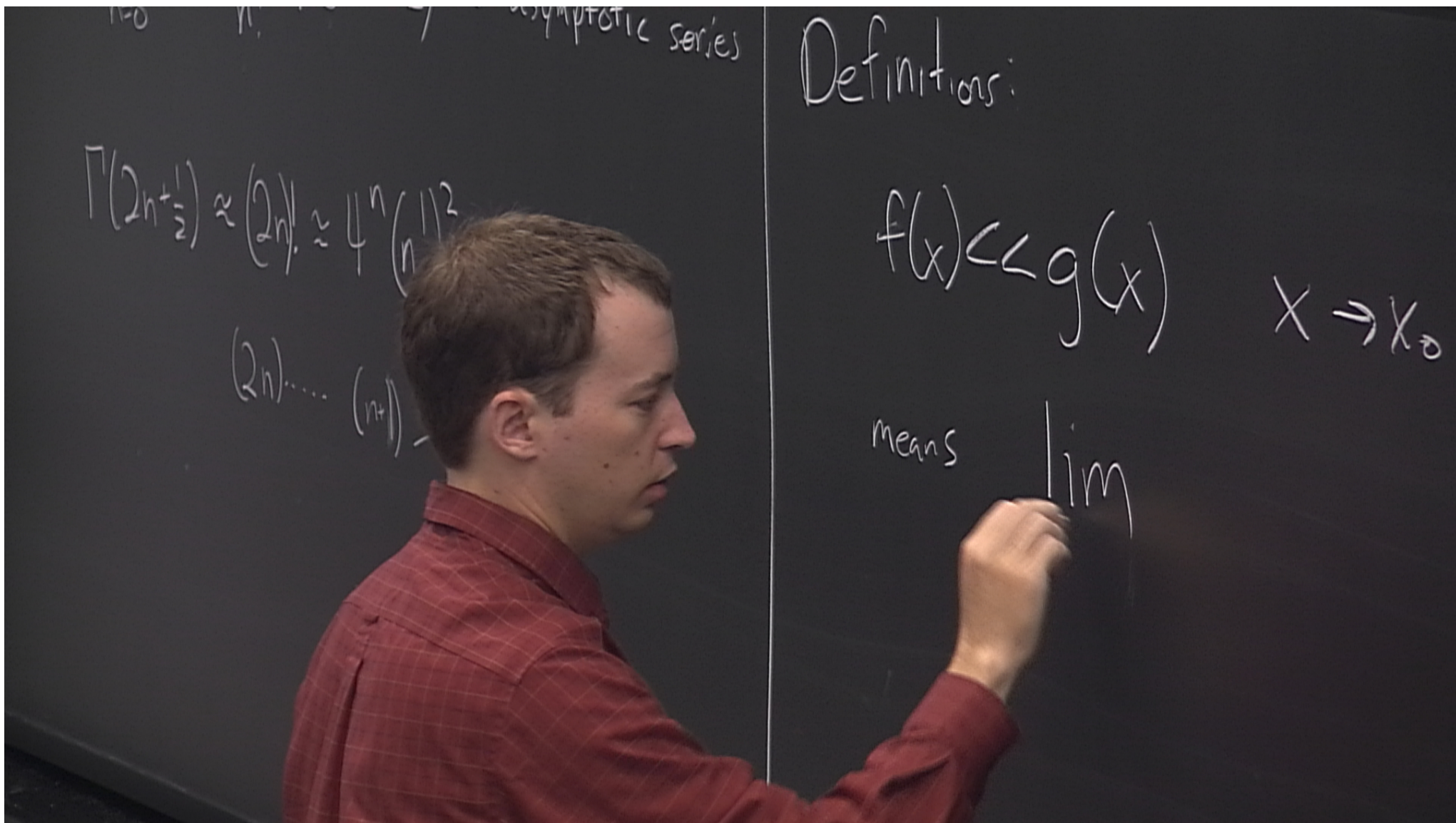
$$\Gamma(2n + \frac{1}{2}) \approx 2n!$$

$$Z(\lambda) = \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n}{n!} \Gamma(2n + \frac{1}{2})$$

$$\Gamma(2n + \frac{1}{2}) \approx (2n)! \approx 4^n (n!)^2$$

$$Z(\lambda) = \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n}{n!} \Gamma(2n + \frac{1}{2})$$

$$\Gamma(2n + \frac{1}{2}) \approx (2n)! \approx 4^n (n!)^2$$



$$(2n + \frac{1}{2}) \approx (2n)! \approx 4^n / n^{1/2}$$

$(2n) \dots$

$$1 \quad f(x) \ll g(x) \quad x \rightarrow x_0$$

means $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0$

$$2. \quad f(x) \sim$$

$$(2n+1) \approx (2n)! \approx 4^n (n!)^2$$

$$(2n) \dots (n+1) > 2^n n!$$

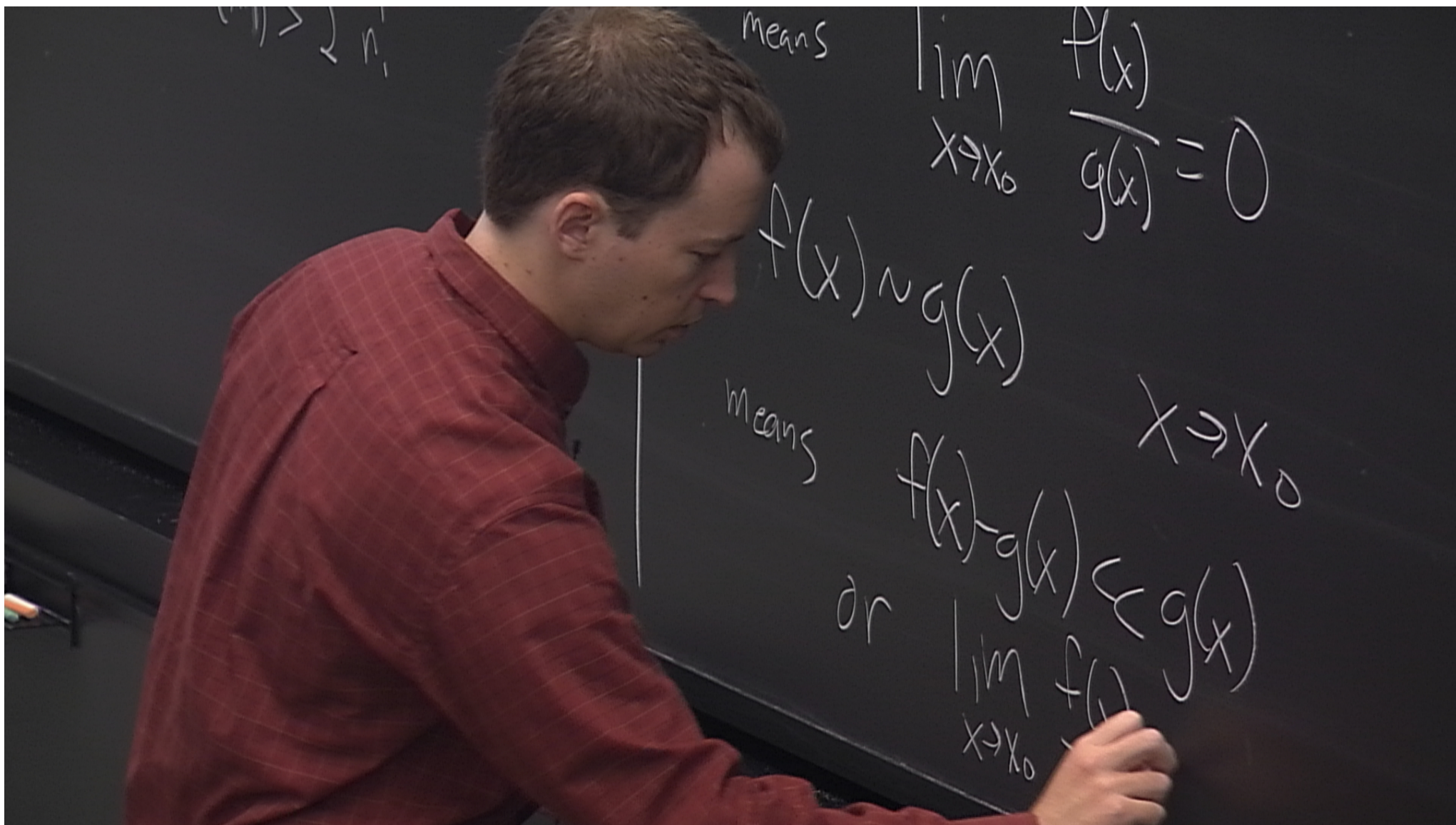
$$1 \quad f(x) \ll g(x)$$

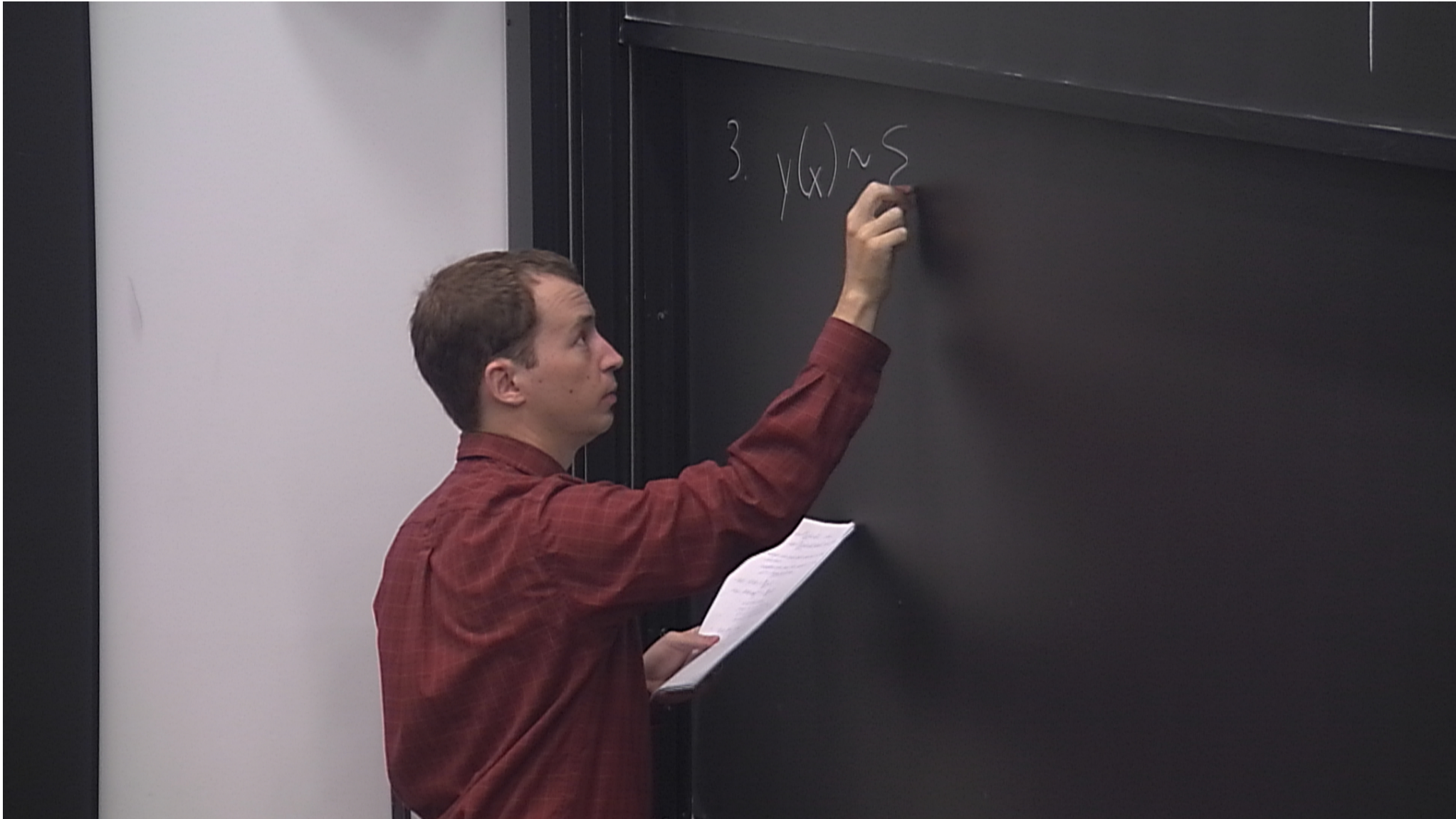
$$x \rightarrow x_0$$

means

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0$$

$$2. \quad f(x) \sim g(x)$$



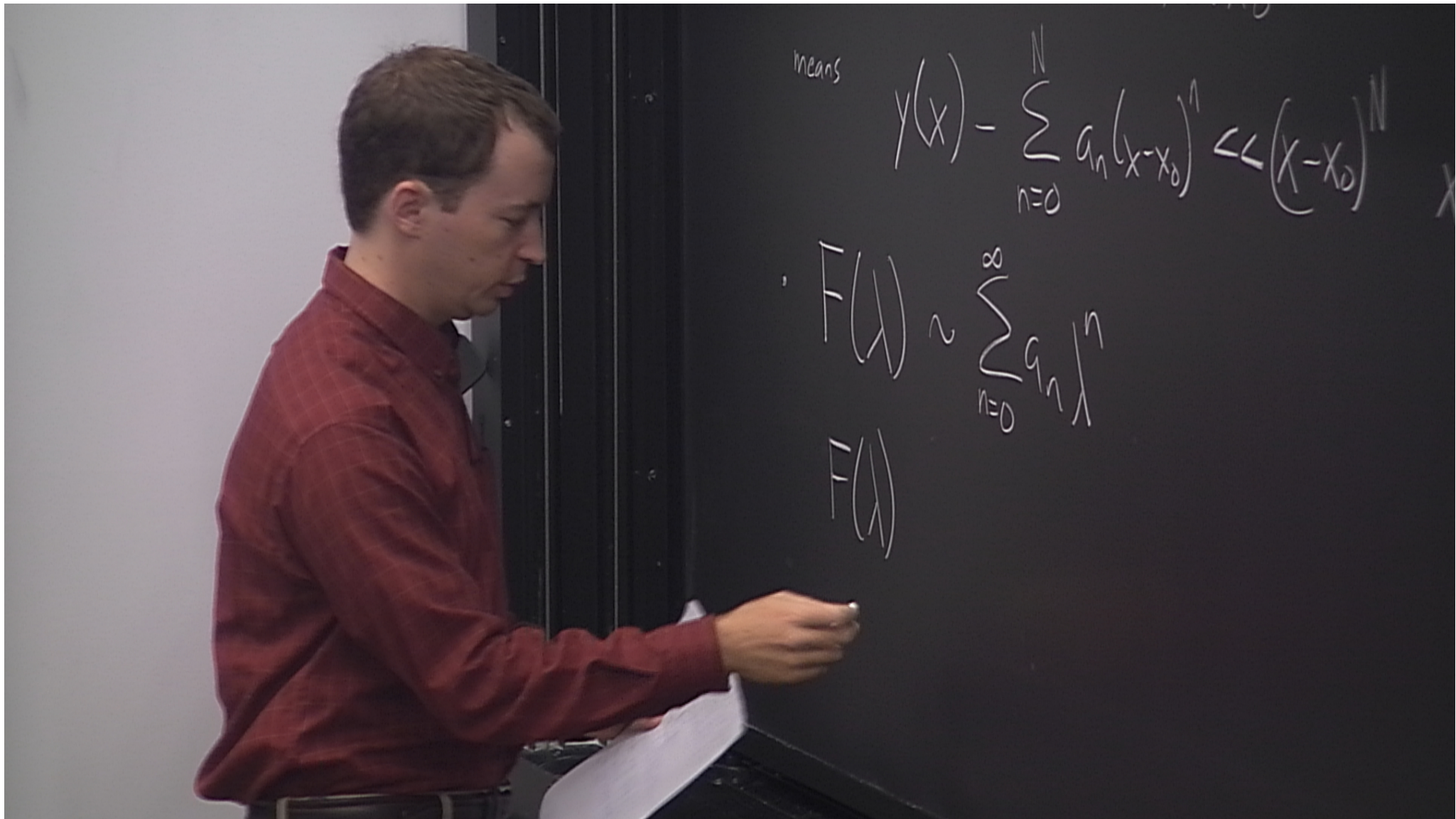


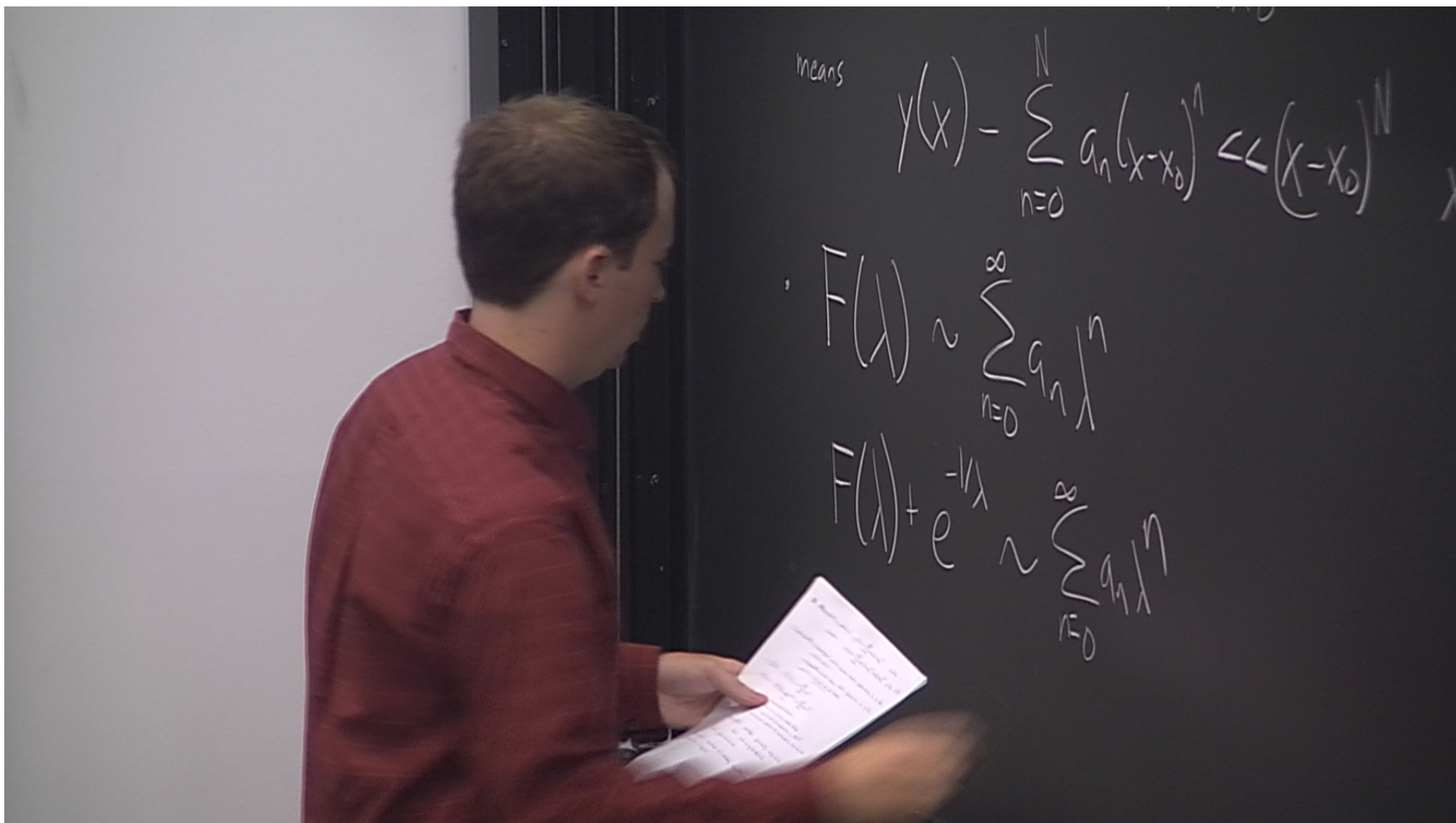
$$3. \quad y(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad x \rightarrow x_0$$

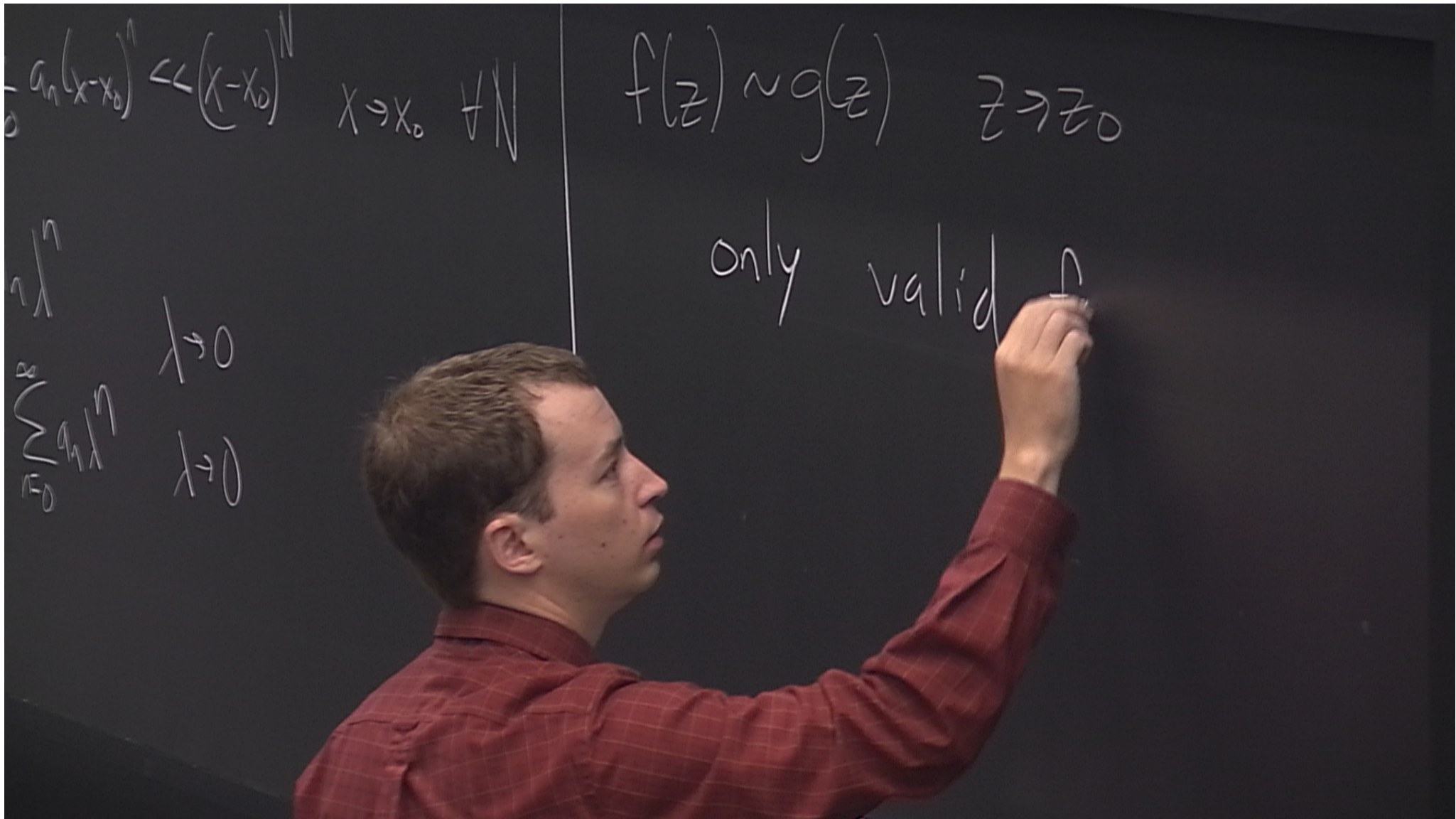
means

$$y(x) - \sum_{n=0}^N a_n (x-x_0)^n \ll (x-x_0)^N \quad x \rightarrow x_0 \quad \forall N$$

3. $y(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad x \rightarrow x_0$
means $y(x) - \sum_{n=0}^N a_n (x-x_0)^n \ll (x-x_0)^N$







$$|a_n(x-x_0)| < (x-x_0)^N \quad x \rightarrow x_0 \quad \forall N$$

$$\sum_{n=0}^{\infty} a_n x^n \quad x \rightarrow 0$$

$$f(z) \sim g(z) \quad z \rightarrow z_0$$

only valid

$$|a_n(x-x_0)^n| < (x-x_0)^N \quad x \rightarrow x_0 \quad \forall N$$

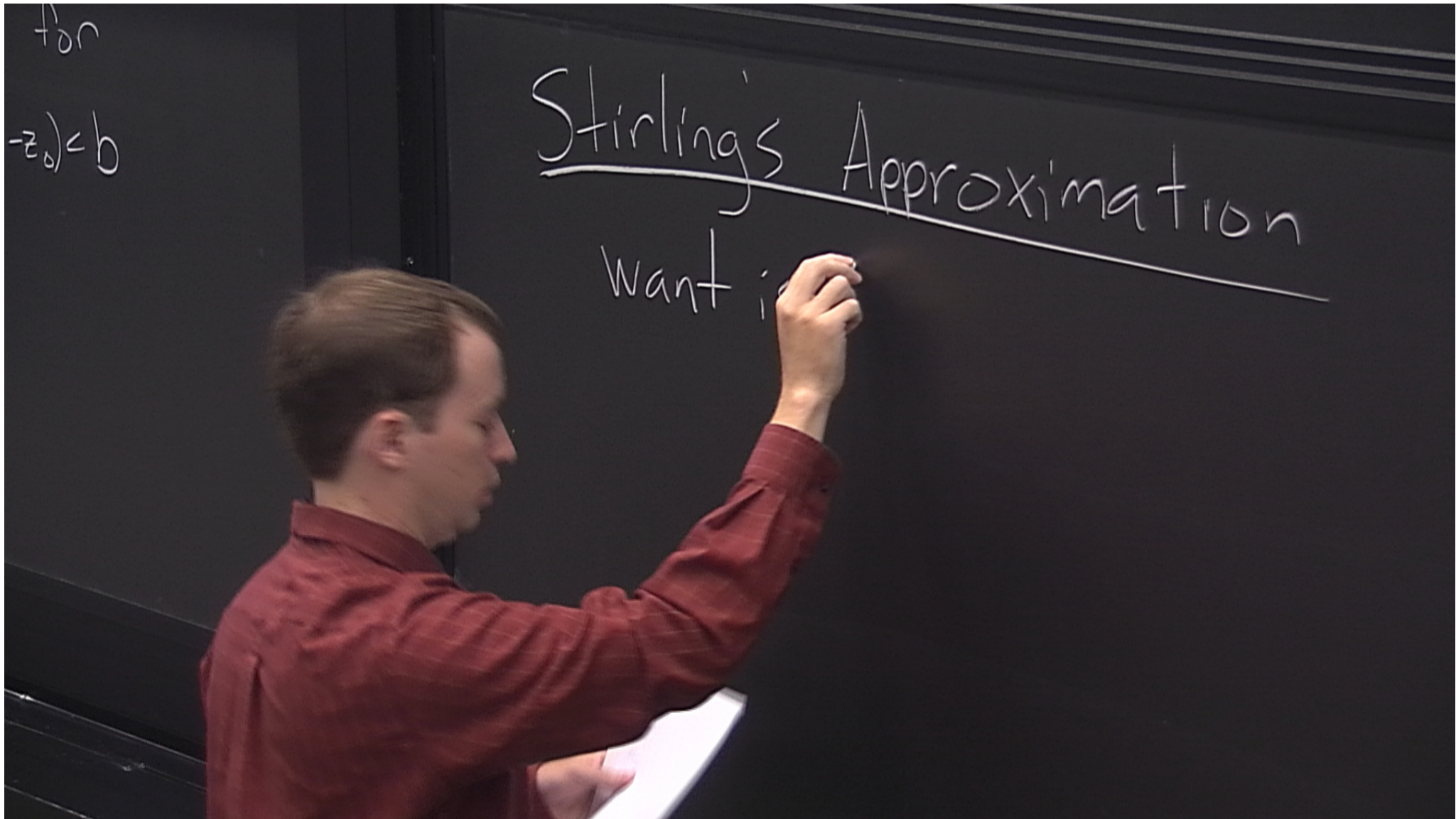
$$\sum_{n=0}^{\infty} a_n x^n \quad x \rightarrow 0$$

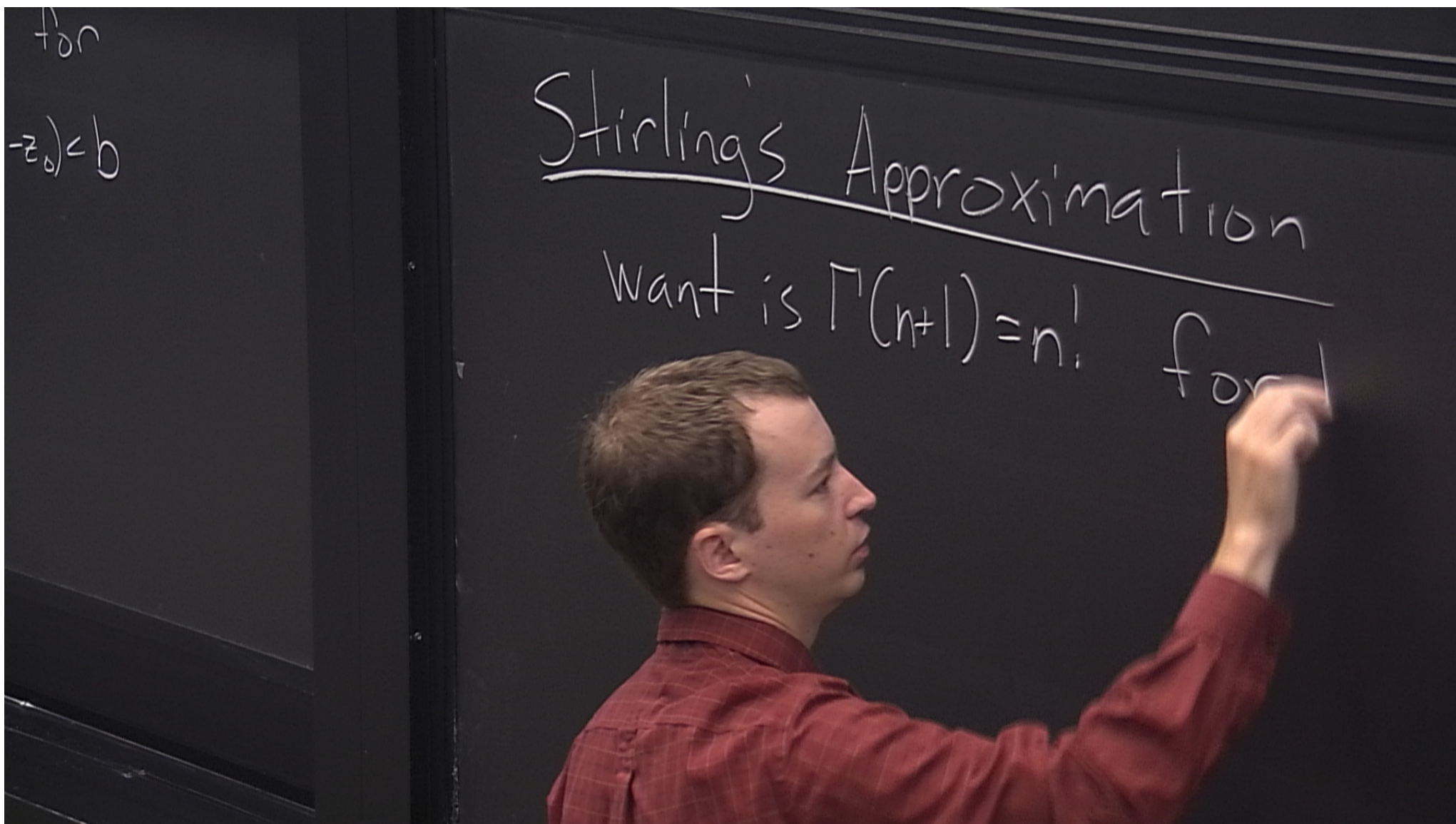
$$f(z) \sim g(z) \quad z \rightarrow z_0$$

only valid for

$$a < \arg(z-z_0) < b$$

Stokes' phenomenon

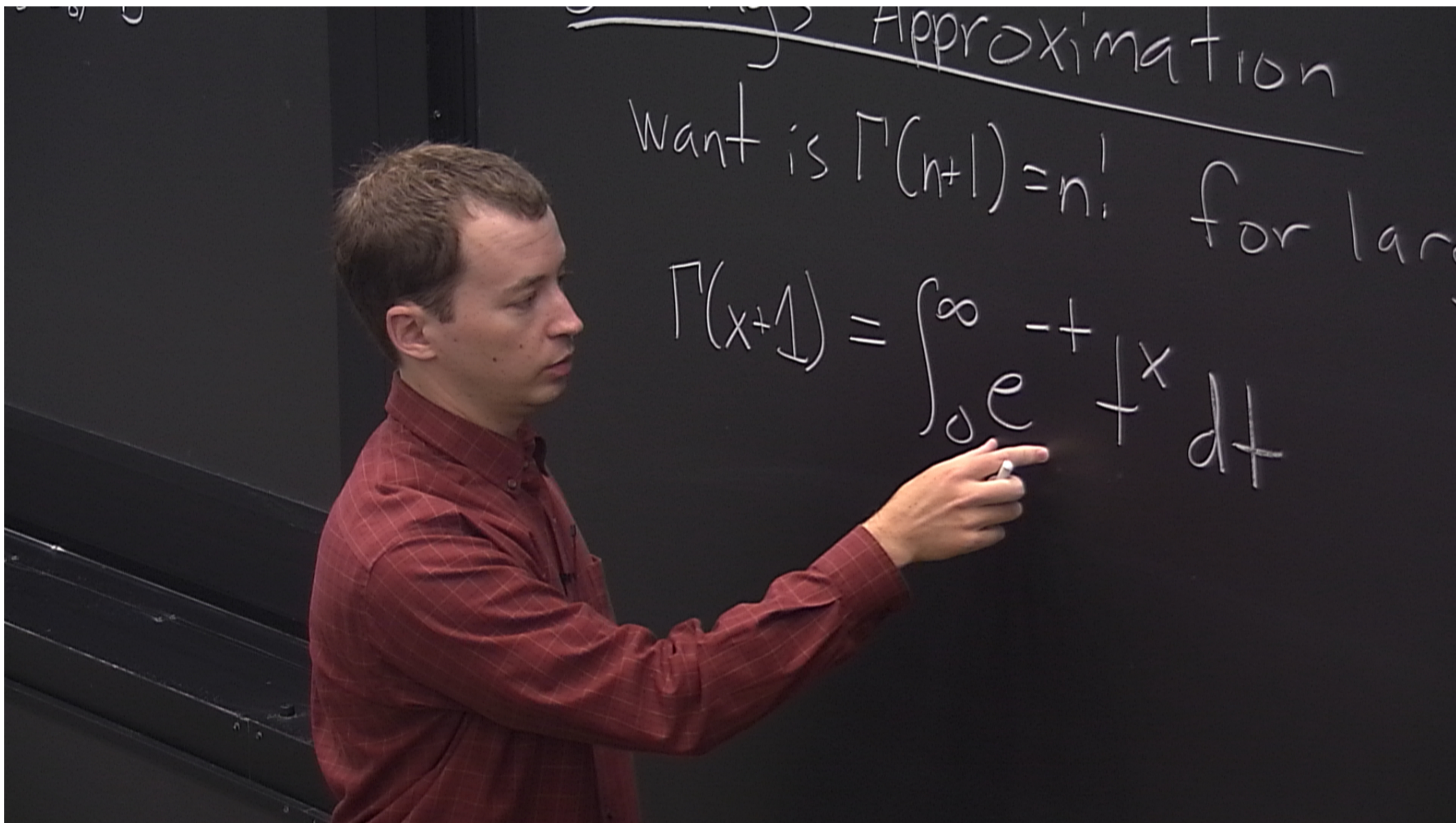




Stirling's Approximation

want is $\Gamma(n+1) = n!$ for large real n

$$\Gamma(x+1) = \frac{e^x}{x^x}$$



$$f(x) \leq g(x)$$

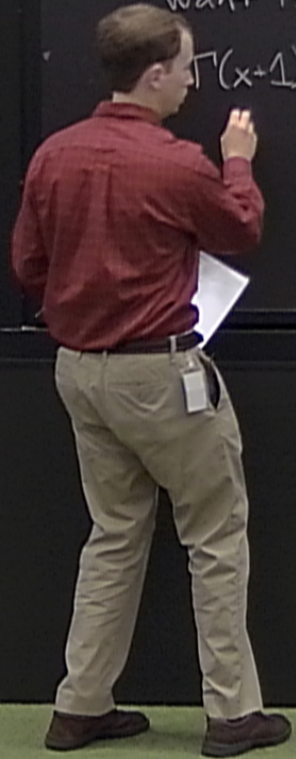
$$\frac{f(x)}{g(x)} = 1$$

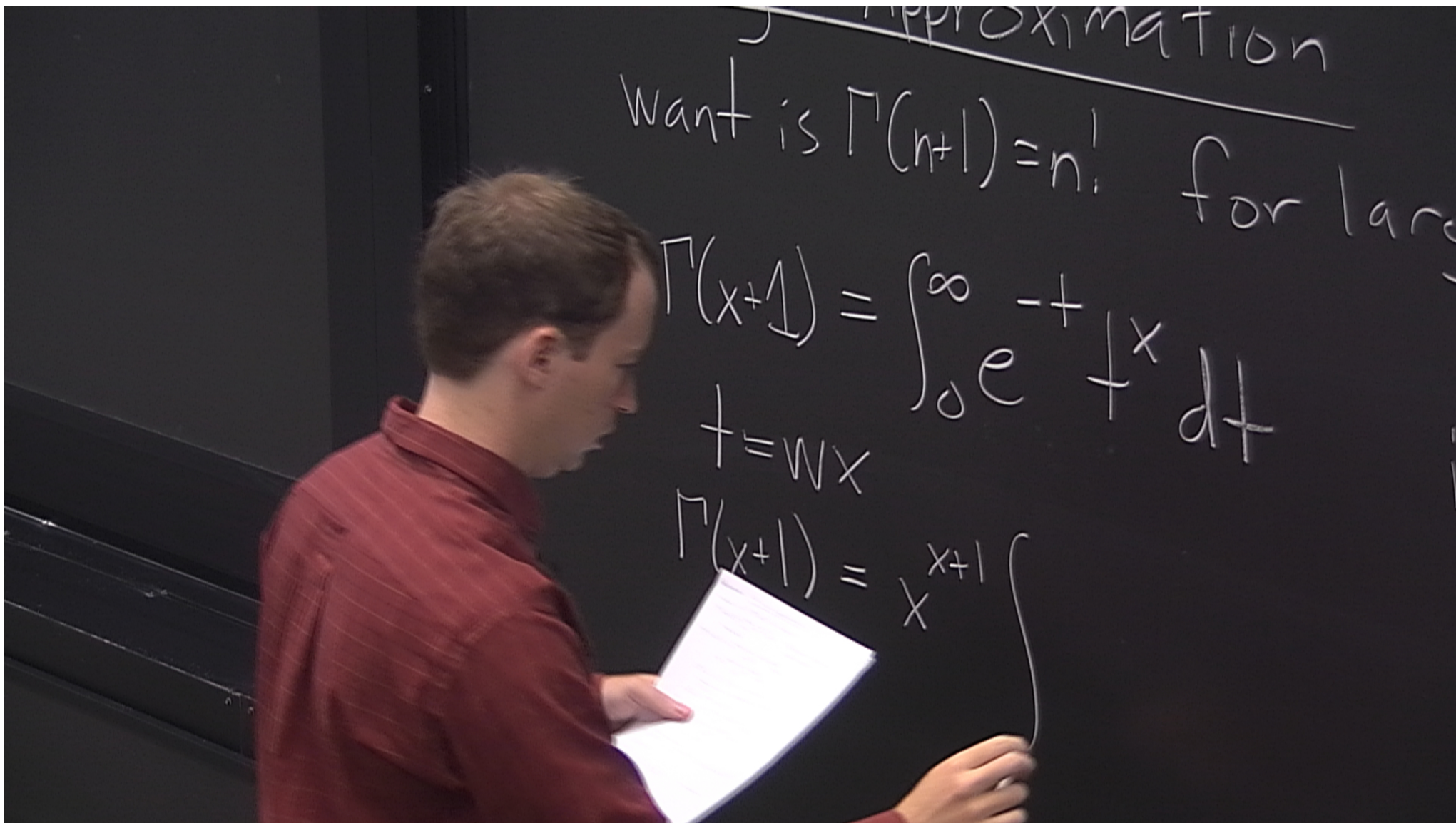
$g(z)$ $z \rightarrow z_0$
 valid for
 $a < \arg(z - z_0) < b$
 phenomenon

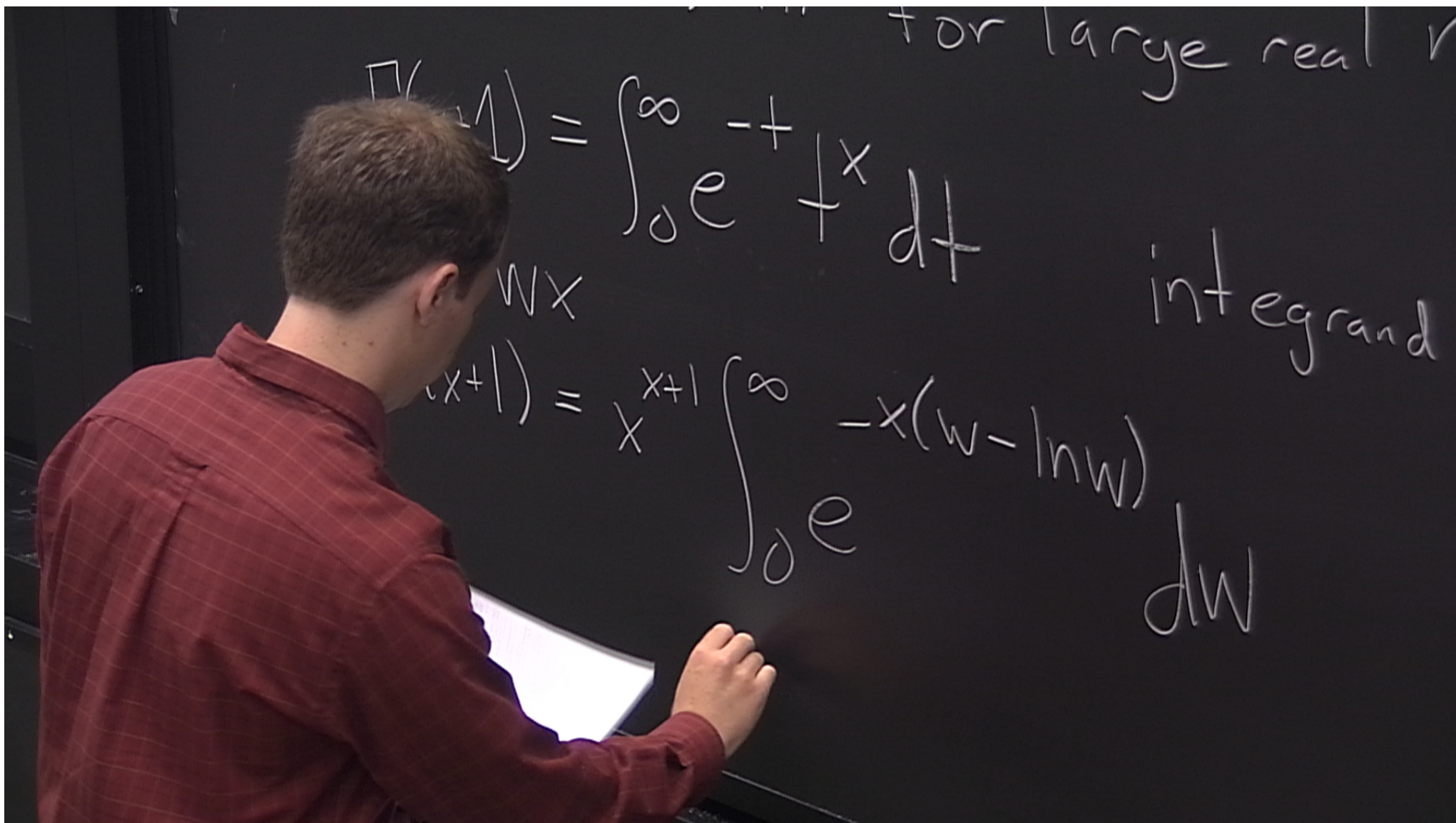
Stirling's Approximation

want is $\Gamma(n+1) = n!$ for large real n

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x dt \quad \text{integrand large only in small region}$$







Stirling's Approximation

want is $\Gamma(n+1) = n!$ for large real n

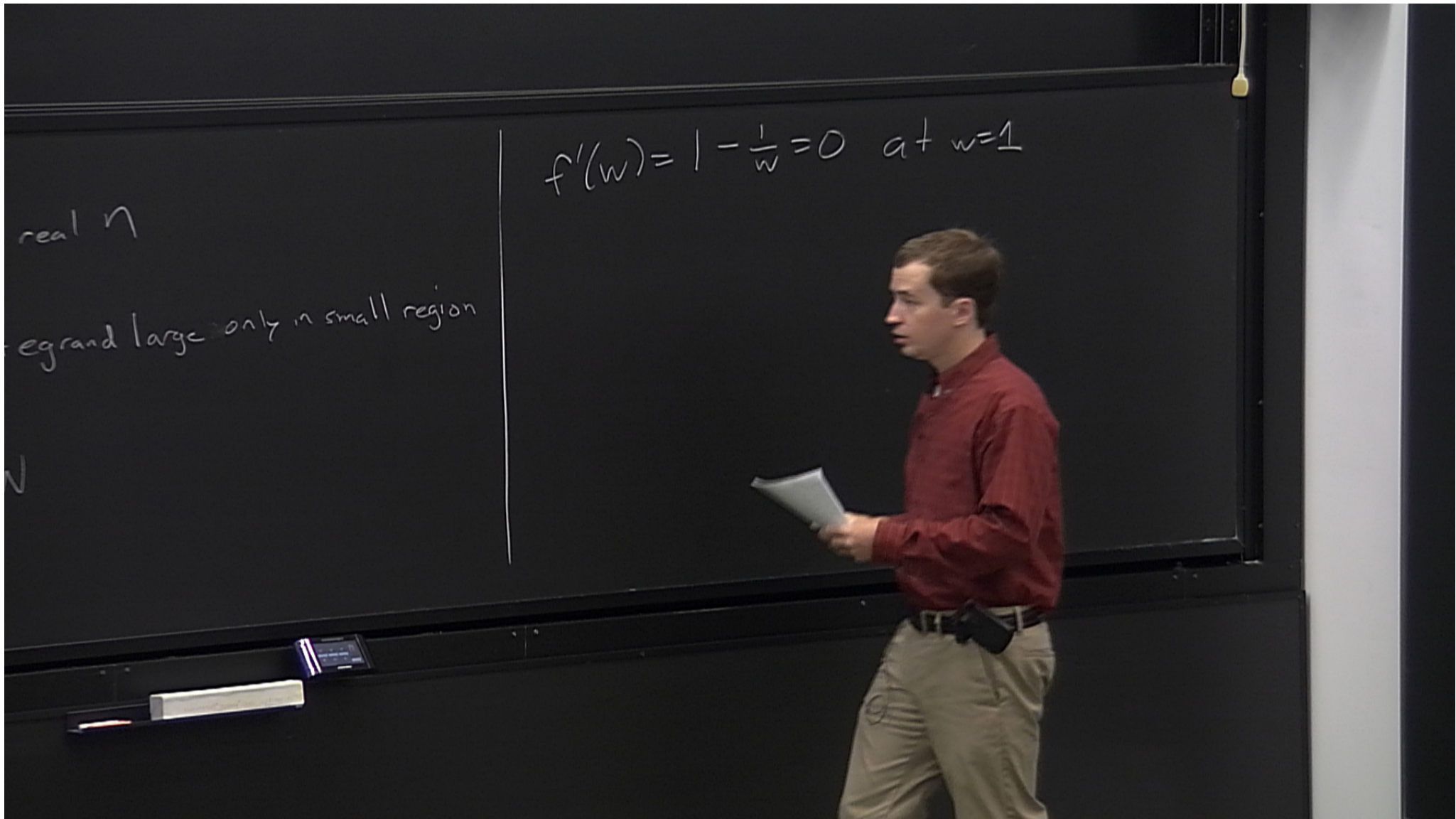
$$\Gamma(x+1) = \int_0^{\infty} e^{-t} t^x dt$$

integrand large small region

$$t = wx$$

$$\Gamma(x+1) = x^{x+1} \int_0^{\infty} e^{-x(w - \ln w)} dw$$

$$w - \ln w = f(w)$$



in small region

$$f'(w) = 1 - \frac{1}{w} = 0 \text{ at } w=1$$

$$f(w) = 1 + \frac{1}{2}(w-1)^2 + \dots$$

$$\Gamma(x+1) = x^{x+1} e^{-x} \int_0^{\infty} e^{-\frac{x}{2}(w-1)^2 + \dots}$$

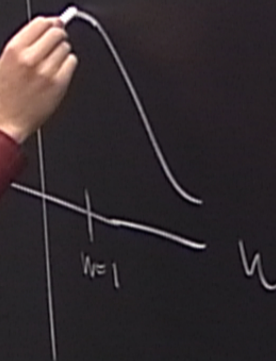
$$\approx x^{x+1} e^{-x} \int e^{-\frac{x}{2}(w-1)^2} dw$$

$$\frac{\Gamma(x+1)}{\sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x}} \sim 1$$

valid for
 $\arg(z-z_b) < b$

$$\frac{\Gamma(x+1)}{\sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x}} \sim 1 + \frac{1}{12x} + \dots \quad x \rightarrow \infty$$

$e^{-x\Gamma(u)}$

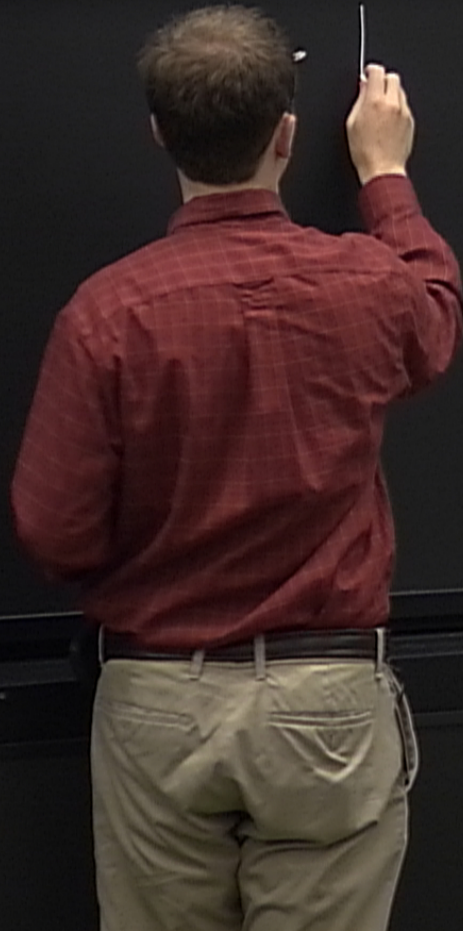


point Method

$$f(z) = \int_C g(w) e^{zf(w)} dw$$

for large real z

$\hat{t}$ tangent vector to curve at w_0



Suppose $\nabla u = 0$

$$w = x + iy$$

$$\frac{\partial u}{\partial z}$$

ary

$$\frac{\partial u}{\partial x} = 0 = -\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$$

near w_0

$$\nabla v \cdot \hat{t} = 0$$

$$\nabla u \parallel \hat{t}$$

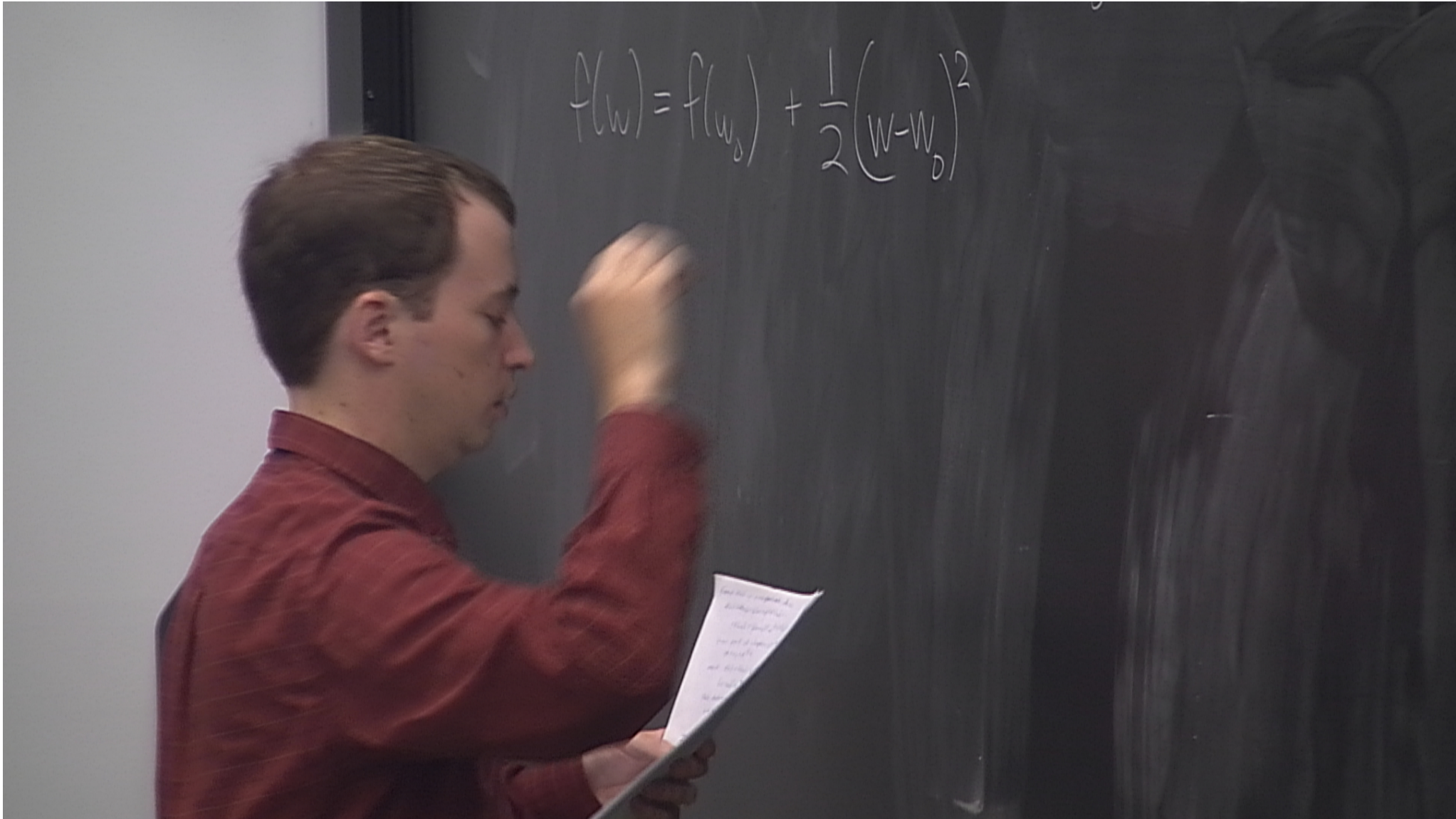
along path

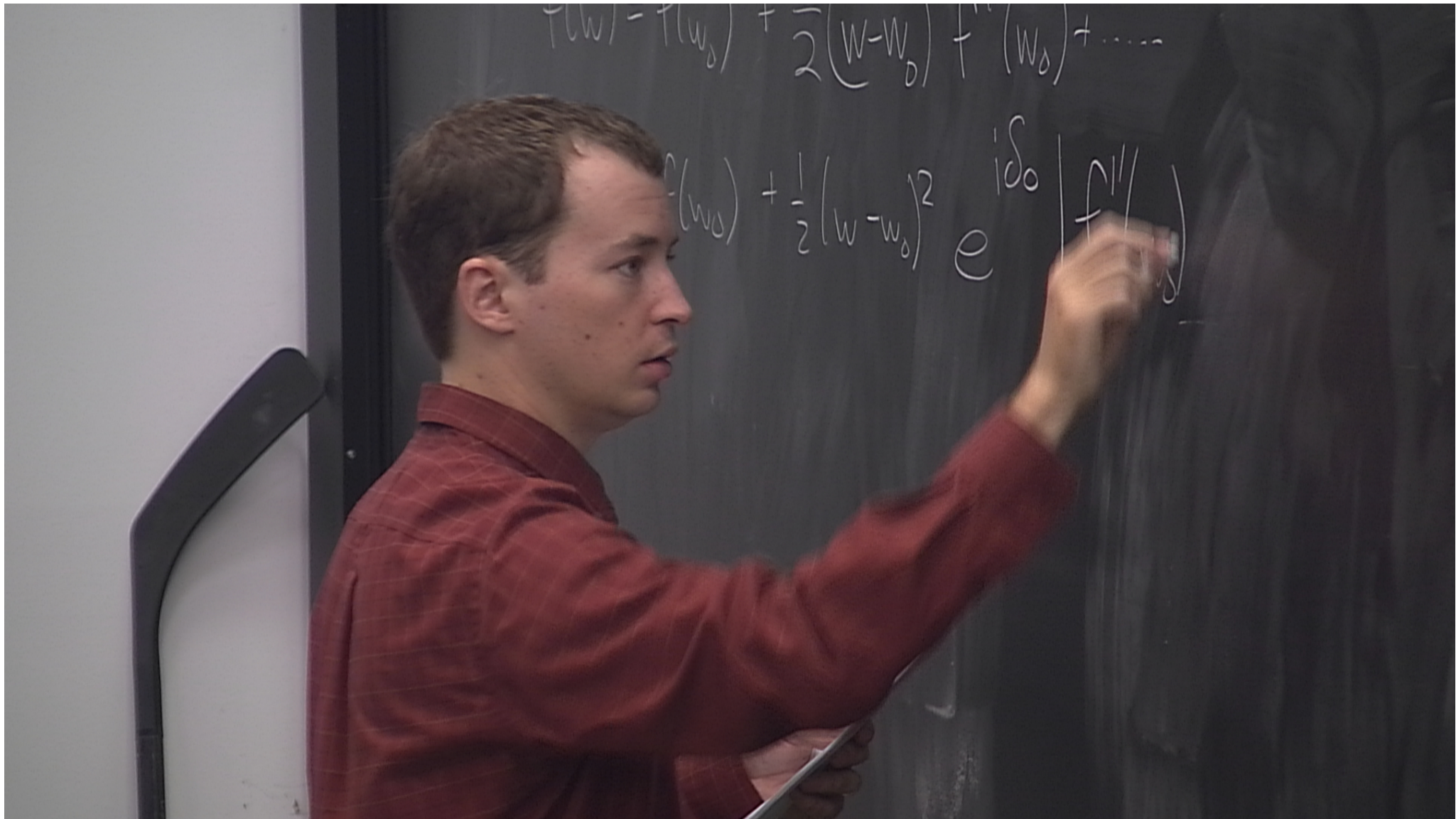
steepest descent

$$f'(w_0) = 0$$

w_0 is a critical point
inflection point

so phase is stationary

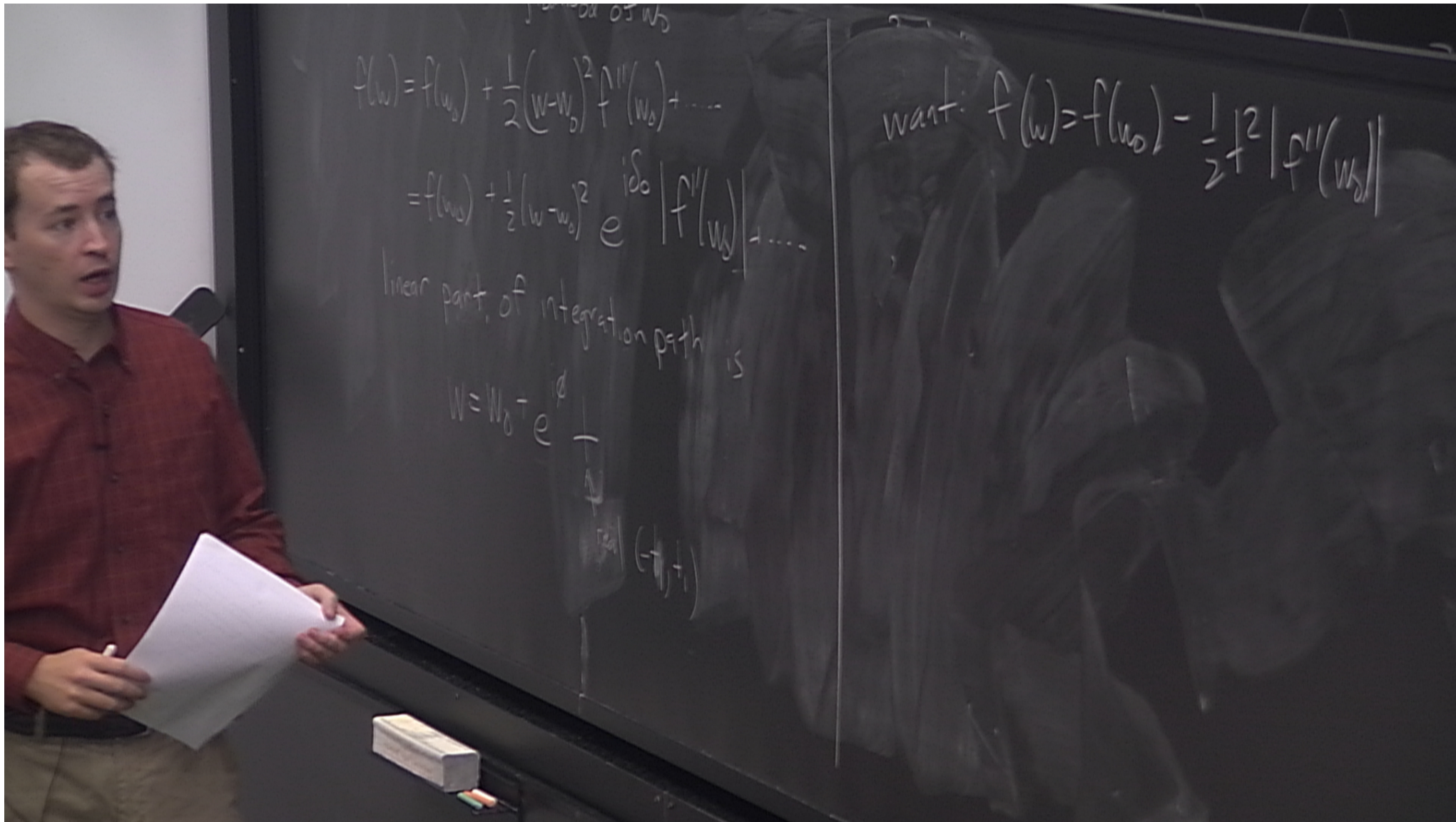


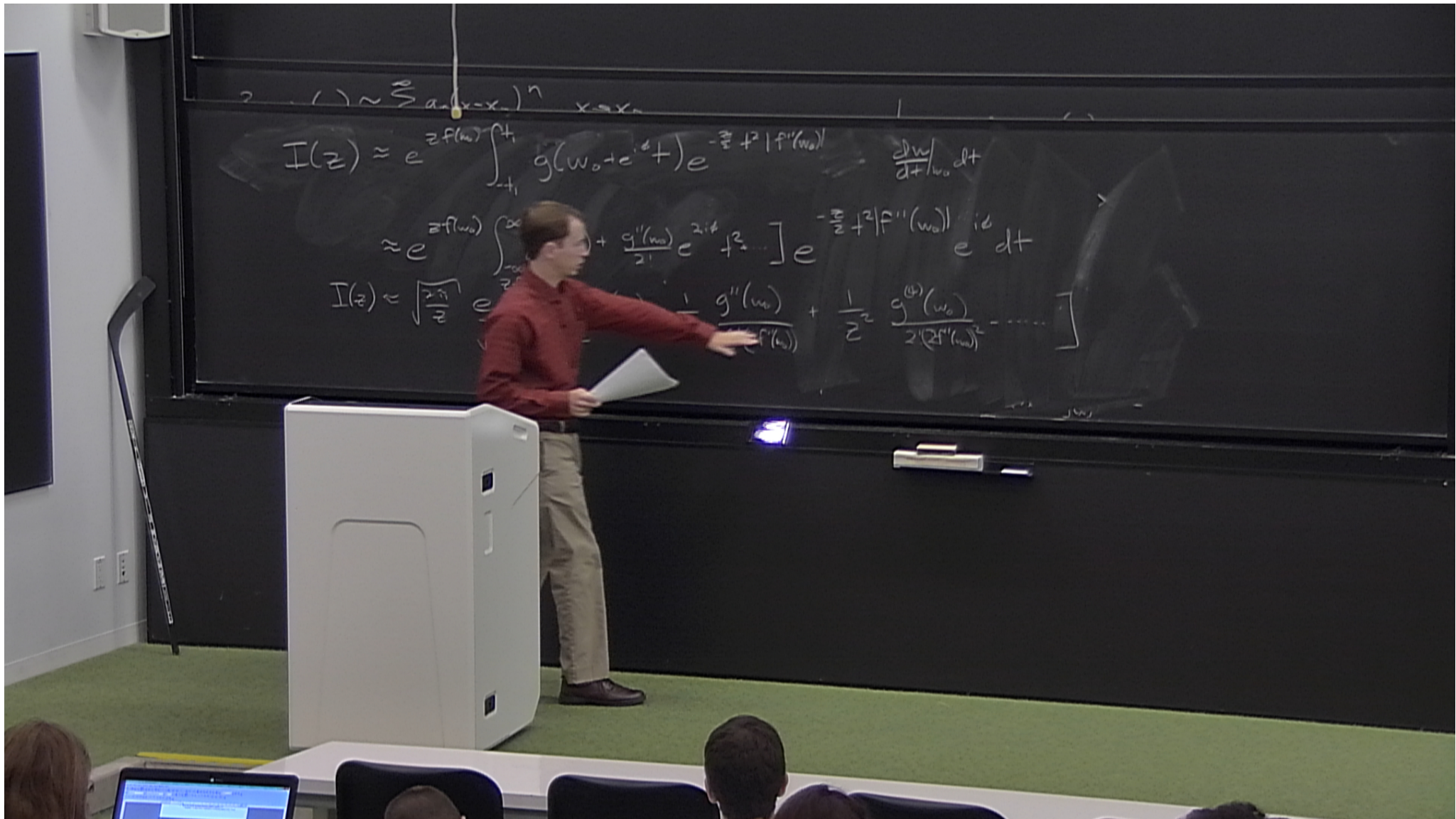


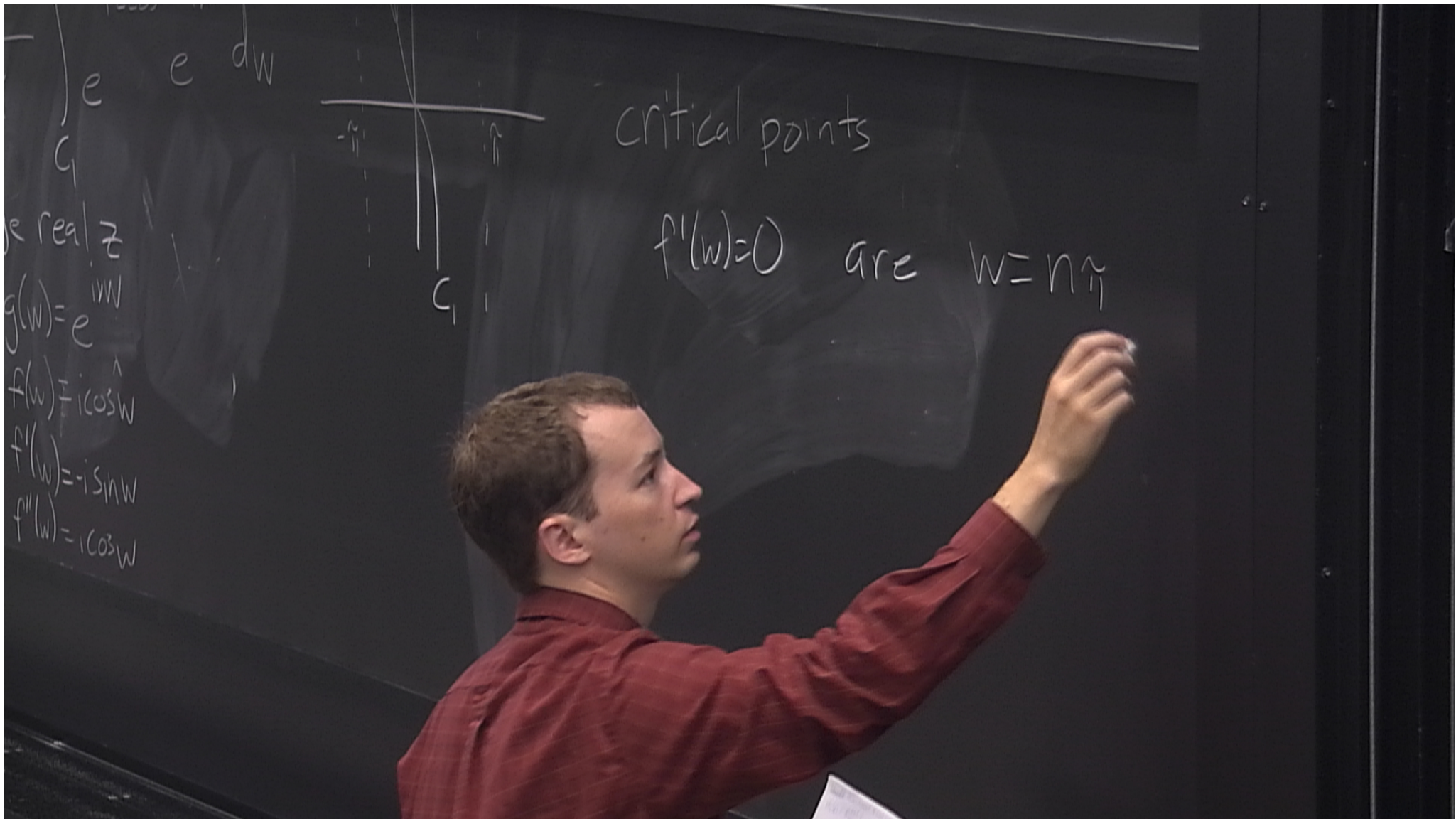
Expand $f(w)$ in a neighborhood of w_0

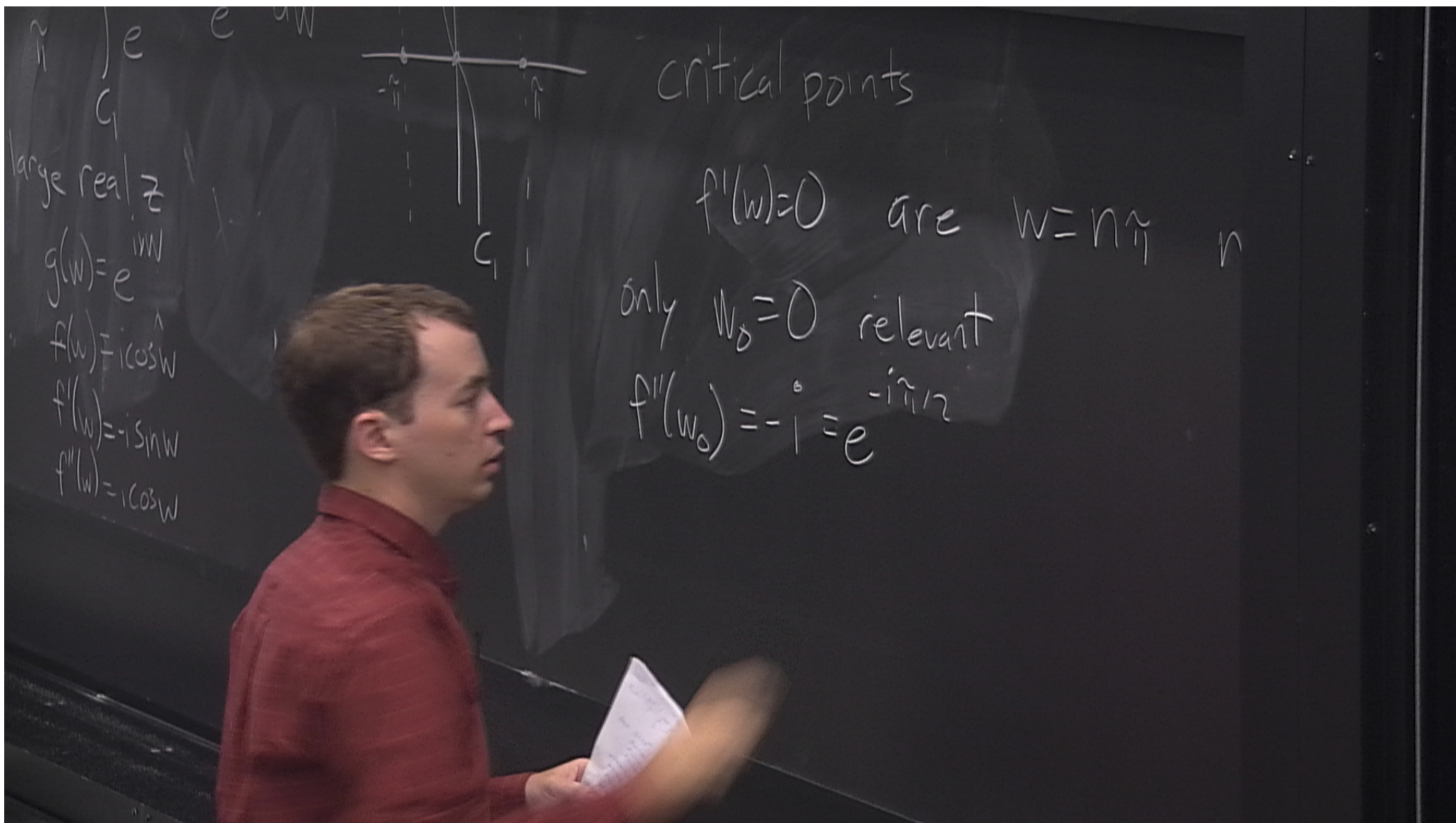
$$f(w) = f(w_0) + \frac{1}{2}(w-w_0)^2 f''(w_0) + \dots$$

$$f(w_0) + \frac{1}{2}(w-w_0)^2 e^{i\delta_0} |f''(w_0)|$$









path angle satisfies

$$e^{-j\pi/2} = 2j\phi \quad \pm j\pi$$

$$e^{j\pi/2} = 2j\phi$$

$$\phi = \frac{1}{2} \left(\frac{\pi}{2} \right)$$

$$\text{or } \frac{3\pi}{4}$$

