

Title: 13/14 PSI - Distributions & Special Functions - Lecture 3

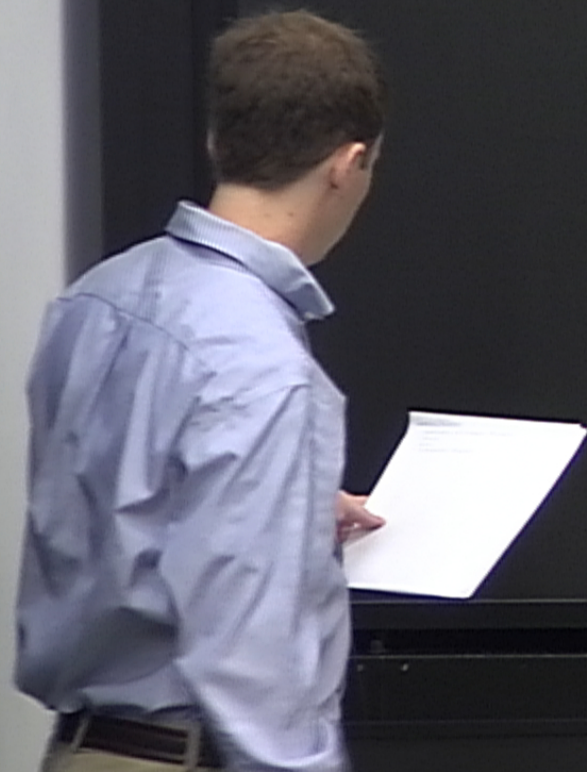
Date: Aug 23, 2013 09:00 AM

URL: <http://pirsa.org/13080018>

Abstract:

Special Functions

Orthogonal Polynomial



Special Functions

Orthogonal Polynomial

Gamma

Zeta

Hypergeometric Functions

Explicit construction

want $\int_a^b P_n x^p w dx = 0$

$$p = 0, \dots, n-1$$

Explicit construction

want $\int_a^b P_n x^p w dx = 0$

$$p = 0, \dots, n-1$$

$$p=0 \quad n=1$$

$$\int_a^b P_n w dx = 0$$

Explicit construction

want $\int_a^b P_n x^p w dx = 0$

$$p = 0, \dots, n-1$$

$$p=0 \quad n=1$$

$$\int_a^b P_1 w dx = 0$$

$$P_1 w \propto \frac{d}{dx}(Y_1) \quad \text{and} \quad Y_1(a) = 0 = Y_1(b)$$

$$= 0 \quad p = 0, \dots, n-1$$

$$\text{and } Y_1(a) = 0 = Y_1(b) \quad \text{choose } Y_1 = (x-a)(x-b)$$

$$n=2 \quad p=1$$

$$w p_2 \propto \frac{d^2}{dx^2} Y_2$$

$$\text{and } Y_2 = (x-a)^2(x-b)^2$$

$$\int_a^b \left(\frac{d^2 Y_2}{dx^2} \right) x dx = \left(\frac{d}{dx} Y_2 \right) x \Big|_a^b - \int_a^b \frac{d}{dx} Y_2 dx$$

In general:
w $P_n \propto \frac{d^n}{dx^n} Y_n$

$$Y_n = (x-a)^n (x-b)^n$$

$$\int_a^b \left(\frac{d^n Y}{dx^n} \right) x^p dx \quad \text{for pen}$$

$$\frac{d}{dx} Y_2 dx$$

In general:
w $P_n \propto \frac{d^n}{dx^n} Y_n$

$$Y_n = (x-a)^n (x-b)^n$$

$$\int_a^b \left(\frac{d^n}{dx^n} Y_n \right) x^p dx \quad \text{for } p \leq n$$

Suppose $a = -\infty$ b finite

$$Y_n = (x-b)^n f_n$$

$$f_n \rightarrow 0 \quad \text{faster than } (x-b)^{-n} \\ a \rightarrow -\infty$$

$$Y_n = (X-b)$$

$$w P_n = \frac{d}{dx^n} Y_n$$

$$Y_n = s^n f_n$$

with

$$s = \begin{cases} (x-a)(x-b) \\ x-a \\ x-b \\ 1 \end{cases}$$

if a, b finite
 if a finite $b = \infty$
 if b finite $a = -\infty$
 $a = -\infty$ $b = \infty$

choose $f_n = w$

Generalized Rodrigues' Formula

$$P_n(x) = \frac{C_n}{w(x)} \frac{d^n}{dx^n} (w(x) s(x))$$

$$\langle P_n, P_m \rangle_w = \delta_{nm}$$

if w is finite + differentiable

$$w(a)s(a) = 0 = w(b)s(b) \quad \text{if } a, b \text{ finite}$$

$w(x)s(x) \rightarrow 0$ faster than any power of x as $x \rightarrow \pm\infty$

case 1:

$$P_1 = \pm 2x C_1$$

$$\frac{d}{dx} \frac{dw}{w} = \pm 2x$$

$$w(x) = C \exp(\pm x^2)$$

case 1:

$$P_1 = \pm 2x c_1$$

$$P(cx+d)$$

$$\frac{d_1}{w} \frac{dw}{dx} = \pm 2x$$

$$w(x) = C \exp(\pm x^2)$$

want $w(x) \rightarrow 0$ as $x \rightarrow \pm\infty$

$$w(x) = \exp(-x^2)$$

$$P_n(x) = c_n \exp(x^2) \frac{d^n}{dx^n} (\exp(-x^2))$$

Hermite polynomials

$$c_n = (-1)^n$$

case 2+3: tutorials

Gamma Function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$

Gamma Function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \quad \operatorname{Re}(z) > 0$$

Gamma Function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \quad \operatorname{Re}(z) > 0$$

Functional Equation

$$\frac{d}{dt}(t^z e^{-t}) = z t^{z-1} e^{-t} - t^z e^{-t}$$

$$t^z e^{-t} \Big|_0^{\infty} = z \int_0^{\infty} t^{z-1} e^{-t} dt - \int_0^{\infty} t^z e^{-t} dt$$

Gamma Function

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \quad \operatorname{Re}(z) > 0$$

Functional Equation

$$\frac{d}{dt}(t^z e^{-t}) = z t^{z-1} e^{-t} - t^z e^{-t}$$

$$t^z e^{-t} \Big|_0^{\infty} = z \int_0^{\infty} t^{z-1} e^{-t} dt - \int_0^{\infty} t^z e^{-t} dt$$

$$0 = z\Gamma(z) - \Gamma(z+1)$$

$$\boxed{\Gamma(z+1) = z\Gamma(z)}$$

$$\Gamma(1) = 1$$

$$\Gamma(n+1) = n!$$

Extend to $\operatorname{Re}(z) < 0$ with Func. Eq.

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}$$

$$\operatorname{Re}(z)+n > 0$$

Extend to $\operatorname{Re}(z) < 0$ with Func. Eq.

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}$$

$$\operatorname{Re}(z)+n > 0$$

Suppose $z = -n + \varepsilon$

$$\Gamma(z) = \frac{\Gamma(1+\varepsilon)}{\varepsilon(\varepsilon-1)\cdots(\varepsilon-n)} \approx \frac{(-1)^n}{\varepsilon n!}$$

$$\operatorname{Re}(z) + n > 0$$

$$\sim \frac{(-1)^n}{n!}$$

pole at $z = 0, -1, \dots$
with residue $\frac{(-1)^n}{n!}$

Extend to Re $z > 0$ with

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}$$

$$\operatorname{Re}(z)+n >$$

Suppose $z = -n + 1 + \varepsilon$

$$\Gamma(z) = \frac{\Gamma(1+\varepsilon)}{\varepsilon(\varepsilon-1)\cdots(\varepsilon-n)} \approx \frac{(-1)^n}{\varepsilon n!}$$

Another def of Γ

$$\Gamma_p(z) = \frac{p! p^z}{z(z+1)\cdots(z+p)} = \frac{p^z}{z(1+\frac{z}{1})\cdots(1+\frac{z}{p})}$$

can show: $\Gamma_p(1)$

Another def of Γ

$$\Gamma_p(z) = \frac{p! p^z}{z(z+1)\cdots(z+p)} = \frac{p^z}{z(1+\frac{1}{z})\cdots(1+\frac{1}{z+p})}$$

can show: $\Gamma_p(1) = \frac{p}{p+1}$

$$\Gamma_p(z+1) = \frac{p}{z+p+1} z \Gamma_p(z)$$

$$\Gamma(z) = \lim_{p \rightarrow \infty} \Gamma_p(z)$$

Weierstrass Formula

$$p^z = e^{z \log p}$$

$$\Gamma_p(z) = \frac{e^{z(\log p - 1 - 1/2 - \dots - 1/p)} e^{z + z/2 + \dots + z/p}}{z(1+z) \dots \left(1 + \frac{z}{p}\right)}$$

$$\lim_{p \rightarrow \infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{p} - \log p \right) = \gamma$$

Weierstrass Formula

$$p^z = e^{z \log p}$$

$$\Gamma_p(z) = \frac{e^{z(\log p - 1 - 1/2 \dots - 1/p)} e^{z + z/2 + \dots + z/p}}{z(1+z) \dots (1 + \frac{z}{p})}$$

$$\lim_{p \rightarrow \infty} (1 + \frac{1}{2} + \dots + \frac{1}{p} - \log p) = \gamma \approx 0.577$$

$$\frac{1}{\Gamma'(z)} = z e^{\gamma z} \prod_{p=1}^{\infty} \left(1 + \frac{z}{p} e^{-z/p}\right)$$

z/p

$$\frac{1}{\Gamma'(z)} = z e^{\gamma z} \prod_{p=1}^{\infty} \left(1 + \frac{z}{p}\right) e^{-z/p}$$

$$\frac{1}{\Gamma'(z)} = z e^{\gamma z} \prod_{p=1}^{\infty} \left(1 + \frac{z}{p}\right) e^{-z/p}$$

$\frac{1}{\Gamma'(z)}$ no poles $\rightarrow \Gamma(z)$ has no zeros

$$\frac{1}{\Gamma'(z)} = z e^{\gamma z} \prod_{p=1}^{\infty} \left(1 + \frac{z}{p}\right) e^{-z/p}$$

$\frac{1}{\Gamma'(z)}$ no poles $\rightarrow \Gamma(z)$ has no zeros

$\frac{1}{\Gamma(z)}$ has zeros at $z=0, -1, -2, \dots$ $\rightarrow \Gamma(z)$ has poles at $z=0, -1, -2, \dots$

Complement Formula

$$\frac{1}{\Gamma(z)} \frac{1}{\Gamma(-z)} = -z^2 e^{\gamma z - \gamma z} \prod_{p=1}^{\infty} \left(1 - \frac{z^2}{p^2}\right) e^{-z/p + z/p}$$

$$\frac{1}{\Gamma(z)} \frac{-z}{\Gamma(1-z)} = -z^z \prod_{p=1}^{\infty} \left(1 - \frac{z^2}{p^2}\right)$$

Complement Formula

$$\frac{1}{\Gamma(z)} \frac{1}{\Gamma(-z)} = -z^2 e^{\gamma z - \gamma z} \prod_{p=1}^{\infty} \left(1 - \frac{z^2}{p^2}\right) e^{-z/p + z/p}$$

$$\begin{aligned} \frac{1}{\Gamma(z)} \frac{-z}{\Gamma(1-z)} &= -z^z \prod_{p=1}^{\infty} \left(1 - \frac{z^2}{p^2}\right) \\ &= \frac{\sin(\pi z)}{\pi} \end{aligned}$$

$$\begin{aligned} \text{Zeta} \\ \zeta(s) &= \sum_{n=1}^{\infty} n^{-s} \quad \operatorname{Re}(s) > 1 \\ &= \left(\prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \right) \end{aligned}$$

Zeta
 $\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad \operatorname{Re}(s) > 1$

$$\left(= \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \right)$$

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

change variables $t = ku$ $k \in \mathbb{N}$

$$\Gamma(x) = k^x \int_0^{\infty} u^{x-1} e^{-ku} du$$

change variables $t = ku$ $k \in \mathbb{N}$

$$\Gamma(x) = K^x \int_0^{\infty} u^{x-1} e^{-ku} du$$

$$\frac{1}{K^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} u^{x-1} e^{-ku} du$$

$$e(s) > 1$$

change variables $t = ku$ $k \in \mathbb{N}$

$$\Gamma(x) = k^x \int_0^{\infty} u^{x-1} e^{-ku} du$$

$$\sum_{k=1}^{\infty} \frac{1}{k^x} = \sum_{k=1}^{\infty} \frac{1}{\Gamma(x)} \int_0^{\infty} u^{x-1} e^{-ku} du$$

change variables $t = ku$ $k \in \mathbb{N}$

$$\Gamma(x) = k^x \int_0^{\infty} e^{-ku} u^{x-1} du$$

$$\sum_{k=1}^{\infty} \frac{1}{k^x} = \sum_{k=1}^{\infty} \frac{1}{\Gamma(x)} \int_0^{\infty} e^{-ku} u^{x-1} du$$

$$= \frac{1}{\Gamma(x)} \int_0^{\infty} u^{x-1} \sum_{k=1}^{\infty} e^{-ku} du$$

$$\sum_{k=1}^{\infty} \frac{1}{k^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} u^{x-1} \frac{e^{-u}}{1-e^{-u}} du$$

$$\zeta(x) \Gamma(x) = \int_0^{\infty} \frac{u^{x-1}}{e^u - 1} du$$

$$\sum_{k=1}^{\infty} \frac{1}{k^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} u^{x-1} \frac{e^{-u}}{1-e^{-u}} du$$

$$\zeta(x) \Gamma(x) = \int_0^{\infty} \frac{u^{x-1}}{e^u - 1} du$$

$$\Lambda(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

$$\Lambda(s) = \Lambda(1-s)$$

$$\sum_{k=1}^{\infty} \frac{1}{k^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} u^{x-1} \frac{e^{-u}}{1-e^{-u}} du$$

$$\zeta(x) \Gamma(x) = \int_0^{\infty} \frac{u^{x-1}}{e^u - 1} du$$

$$\Lambda(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

$$\Lambda(s) = \Lambda(1-s)$$

Λ analytic except at 0

$$\sum_{k=1}^{\infty} \frac{1}{k^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} u^{x-1} \frac{e^{-u}}{1-e^{-u}} du$$

$$\zeta(x) \Gamma(x) = \int_0^{\infty} \frac{u^{x-1}}{e^u - 1} du$$

$$\Lambda(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

$$\Lambda(s) = \Lambda(1-s)$$

Λ analytic except at 0 and 1

$$\sum_{k=1}^{\infty} \frac{1}{k^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} u^{x-1} \frac{e^{-u}}{1-e^{-u}} du$$

$$\zeta(x) \Gamma(x) = \int_0^{\infty} \frac{u^{x-1}}{e^u - 1} du$$

$$\Lambda(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

$$\Lambda(s) = \Lambda(1-s)$$

Λ analytic except at 0 and 1

$S(s)$ has a pole at $s=1$ residue 1

$S(s)$ has zeros at $s=-2, -4, -6, \text{etc}$

except at 0 and 1

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

$$\Gamma(x) = \int_0^\infty u^{x-1} \sum_{k=1}^\infty e^{-ku} du$$

$$\sum_{k=1}^\infty \frac{1}{k^x} = \frac{1}{\Gamma(x)} \int_0^\infty u^{x-1} \frac{e^{-u}}{1-e^{-u}} du$$

$$\zeta(x) \Gamma(x) = \int_0^\infty \frac{u^{x-1}}{e^u - 1} du$$

$$\Lambda(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

$$\Lambda(s) = \Lambda(1-s)$$

Λ analytic except at 0 and 1

Weierstrass Formula
 $z = z \log p$

$$\frac{1}{\Gamma'(z)} = ze$$

Hypergeometric Function

$${}_2F_1(a, b; c; z) \equiv \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} z^n$$

Weierstrass Formula
 $z = z \log p$

$$\frac{1}{\Gamma'(z)} = z e$$

Hypergeometric Function

$${}_2F_1(a, b; c; z) \equiv \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} z^n$$

Hypergeometric Function

$${}_2F_1(a, b; c; z) \equiv \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(1+n)} z^n$$

$$\begin{aligned} {}_2F_1(-n, b; b; -z) &= \frac{\Gamma(b)}{\Gamma(-n)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(-n+k)\Gamma(b+k)}{\Gamma(b+k)\Gamma(1+k)} (-z)^k \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(-n+k)}{\Gamma(-n)} \frac{(-z)^k}{k!} \end{aligned}$$

Weierstrass Formula

$$z = z \log p$$

$$\frac{1}{\Gamma(z)} = z e$$

$$= 1 + (-n)(-z) + \frac{(-n)(-n+1)(-z)^2}{2!}$$

$$= 1 + (-n)(z) + \frac{(-n)(-n+1)(-z)^2}{2!}$$

$$= (1+z)^n$$

Strass formula

$$z \approx \log p$$

$$\frac{1}{\Gamma'(z)} =$$

$$= 1 + (-n)(-z) + \frac{(-n)(-n+1)(-z)^2}{2!}$$

$$= \underline{(1+z)^n}$$

$$P_e^m(x) = \frac{(l+m)!}{(l-m)!} \frac{(1-x^2)^{m/2}}{2^m m!} {}_2F_1\left(m-l, m+l+1, m+1, \frac{1-x}{2}\right)$$

Diff Eq

$$z(1-z)\left(\frac{d}{dz}\right)^2 F_1 + c - (a+b+1)z \frac{d}{dz} F_1 - ab F_1 = 0$$

Diff Eq

$$z(1-z)\left(\frac{d}{dz}\right)^2 {}_2F_1 + c - (a+b+1)z \frac{d}{dz} {}_2F_1 - ab {}_2F_1 = 0$$

Integral Rep

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-t)^{-a+b-1} (1-t)^{c-b-1} dt$$

Diff Eq

$$z(1-z)\left(\frac{d}{dz}\right)^2 {}_2F_1 + c - (a+b+1)z \frac{d}{dz} {}_2F_1 - ab {}_2F_1 = 0$$

Integral Rep

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-tz)^{-a} t^{b-1} (1-t)^{c-b-1} dt$$

Diff Eq

$$z(1-z)\left(\frac{d}{dz}\right)^2 {}_2F_1 + c - (a+b+1)z \frac{d}{dz} {}_2F_1 - ab {}_2F_1 = 0$$

Integral Rep

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-tz)^{-a} t^{b-1} (1-t)^{c-b-1} dt$$