

Title: 13/14 PSI - Distributions & Special Functions - Lecture 2

Date: Aug 22, 2013 09:00 AM

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Abstract:

Composition with a Function

$$(u \circ f)(\varphi) = \int_{-\infty}^{\infty} u(f(x)) \varphi(x) dx$$

$$y = f(x)$$

$(u \circ f)$ only guaranteed to exist if

- f is C^∞
- $y = f(x)$ has

a Function

$$f(x)) \varphi(x) dx$$

exist if - f is C^∞

- $y=f(x)$ has a unique solution $x=g(y) \quad \forall y$
- $g'(y)$ does not change sign

then

$$(u \circ f)(\varphi) \equiv \int_{-\infty}^{\infty} u(y) \underbrace{\varphi(g(y)) |g'(y)| dy}_{\text{test function}}$$

has a unique solution $x = g(y) \quad \forall y$
not change sign

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is not change sign

$$(\delta \circ f)(\varphi) \equiv \sum_i \frac{\delta x_i}{|f'(x_i)|} (\varphi)$$

$\uparrow$
 sum is over roots of f
 works if f has only simple roots

Warn

Warnings: $u(\varphi)$ is only defined if φ is test function
" " " " φ is C^∞
 $u \cdot \varphi = \varphi u$

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$$u \cdot \varphi = \varphi u$$

$u \circ \varphi$ is generally difficult to define

linear theory

$$(\delta \circ f)(\varphi) \equiv \sum_i \frac{\delta_{x_i}}{|f'(x_i)|} (\varphi)$$

↑
sum is over roots of f

works if f has only simple roots

$$\delta_{x_i}(\varphi) = \varphi(x_i)$$

Orthogonal Functions

Sturm-Liouville Theory

Parseval's Theorem

Orthogonal Polynomials

Sepa

Functions

le Theory

neorem

Polynomials

Separation of Variables

$$\nabla^2 \phi = 0$$

$$\frac{1}{r} \frac{\partial^2 (r\phi)}{\partial r^2} - \frac{\hat{L}^2}{r} \phi = 0$$

$$\hat{L}^2 = -\frac{1}{\sin^2 \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

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$$\phi = R(r) Y(\theta, \phi)$$

$$\frac{1}{Rr} \frac{d^2 (rR)}{dr^2} - \frac{1}{r^2} \left(\frac{1}{Y} \hat{L}^2 Y \right) = 0$$

$$\hat{L}^2 Y = \underbrace{\ell(\ell+1)}_{\text{Separation constant}} Y$$

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$$r^2 \frac{d^2(rR)}{dr^2} - \ell(\ell+1)(rR) = 0$$

$$Y) = 0$$

0

$$\int \psi_\lambda(x) + \lambda w(x) \psi_\lambda(x) = 0$$

$$\Omega u_\lambda(x) + \lambda w(x) u_\lambda(x) = 0$$

solve in interval $[a, b]$

$$\int_a^b u_{\lambda'} \Omega u_\lambda dx = -\lambda \int_a^b u_{\lambda'} u_\lambda w dx$$

similarly starting with eq for $u_{\lambda'}$

$$\int_a^b (\Omega u_{\lambda'}) u_\lambda dx = -\lambda' \int_a^b u_{\lambda'} u_\lambda w dx$$

$$\int_a^b u_{\lambda'} \Omega$$

$$\text{If } \Omega = \frac{d}{dx} \left[S(x) \frac{d}{dx} \right] + t(x)$$

S-L

$$\text{If } \Omega = \frac{d}{dx} \left[s(x) \frac{d}{dx} \right] + t(x) \quad \text{S-L form}$$

$$\langle f, \Omega g \rangle - \langle \Omega f, g \rangle = s(fg' - gf') \Big|_a^b$$

$$\int_a^b (u_{\lambda'} \Omega u_{\lambda} - (\Omega u_{\lambda'}) u_{\lambda}) dx = (\lambda' - \lambda) \int_a^b u_{\lambda'} u_{\lambda} w dx$$

$u_{\lambda'}, u_{\lambda}$ orthogonal, if

$$\langle u_{\lambda}, \Omega u_{\lambda'} \rangle = \langle \Omega u_{\lambda}, u_{\lambda'} \rangle$$

Ω is self-adjoint
or Hermitian

means $u_{\lambda'}, u_{\lambda}$ orthogonal

$$\delta_{\lambda\lambda'} = \langle u_{\lambda'}, u_{\lambda} \rangle_w \leftarrow L^2[a, b]_w$$

$w(x)$ is the weight

$$\text{If } \Omega = \frac{d}{dx} \left[s(x) \frac{d}{dx} \right] + t(x) \quad \text{S-L form}$$

$$\langle f, \Omega g \rangle - \langle \Omega f, g \rangle = s(fg' - gf') \Big|_a^b$$

$$\text{If } \Omega = \frac{d}{dx} \left[s(x) \frac{d}{dx} \right] + t(x) \quad \text{S-L form}$$

$$\langle f, \Omega g \rangle - \langle \Omega f, g \rangle = s(fg' - gf') \Big|_a^b$$

Self-adjointness depends on b.d conditions

Simplest case: separated b.d conditions

$$s(fg' - gf')(a) = 0 = s(fg' - gf')(b)$$

L form

$$u|_a^b = 0$$

$$g f'(b)$$

If $s(x) > 0$ and a, b finite
can impose:

$$\alpha_1 u_\lambda(a) + \alpha_2 (u_\lambda)'(a) = 0$$

$$\beta_1 u_\lambda(b) + \beta_2 (u_\lambda)'(b) = 0$$

$$\alpha_1^2 + \alpha_2^2 > 0$$

$$\beta_1^2 + \beta_2^2 > 0$$

- eigenvalues are real $-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \dots$
- u_{λ_n} form an orthonormal basis of $L^2[a, b]_w$

- $x_1, y_1, x_2, y_2, \dots, x_n, y_n$
- the zeros of U_{n+1} are in between the zeros of U_n
 - U_n has exactly n distinct zeros in (a, b)

zeros of u_n

b)

First 3 points hold even if

- $S(x) = 0$ at a or b + u_n is finite at a or b

- a or b is infinite + $\int_a^b u_n^2 \rightarrow 0$ as $x \rightarrow \pm\infty$

$$H_\lambda'' - 2x H_\lambda' = -2\lambda H_\lambda$$

$$g(\Omega u_\lambda + \lambda w u_\lambda) = 0$$

for some g

$$s(x) = g(x)$$

$$s'(x) = -2xg(x) = g'(x)$$

$$g(x) = e^{-x^2}$$

$$w(x) = e^{-x^2}$$

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$$w(x) = e^{-x^2}$$

- γ , does

Best approximation

$$f \approx \sum_{n=1}^N a_n u_n$$

minimize $\Delta_N = \|f - \sum_{n=1}^N a_n u_n\|^2$

then $a_n = \langle u_n, f \rangle$

min value is $\Delta_N = \|f\|^2 - \sum_{n=1}^N |a_n|^2$

u_0

is generally difficult to define

Best approximation

$$f \approx \sum_{n=1}^N a_n u_n$$

minimize $\Delta_N = \|f - \sum_{n=1}^N a_n u_n\|_w^2$

then $a_n = \langle u_n, f \rangle_w$

min value is $\Delta_N = \|f\|_w^2 - \sum_{n=1}^N |a_n|^2$

Parsavat's Theorem

If $\|f\|_w^2 < \infty$ then $\Delta_N \rightarrow 0$ as $N \rightarrow \infty$

$$\|f\|_w^2 = \sum_{n=1}^{\infty} |a_n|^2$$

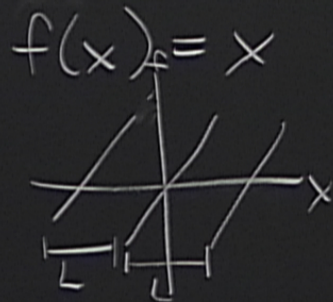
Example: Consider wave/amp $f(x)$ and period L

f

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$$f = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x / L}$$

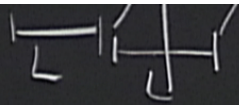
total energy = sum of energy in Fourier components



$$\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |a_n|^2$$

$$\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dx =$$

$$\frac{L^2}{12}$$



$a_0 = 0$ by symmetry

$$n \neq 0 \quad a_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x e^{-2\pi i n x / L} dx$$

$$= -\frac{1}{2\pi i n} \int_{-\frac{L}{2}}^{\frac{L}{2}} x \frac{d}{dx} \left(e^{-2\pi i n x / L} \right) dx$$

$$= -\frac{1}{2\pi i n} \left[x e^{-2\pi i n x / L} \Big|_{-\frac{L}{2}}^{\frac{L}{2}} - \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{-2\pi i n x / L} dx \right]$$

$$= \frac{L}{2\pi i n} (-1)^{n+1}$$

$$\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |a_n|^2$$

$$\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dx = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{L^2}{(2\pi n)^2}$$

$$\frac{L^2}{12} = \frac{L^2}{4\pi^2} 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2)$$

Orthogonal Polynomials $\hookrightarrow$
polynomials form a complete basis of $L^2[a,b]_w$

Orthogonal Polynomials

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$$\text{start with } P_0(x) = \frac{1}{\|1\|_w}$$

$$P_{n+1} = \frac{x P_n - \sum_{i=0}^n P_i \langle P_i, x P_n \rangle_w}{\alpha_{n+1}}$$

$$\alpha_{n+1} = \|x P_n - \sum_{i=0}^n P_i \langle P_i, x P_n \rangle_w\|_w$$

Orthogonal Polynomials

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$$\langle P_i, x P_n \rangle_w = \int_a^b w(x) P_i(x) P_n(x) dx$$

$$= \langle x P_i, P_n \rangle_w$$

$$= 0 \quad \text{for } i=0, \dots, n-2$$

$$P_{n+1} = a_n x P_n + b_n P_n + c_n P_{n-1}$$

