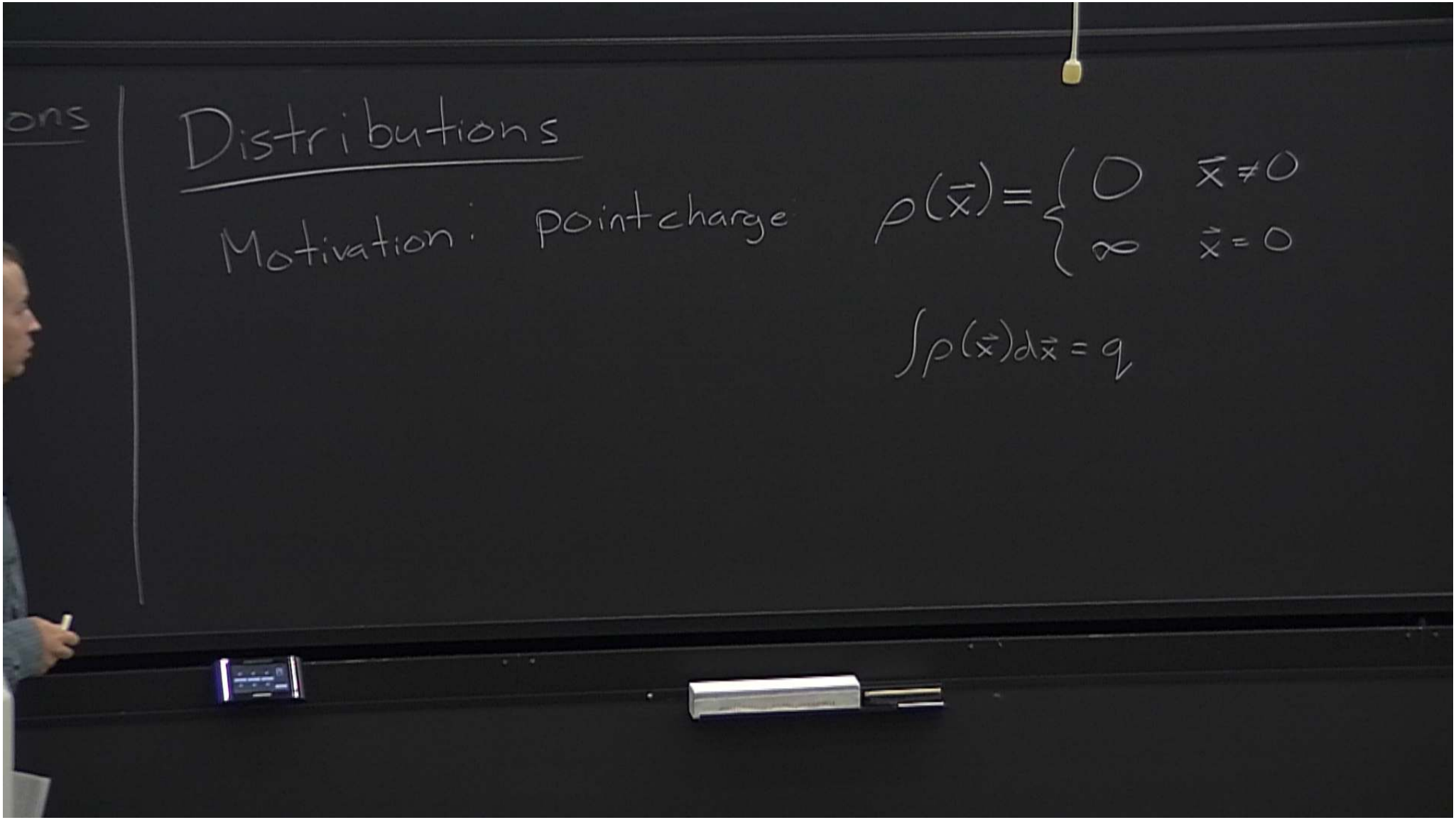
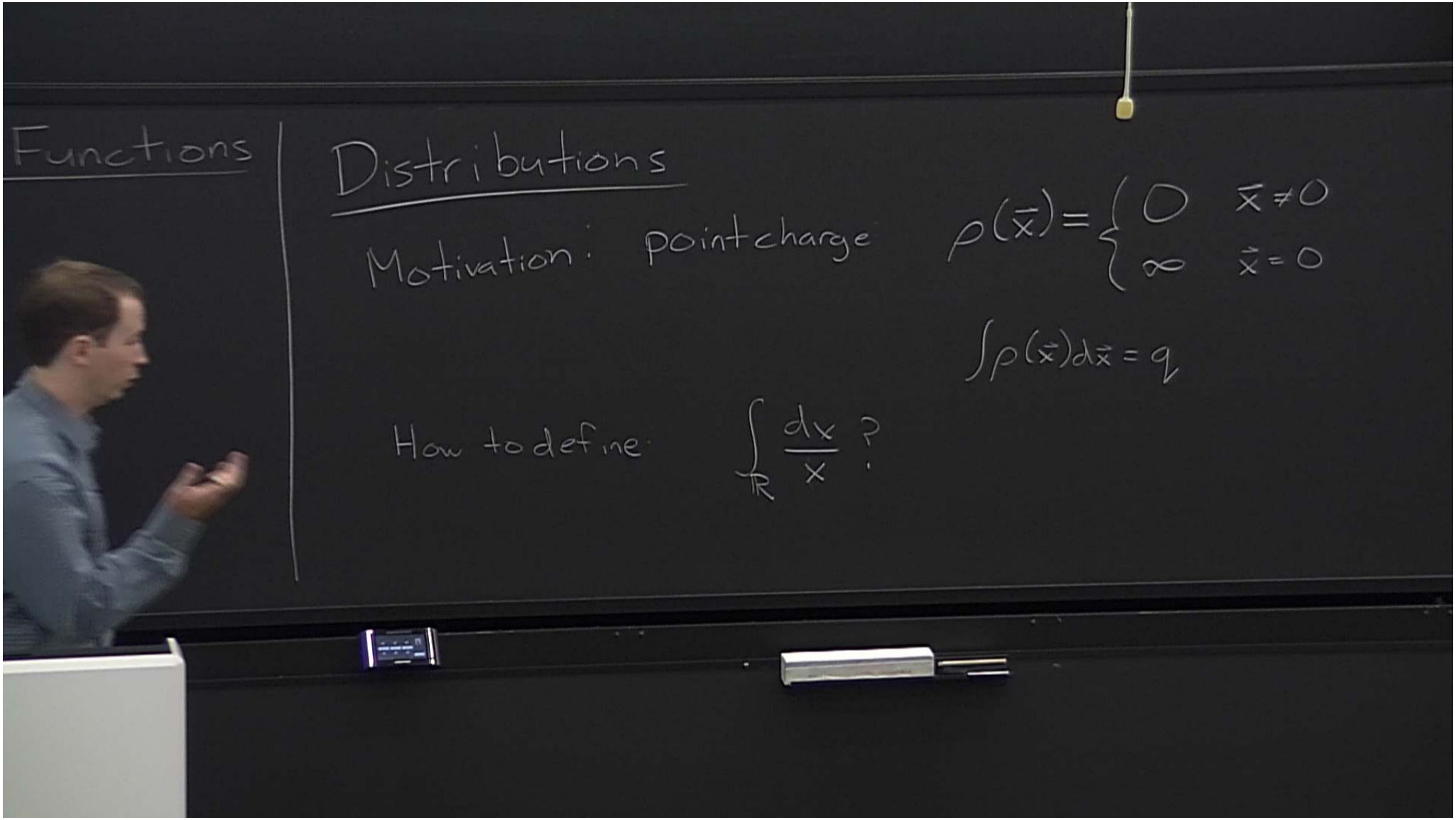
Title: 13/14 PSI - Distribuitions & Special Functions - Lecture 1

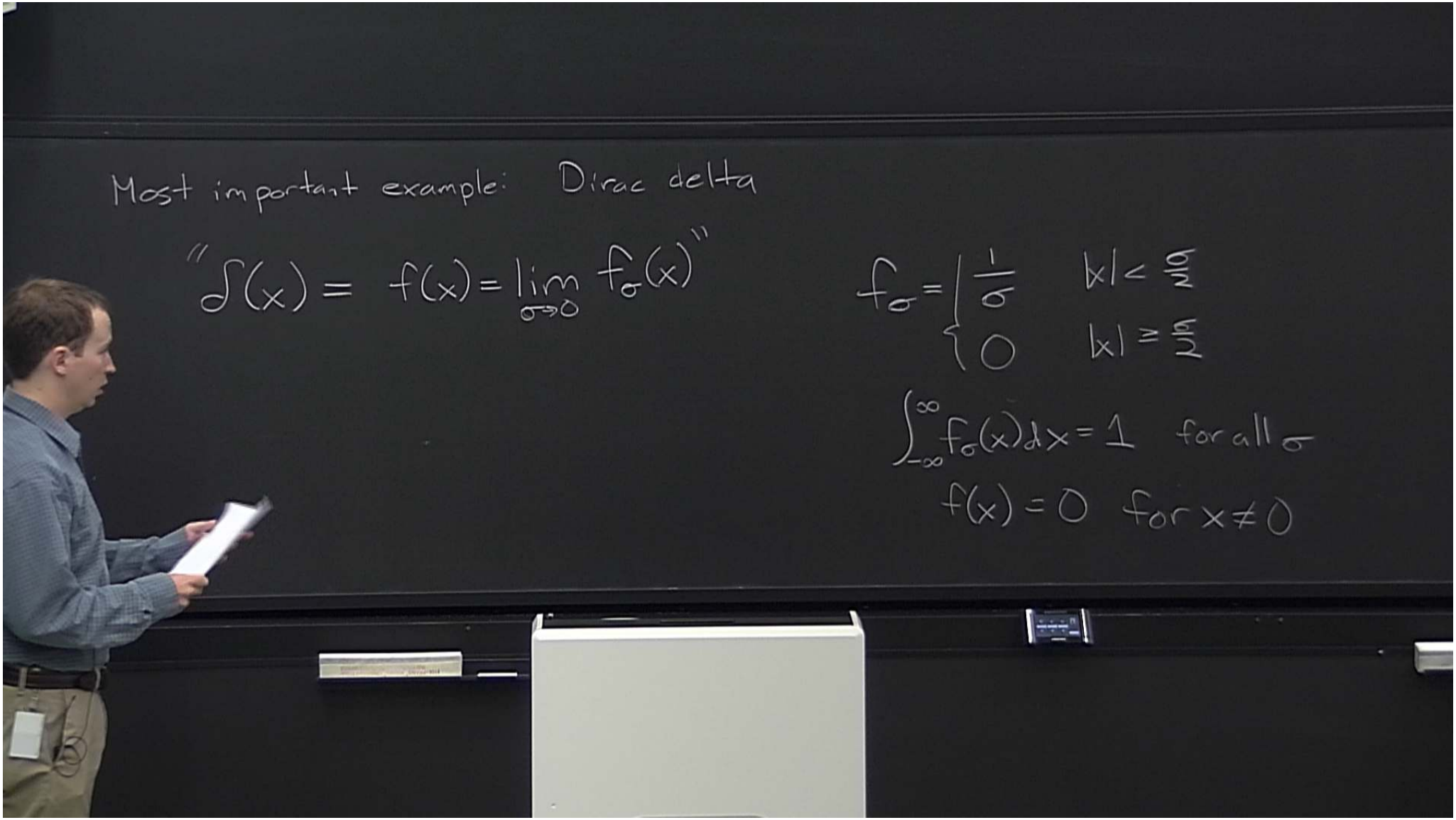
Date: Aug 21, 2013 09:00 AM

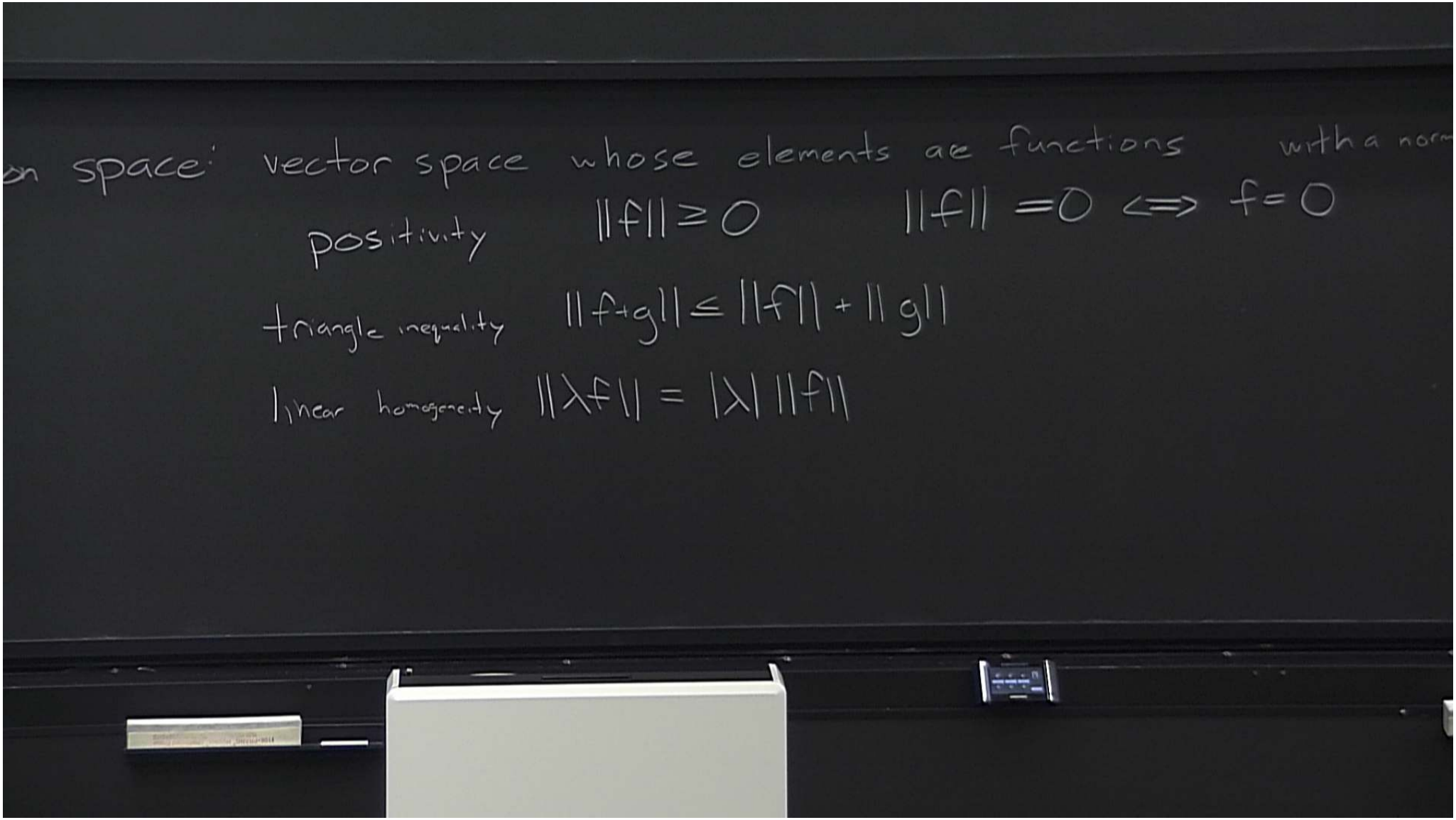
URL: http://pirsa.org/13080016

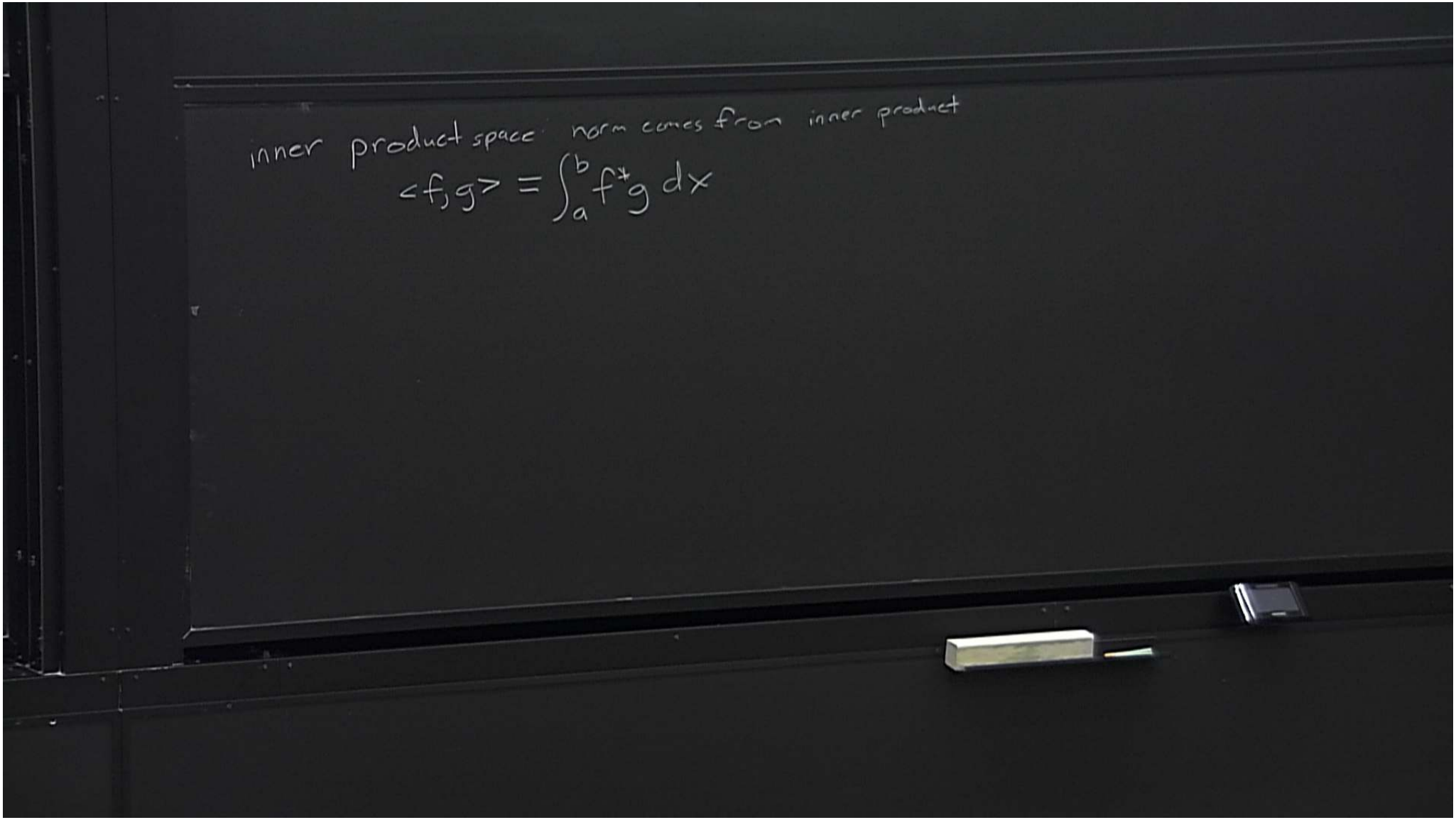
Abstract:

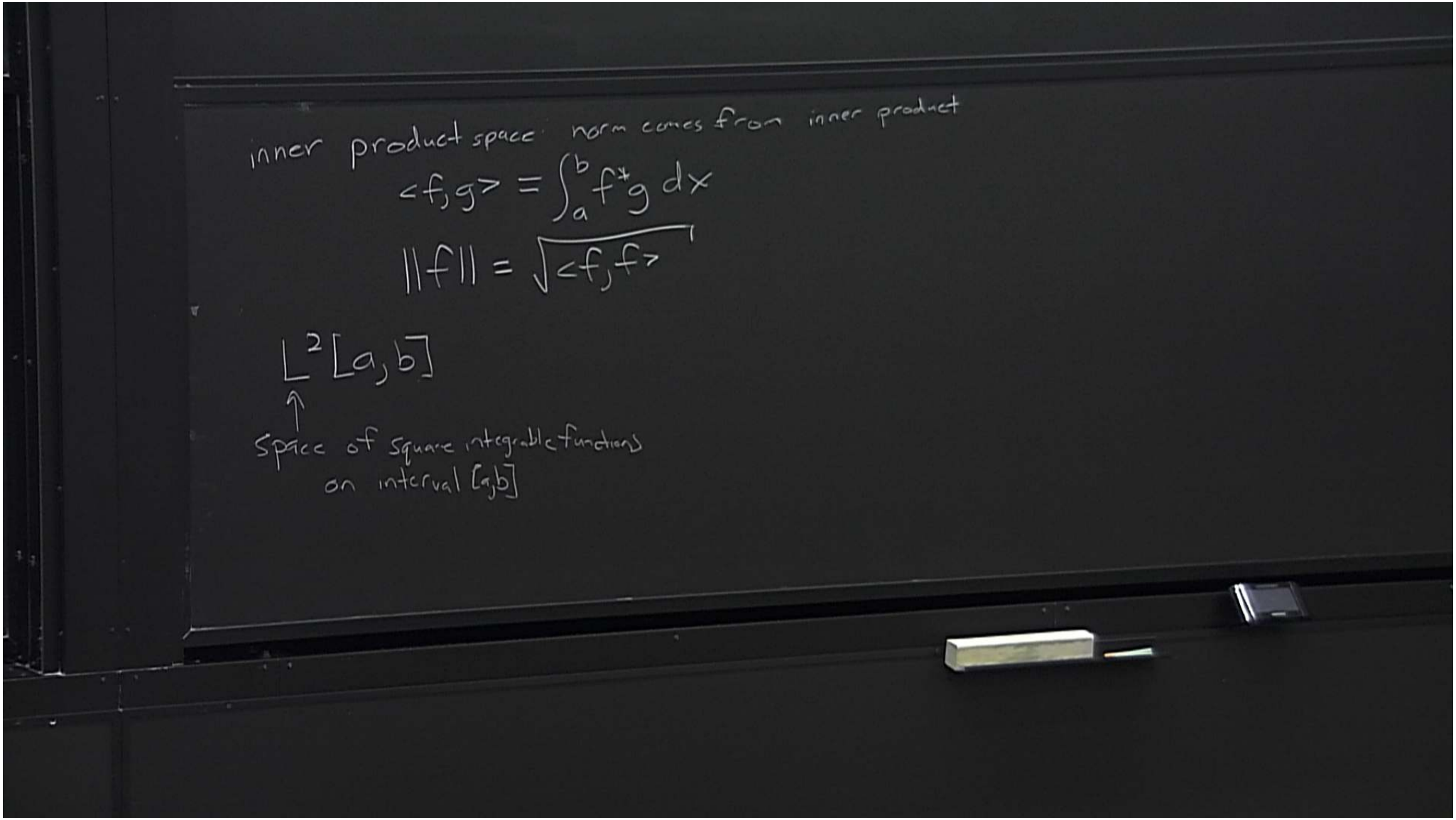

ons
Distributions
Motivation: point charge
$\rho(\vec{x}) = \begin{cases} 0 & \vec{x} \neq 0 \\ \infty & \vec{x} = 0 \end{cases}$
$\int \rho(\vec{x}) d\vec{x} = q$

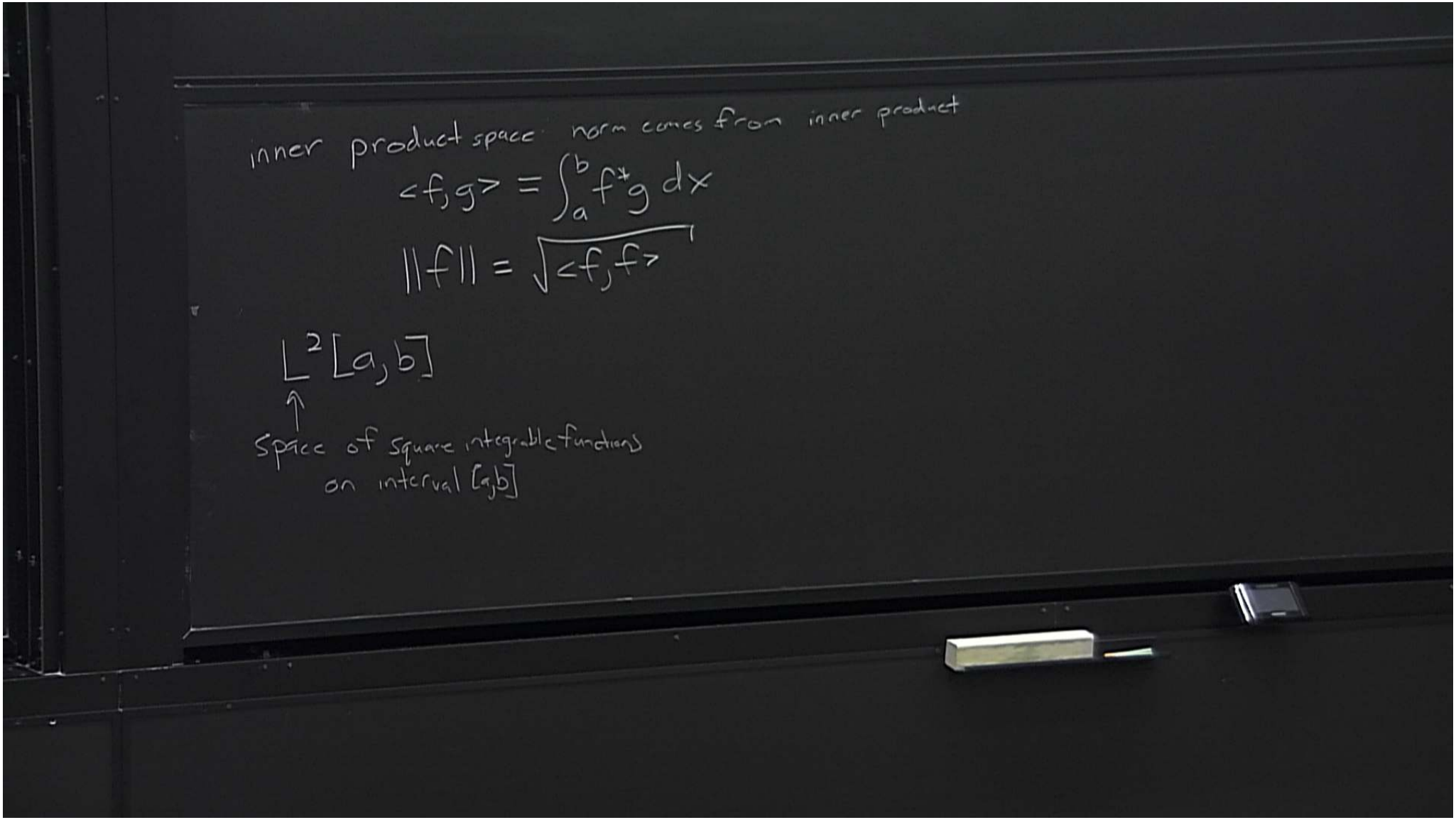

Functions
Distributions
Motivation: pointcharge
$\rho(\vec{x}) = \begin{cases} 0 & \vec{x} \neq 0 \\ \infty & \vec{x} = 0 \end{cases}$
$\int \rho(\vec{x}) d\vec{x} = q$
How to define $\int_{\mathbb{R}} \frac{dx}{x}$?


Most important example: Dirac delta
"$\delta(x) = f(x) = \lim_{\sigma \to 0} f_\sigma(x)$"
$f_\sigma = \begin{cases} \frac{1}{\sigma} & |x| < \frac{\sigma}{2} \\ 0 & |x| \geq \frac{\sigma}{2} \end{cases}$
$\int_{-\infty}^{\infty} f_\sigma(x)\,dx = 1$ for all σ
$f(x) = 0$ for $x \neq 0$


on space: vector space whose elements ae functions with a norm
positivity $\|f\| \geq 0$ $\|f\| = 0 \iff f = 0$
triangle inequality $\|f+g\| \leq \|f\| + \|g\|$
linear homogeneity $\|\lambda f\| = |\lambda| \|f\|$


inner product space norm comes from inner product
$\langle f, g \rangle = \int_a^b f^* g \, dx$


inner product space norm comes from inner product
$\langle f, g \rangle = \int_a^b f^* g \, dx$
$\|f\| = \sqrt{\langle f, f \rangle}$
$L^2[a,b]$
space of square integrable functions on interval $[a,b]$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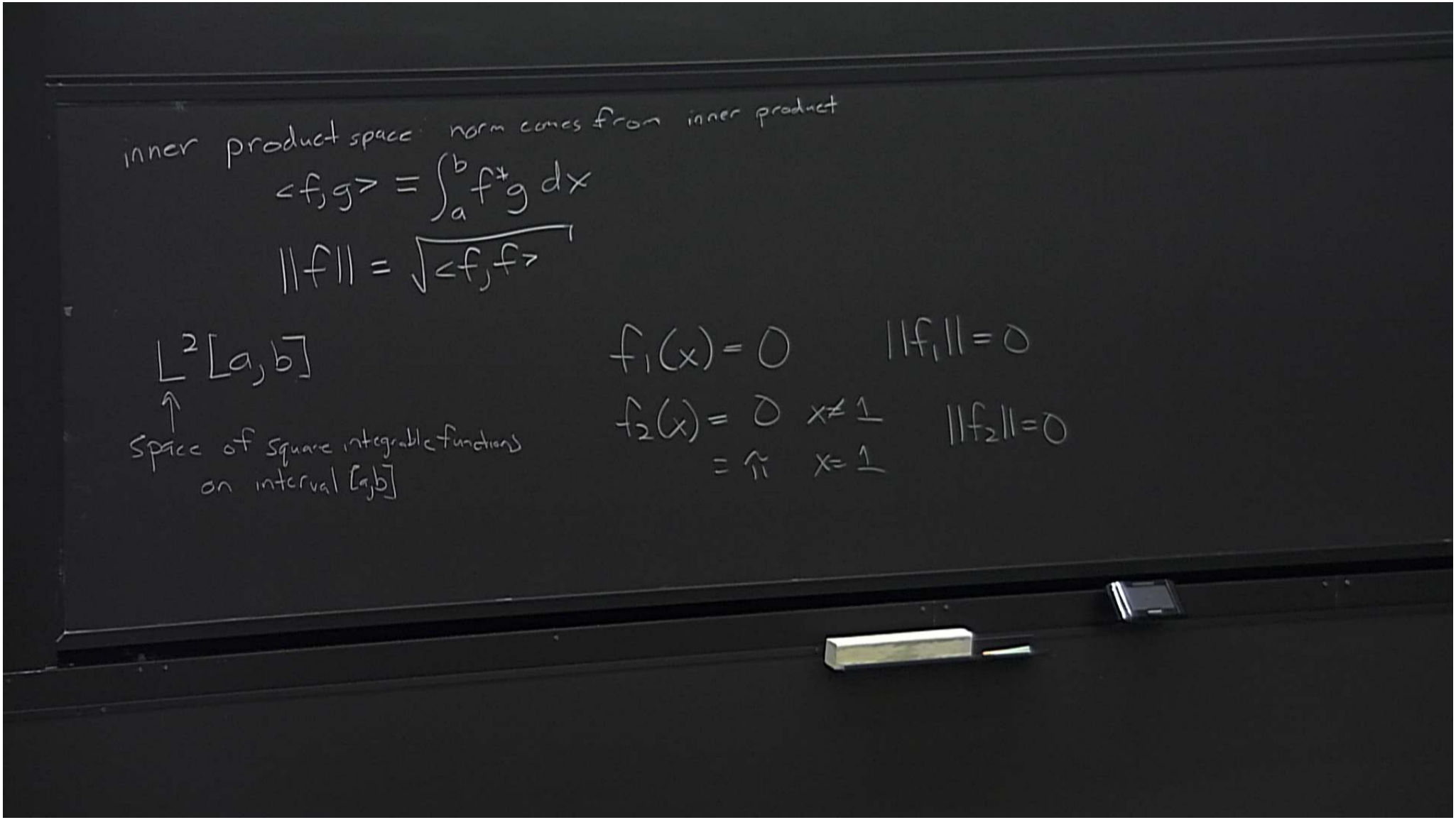
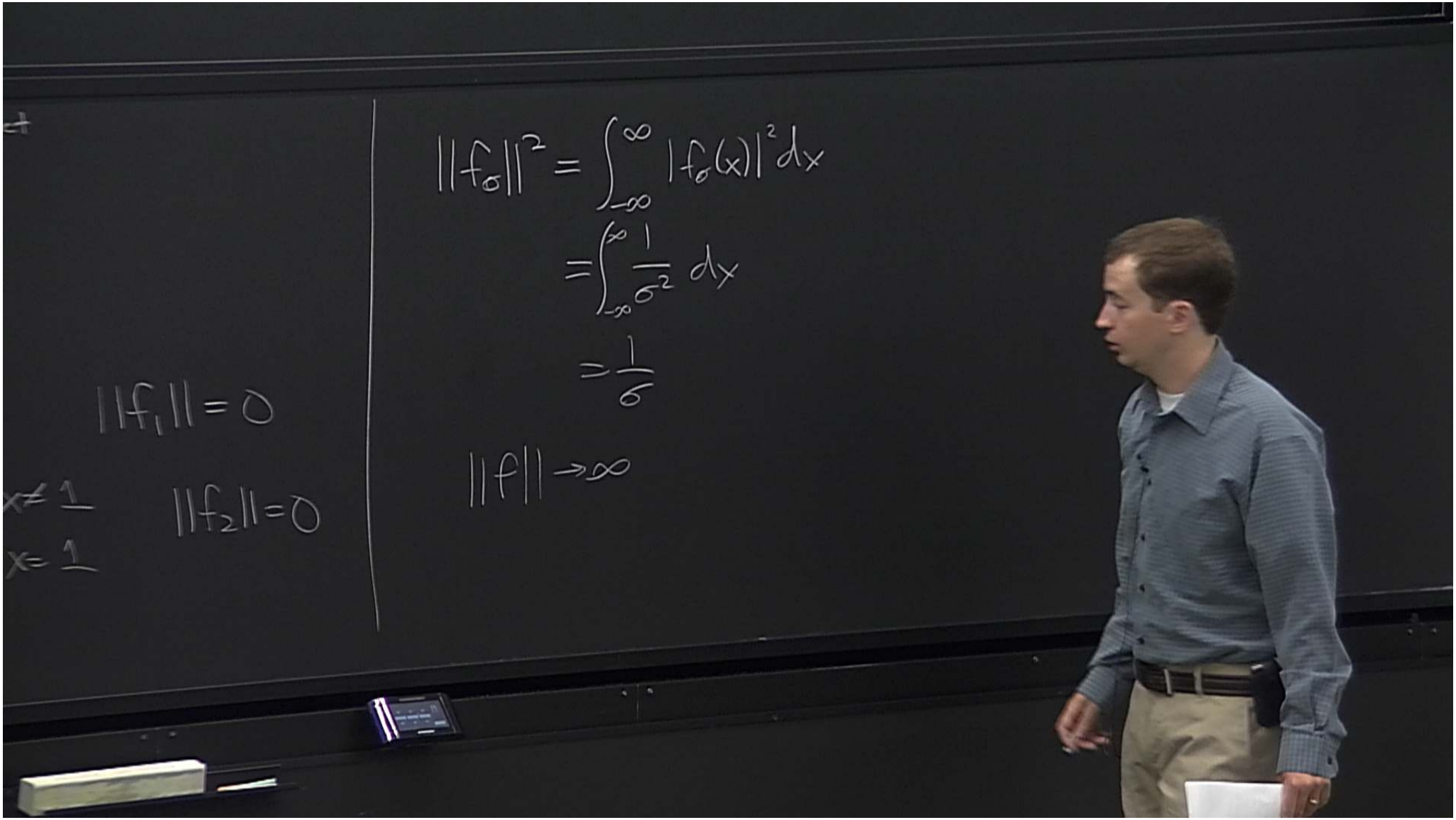
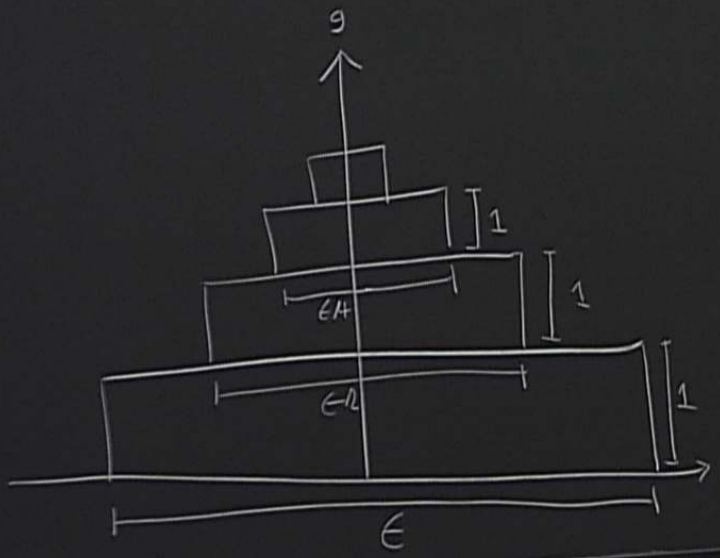

inner product space norm comes from inner product
$\langle f, g \rangle = \int_a^b f^* g \, dx$
$\|f\| = \sqrt{\langle f, f \rangle}$
$L^2[a,b]$
space of square integrable functions on interval $[a,b]$
$f_1(x) = 0$ $\|f_1\| = 0$
$f_2(x) = 0$ $x \neq 1$ $\|f_2\| = 0$
$= \pi$ $x = 1$


$\|f_\sigma\|^2 = \int_{-\infty}^{\infty} |f_\sigma(x)|^2 dx$
$= \int_{-\infty}^{\infty} \frac{1}{\sigma^2} dx$
$= \frac{1}{\sigma}$
$\|f\| \to \infty$
$\|f_1\| = 0$
$\|f_2\| = 0$

Choose a sequence $g_\sigma \to "g = \delta"$ for which $\lim_{\sigma \to 0} g_\sigma(x) = 0 \quad x \neq 0$

$$\int_{-\infty}^{\infty} g_\sigma(x)\, dx = 1 \quad \forall \sigma$$



Choose a sequence $g_\sigma \to "g = \delta"$ for which $\lim_{\sigma \to 0} g_\sigma(x) = 0 \quad x \neq 0$

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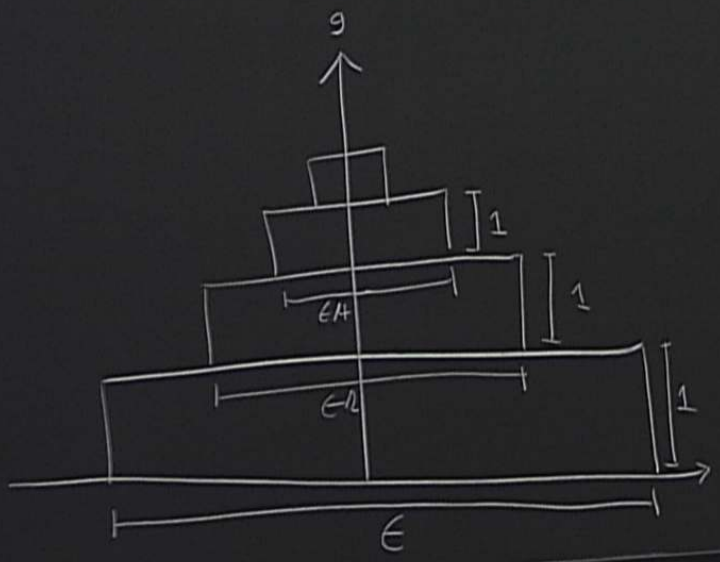
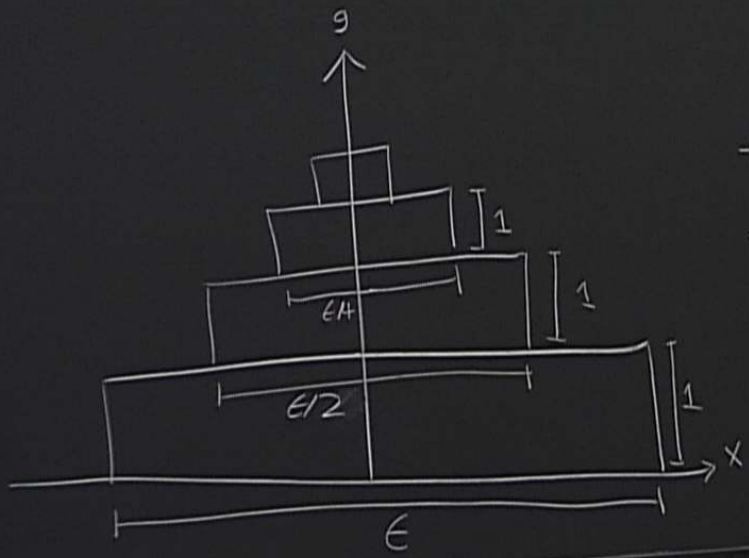


for each σ_*
we can find an ϵ
so that g_σ lies in the boxes
for $\sigma < \sigma_*$

Choose a sequence $g_\sigma \to \ddot{g} = \delta''$ for which $\lim_{\sigma \to 0} g_\sigma(x) = 0 \quad x \neq 0$

$$\int_{-\infty}^{\infty} g_\sigma(x)\,dx = 1 \quad \forall \sigma$$



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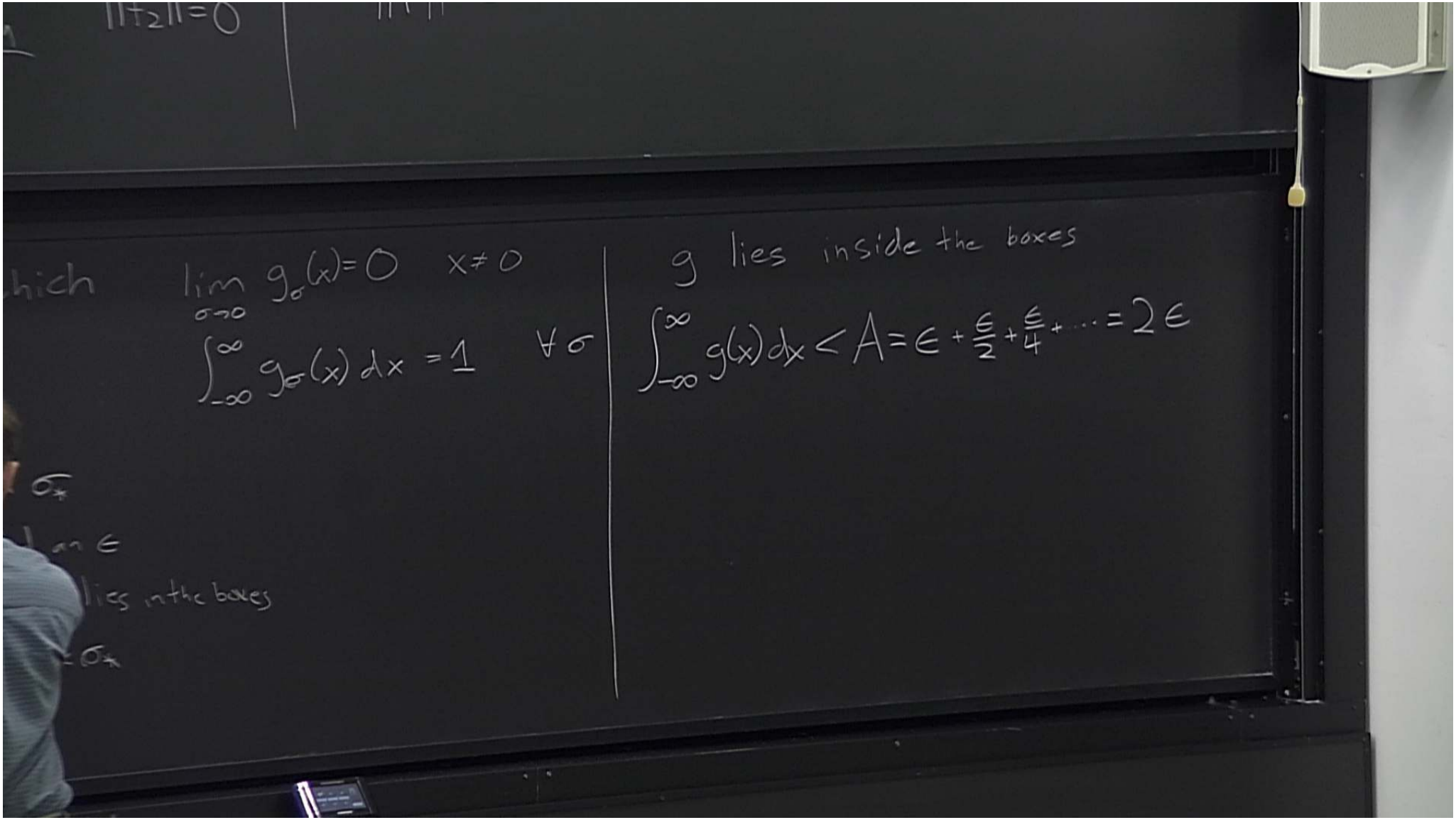
$\|f_2\| = 0$
which
$\lim_{\sigma \to 0} g_\sigma(x) = 0 \quad x \neq 0$
$\int_{-\infty}^{\infty} g_\sigma(x)\,dx = 1 \quad \forall \sigma$
σ_*
an ϵ
lies in the boxes
σ_*
g lies inside the boxes
$\int_{-\infty}^{\infty} g(x)\,dx < A = \epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{4} + \cdots = 2\epsilon$

e a sequence $g_\sigma \to "g = \delta"$ for which $\lim_{\sigma \to 0} g_\sigma(x) = 0 \quad x \neq 0$

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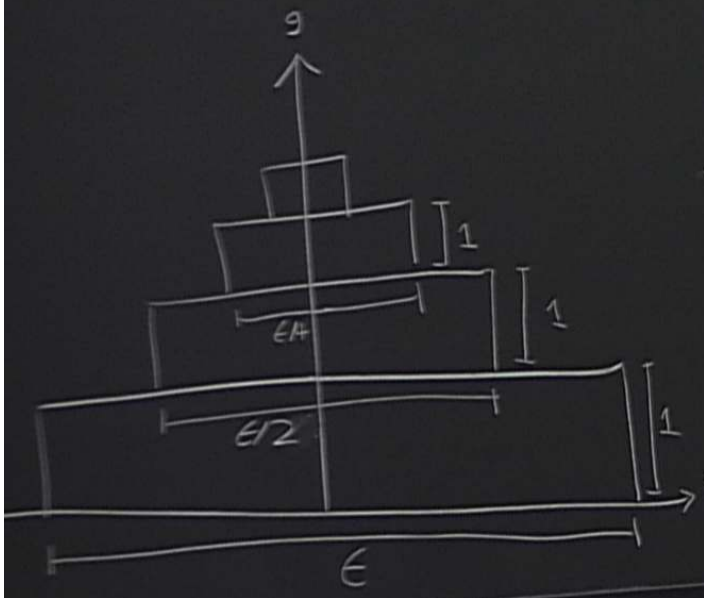
$\int_{-\infty}^{\infty} g(x)\,dx$

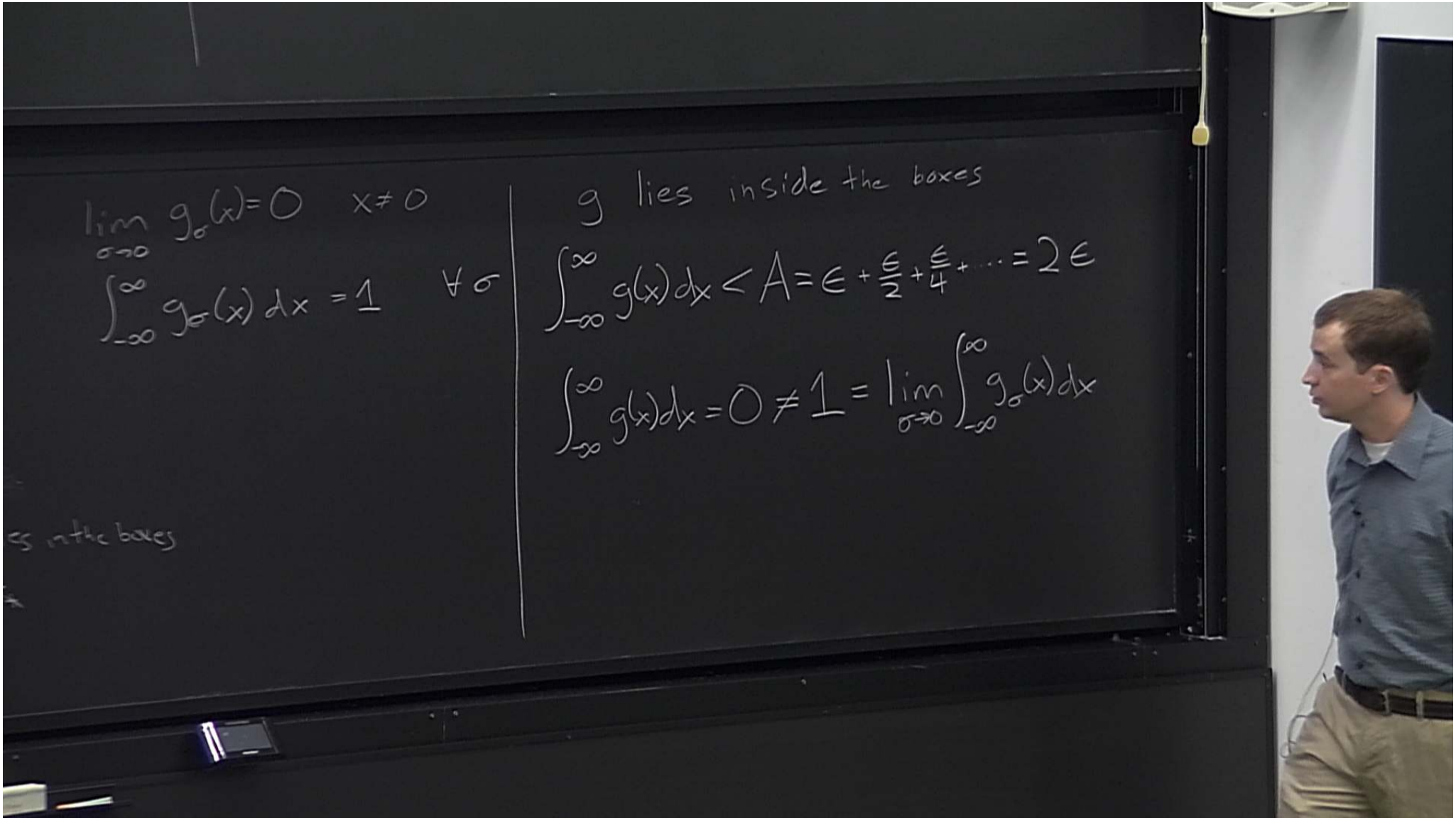


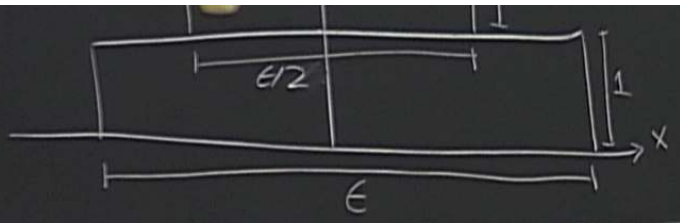
For each ϵ

we can find σ_*

so that g_σ lies in the boxes

for $\sigma < \sigma_*$


$\lim_{\sigma \to 0} g_\sigma(x) = 0 \quad x \neq 0$
$\int_{-\infty}^{\infty} g_\sigma(x)\,dx = 1 \quad \forall \sigma$
g lies inside the boxes
$\int_{-\infty}^{\infty} g(x)\,dx < A = \epsilon + \frac{\epsilon}{2} + \frac{\epsilon}{4} + \cdots = 2\epsilon$
$\int_{-\infty}^{\infty} g(x)\,dx = 0 \neq 1 = \lim_{\sigma \to 0} \int_{-\infty}^{\infty} g_\sigma(x)\,dx$



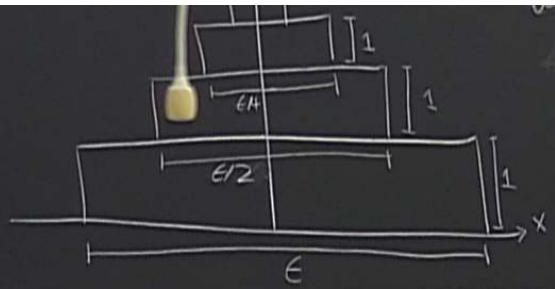
for $\epsilon < \sigma_*$

Test functions: Let's choose $\mathcal{D}$: set of C^∞ functions with

$\mathcal{D}$ is a vector space

$\mathcal{D}^*$ dual vector space of linear functionals from $\mathcal{D}$ to $\mathbb{R}$

$u \in \mathcal{D}^*$ $u(\varphi) \in \mathbb{R}$ for every $\varphi \in \mathcal{D}$



so that g_σ lies in the boxes for $\sigma < \sigma_*$

$$\int_{-\infty}^{\infty} g(x)\,dx = 0 \neq 1 = \lim_{\sigma \to 0} \int_{-\infty}^{\infty}$$

Test functions: Let's choose $\mathcal{D}$: set of C^∞ functions with compact support

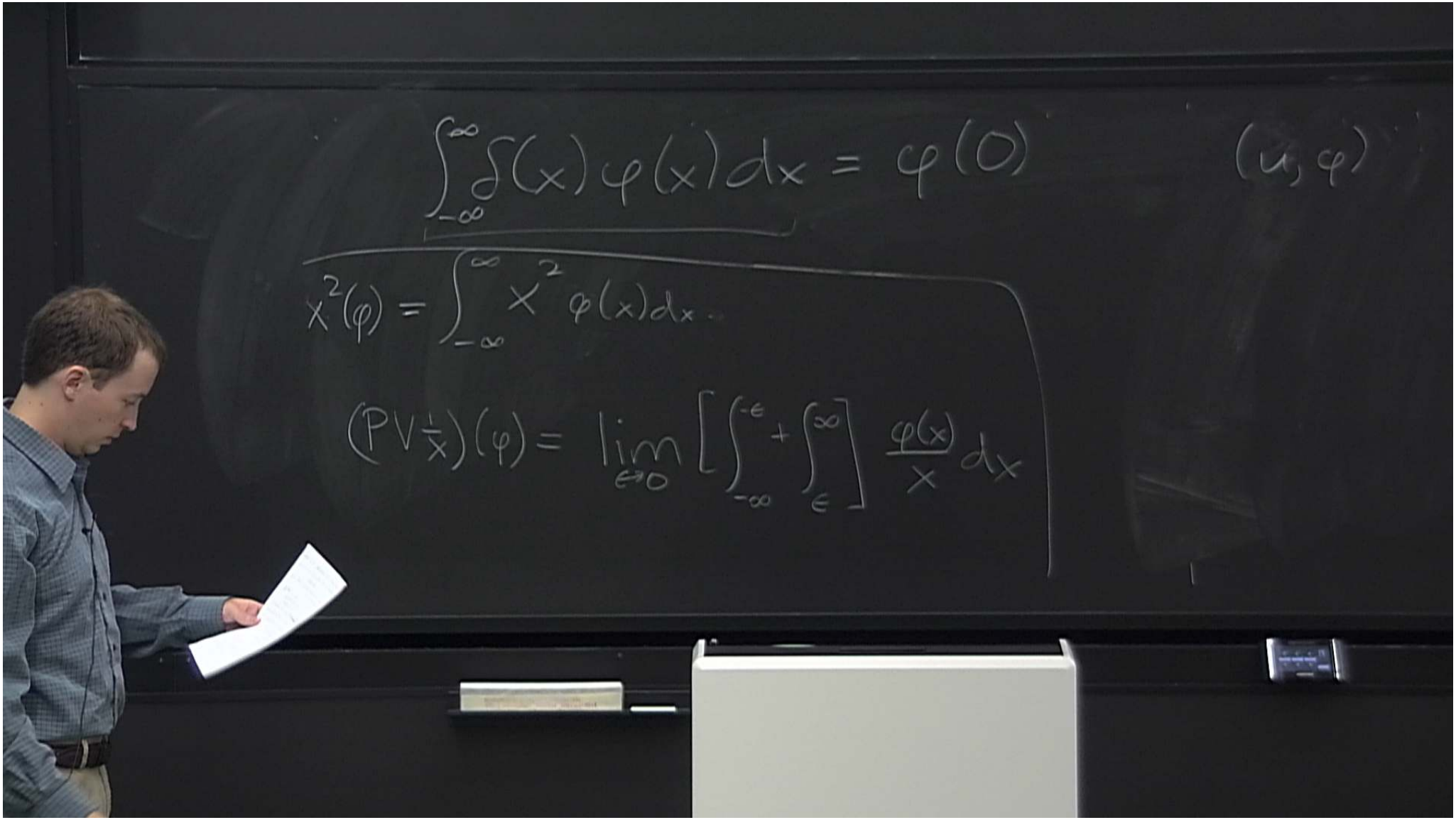
identically zero outside of a finite interval

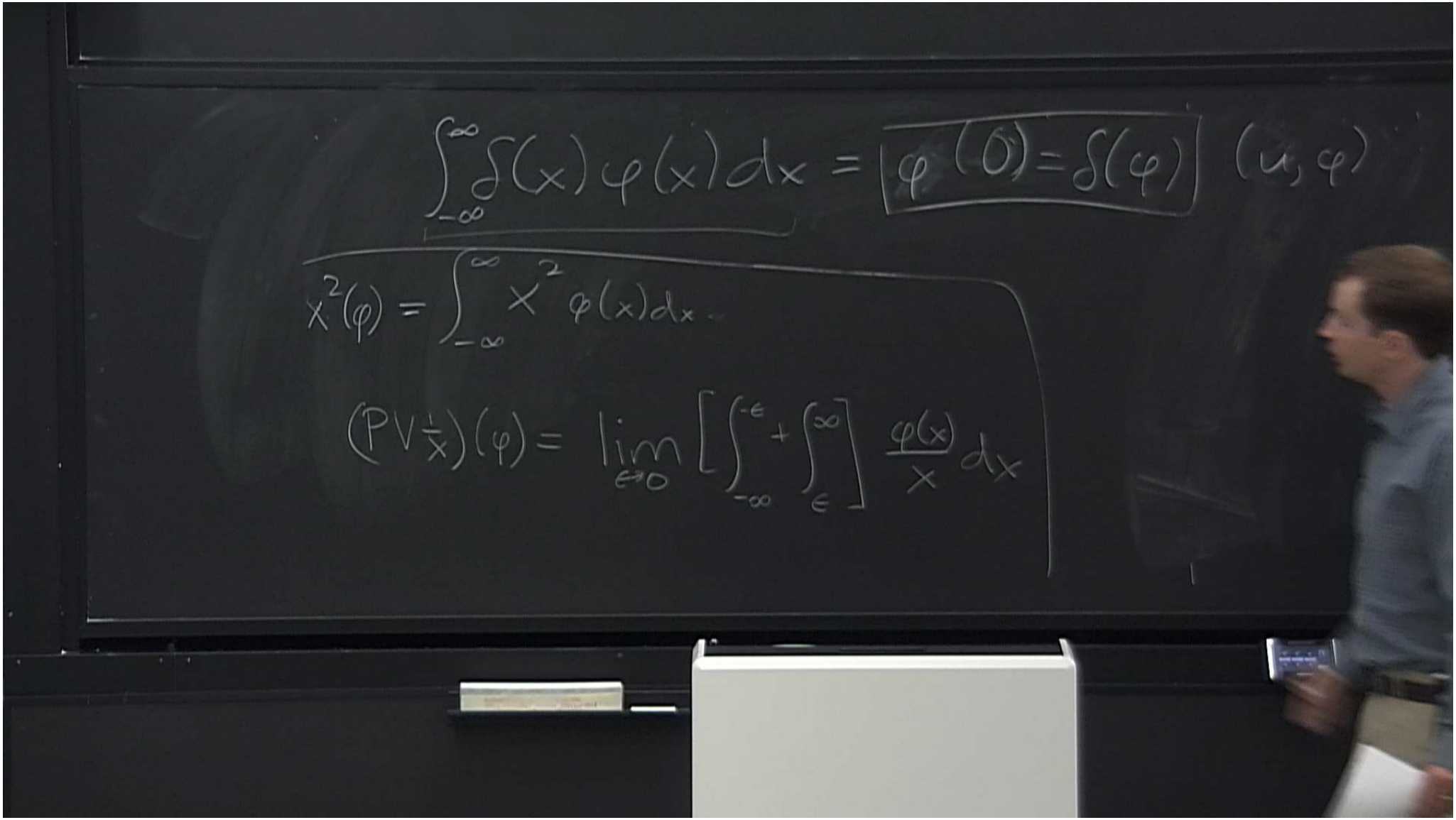
$\mathcal{D}$ is a vector space

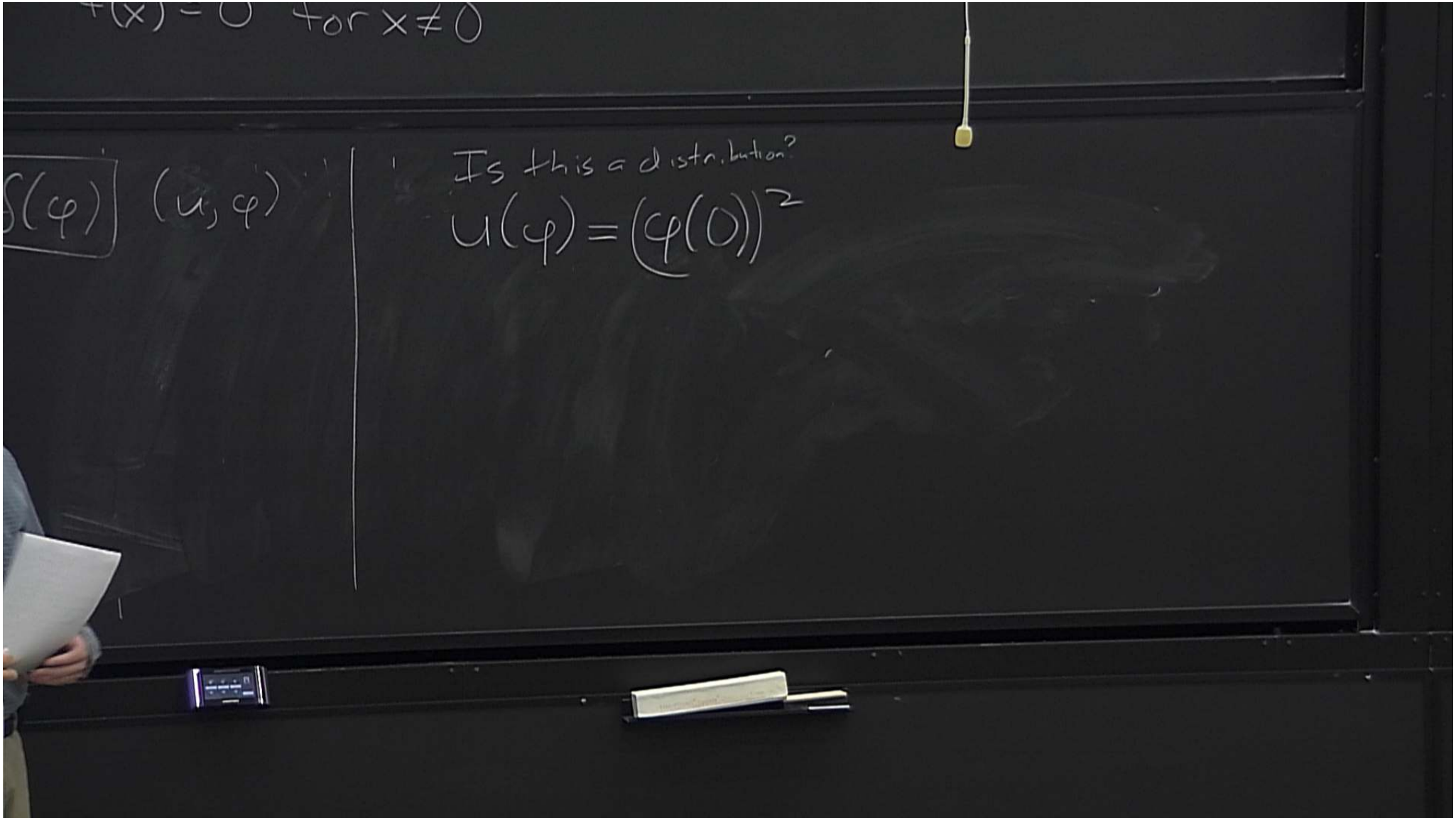
$\mathcal{D}^*$ dual vector space of linear functionals from $\mathcal{D}$ to $\mathbb{R}$

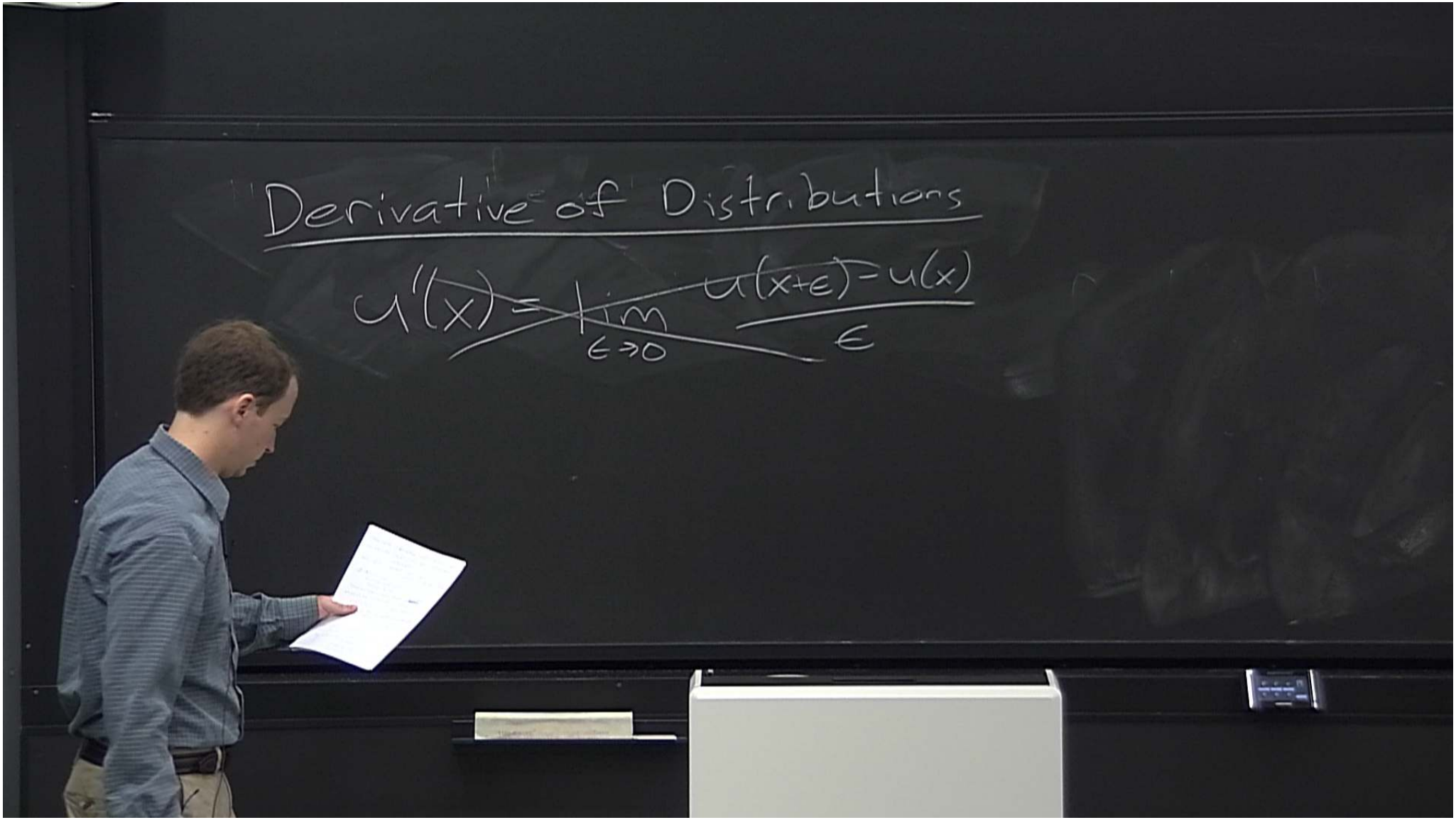
$u \in \mathcal{D}^*$ $u(\varphi) \in \mathbb{R}$ for every $\varphi \in \mathcal{D}$

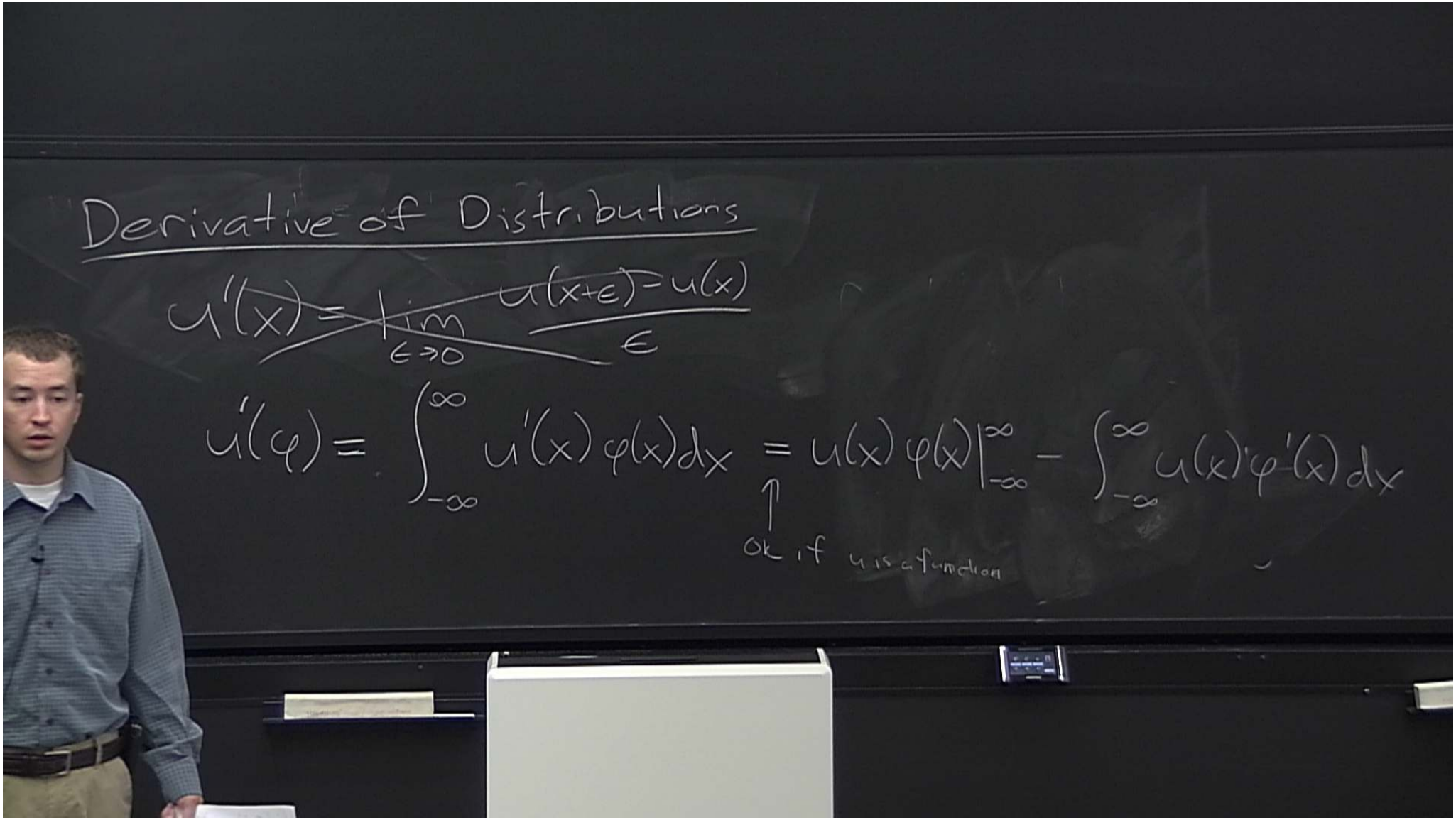
$\mathcal{D}'$ largest continuous dual space continuous: if $\varphi_n \to \varphi$ $u(\varphi_n) \to u(\varphi)$ $n \to \infty$

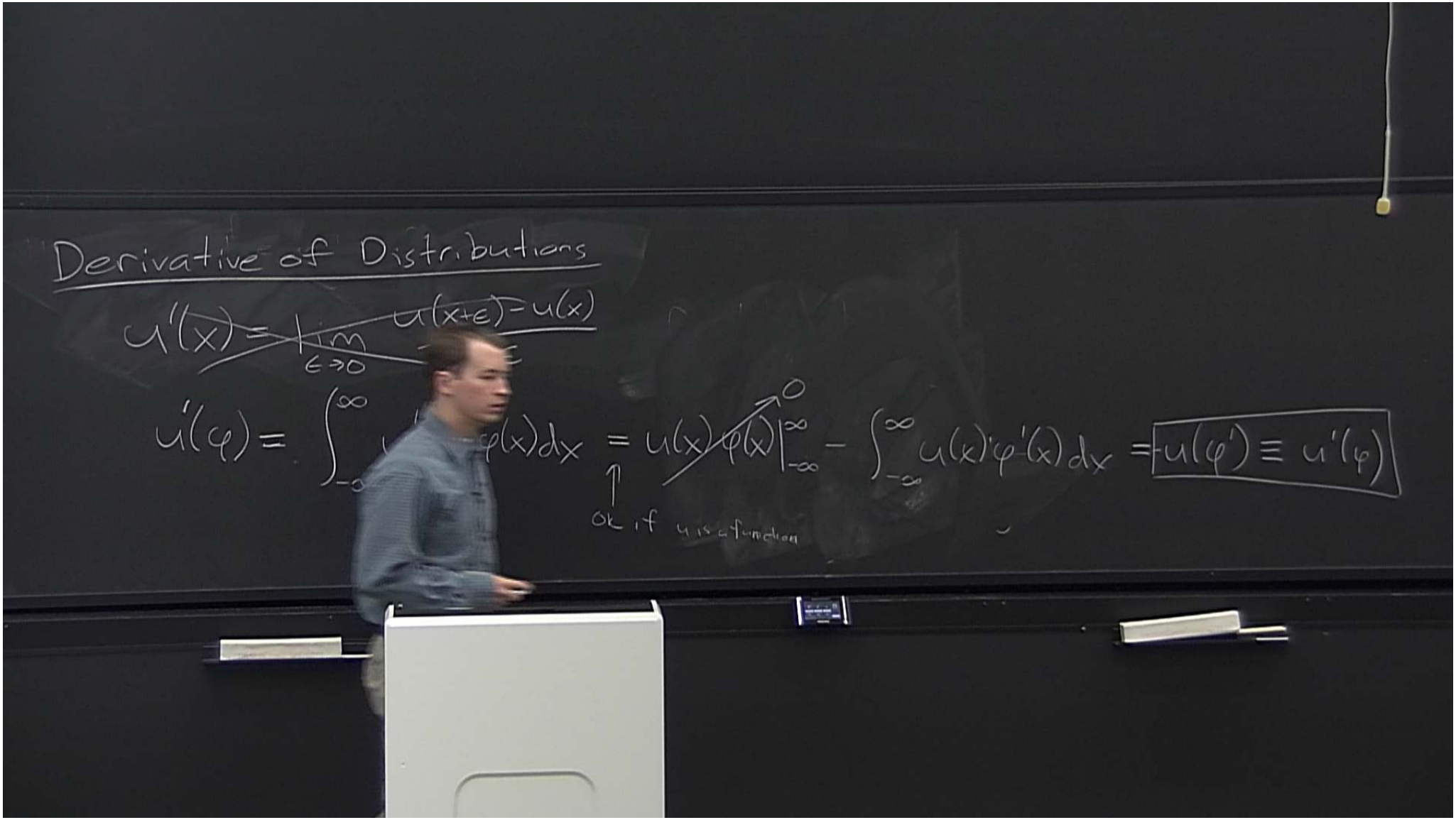

$\int_{-\infty}^{\infty} \delta(x)\varphi(x)dx = \varphi(0)$
(u,φ)
$x^2(\varphi) = \int_{-\infty}^{\infty} x^2 \varphi(x)dx$
$(PV\frac{1}{x})(\varphi) = \lim_{\epsilon \to 0}\left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right] \frac{\varphi(x)}{x}dx$

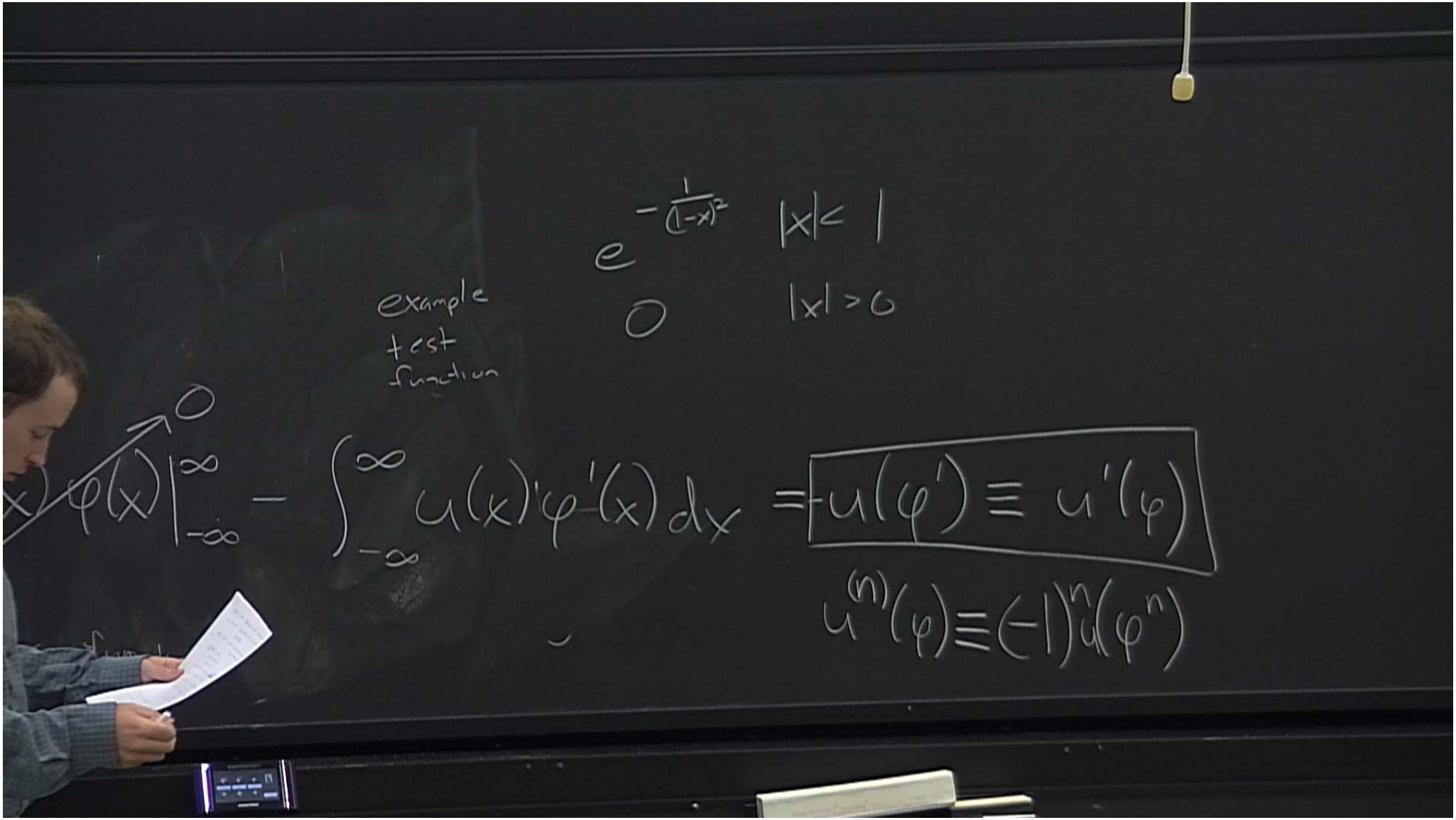

$\int_{-\infty}^{\infty} \delta(x)\,\varphi(x)\,dx = \varphi(0) = \delta(\varphi)$ (u, φ)
$x^2(\varphi) = \int_{-\infty}^{\infty} x^2\,\varphi(x)\,dx$
$(PV\tfrac{1}{x})(\varphi) = \lim_{\epsilon \to 0} \left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right] \frac{\varphi(x)}{x}\,dx$


$f(x) = 0$ for $x \neq 0$
$\delta(\varphi)$ (u, φ)
Is this a distribution?
$u(\varphi) = (\varphi(0))^2$


Derivative of Distributions
u'(x) = lim u(x+ε)-u(x) / ε
ε→0


Derivative of Distributions
$u'(x) = \lim_{\epsilon \to 0} \frac{u(x+\epsilon) - u(x)}{\epsilon}$
$u'(\varphi) = \int_{-\infty}^{\infty} u'(x)\varphi(x)dx = u(x)\varphi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x)\varphi'(x)dx$
ok if u is a function


Derivative of Distributions
$u'(x) = \lim_{\epsilon \to 0} \frac{u(x+\epsilon) - u(x)}{\epsilon}$
$u'(\varphi) = \int_{-\infty}^{\infty} \varphi(x)dx = u(x)\varphi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x)\varphi'(x)dx = -u(\varphi') \equiv u'(\varphi)$
ok if u is a function


$e^{-\frac{1}{(1-x)^2}}$ $|x|<1$
0 $|x|>0$
example
test
function
$\varphi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x)\varphi'(x)\,dx = -u(\varphi') \equiv u'(\varphi)$
$u^{(n)}(\varphi) \equiv (-1)^n u(\varphi^n)$

ok if φ is a function

Example: Heaviside function: $\Theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

$$\Theta(\varphi) = \int_{-\infty}^{\infty} \Theta(x)\varphi(x)\,dx$$

$$= \int_0^{\infty} \varphi(x)\,dx.$$

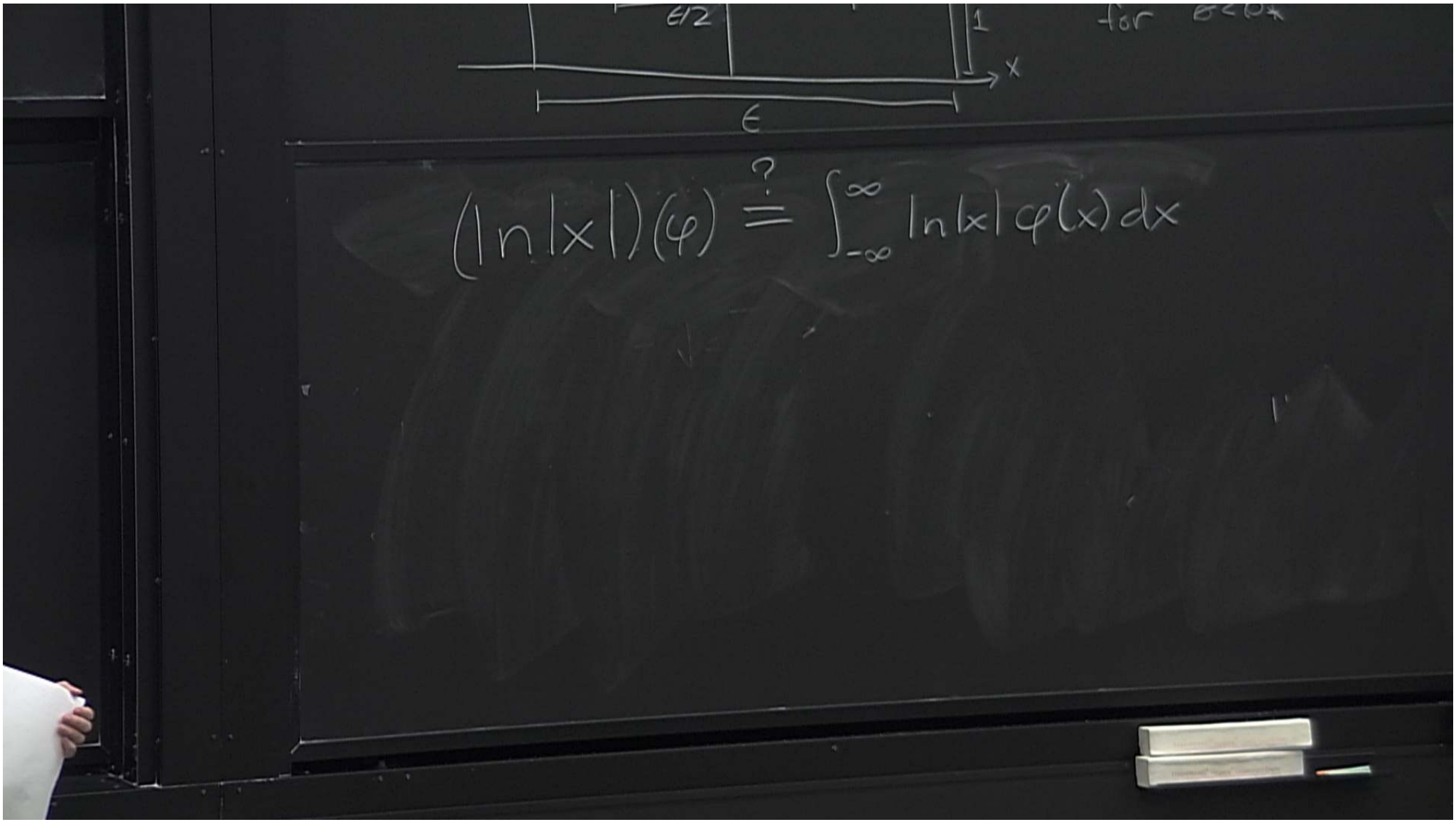
$$\Theta'(\varphi) = -\Theta(\varphi')$$

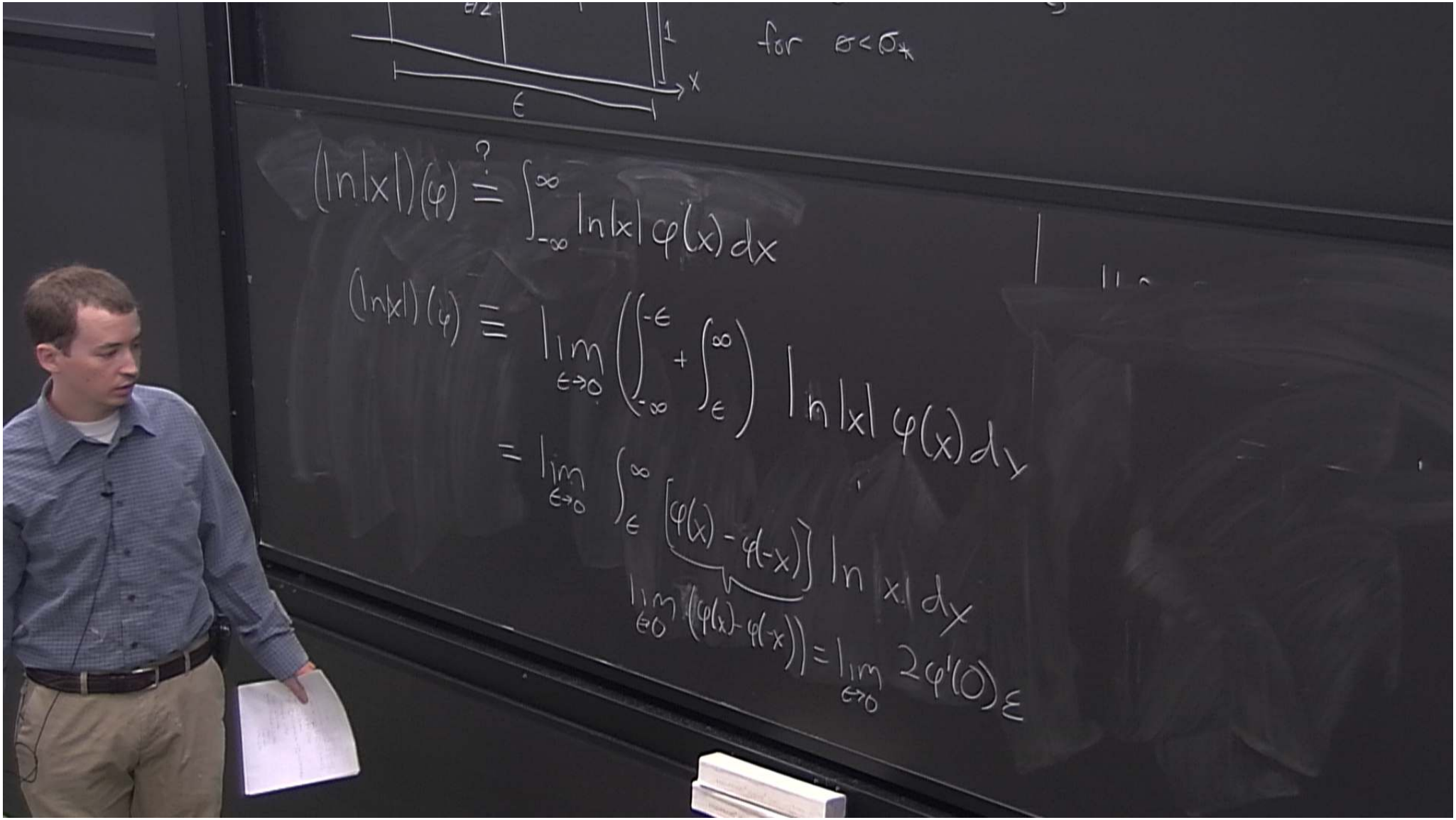
$$= -\int_0^{\infty} \varphi'(x)\,dx$$

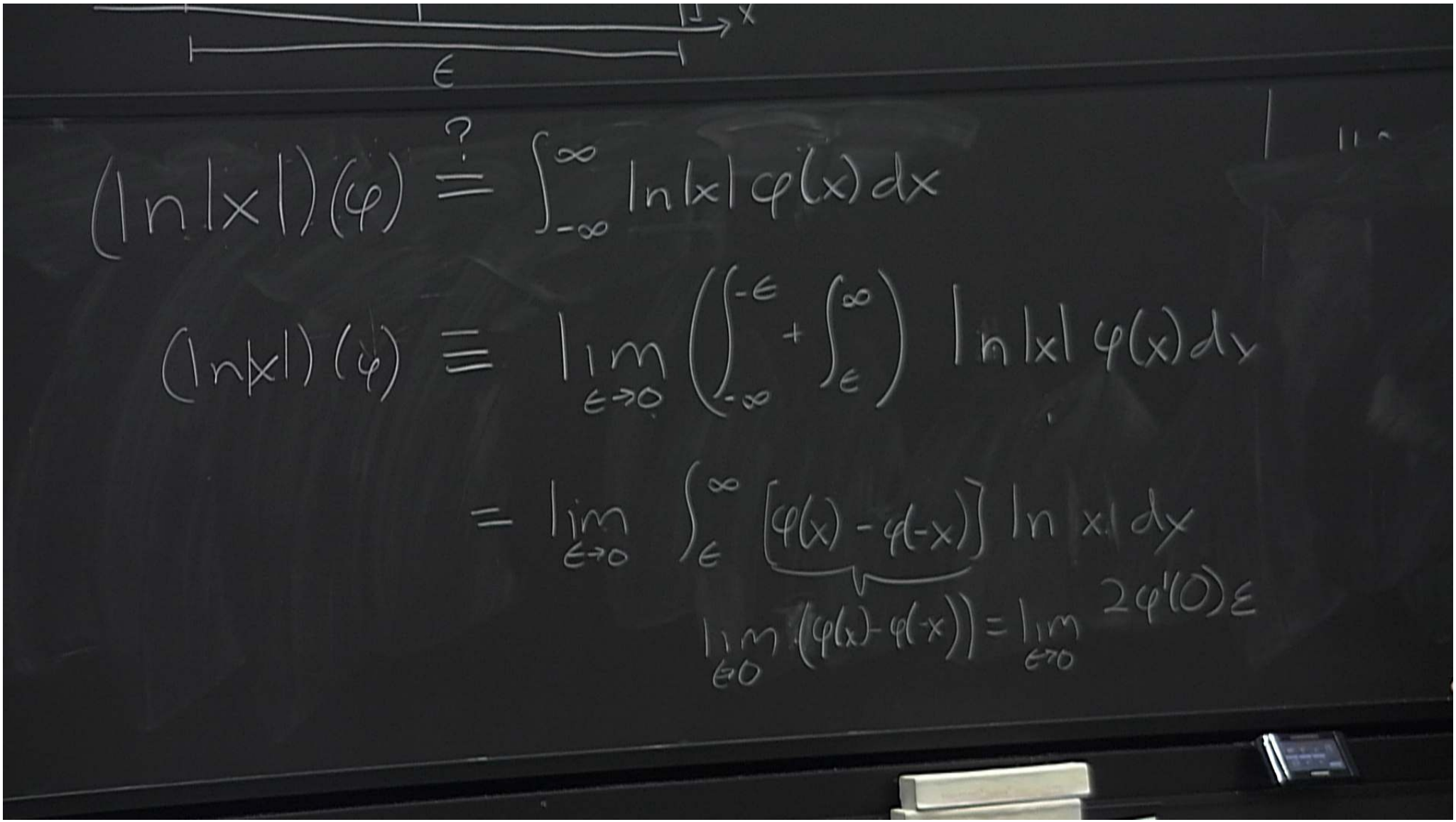
$$= -(\varphi(\infty) - \varphi(0))$$

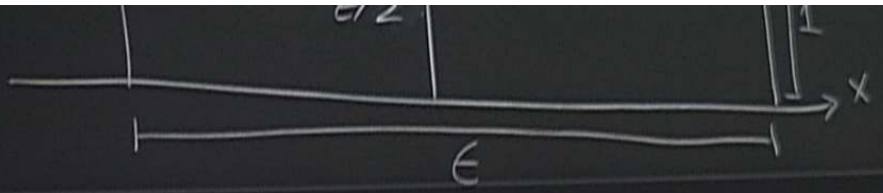
$$= \varphi(0)$$

$$= \delta(\varphi)$$


ε/2
1
x
ε
for
$(\ln|x|)(\varphi) \stackrel{?}{=} \int_{-\infty}^{\infty} \ln|x|\,\varphi(x)\,dx$


for $\epsilon < \epsilon_*$
ϵ
x
$(\ln|x|)(\varphi) \stackrel{?}{=} \int_{-\infty}^{\infty} \ln|x|\,\varphi(x)\,dx$
$(\ln|x|)(\varphi) \equiv \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right) \ln|x|\,\varphi(x)\,dx$
$= \lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} [\varphi(x) - \varphi(-x)] \ln x\, dx$
$\lim_{\epsilon\to 0} (\varphi(x) - \varphi(-x)) = \lim_{\epsilon\to 0} 2\varphi'(0)\epsilon$


x
ε
(ln|x|)(φ) ≟ ∫_{-∞}^{∞} ln|x| φ(x) dx
(ln|x|)(φ) ≡ lim_{ε→0} (∫_{-∞}^{-ε} + ∫_{ε}^{∞}) ln|x| φ(x) dx
= lim_{ε→0} ∫_{ε}^{∞} [φ(x) - φ(-x)] ln x dx
lim_{ε→0} (φ(x) - φ(-x)) = lim_{ε→0} 2φ'(0)ε

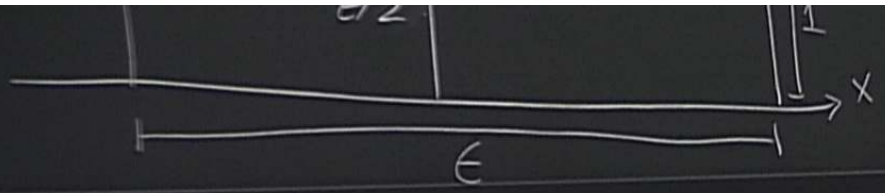


$$(\ln|x|)(\varphi) \overset{?}{=} \int_{-\infty}^{\infty} \ln|x|\,\varphi(x)\,dx$$

$$(\ln|x|)(\varphi) \equiv \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right) \ln|x|\,\varphi(x)\,dx$$

$$= \lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} \underbrace{[\varphi(x)-\varphi(-x)]}_{\approx 2\varphi'(0)\ \text{for } x\ll 1} \ln x\,dx$$

integrand goes as $\frac{(\varphi(x)-\varphi(-x))}{x}$

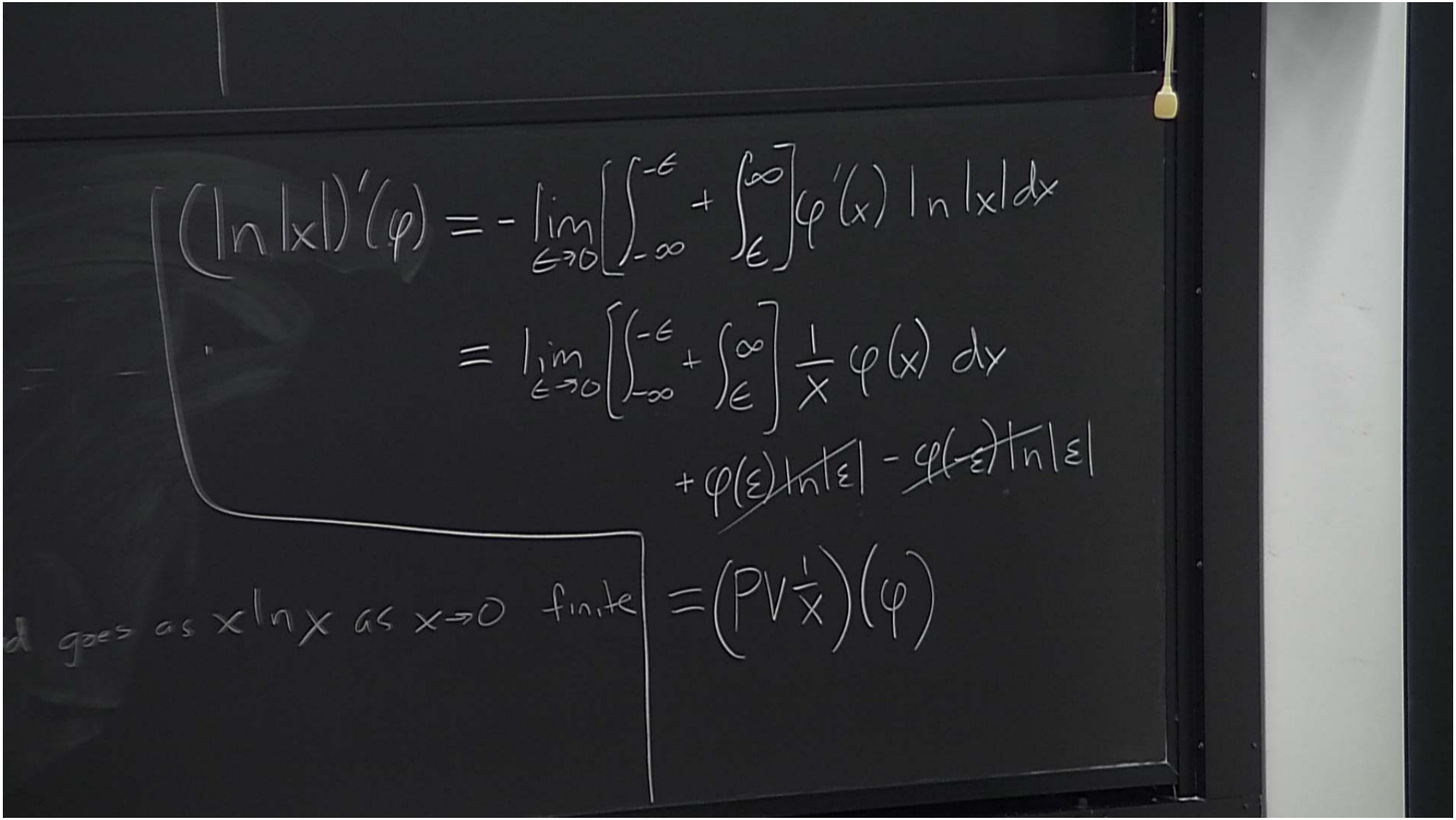


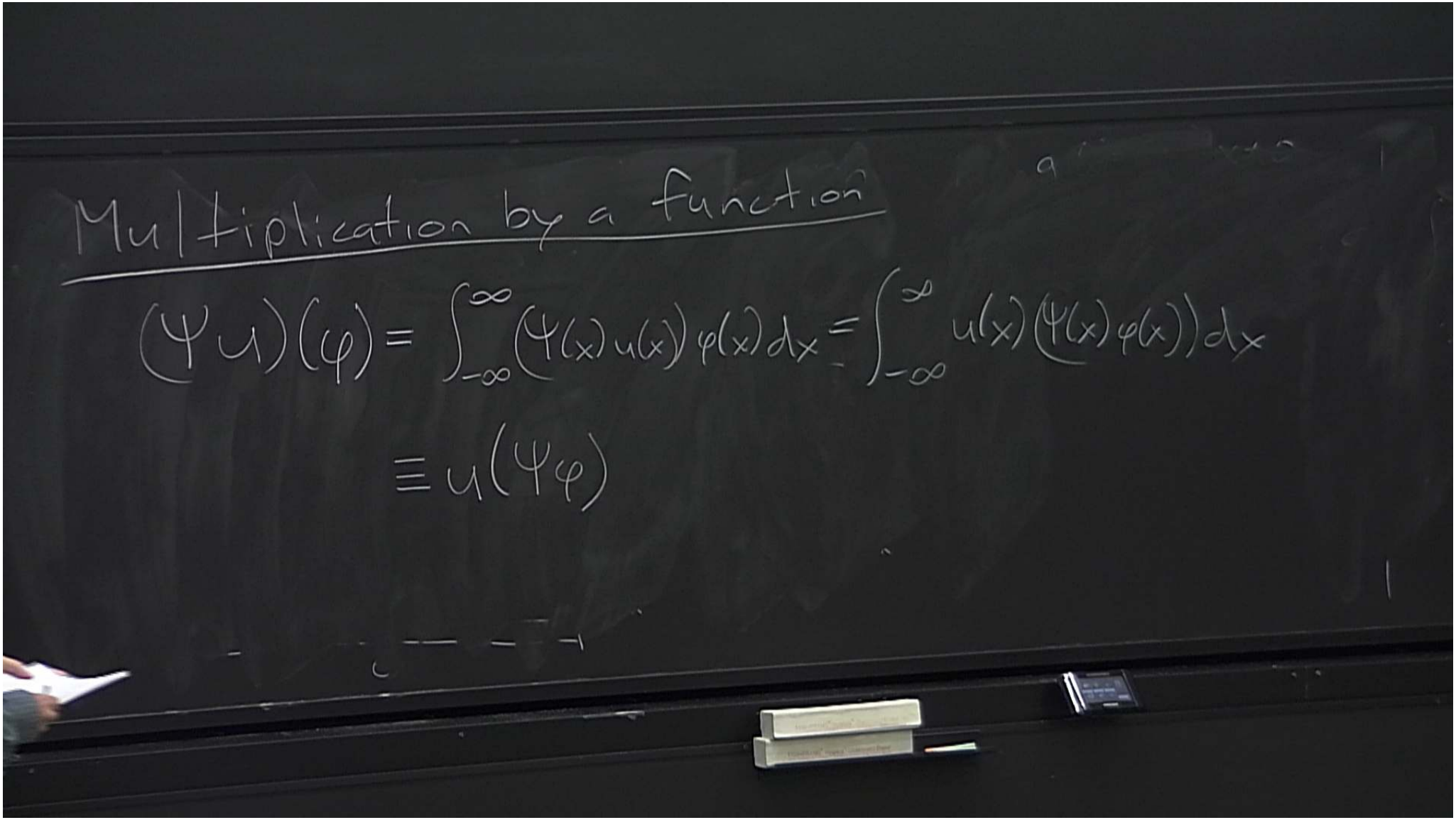
$$(\ln|x|)(\varphi) \stackrel{?}{=} \int_{-\infty}^{\infty} \ln|x|\,\varphi(x)\,dx$$

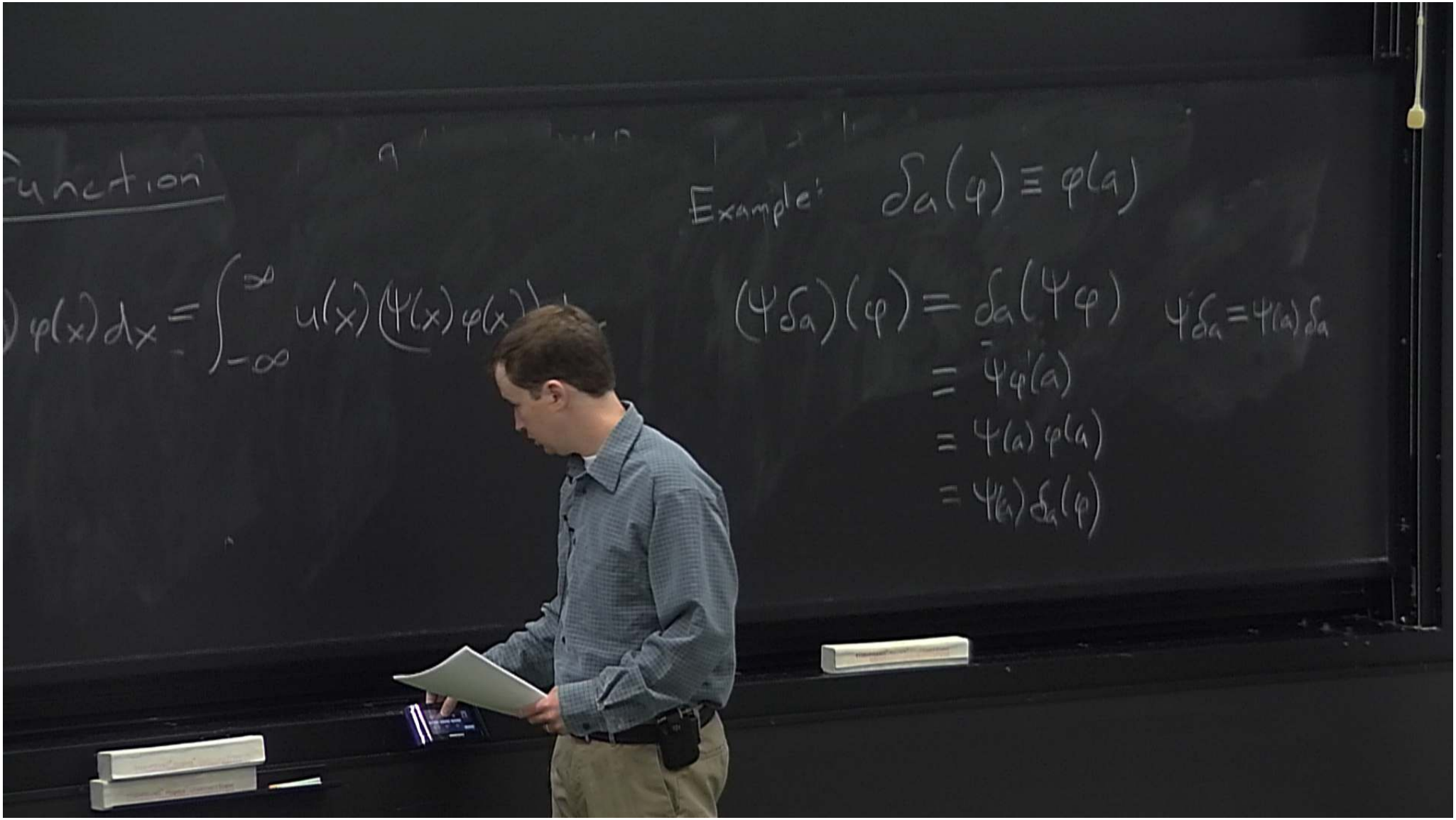
$$(\ln|x|)(\varphi) \equiv \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right) \ln|x|\,\varphi(x)\,dx$$

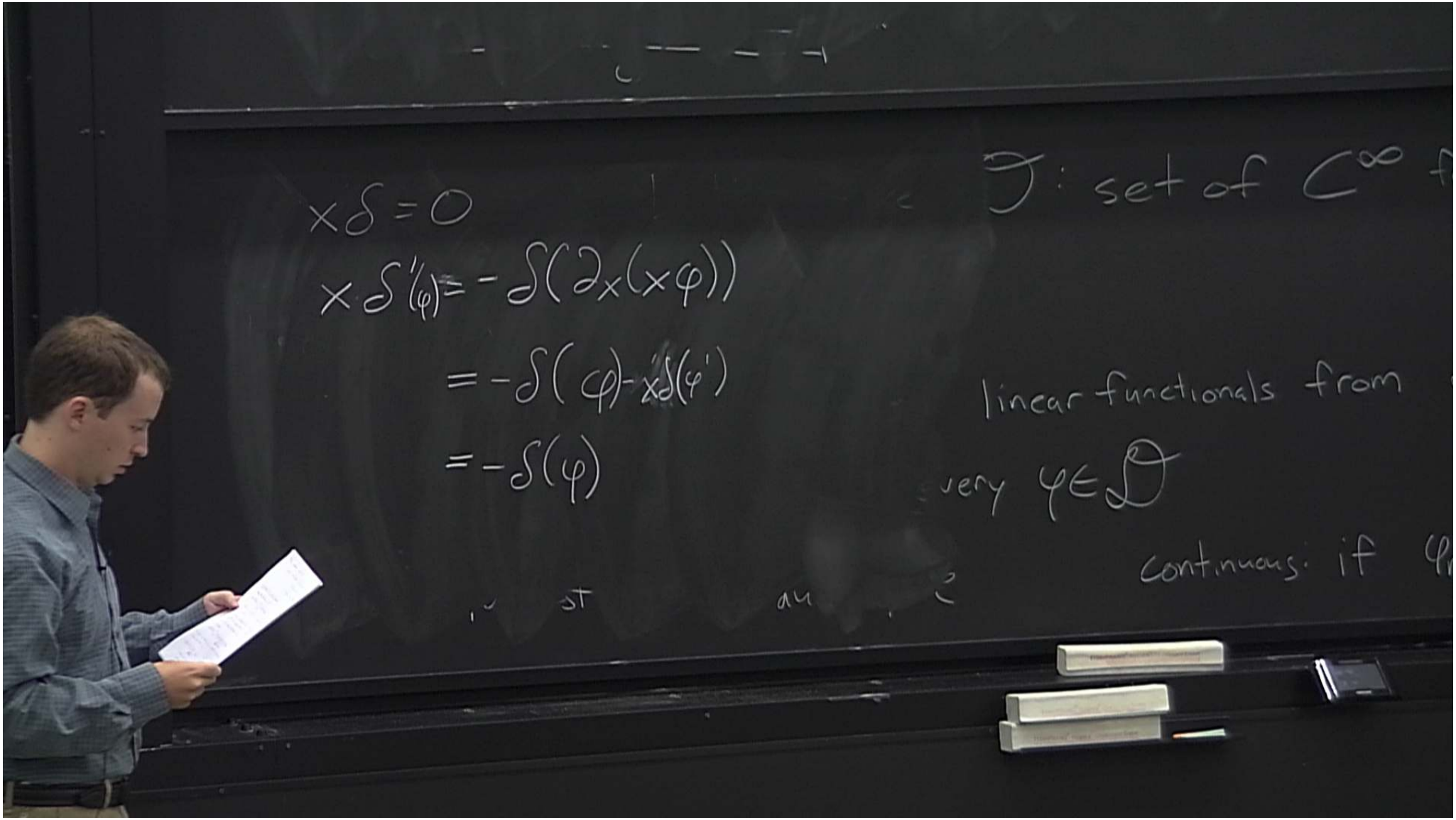
$$= \lim_{\epsilon\to 0^+} \int_{\epsilon}^{\infty} \underbrace{[\varphi(x)-\varphi(-x)]}_{\approx 2\varphi'(0)\cdot x \text{ for } x\ll 1} \ln x\,dx$$

integrand goes as $\frac{(\varphi(x)-\varphi(-x))}{x}$


$(\ln|x|)'(\varphi) = -\lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right]\varphi'(x)\ln|x|\,dx$
$= \lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right]\frac{1}{x}\varphi(x)\,dx$
$+\varphi(\varepsilon)\ln|\varepsilon| - \varphi(-\varepsilon)\ln|\varepsilon|$
d goes as $x\ln x$ as $x\to 0$ finite
$= (PV\frac{1}{x})(\varphi)$


Multiplication by a function
$(\psi u)(\varphi) = \int_{-\infty}^{\infty} (\psi(x)\, u(x))\, \varphi(x)\, dx = \int_{-\infty}^{\infty} u(x)\, (\psi(x)\, \varphi(x))\, dx$
$\equiv u(\psi\varphi)$


Function
Example: $\delta_a(\varphi) \equiv \varphi(a)$
$\varphi(x)\,dx = \int_{-\infty}^{\infty} u(x)\,(\psi(x)\,\varphi(x)$
$(\psi\delta_a)(\varphi) = \delta_a(\psi\varphi)$
$= \psi\varphi(a)$
$= \psi(a)\,\varphi(a)$
$= \psi(a)\,\delta_a(\varphi)$
$\psi\delta_a = \psi(a)\,\delta_a$


$x\delta = 0$
$x\delta'(\varphi) = -\delta(\partial_x(x\varphi))$
$= -\delta(\varphi) - x\delta(\varphi')$
$= -\delta(\varphi)$
$\mathcal{D}$: set of C^∞
linear functionals from
very $\varphi \in \mathcal{D}$
continuous. if