

Title: 13/14 PSI - Algebra - Lecture 3

Date: Aug 19, 2013 10:30 AM

URL: <http://pirsa.org/13080014>

Abstract:

Lie Algebras.

$$[X_a, X_b] = i \underbrace{f_{abc}}_{\text{structure constants}} X_c$$

$a, b = 1, \dots, \# \text{ of generators}$

Lie Algebras.

$$[X_a, X_b] = i \underbrace{f_{abc}}_{\text{structure constants}} X_c$$

$a, b = 1, \dots, \# \text{ of generators}$

In a rep $[X_a]_{ij}$ $i, j = 1, \dots, \text{dim of rep}$

act on states $|\psi\rangle_i$ $i = 1, \dots, \text{dim of rep}$

Jacobi identity: $[X_a, [X_b, X_c]] + \text{cycl. perm.} = 0$
because of defn $[X_a, X_b] = X_a X_b - X_b X_a$

Jacobi identity: $[X_a, [X_b, X_c]] + \text{cycl. perm.} = 0$

because of defn $[X_a, X_b] = X_a X_b - X_b X_a$

$$(\text{check}) \Rightarrow f_{bcd} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0$$

$[X_a, X_b]$

$[T_a]_{bc}$

$a=1, \dots, M$

$b, c=1, \dots, M$

act on a M -dim
vector space

f_{bac}

$$[X_a, X_b]$$

$$\begin{array}{l} \text{ADJOINT} \\ [T_a]_{bc} \\ \downarrow \\ a=1, \dots, M \\ b, c=1, \dots, M \end{array} \left\{ \begin{array}{l} \text{act on a } M\text{-dim} \\ \text{vector space} \end{array} \right.$$

dim of Adjoint rep = # of generators

$$[X_b, X_a] = -[X_a, X_b]$$



$$f_{abc} = -f_{bac}$$

$$\begin{array}{l} \text{ADJOINT} \\ [T_a]_{bc} \left\{ \begin{array}{l} \text{act on a } M\text{-dim} \\ \text{vector space} \end{array} \right. \\ \downarrow \\ a = 1, \dots, M \\ b, c = 1, \dots, M \end{array}$$

dim of Adjoint rep = # of generators

$$su(N): M = N^2 - 1$$

$$[X_b, X_a]$$

$$\downarrow$$

$$f_{abc} =$$

$$\begin{array}{l} \text{ADJOINT} \\ [T_a]_{bc} \\ \downarrow \\ a=1, \dots, M \\ b, c=1, \dots, M \end{array} \left\{ \begin{array}{l} \text{act on a } M\text{-dim} \\ \text{vector space} \end{array} \right.$$

dim of Adjoint rep = # of generators

$$su(N): M = N^2 - 1$$

$$a, b, c = 1, \dots, N^2 - 1$$

n a M-dim
or space

of generators

Define a scalar product on the space of generators

$$(T_a, T_b) = \text{Tr}(T_a T_b) \\ = [T_a]_{ij} [T_b]_{ji}$$

similarity $\rightarrow$

$$= (\pm 1 \text{ or } 0) \delta_{ab}$$

in a M -dim
or space

$p = \#$ of generators

Define a scalar product on the space of generators

$$(T_a, T_b) = \text{Tr}(T_a T_b) \\ = [T_a]_{ij} [T_b]_{ji}$$

similarity $\rightarrow$

$$= (\pm 1 \text{ or } 0) \delta_{ab}$$

"Compact" Lie Algebras ± 1 $\text{Tr}(T_a T_b) = \delta_{ab}$

on the space of generators

$T_b)$

$[T_b]_{ji}$

Compact Lie Algebras:

$$\text{Tr}([T_a, T_b], T_c) = i f_{abd} \text{Tr}(T_d, T_c) \\ = i f_{abc}$$

$$\Rightarrow f_{abc} = -i \text{Tr}([T_a, T_b], T_c)$$

of generators

Compact Lie Algebras

$$\text{Tr}([T_a, T_b] T_c) = i f_{abc} \text{Tr}(T_a T_c) = i f_{abc}$$

$$\Rightarrow f_{abc} = -i \text{Tr}([T_a, T_b] T_c)$$

$$= \dots = i \text{Tr}([T_a, T_c] T_b) = -f_{acb}$$

$$\Rightarrow \boxed{f_{abc} = -f_{acb}}$$

δ_{ab}

of generators

Compact Lie Algebras

$$\text{Tr}([T_a, T_b] T_c) = i f_{abd} \text{Tr}(T_d T_c) \\ = i f_{abc}$$

$$\Rightarrow f_{abc} = -i \text{Tr}([T_a, T_b] T_c)$$

$$= \dots = i \text{Tr}([T_a, T_c] T_b) = -f_{acb}$$

$$\Rightarrow \boxed{f_{abc} = -f_{acb}} \text{ FULLY ANTISYMMETRIC } f_{abc}'\text{'s}$$

Invariant subalgebra $A_1 \subset A$
 $X, Y \in A_1 \Rightarrow [X, Y] \in A_1$

Invariant subalgebra $A_1 \subset A$

$$X, Y \in A_1 \Rightarrow [X, Y] \in A_1$$

"Simple" Algebras no invariant subalgebras

Invariant subalgebra $A_1 \subset A$

$$X, Y \in A_1 \Rightarrow [X, Y] \in A_1$$

"Simple" Algebras no invariant subalgebras

"semi-simple" " " " Abelian "

$U(1)$: invariant subalgebra with a single generator
that commutes with all other generators

$$\Rightarrow \boxed{f_{abc} = -f_{acb}} \text{ FULLY ANTISYMMETRIC } f$$

$U(1)$: invariant subalgebra with a single generator
that commutes with all other generators

$$[J_i, J_j] = i \epsilon_{ijk} J_k$$

simplest compact Lie Alg.

$$\Rightarrow \boxed{f_{abc} = -f_{acb}} \text{ FULLY ANTISYMMETRIC } f$$

$U(1)$: invariant subalgebra with a single generator
that commutes with all other generators

$$[J_i, J_j] = i \epsilon_{ijk} J_k$$

simplest compact Lie Alg.

$$i, j = 1, 2, 3$$

$$A_1 = su(2) \simeq so(3)$$

$$\mathfrak{su}(2) \quad [T_i, T_j] = i \epsilon_{ijk} T_k$$

→ IRREPS

$$\mathfrak{su}(2) \quad [J_i, J_j] = i \epsilon_{ijk} J_k$$

→ IRREPS

→ can diagonalize 1 of J_i 's → $\boxed{J_3}$

$$\mathfrak{su}(2) \quad [J_i, J_j] = i \epsilon_{ijk} J_k$$

→ IRREPS

→ can diagonalize 1 of J_i 's → $\boxed{J_3}$

$J_i^\dagger = J_i \Rightarrow$ real eigenvalues
orthogonal eigenstates

pick eigenstate of J

$$[J_i, J_j] = i \epsilon_{ijk} J_k$$

S

normalize 1 of J_i 's $\rightarrow \boxed{J_3}$

pick eigenstate
of J_3 with max eigenvalue
common eigenstates

pick eigenstate of J_3 with max eigenvalue
 $|j\rangle, J_3|j\rangle = j|j\rangle$

Raising & lowering ops. $\mathcal{T}_{\pm} = \mathcal{T}_1 \pm i\mathcal{T}_2$

$$[\mathcal{T}_3, \mathcal{T}_{\pm}] = \pm \mathcal{T}_{\pm}$$

$$[\mathcal{T}_+, \mathcal{T}_-] = \mathcal{T}_3$$

Raising & lowering ops. $J_{\pm} = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2)$

$$[J_3, J_{\pm}] = \pm J_{\pm}$$

$$[J_+, J_-] = J_3$$

Raising & lowering ops. $J_{\pm} = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2)$

$$[J_3, J_{\pm}] = \pm J_{\pm}$$

$$[J_+, J_-] = J_3$$

If $|m\rangle$ eigenstate of J_3 $J_3|m\rangle = m|m\rangle$

$$J_-|m\rangle = (m+1)J_+|m\rangle$$

Raising & lowering ops. $J_{\pm} = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2)$

$$[J_3, J_{\pm}] = \pm J_{\pm}$$

$$[J_+, J_-] = J_3$$

If $|m\rangle$ eigenstate of J_3 : $J_3|m\rangle = m|m\rangle$

$$J_3 J_{\pm}|m\rangle = (m \pm 1) J_{\pm}|m\rangle$$

$$\text{ops. } \mathcal{J}_{\pm} = \frac{1}{\sqrt{2}}(\mathcal{J}_1 \pm i\mathcal{J}_2)$$

$$[\mathcal{J}_3, \mathcal{J}_{\pm}] = \pm \mathcal{J}_{\pm}$$

$$[\mathcal{J}_3, \mathcal{J}] = 0$$

$$[\mathcal{J}_+, \mathcal{J}_-] = \mathcal{J}_3$$

$$\text{eigenstate of } \mathcal{J}_3: \mathcal{J}_3|m\rangle = m|m\rangle$$

$$\mathcal{J}_{\pm}|m\rangle = (m \pm 1)\mathcal{J}_{\pm}|m\rangle$$

$$J_{\pm} = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2)$$

$$[J_3, J_{\pm}] = \pm J_{\pm}$$

$$[J_3, \cdot] = (\cdot)$$

$$[J_+, J_-] = J_3$$

$$\text{eig of } J_3: J_3|m\rangle = m|m\rangle$$

$$[J_3, X] = \alpha X$$

$$J_3(X|m\rangle) = (\alpha + m)X|m\rangle$$

$$= (m \pm 1) J_{\pm}|m\rangle$$

Raising & lowering ops. $J_{\pm} = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2)$

$$[J_3, J_{\pm}] = \pm J_{\pm}$$

$$[J_3, J_3] = 0$$

$$[J_+, J_-] = J_3$$

If $|m\rangle$ eigenstate of J_3 $J_3|m\rangle = m|m\rangle$

$$J_3 J_{\pm}|m\rangle = (m \pm 1) J_{\pm}|m\rangle$$

Raising & lowering ops. $J_{\pm} = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2)$

$$[J_3, J_{\pm}] = \pm J_{\pm}$$

$$[J_3, J_3] = 0$$

$$\rightarrow [J_+, J_-] = J_3$$

If $|m\rangle$ eigenstate of J_3 : $J_3|m\rangle = m|m\rangle$

$$J_3 J_{\pm}|m\rangle = (m \pm 1) J_{\pm}|m\rangle$$

$$J_{\pm} = \frac{1}{\sqrt{2}} (J_1 \pm i J_2)$$

j : "highest weight"

$$[J_3, J_{\pm}] = \pm J_{\pm}$$

$$[J_3, \cdot] = (\cdot)$$

$$[J_+, J_-] = J_3$$

$$[J_3, X] = \alpha X$$

$$\text{e of } J_3: J_3 |m\rangle = m |m\rangle$$

$$J_3 (X |m\rangle) = (\alpha + m) X |m\rangle$$

$$(m \pm 1) J_{\pm} |m\rangle$$

$\{ |m\rangle \}$

$$\begin{aligned} \langle m' | J_3 | m \rangle &= [J_3]_{m'm} \\ &= \langle m' | m | m \rangle = m \langle m' | m \rangle = m \delta_{m'm} \end{aligned}$$

$$\begin{aligned}
 \{ |m\rangle \} \quad \langle m' | J_3 | m \rangle &= [J_3]_{m'm} \\
 &= \langle m' | m | m \rangle = m \langle m' | m \rangle = m \delta_{m'm} \\
 \langle m' | J_{\pm} | m \rangle &= [J_{\pm}]_{m'm} = ?
 \end{aligned}$$

raising & lowering ops. $J_{\pm} = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2)$

j: "highest w"

$$[J_3, J_{\pm}] = \pm J_{\pm}$$

$$[J_3, \cdot] = ()$$

check! $\rightarrow [J_+, J_-] = J_3$

If $|m\rangle$ eigenstate of J_3 $J_3|m\rangle = m|m\rangle$

$$[J_3, J_{\pm}] = \pm J_{\pm}$$

$$J_3 J_{\pm} |m\rangle = (m \pm 1) J_{\pm} |m\rangle \Rightarrow J_{\pm} |m\rangle$$

lowering ops. $J_{\pm} = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2)$

j : "highest weight"

$$[J_3, J_{\pm}] = \pm J_{\pm}$$

$$[J_3, \cdot] = (\cdot)$$

check! $\rightarrow [J_+, J_-] = J_3$

$$[J_3, X] = \alpha X$$

$|m\rangle$ eigenstate of J_3 $J_3|m\rangle = m|m\rangle$

$$J_3(X|m\rangle) = (\alpha + m)X|m\rangle$$

$$J_3 J_{\pm}|m\rangle = (m \pm 1) J_{\pm}|m\rangle \Rightarrow J_{\pm}|m\rangle \propto |m \pm 1\rangle$$

We can choose orthon. $\{|m\rangle\}$ $\langle m' | m \rangle = \delta_{m'm}$

$$J_- |m\rangle = N(m) |m-1\rangle$$

① $m = j = \text{highest weight} : N(j) = \sqrt{j}$

② $J_+ |m-1\rangle = N(m) |m\rangle$

We can choose orthonormal $\{|m\rangle\}$ $\langle m' | m \rangle = \delta_{m'm}$

$$J_- |m\rangle = N(m) |m-1\rangle$$

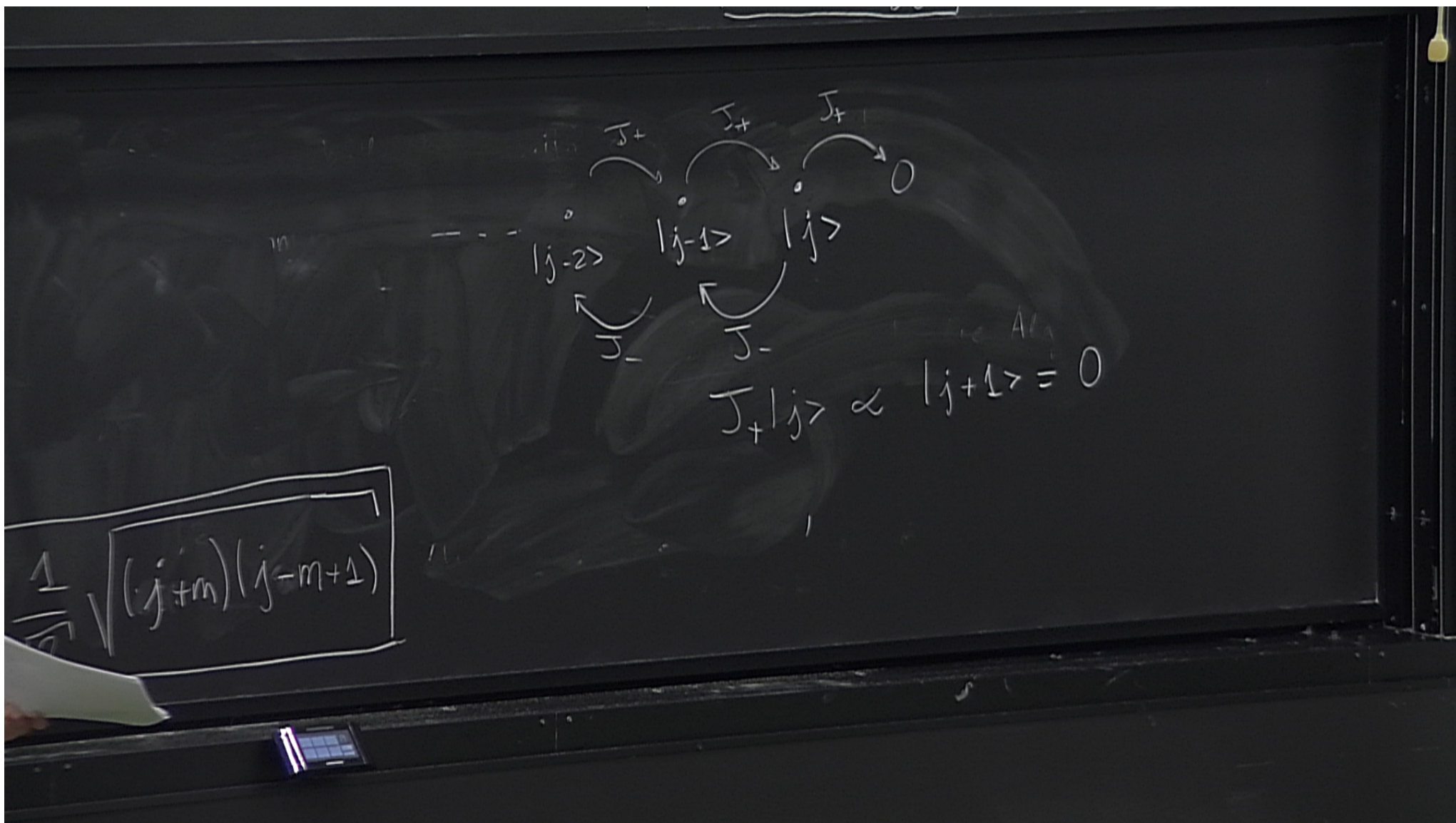
① $m = j = \text{highest weight} : N(j) = \sqrt{j}$

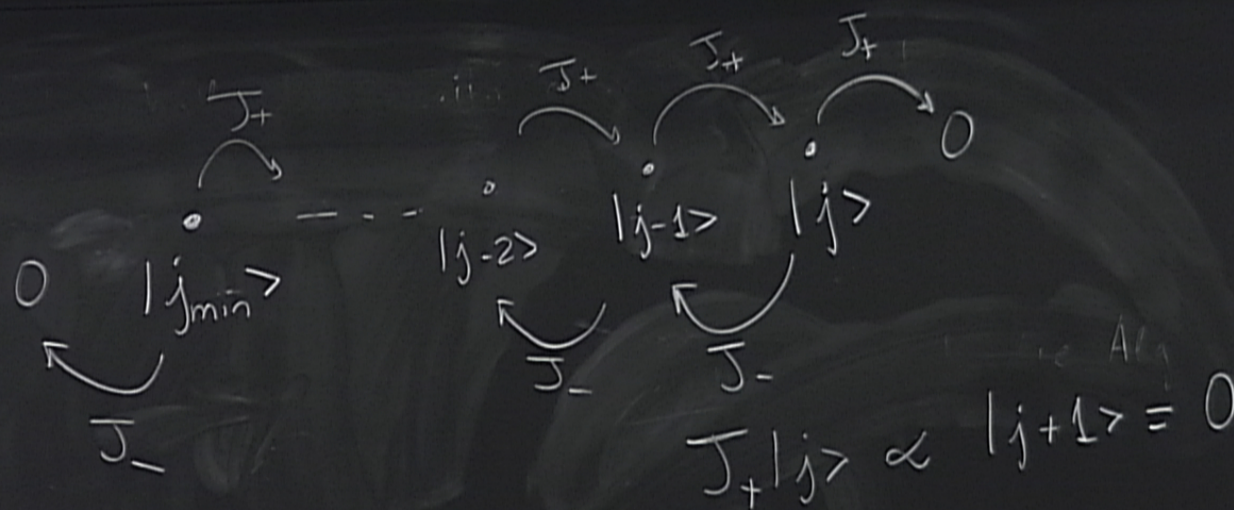
② $J_+ |m-1\rangle = N(m) |m\rangle$

③ $N(m)^2 = m + N(m+1)^2$

$\Rightarrow$

$$N(m) = \frac{1}{\sqrt{2}} \sqrt{(j+m)(j-m+1)}$$





$$\frac{1}{\sqrt{2}} \sqrt{(j+m)(j-m+1)}$$

$$J_- |j_{\min}\rangle = 0$$

$$j_{\min} = j - l$$

$$J_+ |j\rangle = N(j+1) |j+1\rangle = 0$$

Define
(Ta

"Compa

$$J_- |j_{\min}\rangle = 0$$

$$j_{\min} = j - l$$

$$\Rightarrow N(j-l) = 0$$

$$\Rightarrow \boxed{l = 2j} \Leftrightarrow \boxed{j = \frac{l}{2}}$$

$$\begin{aligned} J_+ |j\rangle &= \\ N(j+1) |j+1\rangle \\ &= 0 \end{aligned}$$

Define

(Ta

"Compa

$$J_- |j_{\min}\rangle = 0$$

$$\begin{aligned} J_+ |j\rangle &= \\ N(j+1) |j+1\rangle \\ &= 0 \end{aligned}$$

$$j_{\min} = j - l$$

$$\Rightarrow N(j-l) = 0$$

$$\Rightarrow \boxed{l = 2j} \Leftrightarrow \boxed{j = \frac{l}{2}} \quad l \in \mathbb{N}$$

Define

(Ta

"Compa

$$\Rightarrow \left[\begin{matrix} \sqrt{2} & 1 \\ 0 & 0 \end{matrix} \right]$$

$$J_+ |j\rangle = \sqrt{j(j+1)} |j+1\rangle$$

highest weight
 $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

$$\boxed{l = \frac{l}{2}} \quad l \in \mathbb{N}$$

the space of generators

Compact Lie
 $\text{Tr}([T_a, T_b])$

$$\Rightarrow f_{abc}$$

$$\Rightarrow \boxed{f_{abc}}$$

$$\Rightarrow \left[\begin{matrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{matrix} \right]$$

$$J_+ |j\rangle = \sqrt{j(j+1)} |j+1\rangle$$

$$J_- |j\rangle = \sqrt{j(j-1)} |j-1\rangle$$

$$J_z |j\rangle = j |j\rangle$$

highest weight
 $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

$$\dim \text{ of rep} = l+1 = 2j+1$$

$$l = \frac{l}{2} \in \mathbb{N}$$

the space of generators

Compact Lie
 $\text{Tr}([T_a, T_b])$

$$\Rightarrow f_{abc} = -c$$

$$f_{abc} = i T$$

$$\Rightarrow f_{abc} = -$$

$$j+1, j \rangle =$$

$$N(j+1) |j+1\rangle$$

$$= 0$$

$$j = l$$

$$j-l = 0$$

$$l = 2j \iff j = \frac{l}{2} \quad l \in \mathbb{N}$$

highest weight

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$$

dim of rep = $l+1 = 2j+1$

$|j, m\rangle$ WEIGHT

HIGHEST w

is source of

$$\Rightarrow \left[\dots \sqrt{2} \dots \right]$$

the source of generators

$$|3/2, 1\rangle$$

highest weight
 $j = 0, 1/2, 1, 3/2, \dots$

$$\dim \text{ of rep} = l+1 = 2j+1$$

$|j, m\rangle$ ^{WEIGHT}
 HIGHEST w

compact Lie Alg
 $\text{Tr}([T_a, T_b], T_c)$

$$\Rightarrow f_{abc} = -i \text{Tr}([T_a, T_b] T_c)$$

$$= \dots = i \text{Tr}([T_b, T_c] T_a)$$

$$\Rightarrow \boxed{f_{abc} = -f_{acb}}$$

$$\{ |m\rangle \} \quad \langle m' | J_3 | m \rangle = [J_3]_{m'm}$$

$$= \langle m' | m | m \rangle = m \langle m' | m \rangle = m \delta_{m'm}$$

$$\langle m' | J_1 | m \rangle = [J_1]_{m'm} = \frac{1}{\sqrt{2}} [J_+ + J_-]_{mm'}$$

$$[J_+]_{m'm} = \langle m' | J_+ | m \rangle = N(m+1) \langle m' | m+1 \rangle$$

$$\langle m' | J_3 | m \rangle = [J_3]_{m'm}$$

$$= \langle m' | m | m \rangle = m \langle m' | m \rangle = m \delta_{m'm}$$

$$\langle m | J_1 | m \rangle = [J_1]_{mm} = \frac{1}{\sqrt{2}} [J_+ + J_-]_{mm}$$

$$= \langle m' | J_+ | m \rangle = N(m+1) \langle m' | m+1 \rangle = N(m+1) \delta_{m', m+1}$$

$$[J_3]_{m'm}$$

$$\langle m' | m \rangle = \delta_{m'm}$$

$$m = -j, \dots, j$$

$$J_y = \frac{1}{i\hbar} [J_+ - J_-]_{mm'}$$

$$[J_-]_{-1/2, 1/2}$$

$$\langle m' | m+1 \rangle = N(m+1) \delta_{m', m+1}$$

$$J_- | j, m \rangle$$

$$j_{\min} =$$

$$N(j)$$

$$\Rightarrow$$