Title: 13/14 PSI - Algebra - Lecture 2

Date: Aug 15, 2013 10:30 AM

URL: http://pirsa.org/13080010

Abstract:

ng an operator A in o/n basis $\{|e_i\rangle\}$

matrix compts a_{ij}

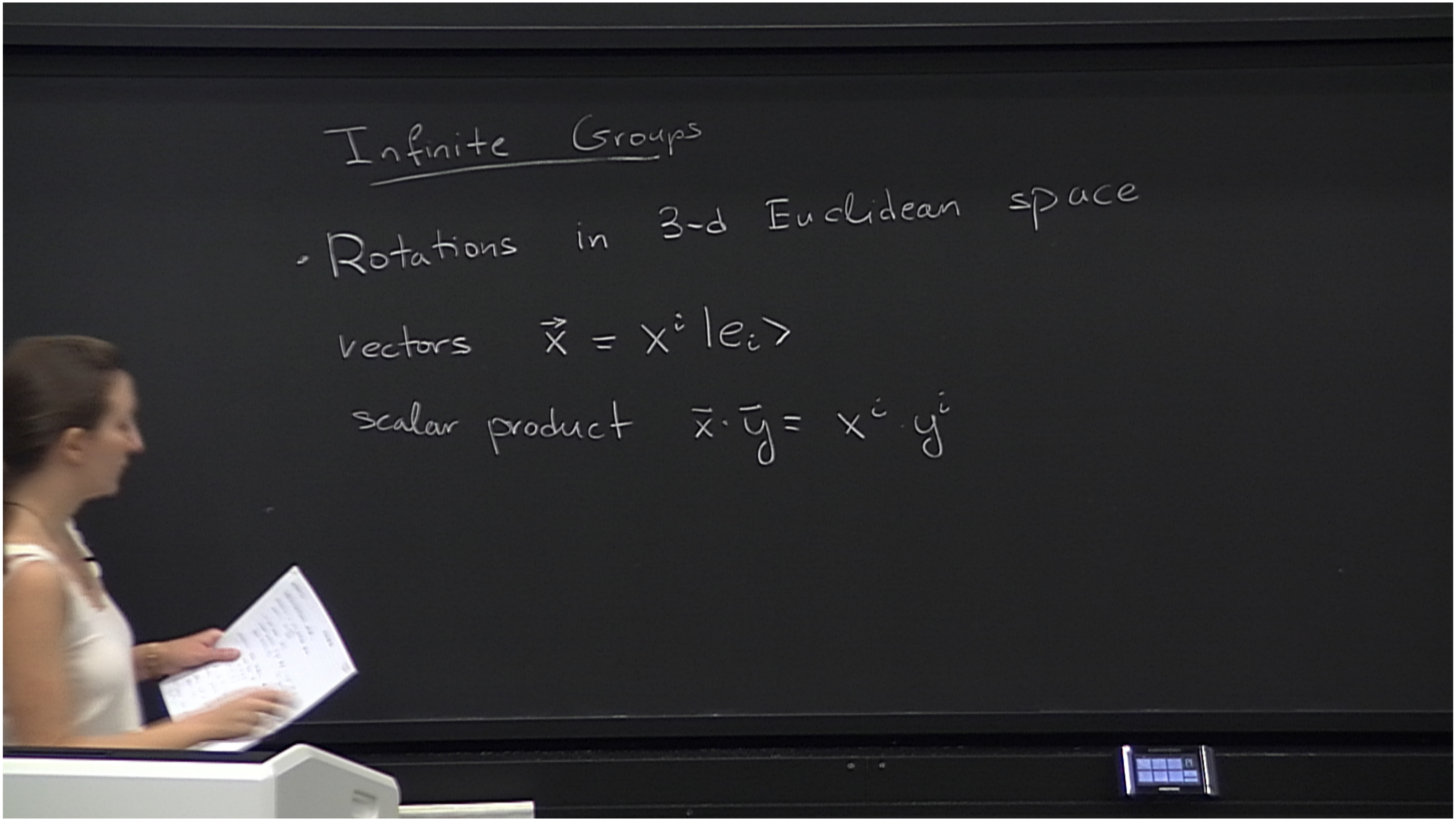
$$\langle e_i | A | e_j \rangle = a_{ij} \quad \Longleftrightarrow \quad \langle e^i | A | e_j \rangle = a^i{}_j$$

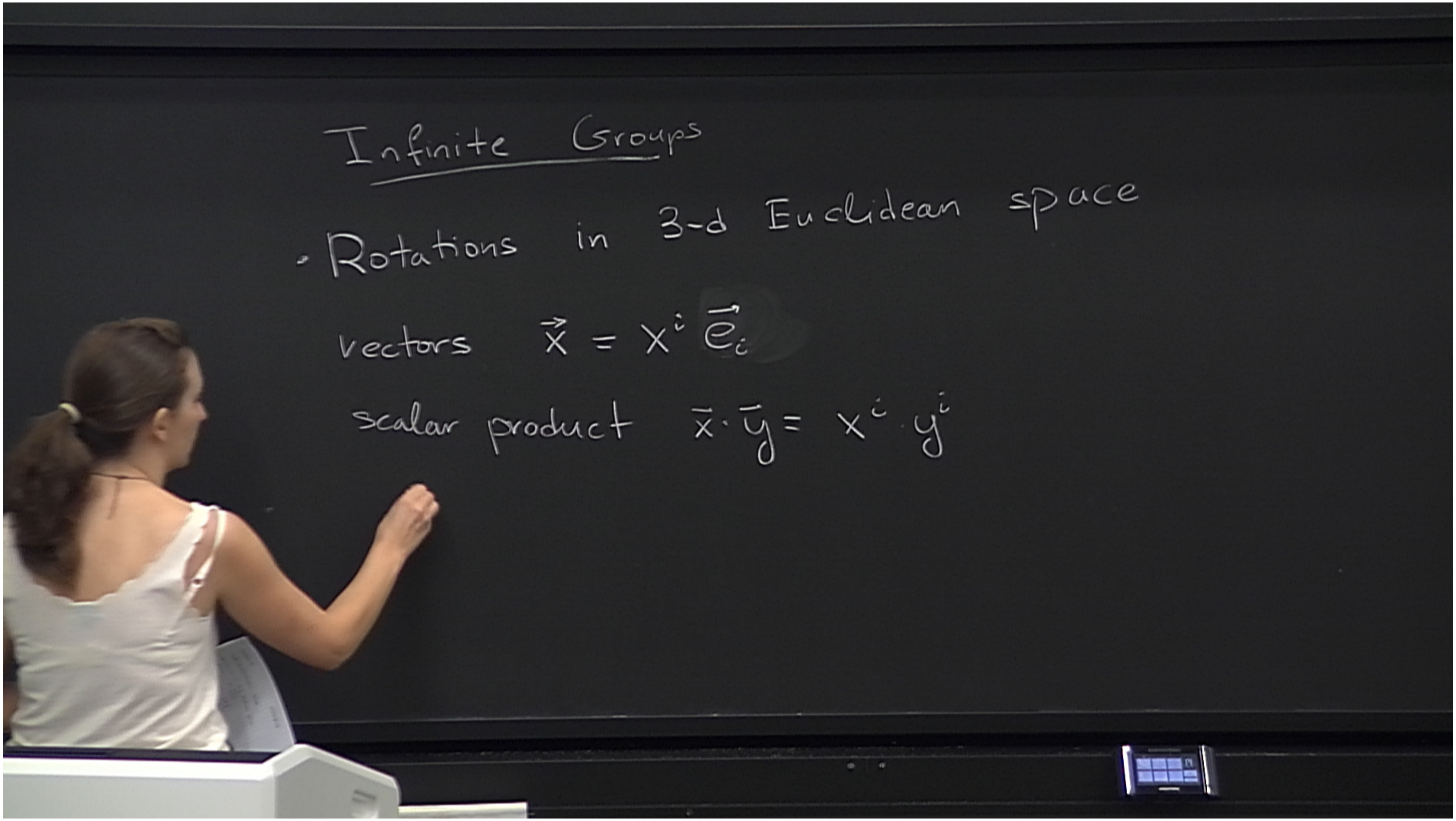
on. group element g $\longrightarrow$ operator $D(g)$

$$\left(\begin{array}{l} D(g_1 g_2) = D(g_1)\, D(\\ D(e) = 1 \end{array} \right.$$

dim of V on which $D(g)$ act

$$\langle e_i | D(g) | e_j \rangle = [D(g)]_{ij}$$


Infinite Groups
• Rotations in 3-d Euclidean space
vectors $\vec{x} = x^i |e_i\rangle$
scalar product $\bar{x} \cdot \bar{y} = x^i \cdot y^i$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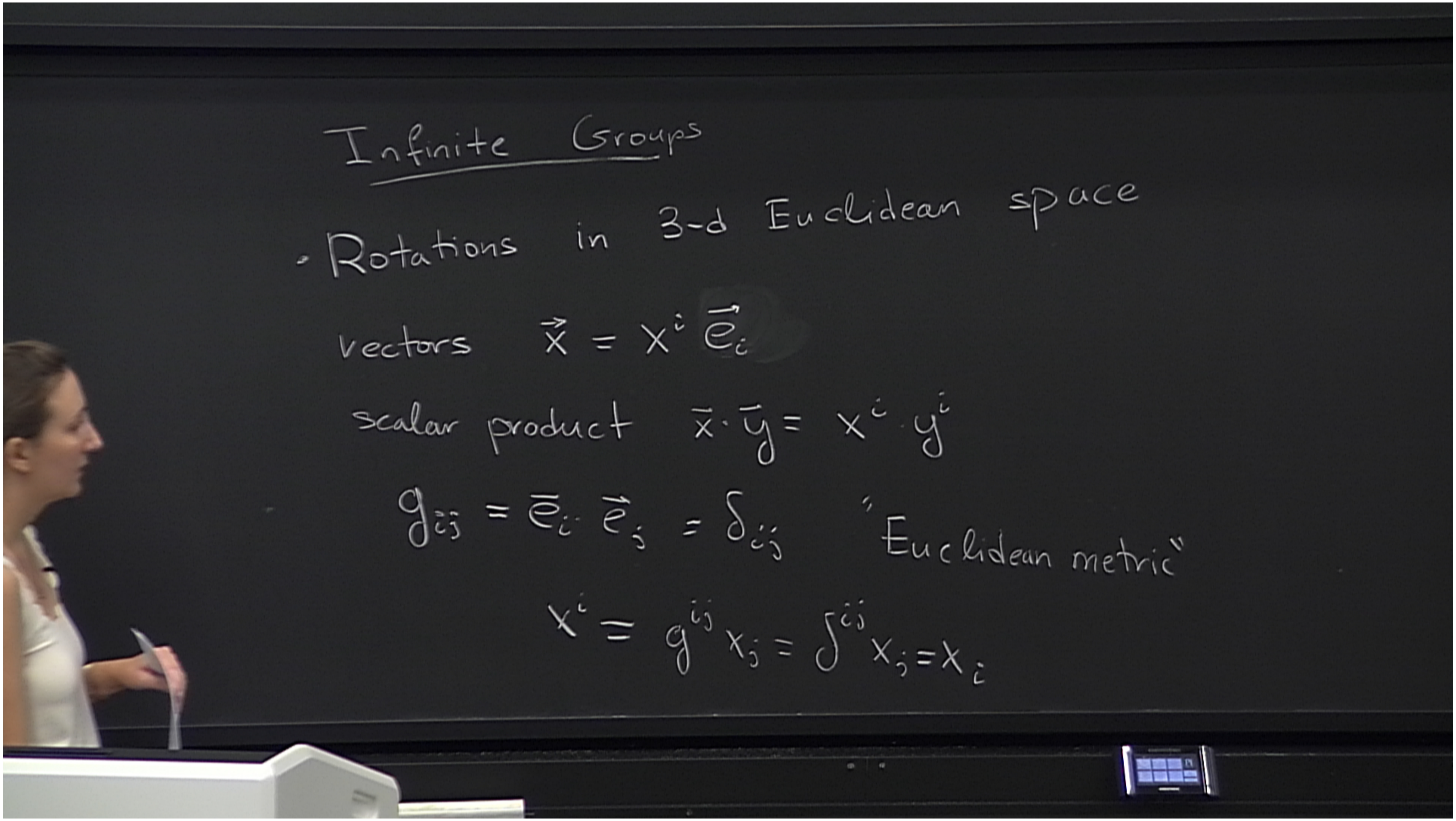
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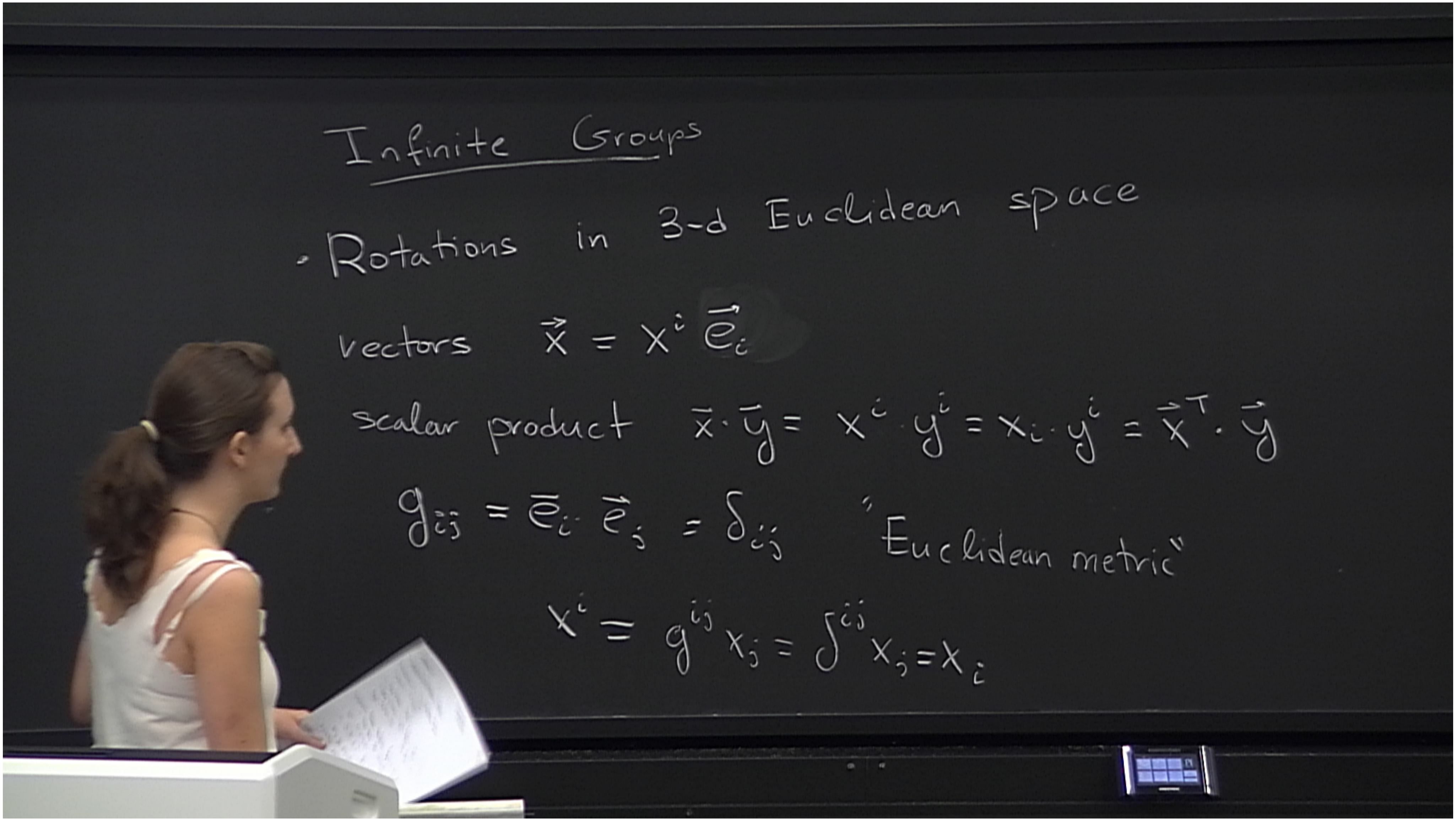
$g_{ij} = \vec{e}_i \cdot \vec{e}_j = \delta_{ij}$; "Euclidean metric"
$x^i = g^{ij} x_j = \delta^{ij} x_j = x_i$


Infinite Groups
• Rotations in 3-d Euclidean space
vectors $\vec{x} = x^i \vec{e}_i$
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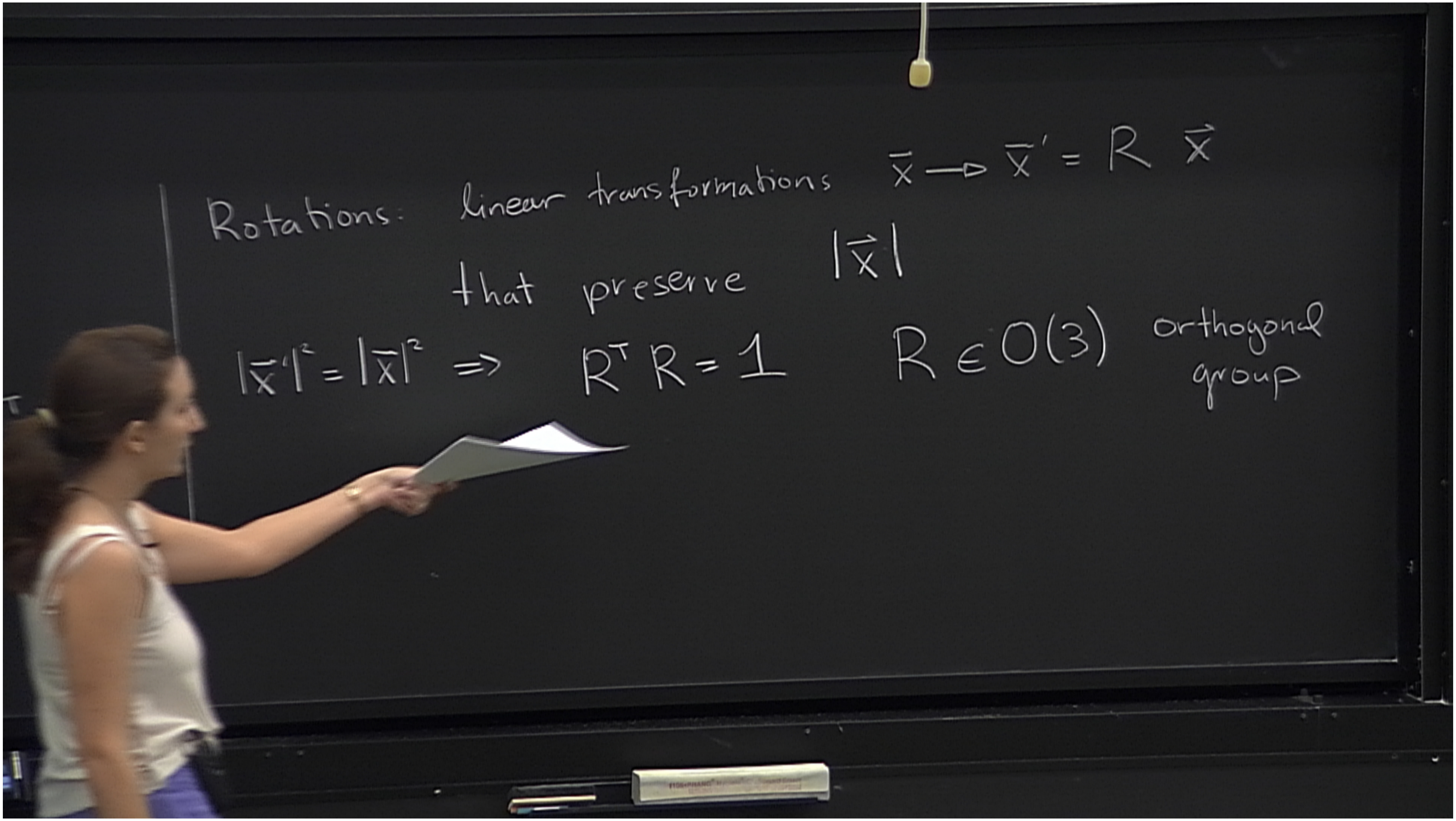
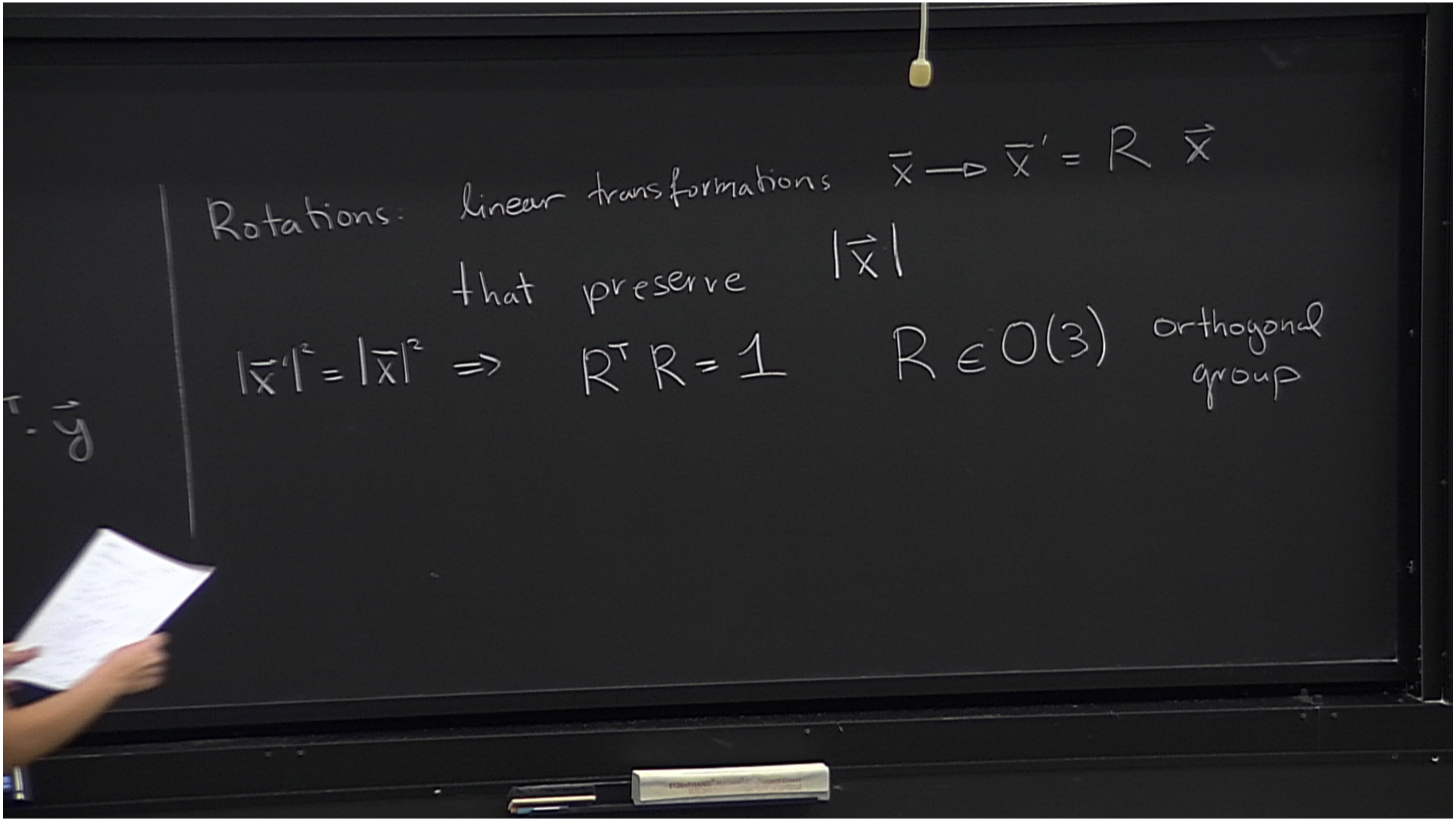

Rotations: linear transformations $\vec{x} \longrightarrow \vec{x}\,' = R\,\vec{x}$
that preserve $|\vec{x}|$
$|\vec{x}\,'|^2 = |\vec{x}|^2 \Rightarrow R^T R = 1$
$R \in O(3)$ orthogonal group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$^T \cdot \vec{y}$

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$$R^i{}_k \, g_{ij} \, R^k{}_l = g_{kl}$$

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$\Rightarrow$

$\vec{x}^T \cdot \vec{y}$

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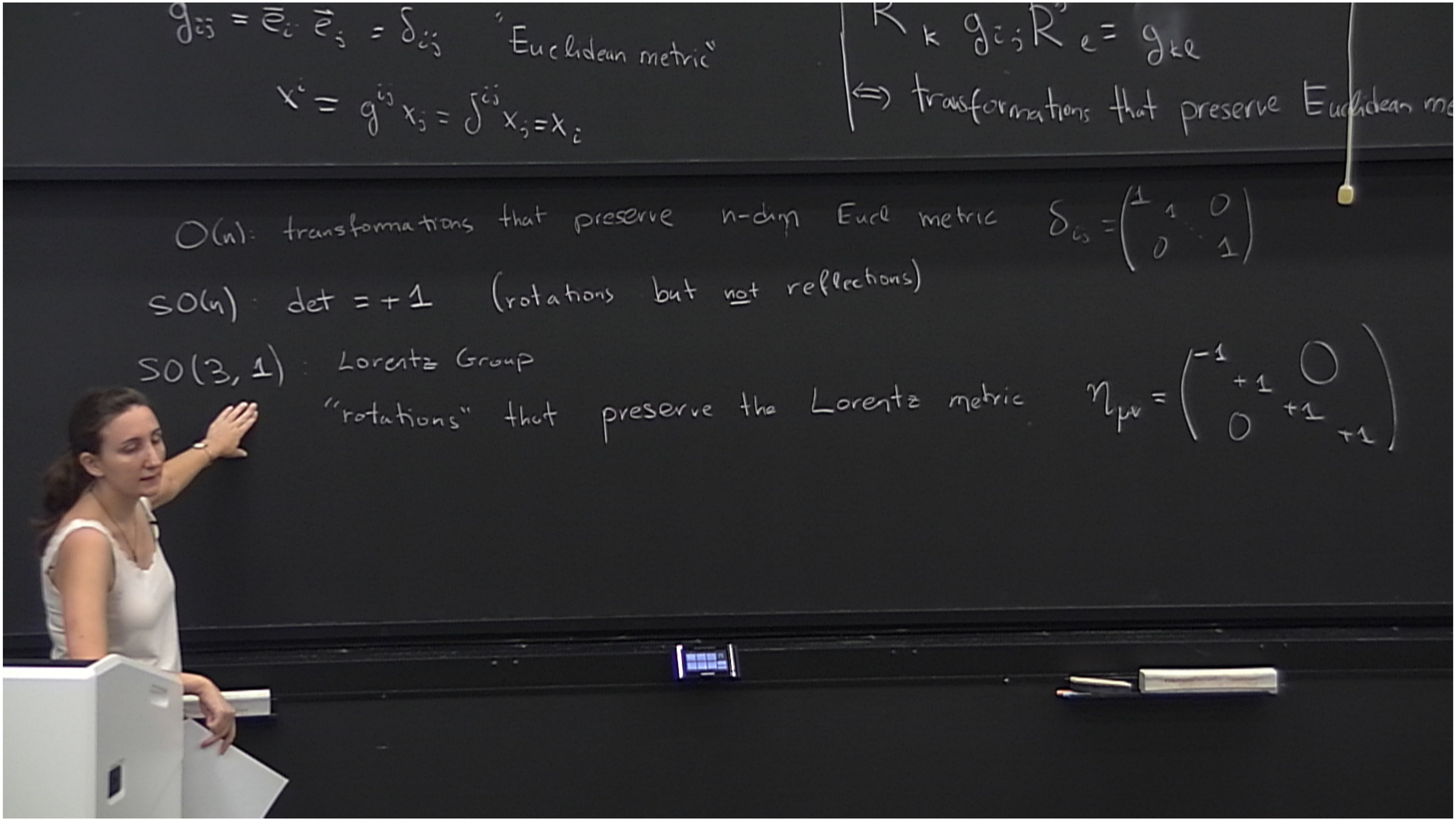
$$R^i{}_k \, g_{ij} \, R^j{}_l = g_{kl}$$

$\Longleftrightarrow$ transformations that preserve Euclidean metric $\delta_{ij} = g_{ij}$



$g_{ij} = \vec{e}_i \cdot \vec{e}_j = \delta_{ij}$ "Euclidean metric"
$x^i = g^{ij} x_j = \delta^{ij} x_j = x_i$
$R^i{}_k \, g_{ij} R^j{}_l = g_{kl}$
$\Leftrightarrow$ transformations that preserve Euclidean me
$O(n)$: transformations that preserve n-dim Eucl metric
$\delta_{ij} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$
$SO(n)$: det = +1 (rotations but not reflections)
$SO(3,1)$: Lorentz Group
"rotations" that preserve the Lorentz metric
$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & +1 \end{pmatrix}$

$g_{ij} = e_i \cdot e_j = \delta_{ij}$ "Euclidean metric"

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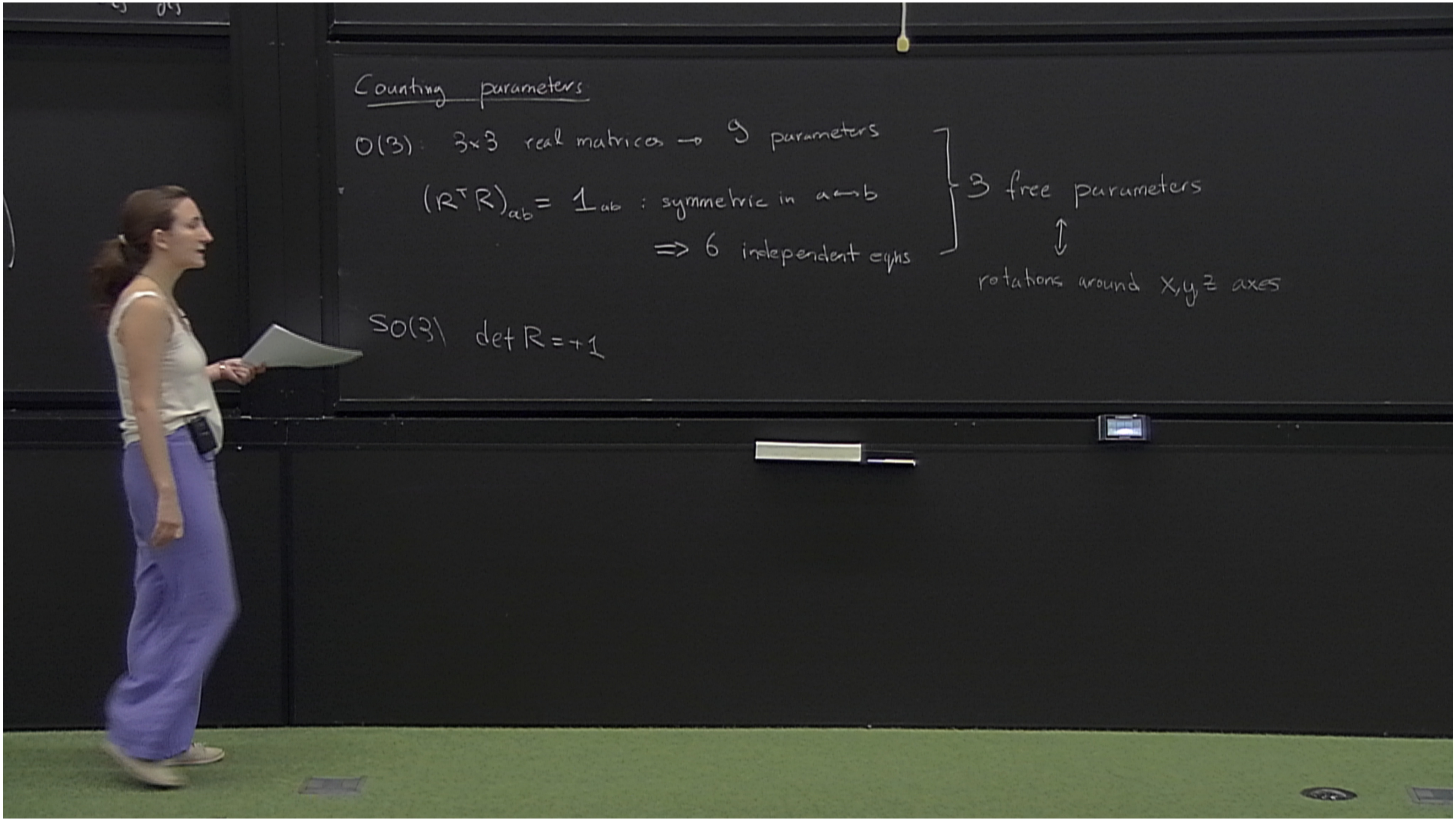
$SO(3,1)$: Lorentz Group

"rotations" that preserve the Lorentz metric $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & +1 \end{pmatrix}$

$U(n) / SU(n)$: Rotations in complex space that preserve the identity

$U^+ U = 1$ (Special) $\det U = +1$


Counting parameters
O(3): 3×3 real matrices → 9 parameters
$(R^T R)_{ab} = 1_{ab}$: symmetric in a↔b
⇒ 6 independent eqns
3 free parameters
rotations around x,y,z axes
SO(3) det R = +1

$U(2)$: 2×2 complex matrices $\longrightarrow$ 4 complex / 8 real parameters

$U^\dagger U = \mathbb{1}$ $\rightarrow$ eqns can be put in hermitian matrix $\Big\}$ 4 real eqns

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$\det U = e^{i\phi}$

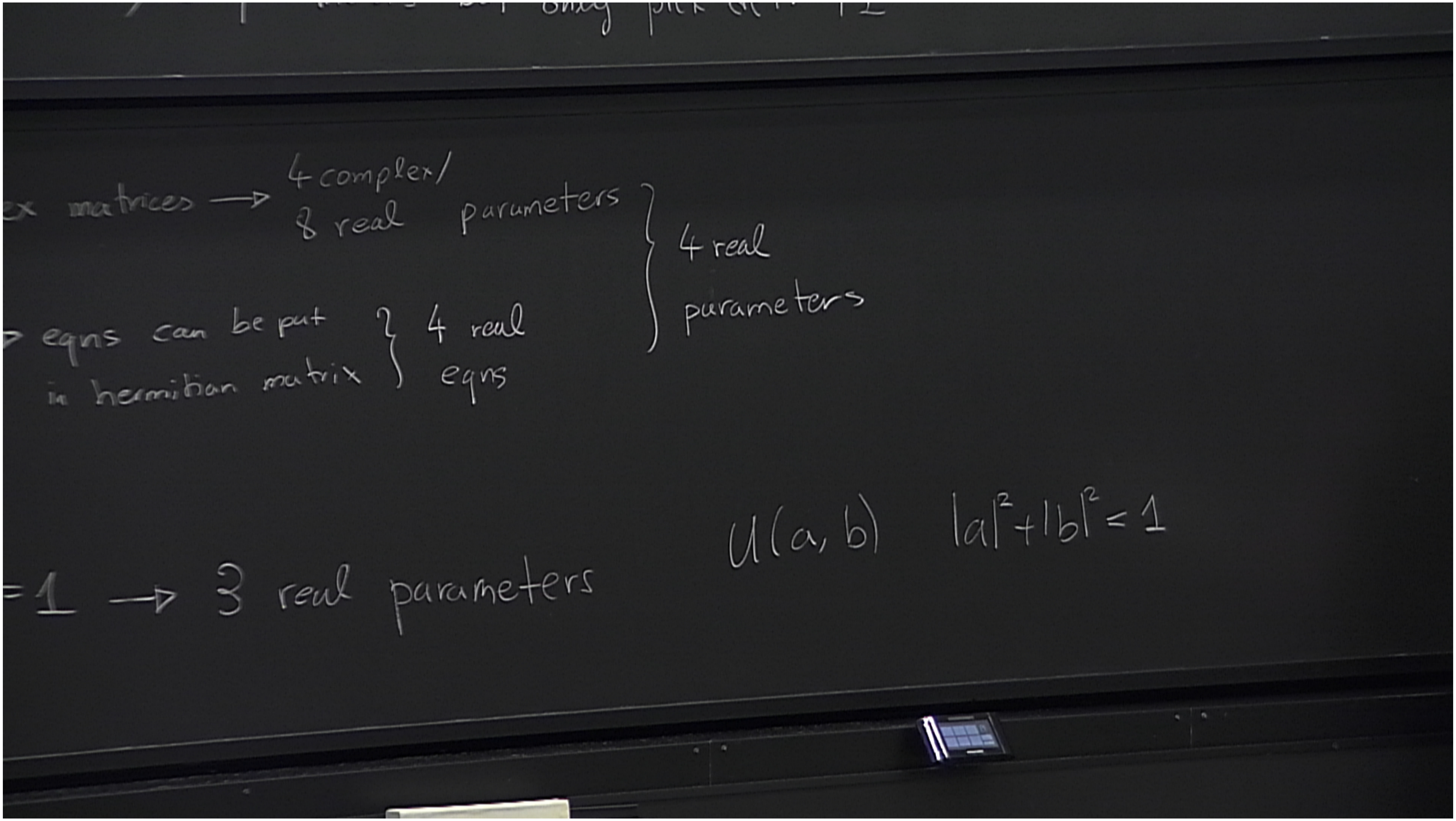
$U(2)$: 2×2 complex matrices $\longrightarrow$ 4 complex / 8 real parameters

$U^{\dagger}U = \mathbb{1}$ $\longrightarrow$ eqns can be put in hermitian matrix → 4 real eqns

→ 4 real parameters

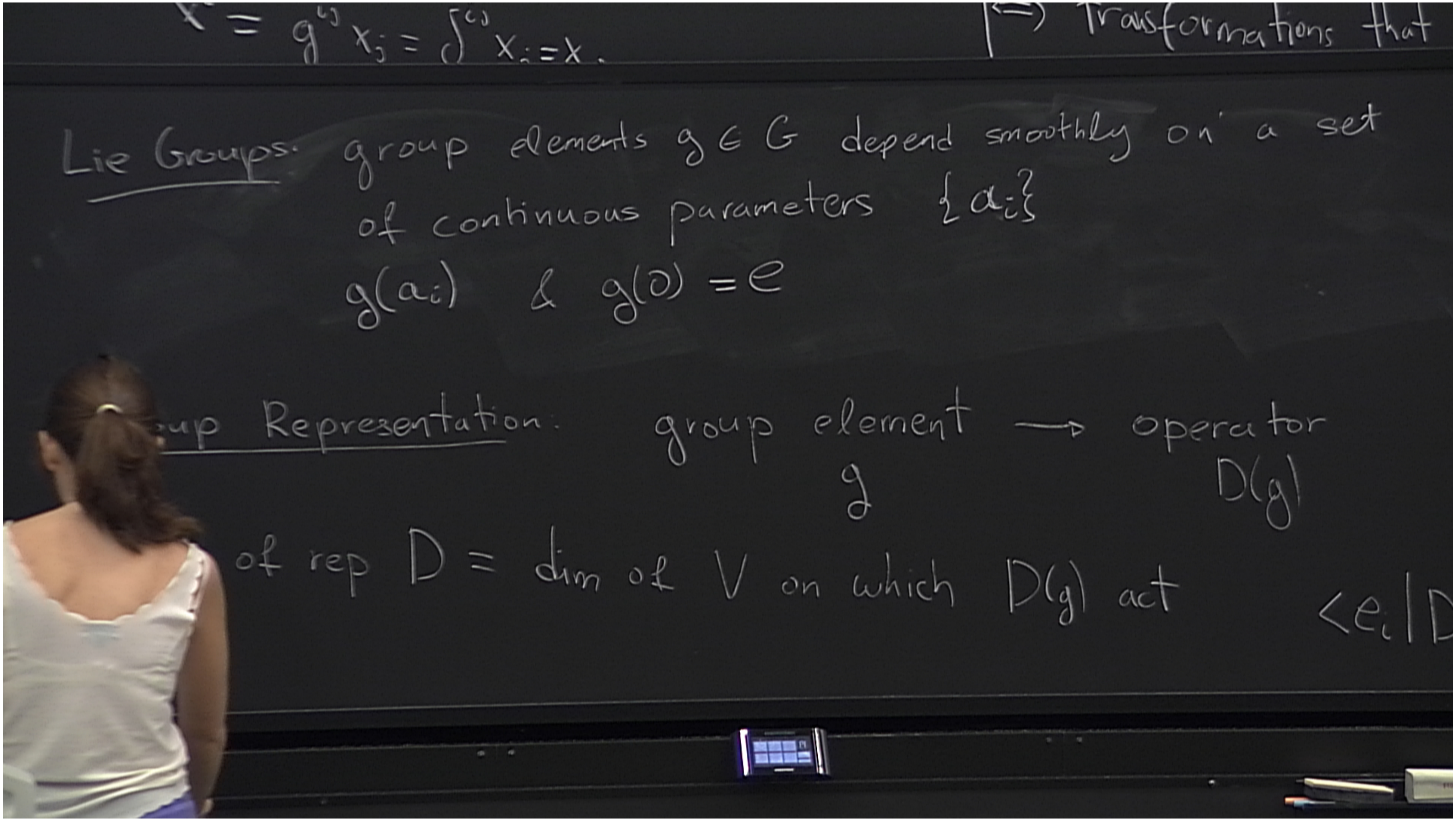
$\det U = e^{i\phi}$

$SU(2)$: $\det U = \mathbb{1}$ $\longrightarrow$ 3 real parameters


ex matrices → 4 complex/ 8 real parameters
4 real parameters
eqns can be put in hermitian matrix
4 real eqns
U(a, b) |a|²+|b|²=1
=1 → 3 real parameters

$Sp(2n)$: "symplectic" transformations that leave

$$\begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \quad \text{invariant}$$


x = g x_j = δ x_j = x.
⟺ Transformations that
Lie Groups: group elements g ∈ G depend smoothly on a set of continuous parameters {a_i}
g(a_i) & g(0) = e
oup Representation: group element g → operator D(g)
of rep D = dim of V on which D(g) act
<e_i|D

$U(2)$: 2×2 complex matrices $\longrightarrow$ 4 complex / 8 real parameters } 4 re… para…

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$\det U = e^{i\phi}$

$SU(2)$: $\det U = 1 \longrightarrow 3$ real parameters

$SO(3) \simeq SU(2)/\mathbb{Z}_2$

U(2): 2×2 complex matrices $\longrightarrow$ 4 complex / 8 real parameters $\Big\}$ 4 real parameters

$U^\dagger U = 1$ $\rightarrow$ eqns can be put in hermitian matrix $\Big\}$ 4 real eqns

$\det U = e^{i\phi}$

$U(a, b)$ $|a|^2$

SU(2): $\det U = 1$ $\longrightarrow$ 3 real parameters

$SO(3) \simeq SU(2)/Z_2$ $so(3) \simeq su(2)$

(4 complex)

→ 8 real parameters

4 real parameters

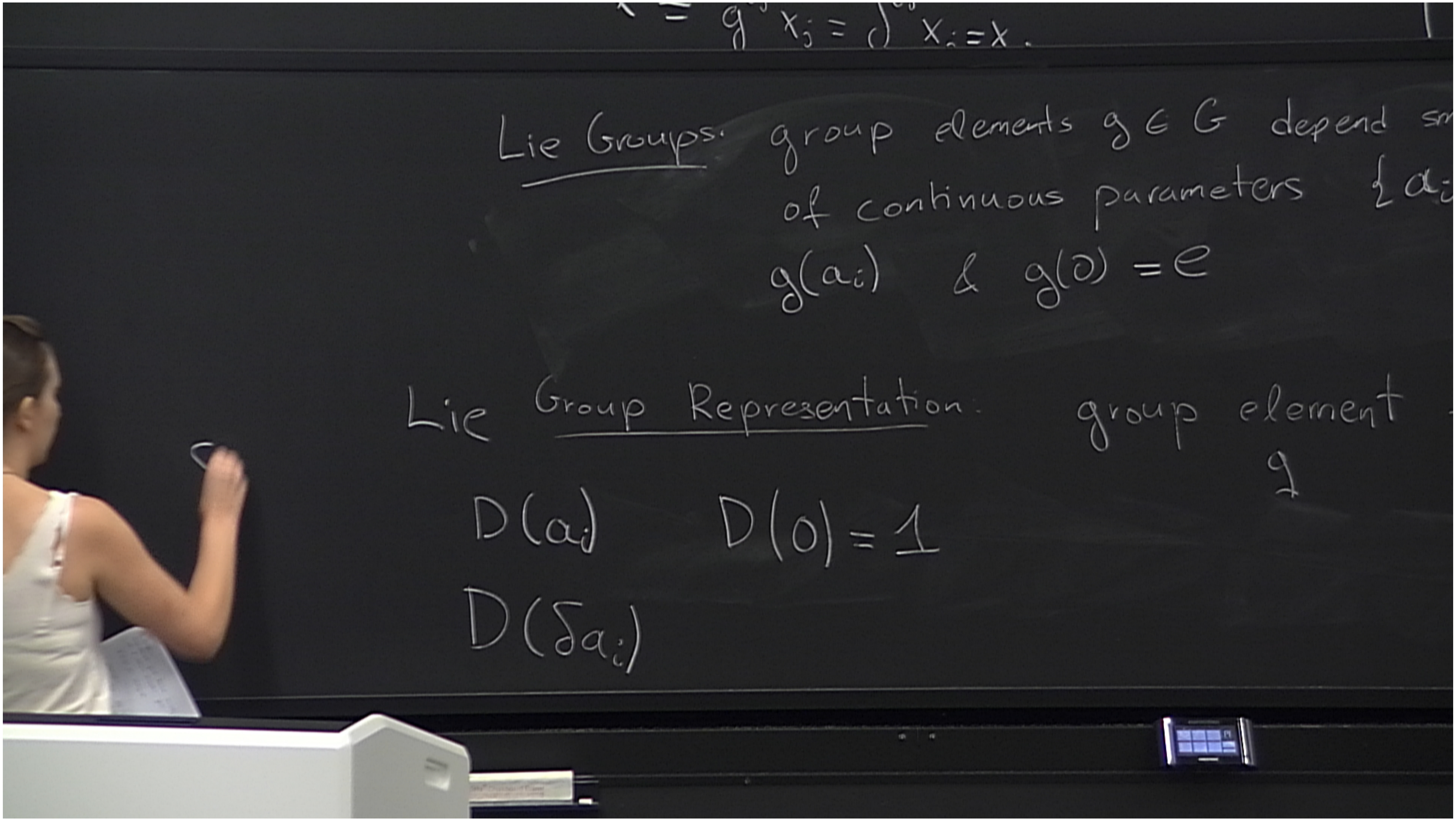
can be put ... ition matrix — 4 real eqns

→ 3 real parameters

$U(a,b)$ $\quad |a|^2+|b|^2=1$ $\quad U(a,b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$

$a,b\in\mathbb{C}$

$so(3)\simeq su(2)$



Lie Groups: group elements $g \in G$ depend
of continuous parameters
$g(a_i)$ & $g(0) = e$
Lie Group Representation: group element
g
$D(a_i)$ $D(0) = 1$
$D(\delta a_i)$

depend smoothly on a set

ters $\{a_i\}$ $\quad (i = 1, \ldots N)$

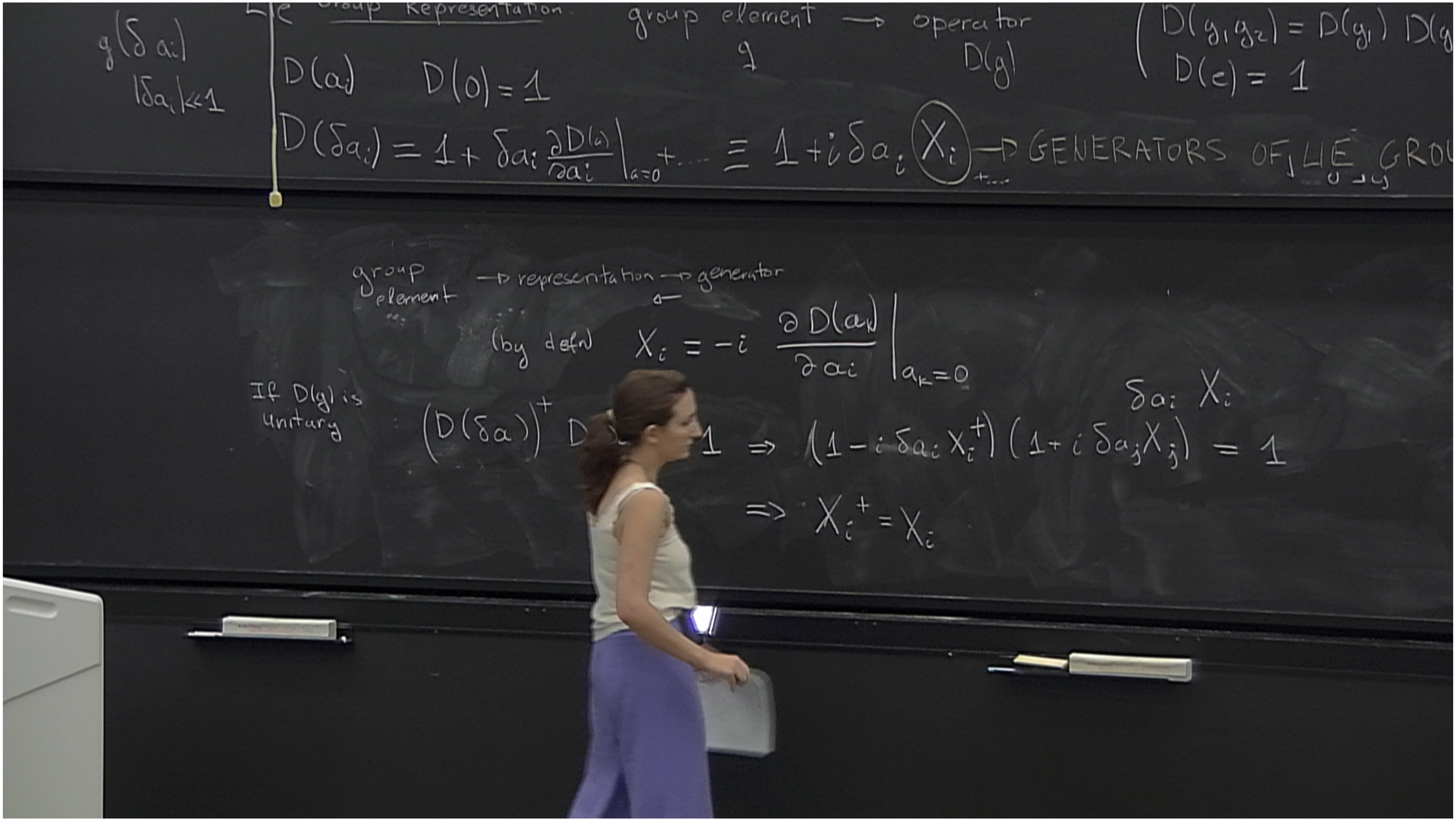
$(a_i \in \mathbb{R})$

element $\longrightarrow$ operator

$g \qquad\qquad\qquad D(g)$

$$\begin{pmatrix} D(g_1 g_2) = D(g_1)\, D(g_2) \\ D(e) = 1 \end{pmatrix}$$

$$\equiv 1 + i\,\delta a_i\, X_i \longrightarrow \text{GENERATORS OF LIE GROUP}$$


g(δ a_i)
|δa_i| ≪ 1
Group Representation
group element g → operator D(g)
D(g_1 g_2) = D(g_1) D(g_2)
D(e) = 1
D(a_i)   D(0) = 1
D(δa_i) = 1 + δa_i ∂D(a)/∂a_i |_{a=0} + ... ≡ 1 + i δa_i X_i + ...
→ GENERATORS OF LIE GROUP
group element → representation → generator
(by defn) X_i = −i ∂D(a_k)/∂a_i |_{a_k=0}
δa_i X_i
If D(g) is unitary
(D(δa))^+ D(δa) = 1 ⇒ (1 − i δa_i X_i^+)(1 + i δa_j X_j) = 1
⇒ X_i^+ = X_i

presentation $\rightarrow$ generator

$\leftarrow$

defn $\quad X_i = -i \left. \frac{\partial D(a_k)}{\partial a_i} \right|_{a_k = 0}$

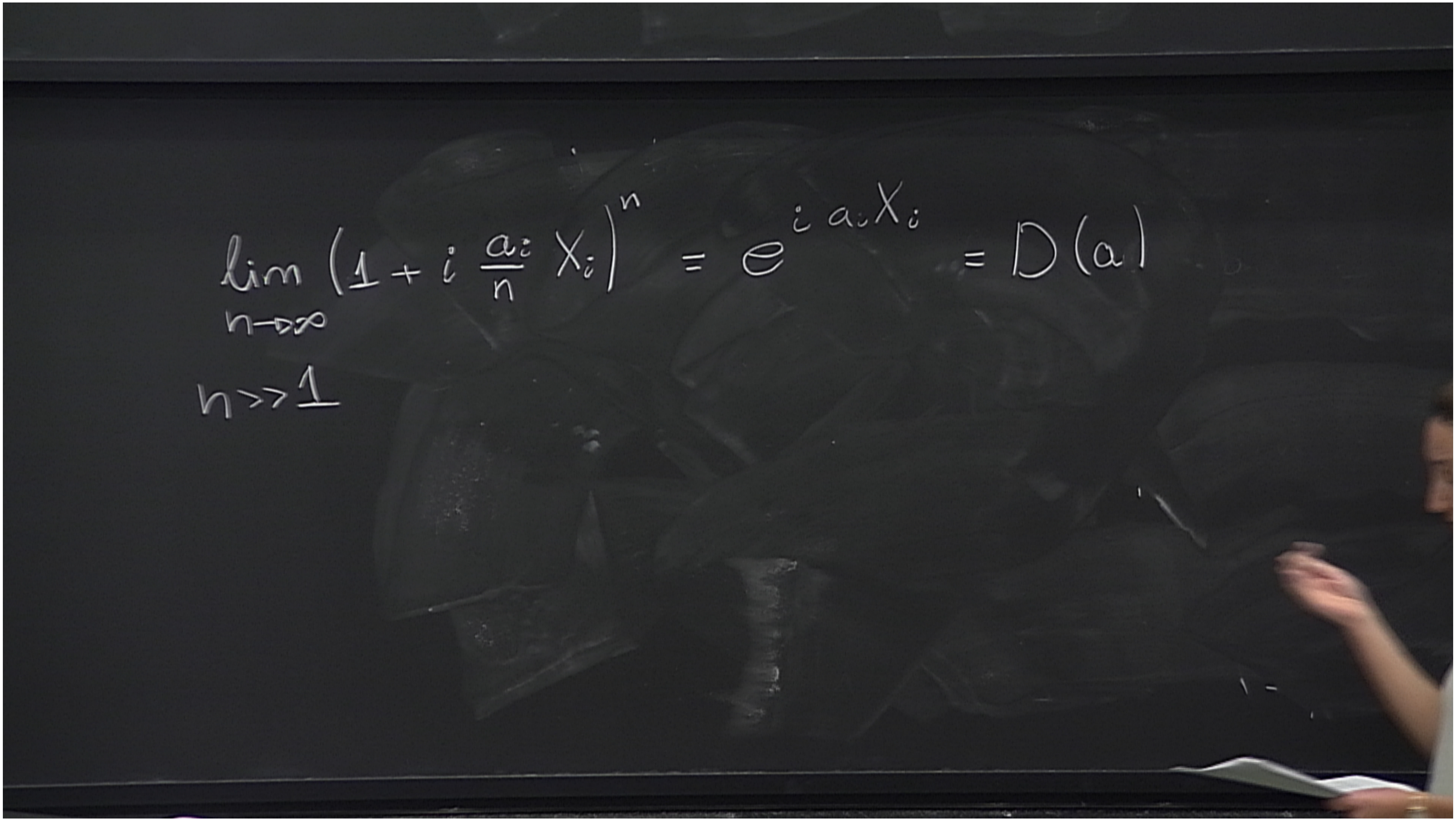
$\delta a_i \; X_i$

$D(\delta a) = 1 \Rightarrow \left(1 - i\,\delta a_i X_i^{+}\right)\left(1 + i\,\delta a_j X_j\right) = 1$

$\Rightarrow \boxed{X_i^{+} = X_i}$

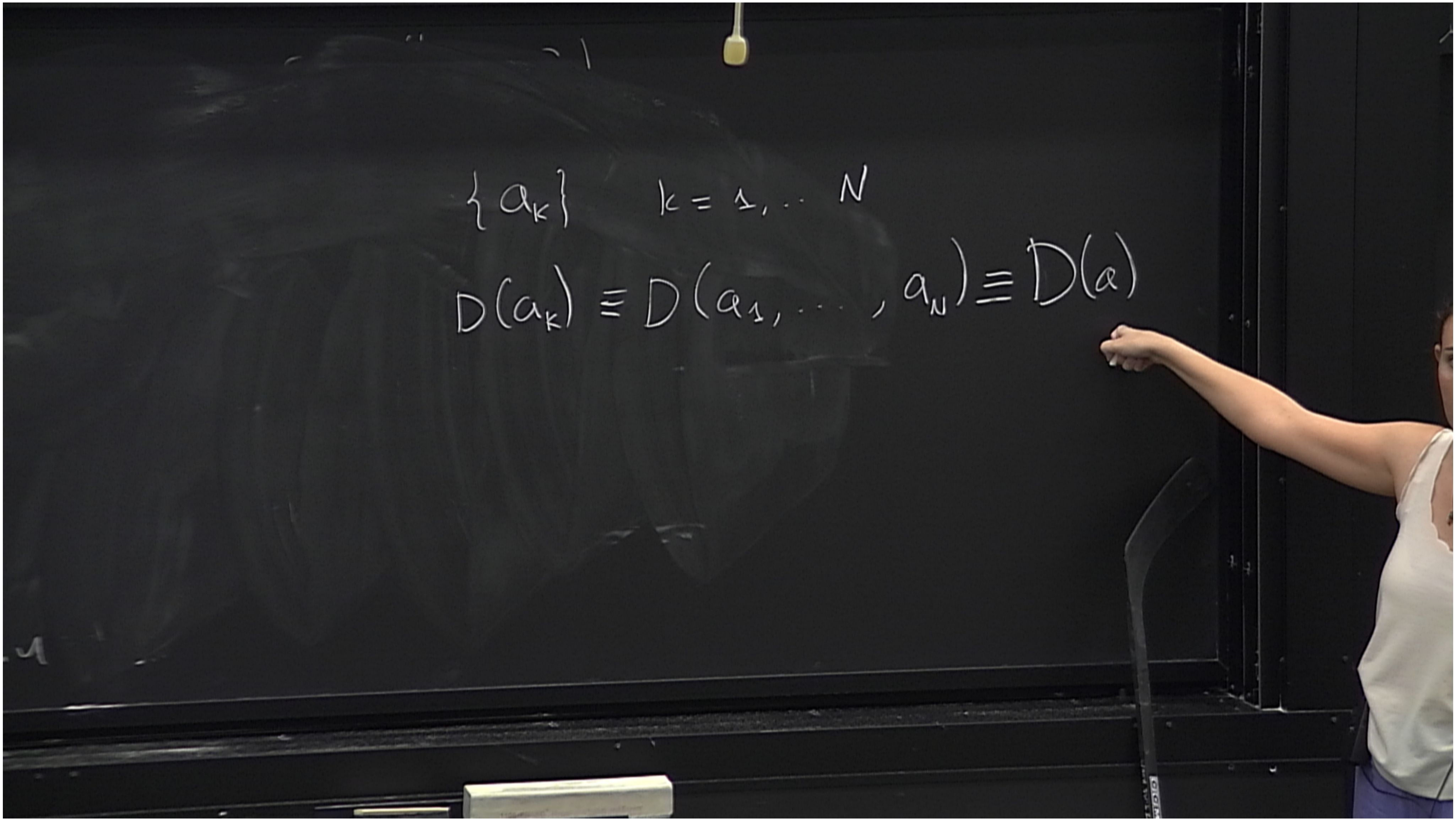
$$\lim_{n \to \infty} \left(1 + i \frac{a_i}{n} X_i\right)^n = e^{i a_i X_i}$$

$n \gg 1$


$\lim_{n\to\infty}\left(1 + i\,\frac{a_i}{n}X_i\right)^n = e^{i a_i X_i} = D(a)$
$n \gg 1$

$$\{a_k\} \qquad k = 1, \ldots N$$

$$D(a_k) \equiv D(a_1, \ldots, a_N)$$

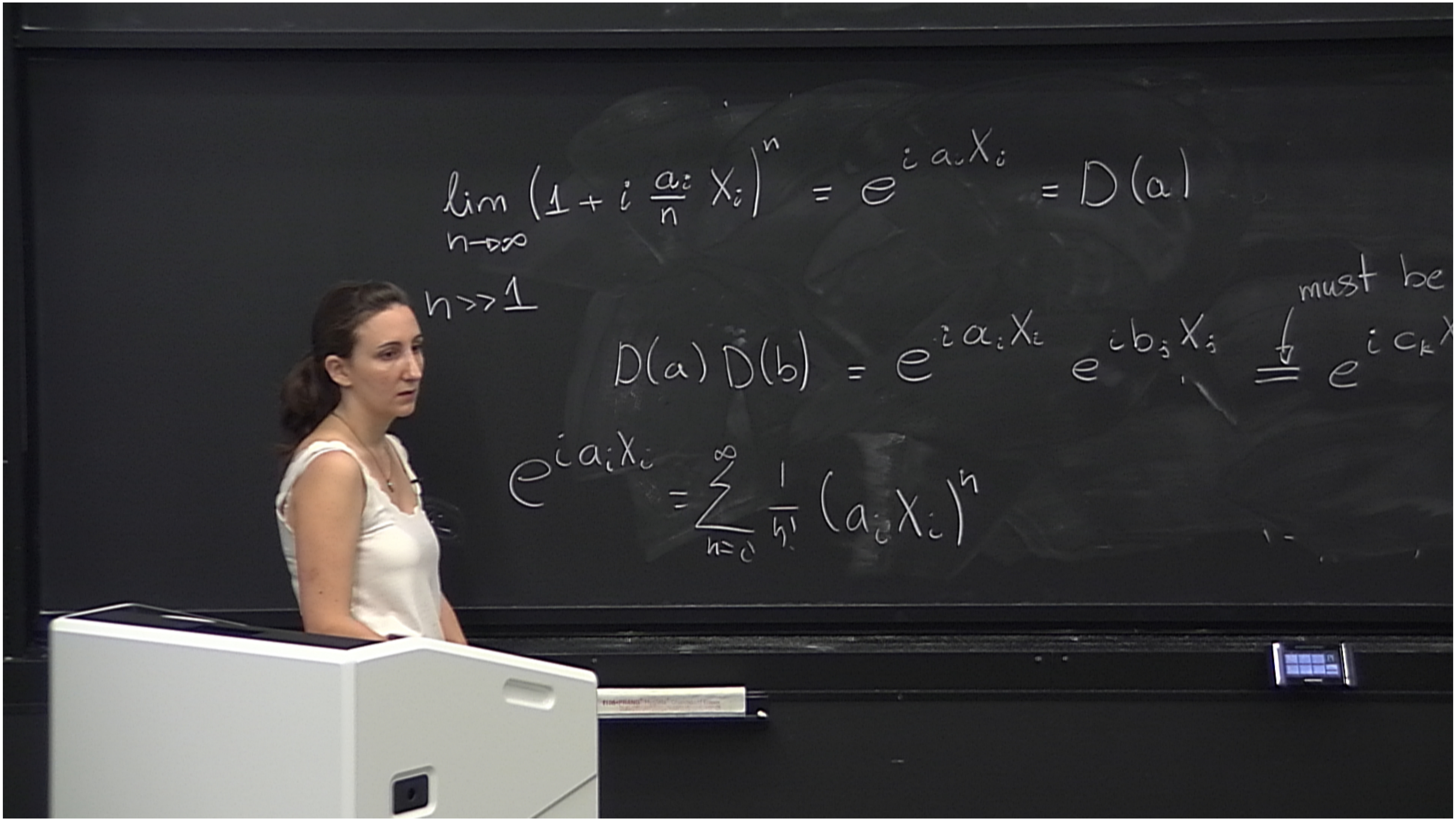

$\{a_k\} \quad k = 1, \ldots N$
$D(a_k) \equiv D(a_1, \ldots, a_N) \equiv D(a)$

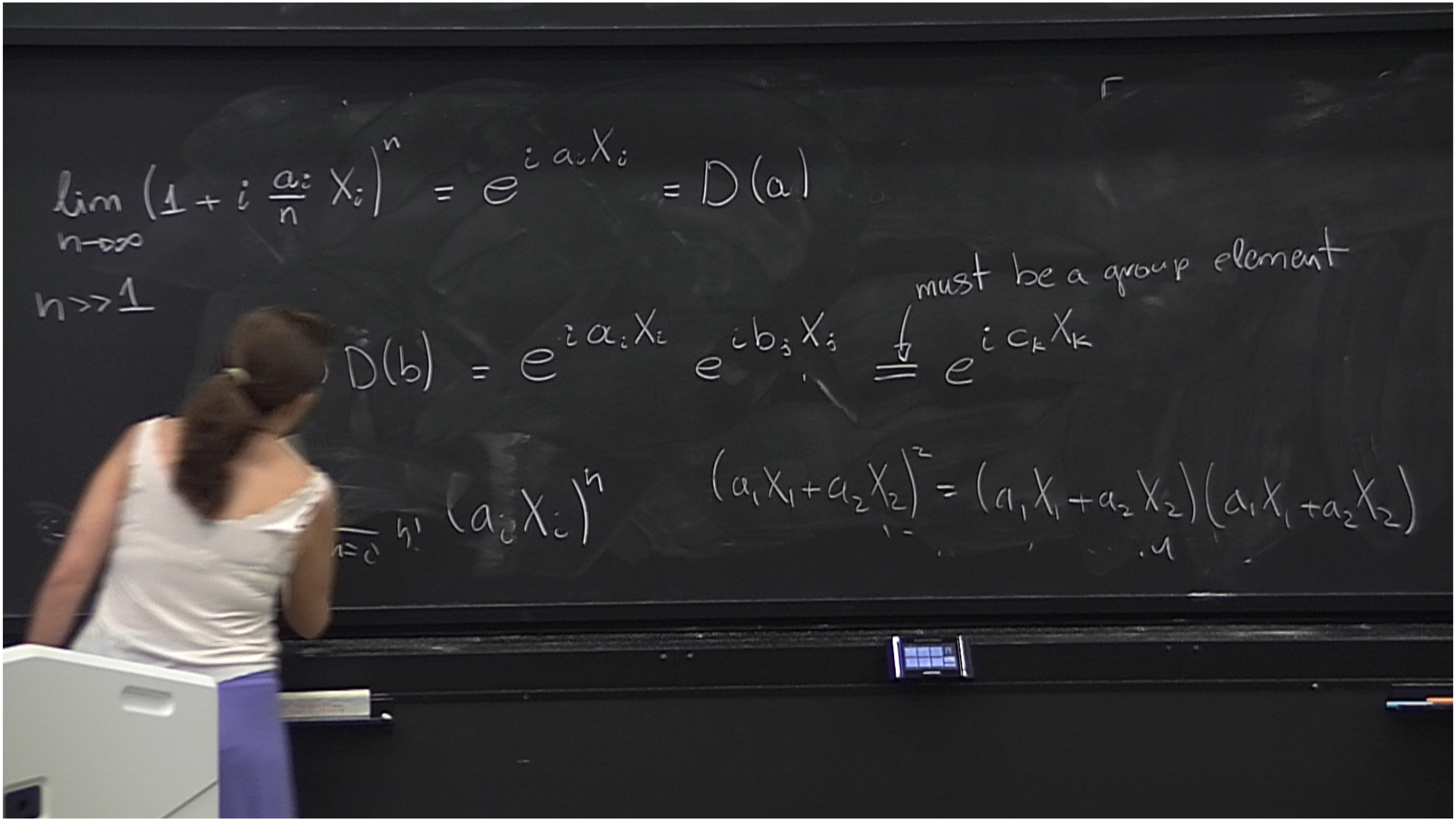
$$\lim_{n\to\infty}\left(1 + i\,\frac{a_i}{n} X_i\right)^n = e^{i a_i X_i} = D(a)$$

$n \gg 1$

must be a group element

$$D(a)\,D(b) = e^{i a_i X_i}\, e^{i b_j X_j} = e^{i c_k X_k}$$


$\lim_{n\to\infty} \left(1 + i \frac{a_i}{n} X_i\right)^n = e^{i a_i X_i} = D(a)$
$n >> 1$
$D(a) D(b) = e^{i a_i X_i} e^{i b_j X_j}$
must be
$= e^{i c_k X}$
$e^{i a_i X_i} = \sum_{n=0}^{\infty} \frac{1}{n!} (a_i X_i)^n$



$\lim_{n\to\infty} \left(1 + i \frac{a_i}{n} X_i\right)^n = e^{i a_i X_i} = D(a)$
$n \gg 1$
must be a group element
$D(b) = e^{i a_i X_i} \, e^{i b_j X_j} = e^{i c_k X_k}$
$(a_i X_i)^n$
$(a_1 X_1 + a_2 X_2)^2 = (a_1 X_1 + a_2 X_2)(a_1 X_1 + a_2 X_2)$

must be a group element

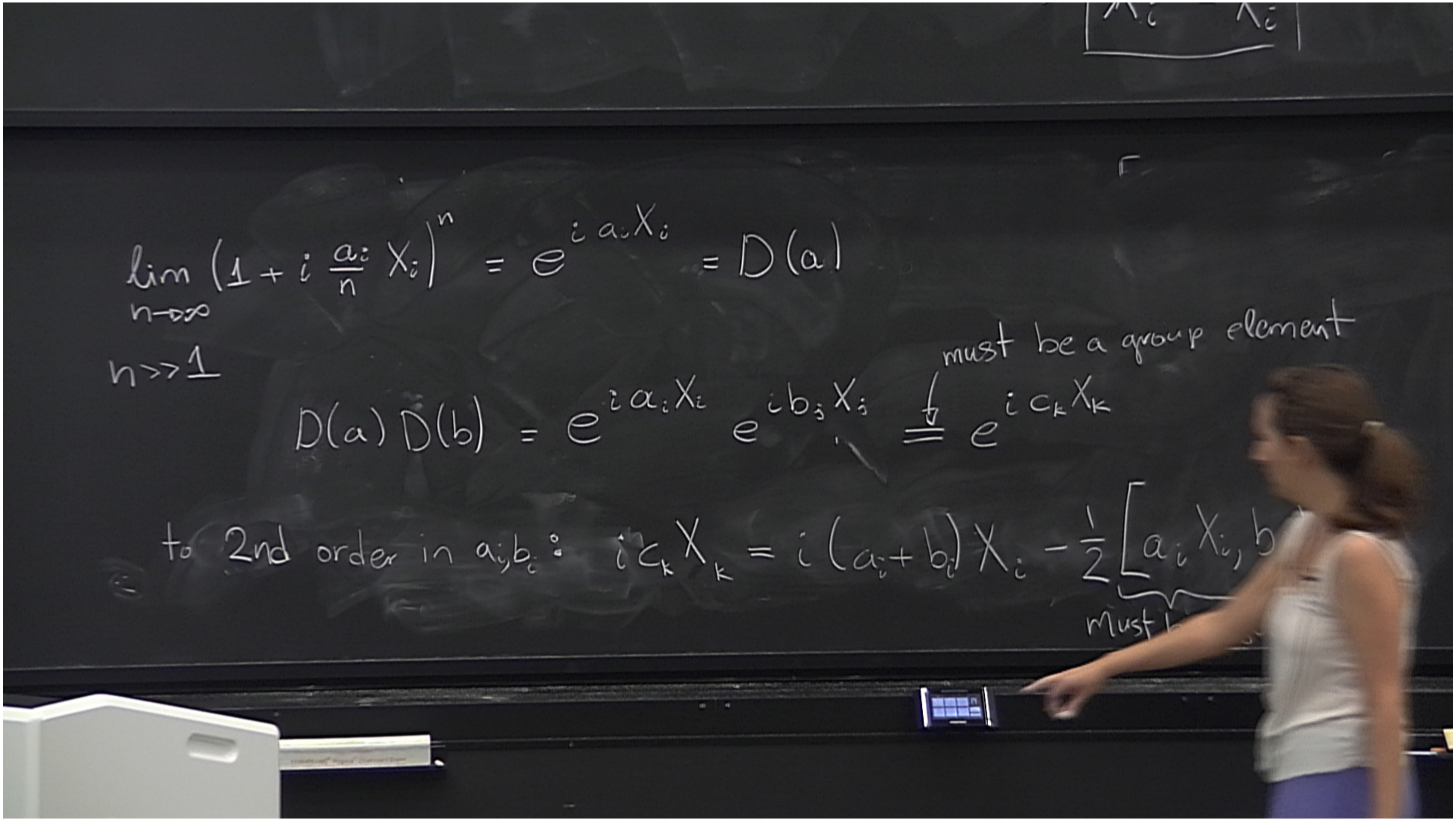
$$\equiv e^{i c_k X_k}$$

$$b_i) X_i - \frac{1}{2}\left[a_i X_i, b_j X_j\right]$$

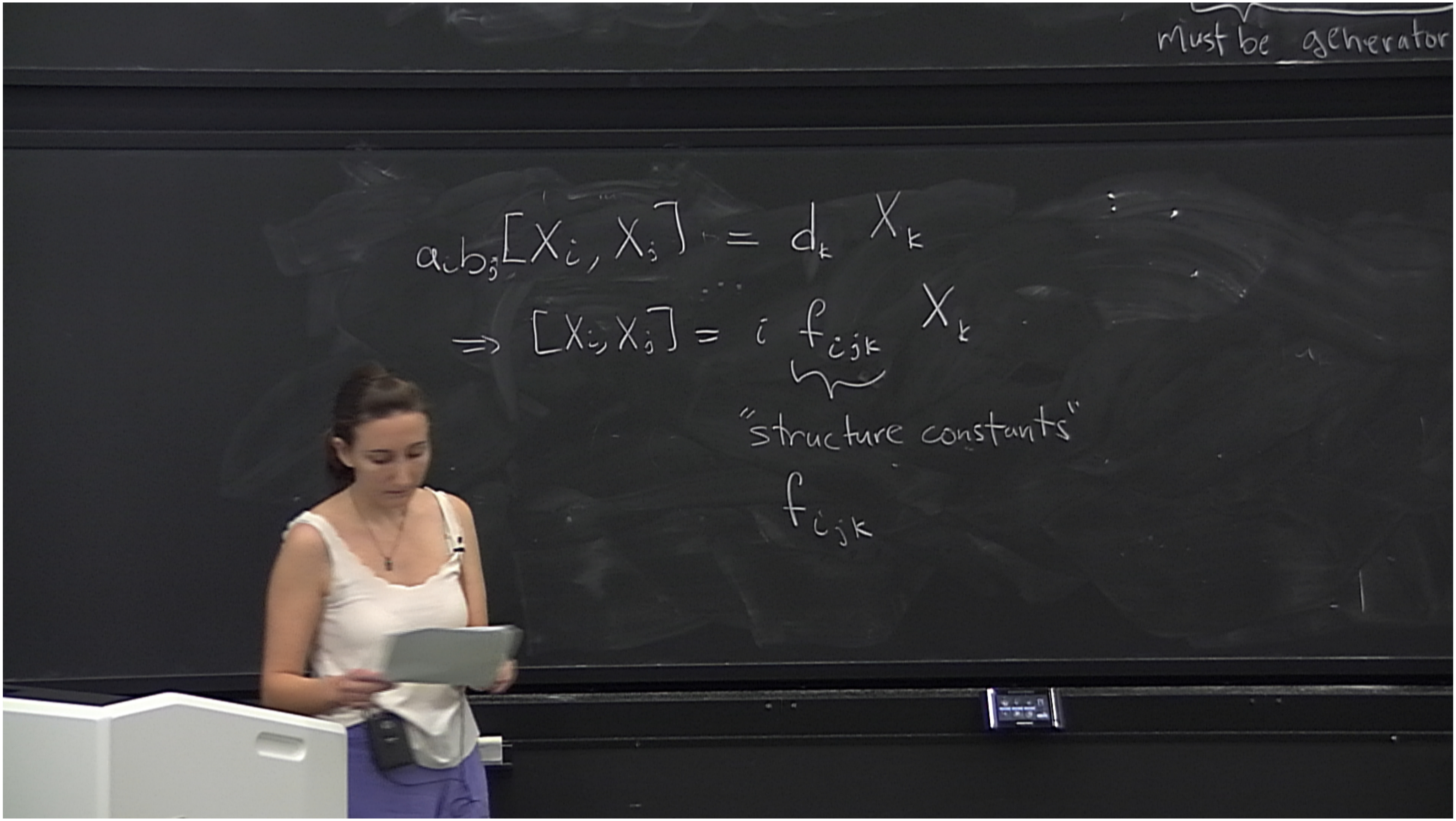
$$\{a_k\} \qquad k = 1, \dots N$$

$$D(a_k) \equiv D(a_1, \dots, a_N) \equiv D(a)$$

$$e^A e^B = e^{\left(A+B-\frac{1}{2}[A,B]+\dots\right)}$$



$\lim_{n\to\infty} \left(1 + i \frac{a_i}{n} X_i\right)^n = e^{i a_i X_i} = D(a)$
$n \gg 1$
must be a group element
$D(a) D(b) = e^{i a_i X_i} e^{i b_j X_j} \stackrel{!}{=} e^{i c_k X_k}$
to 2nd order in a_i, b_i: $i c_k X_k = i(a_i + b_i) X_i - \frac{1}{2}[a_i X_i, b$
must


must be generator
a_i b_j [X_i, X_j] = d_k X_k
⇒ [X_i, X_j] = i f_ijk X_k
"structure constants"
f_ijk

Must be generator

$[X_i, X_j] = d_k \; X_k$

$[X_i, X_j] = i \underbrace{f_{ijk}}_{\text{"structure constants"}} X_k$ LIE ALGEBRA

"structure constants" summarize ALL group multiplication

f_{ijk}


FOOTBALL
STADIUM