

Title: 13/14 PSI - Algebra - Lecture 1

Date: Aug 14, 2013 10:30 AM

URL: <http://pirsa.org/13080009>

Abstract:

Quantum Mechanics

- Physical system $\rightarrow$ vector in Hilbert Space
 $|\psi\rangle \in \mathcal{H}$
- Physical observables $\leftrightarrow$ hermitian operators $A=A^\dagger$

Algebras

Quantum Mechanics

- Physical system $\rightarrow$ vector in Hilbert Space
 $|\psi\rangle \in \mathcal{H}$
- Physical observables $\leftrightarrow$ hermitian operators $A=A^\dagger$
- Physical information $\leftrightarrow$ probabilities

$$P(|\psi_1\rangle \rightarrow |\psi_2\rangle) = |\langle \psi_1 | \psi_2 \rangle|^2$$

Symmetry.

$$|\psi_i\rangle \rightarrow U|\psi_i\rangle \equiv |\psi_i'\rangle$$

$$|\langle\psi_i'|\psi_j'\rangle|^2 = |\langle\psi_i|\psi_j\rangle|^2$$

$$P(|\psi_i\rangle \rightarrow |\psi_j\rangle) = P(|\psi_i'\rangle \rightarrow |\psi_j'\rangle)$$

$$|\psi_i\rangle \xrightarrow{U_1} |\psi_i'\rangle \xrightarrow{U_2} |\psi_i''\rangle$$

$$|\psi_i''\rangle = U_2 U_1 |\psi_i\rangle$$



$\Rightarrow |\psi_0''\rangle$

$|\psi_0\rangle$

$U = \mathbb{1}$ is a transformation that
leaves $|\langle\psi_i|\psi_j\rangle|^2$ invariant

→ Any group that is continuously connected
to the identity must be
unitary & linear

$\Rightarrow |\psi_0\rangle$

$|\psi_1\rangle$

U_3

$U = \mathbb{1}$ is a transformation that
leaves $|\langle \psi_i | \psi_j \rangle|^2$ invariant

→ Any group that is continuously connected
to the identity must be
unitary & linear

→ Lie Groups

Quick Review of Linear Algebra

- Vector space V
- $|x\rangle, |y\rangle, \dots \in V$
- Basis $\{|e_i\rangle\}$ $i=1, \dots, N \equiv \dim V$
- Scalar product $\langle x|y\rangle$
- Dual space V^*

Quick Review of Linear Algebra

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- $|x\rangle, |y\rangle, \dots \in V$
- Basis $\{|e_i\rangle\}$ $i=1, \dots, N \equiv \dim V$
- Scalar product $\langle x|y\rangle$
- Dual space V^* space of maps: $V \rightarrow \mathbb{C}$

map $V \rightarrow \mathbb{C} \quad \langle x|y \rangle$

$$\langle y| : \quad |x\rangle \xrightarrow{\langle y|} \langle y|x\rangle = \langle y|(|x\rangle)$$

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$\langle y| \in V^*$ "corector" dual to $|y\rangle$

• Dual basis: $\{ \langle w^i| \}$ s.t. $\langle w^i|e_j\rangle = \delta^i_j$

• Adjoint: $A^\dagger$ s.t. $\langle x|A^\dagger|y\rangle = \langle y|A|x\rangle^*$

map $V \rightarrow \mathbb{C} \quad \langle x|y \rangle$

$$\langle y| : |x\rangle \xrightarrow{\langle y|} \langle y|x\rangle = \langle y|(|x\rangle)$$

$\omega \in V^*$ "covector" dual to $|y\rangle$

• Dual basis: $\{ \langle \omega^i| \}$ s.t. $\langle \omega^i|e_j\rangle = \delta^i_j$

• Adjoint: $A^\dagger$ s.t. $\langle x|A^\dagger|y\rangle = \langle y|A|x\rangle^*$

- Hermitian ops: $H = H^\dagger$
 - real eigenvalues
 - orthogonal eigenstates

- Unitary: $U^\dagger U = U U^\dagger = 1$

- Matrix Representation:

ket $\leftrightarrow$ column $\begin{pmatrix} \end{pmatrix}$
bra $\leftrightarrow$ line $\begin{pmatrix} \end{pmatrix}$
ops $\leftrightarrow$ matrix $\begin{pmatrix} \end{pmatrix}$

$$|\psi\rangle = \psi^i |e_i\rangle \equiv \sum_{i=1}^N \psi^i |e_i\rangle$$

$$= (|e_1\rangle |e_2\rangle \dots |e_n\rangle) \begin{pmatrix} \psi^1 \\ \psi^2 \\ \vdots \\ \psi^N \end{pmatrix}$$

In basis $|e_i\rangle$, $|\psi\rangle \leftrightarrow \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^N \end{pmatrix}$

ψ^i : transform "contravariantly" under change of basis.

ops $\leftrightarrow$ matrix ()

If $\langle e_i, e_s \rangle \in \mathbb{R}$, $g_{ij} = g_{ji}$

defn inverse metric: $g^{ij} \sum_{j=1}^N g^{ij} g_{jk} = \delta^i_k$

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$$\psi_i \equiv g_{ij} \psi^j \quad (\Leftrightarrow \psi^i = g^{is} \psi_s)$$

$$|e^i\rangle \equiv g^{ij} |e_j\rangle$$

$$\langle e^i | e_j \rangle = (|e^i\rangle, |e_j\rangle) = g^{ik} (|e_k\rangle, |e_j\rangle)$$

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$$g_{ij} = g_{ji}$$

$$\sum_{j=1}^N g^{ij} g_{jk} = \delta^i_k$$

$$\psi^i = g^{is} \psi_s$$

$$g^{ik} (|e_k\rangle, |e_i\rangle) = g^{ik} g_{kj} = \delta^i_j \Rightarrow \langle e^i| = \langle w^i| \text{ dual basis vector!}$$

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$$|e_i\rangle$$

$$\langle e_i | : \text{ covector dual to } |e_i\rangle$$

$$|e_i\rangle \quad \langle e_i|: |x\rangle \rightarrow \langle e_i|x\rangle = (|e_i\rangle, |x\rangle)$$

$|e_i\rangle$
 $\langle e_i|$: covector dual to $|e_i\rangle$

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$$|\psi\rangle = \psi^i |e_i\rangle$$

$$\langle\psi| = \langle e_i | \psi^{i*}$$

$$\Rightarrow \langle e^i | = \langle \omega^i | \quad \text{dual basis}$$

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$$(|\psi\rangle, |\chi\rangle) = (\psi^i |e_i\rangle, |\chi\rangle) = \psi^{i*} (|e_i\rangle, |\chi\rangle) = \underbrace{\psi^{i*}}_{\langle\psi|} \langle e_i| \chi\rangle$$

$$g_{ik} g_{kj} = \delta_{ij} \Rightarrow \langle e^i| = \langle \omega^i| \text{ dual basis vector!}$$

$$\begin{aligned}
 g_{jk} &= \delta_{jk} \\
 |\psi\rangle &= \psi^i |e_i\rangle \\
 \langle\psi| &= \langle e_i| \psi^{i*} \quad (1) \\
 (|\psi\rangle, |\chi\rangle) &= (\psi^i |e_i\rangle, |\chi\rangle) = \psi^{i*} (|e_i\rangle, |\chi\rangle) = \underbrace{\psi^{i*}}_{\langle\psi|} \langle e_i| \chi\rangle \\
 \langle\psi| &= \bar{\psi}_i \langle e^i| = \bar{\psi}_i g^{ij} \langle e_j| \quad (2)
 \end{aligned}$$

$$(|e_k\rangle, |e_i\rangle) = g^{ik} g_{kj} = \delta^i_j \Rightarrow \langle e^i| = \langle \omega^i| \text{ dual basis vector!}$$

(1)

$$\langle e_i | \chi \rangle = \psi^{i*} (\langle e_i |, |\chi \rangle) = \underbrace{\psi^{i*}}_{\langle \psi |} \langle e_i | \chi \rangle$$

$$= \bar{\psi}_i g^{ij} \langle e_j | \quad (2)$$

$\Rightarrow \langle e^i | = \langle \omega^i |$ dual basis vector!

$$(1), (2) \Rightarrow \bar{\psi}^{i*} = g^{ij} \bar{\psi}_j$$
$$\Leftrightarrow \bar{\psi}_i = g_{jk} \psi^{k*}$$

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$$(1), (2) \Rightarrow \bar{\psi}_i^* = g^{ij} \bar{\psi}_j$$

$$\Leftrightarrow \bar{\psi}_i = g_{jk} \psi_k^* \equiv \psi_i^*$$

$\Rightarrow \langle e^i | = \langle \omega^i |$ dual basis vector!

Linear ops $\leftrightarrow$ matrices

(e.g. rotations on x-y plane $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$)

$$A: V \rightarrow W$$

$$\dim V = N$$

$$\{|e_i\rangle\}$$

$$\dim W = M$$

$$\{|f_i\rangle\}$$

$$A|e_i\rangle = \alpha^i_j |f_j\rangle = |f_j\rangle \alpha^j_i$$

$$\begin{pmatrix} A|e_1\rangle \\ \vdots \\ A|e_n\rangle \end{pmatrix} = \begin{pmatrix} |f_1\rangle & \dots & |f_M\rangle \end{pmatrix} \begin{pmatrix} \alpha^1_1 & \alpha^1_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Linear ops $\leftrightarrow$ matrices

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$$A|\psi\rangle = A(\psi_i |e_i\rangle) = |f_j\rangle \underbrace{\alpha^j_i}_{\psi^i}$$

Linear ops $\leftrightarrow$ matrices

(eg. rotations on x-y plane

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

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$$\psi^i \rightarrow a^i_j \psi^j$$

Group Theory:

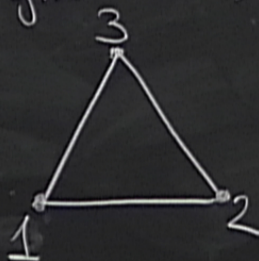
$$g \in G$$

$$g_1 g_2 \in G$$

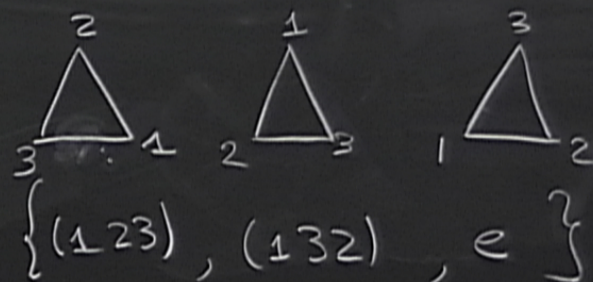
Finite groups

- permutation
- cyclic

Symmetries



- rotations



Group Theory:

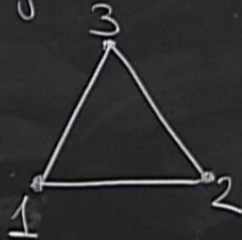
$$g \in G$$

$$g, a \in G$$

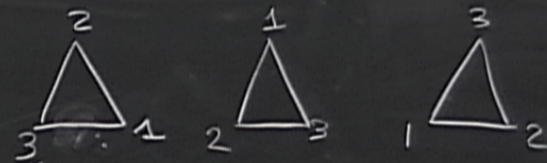
Finite groups

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- rotations



$$\{(123), (132), e\} = C_3$$

$$(123)(123) = (132)$$

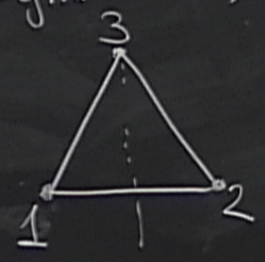
$\pi(\varphi) = \pi(\varphi|_{\langle e, \gamma \rangle}) = 1 + \underbrace{\alpha \cdot \varphi}_{\varphi \rightarrow \alpha \cdot \varphi} \dots$

Group Theory:

Finite groups

- permutation
- cyclic

Symmetries



- rotations

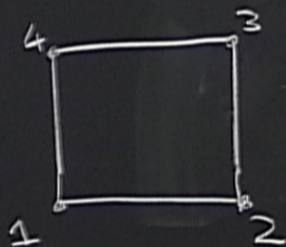


$$\{(123), (132), e\} = C_3 \text{ cyclic}$$

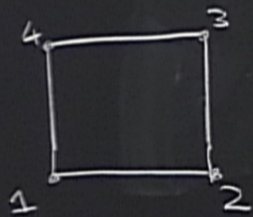
$$(123)(123) = (132)$$

$$C_3 \cup \{(12), (23), (13)\} = S_3 \text{ permutation}$$

$$(123) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$



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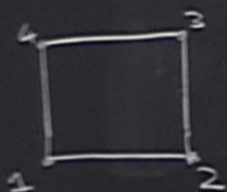


rotations

$$\left\{ \begin{array}{c} \begin{array}{c} 3 \quad 2 \\ \square \\ 4 \quad 1 \end{array} \quad \begin{array}{c} 2 \quad 1 \\ \square \\ 3 \quad 4 \end{array} \quad \begin{array}{c} 1 \quad 4 \\ \square \\ 2 \quad 3 \end{array} \\ (1234), (13)(24), (1432), e \end{array} \right\} = C_4$$

$$C_3 \cup \{(12), (23), (13)\} = S_3 \text{ permu}$$

$$(123) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$



rotations $\left\{ \begin{array}{c} \text{3} \quad \text{2} \\ \text{4} \quad \text{1} \\ \text{1} \quad \text{4} \\ \text{2} \quad \text{3} \end{array} \right\} = C_4$

permutations: $S_4 = C_4 \cup \{(13), (24), \dots\}$ 24 elements

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ s(1) & s(2) & s(3) & s(4) \end{pmatrix}$$

choices $4 \times 3 \times 2 \times 1 = 24$



$(24), (1432), e\} = C_4$

$\cup \{(13), (24), \dots\}$

$$S_n = \left\{ \begin{pmatrix} 1 & 2 & \dots & n \\ s(1) & \dots & s(n) \end{pmatrix} \right\}$$

$n!$ elements

24 elements