Title: Quantum Foundations - 2

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Abstract:

Decoherent Histories of Spin Networks

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LOOPS 13, Perimeter Institute for Theoretical Physics

July 26, 2013

Agenda | Decoherent Histories of Spin Networks

- Motivation
- Formalism of Decoherent Histories
- Application to Covariant Loop Gravity
- Recent Results & Open Questions

We need a rule for assigning probabilities to histories of closed quantum systems

Motivation I: Quantum Theory of Closed Systems

INTRODUCTION

Quelle: Hartle 2004 (picture credit), Isham 1992

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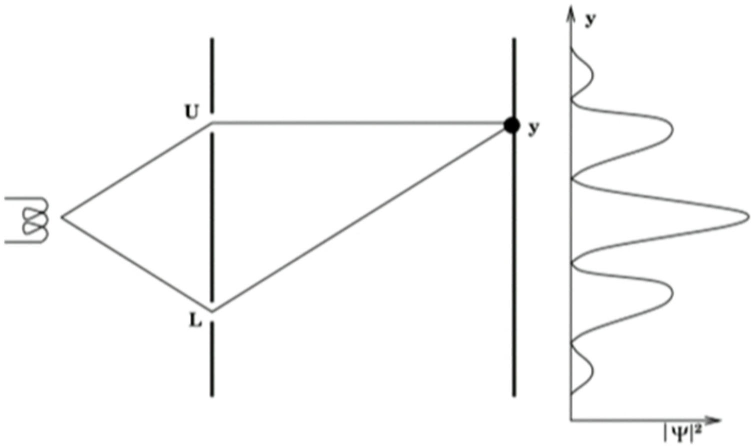
Motivation I: Quantum Theory of Closed Systems

INTRODUCTION

Prediction in physics consist in giving **probabilities of histories** of *single* systems.

Assigning probabilities to histories of quantum systems is problematic due to *interference*.



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Motivation II: "The Problem of Time"

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Motivation II: "The Problem of Time"

General Relativity

General Covariance!

- Einstein Equations transform covariantly under co-ordinate transformations
- All physically meaningful quantities are invariant under diffeomorphisms of the manifold
- In Hamiltonian terms: the fully constrained system is frozen.

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Non-Rel Quantum Theory

- Defined relative to a fixed background time

Empirical Reality

- Physically Meaningful Quantities appear to change over time
- Time is a parameter in experimental setups etc.

Quelle: Isham 1992, Rovelli 2004

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Re-formulate Quantum Theory in terms of *histories*

Formalism I: Histories Theory

BACKGROUND

$$\left|\psi(t)\right\rangle = e^{iHt}\left|\psi(0)\right\rangle$$

$$\hat{I}_k = \sum_{\alpha_k} P^k_{\alpha_k}$$

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$$\left|\psi(t)\right\rangle = e^{iH\frac{t}{N}}\,\hat{I}_N\,e^{iH\frac{t}{N}}\ldots e^{iH\frac{t}{N}}\,\hat{I}_1\,e^{iH\frac{t}{N}}\left|\psi(0)\right\rangle$$

$$\left|\psi(t)\right\rangle = \sum_{\alpha_1\ldots\alpha_N} e^{iH\frac{t}{N}}\,P^N_{\alpha_N}\ldots P^1_{\alpha_1} e^{iH\frac{t}{N}}\left|\psi(0)\right\rangle$$

Quelle:

Decoherent Histories theory is defined by a Class Operator and a Decoherence Functional

Formalism II: Class Operator and Decoherence Functional

BACKGROUND

In the Heisenberg picture, **class operators** are given as follows: where $\alpha = (\alpha_n, \ldots, \alpha_1)$

$$C_\alpha \equiv P^n_{\alpha_n}(t_n) \ldots P^1_{\alpha_1}(t_1)$$

For a pure initial state ψ, the **branch state vector** is given by

$$\left|\psi_\alpha\right\rangle \equiv C_\alpha \left|\psi\right\rangle$$

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$$D(\alpha, \alpha') = tr(\rho_f C_\alpha^{\dagger} \rho_i C_{\alpha'}) \approx 0$$

$$= \sum_{ij} p_i p_j \left\langle \psi_i \middle| C_\alpha \middle| \psi_j \right\rangle \left\langle \psi_j \middle| C_{\alpha'} \middle| \psi_i \right\rangle$$

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Decoherent Histories Theory can be put in terms of Path Integrals and generalised to GR

Formalism V: Field Theory and GR

BACKGROUND

For a coarse-graining c_α of fine-grained histories of field configurations ϕ and boundary states ψ_i and ψ_f, the class operator is given by the expression

$$\langle \psi_f | C_\alpha | \psi_i \rangle := \int_{\psi_f, \alpha, \psi_i} D\phi e^{iS(\phi)}$$

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In Hamiltonian GR, the configuration space is given in terms of the variables $N^\beta = \{N, N^i\}$, q_{ij} with conjugated momentum π_{ij} with Einstein-Hilbert action S. For initial and final three-geometries q'_{ij} q_{ij} on boundary S_f ,S_i respectively, the class operator is given by

Quelle: Hartle 1992

Fine-Grained Histories are given by Individual Spin Foams

Application to LQG I

Single-Particle NRQM
• (Disconnected) Boundary state space given by $L_2[\mathbb{R}] \otimes L_2[\mathbb{R}]$

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Quelle: Hartle 1992, Schroeren 2013

Coarse-Grainings of Spin Network histories are imposed on 'bulk configurations'

Application to LQG II

Single-Particle NRQM

- We can coarse-grain by partitioning the set of fine grained histories into those that pass through an interval $\Delta \in \mathbb{R}$ of the real line at a fixed time t and those that do not.
- The associated class operator is given by

$$\langle q_f, T | C_\alpha | q_i, 0 \rangle = \int_{q_f, \alpha, q_i} \delta q e^{iS[q(\tau)]}$$

$$= \int_{\Delta} \delta q' \int_{q_f, q'} \delta q e^{iS[q(\tau)]} \int_{q', q_i} \delta q e^{iS[q(\tau)]}$$

- That is, we coarse-grain by imposing a condition on the configuration of the "bulk".

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Single-Particle NRQM	Covariant Loop Gravity
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Quelle: Schroeren 2013

Possible coarse-grainings involve extrinsic/instrinsic curvature and volume partitions

Application to LQG III

Coarse-Grainings

Quelle: Schroeren 2013, Craig 2011, Hartle 1992

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Application to LQG III

Coarse-Grainings

- **Intrinsic/Extrinsic Curvature**
- **Volume**

Recall: Coherent states are peaked on certain values of intrinsic and extrinsic curvature, encoded in the labels $H_l \in SL(2, C)$

Coarse-graining: partition set of spin foams associated with a boundary into those whose bulk configuration takes on values H_l and those that do not.

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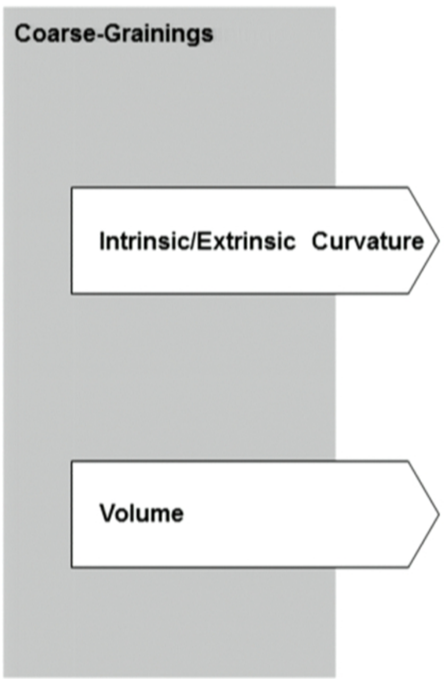

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The Decoherence Functional of Loop Gravity is given by the sum over complex squares of class operator matrix elements

Application to LQG IV

For a suitable normalisation N, the decoherence functional is given by the product of amplitudes restricted to paths satisfying the conditions α, α' respectively.

$$D(\alpha,\alpha') = N \sum_{\psi_i \otimes \psi_j} p_i p_j \langle \psi_i | C_\alpha | \psi_j \rangle \langle \psi_j | C_{\alpha'} | \psi_i \rangle$$

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This functional obeys hermiticity, positivity, normalisation, and superposition.

Which history-spaces of loop gravity decohere under the medium decoherence condition? Answering this would involve computing transition amplitudes involving large sums over two-complices which are extremely difficult to do. Need to resort to approximation techniques.

Quelle: Schroeren 2013

Bianchi, Rovelli, Vidotto (2010) demonstrate that coarse-grained histories behave semi-classically (with caveats)

Quasiclassical Trajectories

RESULTS

Δ_2^*

Consider a **Coherent Spin Network** on a 'dipole graph Δ_2^* composed of four links and two nodes.[1)]

Δ_2^*

1) This is the dual to the cellular decomposition of a manifold that has the topology of a three-sphere; cf. Bianchi et. al. (2010)

Quelle: Bianchi, Rovelli, Vidotto (2010), Freidel, Speziale (2010)

There remain (at least) two major puzzles

Epilogue

Dynamical: What physical process leads to decoherence?
- Equivalently: What are the good coarse-grainings?
- Currently cannot solve the sum over two-complices
- Need to improve understanding of approximation techniques (vertex expansion of spin foam amplitude)

Conceptual: Does spin foam LQG resolve the problem of time?
- We do not understand the causal structure of spin foams
- The case of a spin foam boundary with two disconnected components is merely a special case; spin foams with more/less disconnected boundary components are not *prima facie* unphysical.

The Topos Approach to Quantum Theory and Quantum Gravity

Loops 13
Perimeter Institute, Waterloo
26. July 2013

Andreas Döring

University of Oxford

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Plan of the talk

- Motivation
- Definitions and results
- Summary and outlook

Thrust of ideas

CONCEPTS: Quantum gravity is a problem in the foundations of physics.

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CONCEPTS: Quantum gravity is a problem in the foundations of physics.

WHY QUANTISE?: Quantum theory is more problematic than general relativity, so attempts at quantising gravity may be misguided.

GEOMETRY: We need new geometric ideas, not based on continuum concepts. Noncommutative, pointfree spaces will be key.

LOGIC: We need a formulation of QT and QG that can be interpreted in a realist manner, without referring to measurements and observers.

The topos approach

The **topos approach to the formulation of physical theories**

- was initiated by Chris Isham '97 and Isham/Butterfield '98–'02,
- Other researchers include: Landsman, Heunen, Spitters, Nakayama, Vickers, Fauser, Flori, ...

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- Other researchers include: Landsman, Heunen, Spitters, Nakayama, Vickers, Fauser, Flori, ...
- The approach (and this talk) are about the *architecture* of physical theories, not about specific models.
- Most work so far is on standard, non-relativistic quantum theory – natural starting point, testing ground.

Hilbert spaces be gone

"I would like to make a confession which may seem immoral: I do not believe in Hilbert space anymore."
John von Neumann, in a letter to George David Birkhoff (1935)

- The Hilbert space formalism practically forces an instrumentalist interpretation upon us (Born rule, Kochen-Specker theorem, ...). Makes no sense in QG and QC: system is the whole universe, no external observer who could perform measurements.

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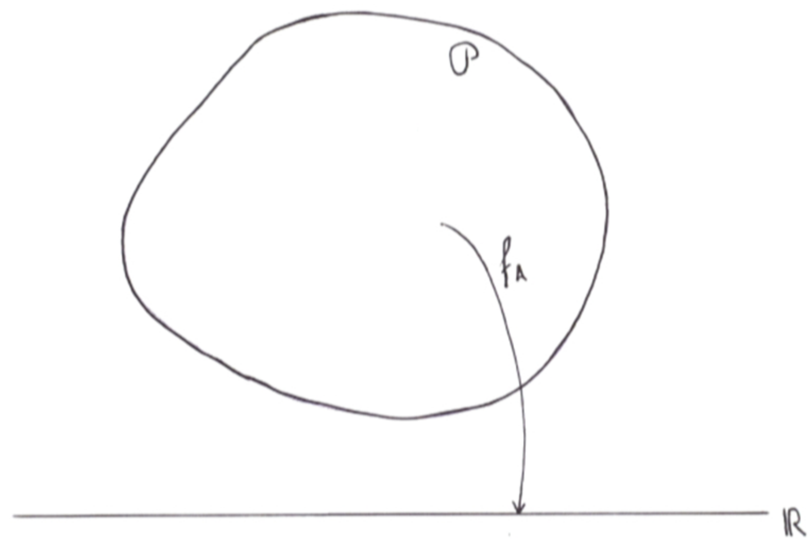
- Continuum ideas are built in (complex numbers, inner products as angles, probabilities, quantising algebras of functions on smooth/continuous spaces, ...). Conceptually dubious for QG: continuum picture expected to break down at Planck scale.

Reminder

Structure of classical physics: *state space (phase space)* $\mathcal{P}$, each physical quantity A is represented by a real-valued function f_A on $\mathcal{P}$:



Towards topoi

Kochen-Specker ('67): No such state space model for quantum theory.

Key insight by Isham: we can generalise state spaces from being *sets* (as in classical physics) to being *objects in a suitable category* (for quantum theory and beyond).

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Such a suitable category is a **topos**. Gives generalised sets/spaces and, at the same time, generalised logic: each topos has an **internal logic** of intuitionistic type. One can talk about *partial truth* in a topos.

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This opens the way to a realist formulation of quantum theory based on a new kind of quantum state spaces, despite the Kochen-Specker theorem.

Contexts

For a start, take a quantum system S, described by the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$.

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Concretely, we consider the **context category** $\mathcal{V}(\mathcal{H})$, the set of all contexts, partially ordered under inclusion.

Example

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Clearly, the intersection of V_x and V_z contains total spin. The context category $\mathcal{V}(\mathcal{H})$ keeps track of how contexts (classical perspectives) overlap, i.e., intersect.

Note that we (indirectly) relate the noncommuting physical quantities spin-x and spin-z.

The quantum state space

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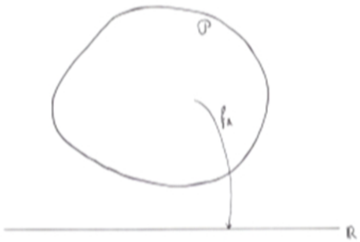

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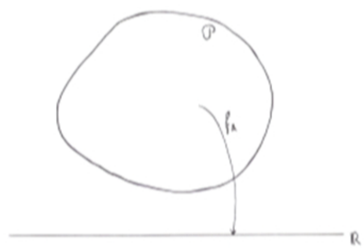
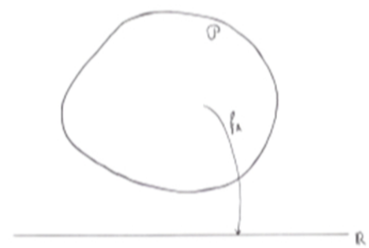


- If $i_{V'V} : V' \hookrightarrow V$, there is a canonical function $\underline{\Sigma}(i_{V'V}) : \underline{\Sigma}_V \to \underline{\Sigma}_{V'}$, $\lambda \mapsto \lambda|_{V'}$.

The quantum state space

- Each context V provides a classical, **'local' state space** by Gelfand duality: the Gelfand spectrum $\underline{\Sigma}_V$ (such that $V \simeq C(\underline{\Sigma}_V)$).
- Self-adjoint operators in V correspond to continuous, real-valued functions on $\underline{\Sigma}_V$. Think



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Definition

The collection $(\underline{\Sigma}_V)_{V \in \mathcal{V}(\mathcal{H})}$ of local state spaces, together with the functions $\underline{\Sigma}(i_{V'V})$, is called the **spectral presheaf** $\underline{\Sigma}$.

The quantum state space (2)

- $\underline{\Sigma}$ is the quantum state space of the system. It is *not a set*.

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- Interpretation: $\underline{\Sigma}$ is a kind of *noncommutative space*.
- $\underline{\Sigma}$ is an object in the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$ of presheaves over $\mathcal{V}(\mathcal{H})$ (no details needed here).

Propositions as subobjects

Recall: in classical physics, a proposition "$A \varepsilon \Delta$" is represented by the subset $f_A^{-1}(\Delta)$ of the state space.

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Proposition

$\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$ *is a complete bi-Heyting algebra.*

This generalises Boolean logic by keeping distributivity, but splitting negation into two concepts. *[arXiv:1202.2750]*

Pure states

In classical physics, (pure) states are points of the state space. But the quantum state space $\underline{\Sigma}$ has no points!

Assigning truth values

In classical physics, a proposition "$A \varepsilon \Delta$" is *true* in a given state $s \in \mathcal{P}$ if $s \in f_A^{-1}(\Delta)$. The truth value can be expressed as a (Boolean) **formula**:

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In the topos formulation, the corresponding formula

$$v(\text{“}A \varepsilon \Delta\text{”}; \underline{\mathfrak{w}}^{\psi}) = (\underline{\mathfrak{w}}^{\psi} \subset \underline{\delta^o}(\hat{E}[A \varepsilon \Delta]))$$

can be interpreted *within the logic of the topos* and gives a truth value, too.

In general, this is neither totally true nor totally false, but something in between. There are uncountably many truth values in the topos. *[arXiv:quant-ph/0703062]*

Neo-realism

But what is this good for?

- In the topos formulation of quantum theory, *every* proposition has a truth value in *every* pure state.

Mixed states

So far, only pure states. What about mixed states?

In classical physics, a mixed state is a *probability measure* $\mu : \mathcal{P} \to \mathbb{R}$ *on the state space*. One can show: *[arXiv:0809.4847]*

Theorem

If $\dim \mathcal{H} \geq 3$, *each quantum state* ρ *(pure or mixed) determines a unique probability measure* μ_ρ *on the quantum state space* $\underline{\Sigma}$ *and vice versa.*

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Proposition

$\underline{\Sigma}$ *is a joint sample space for all quantum observables.*

[arXiv:1210.5747]

Time evolution

Classical physics: flows = one-parameter groups of symplectomorphisms of the state space.

Analogously, one can define flows on the quantum state space $\underline{\Sigma}$. As usual, a Hamiltonian generates a flow.

Physical quantities as generalised functions

In classical physics, a physical quantity A is represented by a function $f_A : \mathcal{P} \to \mathbb{R}$.

In the topos approch, a physical quantity is represented by an arrow

$$\breve{\delta}(\hat{A}) : \underline{\Sigma} \longrightarrow \underline{\mathbb{R}^{\leftrightarrow}}$$

in the topos from the quantum state space to the space of *generalised* values $\underline{\mathbb{R}^{\leftrightarrow}}$. *[arXiv:quant-ph/0703064]*

Summary and outlook

Slogan: *Quantum physics, when formulated in the topos* $\mathbf{Set}^{\mathcal{V}(\mathcal{H})^{\mathrm{op}}}$, *looks like classical physics.*

- By generalising from sets to presheaves, we circumvented the Kochen-Specker no-go theorem. There is a quantum state space, time evolution by flows, states as probability measures, etc.
- There also is a new form of logic for quantum systems; truth values instead of probabilities; neo-realism.
- General structure is 'Hamiltonian': state space and time evolution described by flows. But: generalised, noncommutative spaces.

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- There also is a new form of logic for quantum systems; truth values instead of probabilities; neo-realism.
- General structure is 'Hamiltonian': state space and time evolution described by flows. But: generalised, noncommutative spaces.
- The setup allows many generalisations beyond standard quantum theory. E.g. nets of local algebras as in AQFT give a context category with additional space-time labels.

Summary and outlook

Rough conjecture: a quantum particle (that is not localised perfectly) may 'see the world' as $\underline{\mathbb{R}^{\leftrightarrow}}^4$ rather than as $\mathbb{R}^4$.

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Note: this amounts to embedding space-time into a richer structure. Opposite to ideas of discretisation.

A (Frightening Speculative) Higher C*-categorical Formalism for Relational Quantum Theory

Paolo Bertozzini

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Loops 13

Perimeter Institute, Ontario, Canada
26 July 2013

Outline

- Mathematical Preliminaries ▸ 3
  - Categories and (Non-commutative) Higher Categories
  - Non-commutative n-C*-categories
  - Examples: Hypermatrices
- Motivations from Physics ▸ 2
  - Categorical Covariance
  - Rovelli's Relational Quantum Theory
  - Relativistic and History Formulation
  - Weak Measurements
- Higher C*-categories in Relational Quantum Theory ▸ 4
  - Observers, Symmmetries, Localization, States, Expectations
  - Categories of Correlations as Physical Systems
  - Observers of Observers of ...
- Modular Algebraic Quantum Theory ▸ 1

Ideology of Modular Algebraic Quantum Theory

- **quantum theory** *is a fundamental theory of physics and should not come from a quantization;*
- *geometry should be spectrally reconstructed a posteriori from a basic* **operational theory** *of observables and states;*
- **A.Connes' non-commutative geometry** *provides the natural environment where to attempt an implementation of the spectral reconstruction of a "quantum" space-time;*
- **Tomita-Takesaki modular theory** *should be the main tool to achieve the previous goals, associating to operational data, spectral non-commutative geometries;*
- **categories of operational data** *provide the general framework for the formulation of covariance in this context ... and ultimately for the identification of the geometric degrees of freedom (space-time) hidden in the theory.*

Connes' Spectral Triples

- A naive compact **spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ is a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of a C*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ equipped with a (possibly unbounded) self-adjoint operator D on $\mathcal{H}$, with compact resolvent, such that $[D, \pi(a)]$ extends to a bounded operator on $\mathcal{H}$, for all a in a dense $*$-subalgebra of $\mathcal{A}$, leaving invariant the domain of D.
- Every compact oriented Riemannian spin manifold M is uniquely algebraically encoded as a spectral triple $(C(M), \Gamma(S(M)), D_M)$ where $\Gamma(S(M))$ is the Hilbert space of spinorial fields and D_M the usual Atiyah-Singer Dirac operator.
- When the C*-algebra $\mathcal{A}$ is non-commutative a spectral triple describes a compact "quantum spinorial geometry".
- A.Carey-J.Phillips-A.Rennie-F.Sukochev defined semi-finite and **modular spectral triples** to deal with non-commutative geometries originated from Tomita-Takesaki modular theory.

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Tomita-Takesaki Modular Theory 1

- For every von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ acting on a Hilbert space $\mathcal{H}$, and for every vector $\xi \in \mathcal{H}$ that is cyclic separating,[4] there is a one-parameter unitary group $t \mapsto \Delta_\xi^{it} \in \mathcal{B}(\mathcal{H})$ and a conjugate-linear isometry $J_\xi : \mathcal{H} \to \mathcal{H}$, with $J_\xi \circ J_\xi = \mathrm{Id}_{\mathcal{H}}, \quad J_\xi \circ \Delta_\xi = \Delta_\xi^{-1} \circ J_\xi$, such that:

$$\Delta_\xi^{it} \mathcal{M} \Delta_\xi^{-it} = \mathcal{M}, \forall t \in \mathbb{R},$$
$$J_\xi \mathcal{M} J_\xi = \mathcal{M}'.$$

[4] Meaning that $\overline{(\mathcal{M}\xi)} = \mathcal{H}$ and for $a \in \mathcal{M}$, $a\xi = 0 \Rightarrow a = 0$.

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Tomita-Takesaki Modular Theory 2

- More generally, given a faithful normal state ω on a von Neumann algebra $\mathcal{M}$, there is a one-parameter group of $*$-automorphisms $t \mapsto \sigma_t^\omega \in \text{Aut}(\mathcal{M})$, spatially implemented, in the GNS-representation π_ω induced by ω, by a unitary one-parameter group $t \mapsto \Delta_\omega^{it} \in \mathcal{B}(\mathcal{H})$:

$$\pi_\omega(\sigma_t^\omega(x)) = \Delta_\omega^{it}\pi_\omega(x)\Delta_\omega^{-it}, \quad x \in \mathcal{M}, \ t \in \mathbb{R};$$

and there is a conjugate-linear isometry $J_\omega : \mathcal{H} \to \mathcal{H}$, with $J_\omega^2 = \text{Id}_{\mathcal{H}_\omega}$ and $J_\omega \Delta_\omega = \Delta_\omega^{-1} J_\omega$, whose adjoint action spatially implements a conjugate-linear $*$-isomorphism $\gamma_\omega : \pi_\omega(\mathcal{M}) \to \pi_\omega(\mathcal{M})'$, between $\pi_\omega(\mathcal{M})$ and its commutant:

$$\gamma_\omega(\pi_\omega(x)) = J_\omega \pi_\omega(x) J_\omega, \quad \forall x \in \mathcal{M}.$$

Modular Algebraic Quantum Gravity 2

- Tomita-Takesaki modular theory is here taking the role of the quantum version of Einstein's equation associating "geometries" to "matter content" where:
  - "geometries" are spectrally described by variants of **modular spectral triples** (see A.Carey-A.Rennie-J.Phillips-F.Sukochev),
  - "matter content" is described by the set of quantum correlations between observables specified by the state.
- Every pair $(\mathcal{O}, \omega)$ gives a different "net" of modular spectral geometries $(\mathcal{A}_\omega, \mathcal{H}_\omega, \xi_\omega, K_\omega, J_\omega)_{\mathcal{A}\subset\mathcal{O}}$ that are:
  - **quantum**, since $\mathcal{A} \subset \mathcal{O}$ are non-commutative,
  - **state-dependent** on ω,
  - **relative to observers** $\mathcal{O}$.

Modular Algebraic Quantum Gravity 3

- Tentatively (see C.Rovelli's relativistic quantum mechanics): ξ_ω represents a covariant vacuum, $\mathcal{H}_\omega$ a boundary Hilbert space, K_ω a covariant constraint.
- Modular spectral geometries should be **phase-space geometries of a (free) field-theory**:[7] ... all the "interactions" will be finally codified via correlations in the base of a categorical bundle!
- We did "assume" that partial observables $\mathcal{O}$ are a C*-algebra. This is in line with usual algebraic quantum theory ... but might be reconsidered in the light of better understanding of the foundations of algebraic quantum theory!

[7] They might also be a possible link to relative locality (G.Amelino-Camelia-L.Friedel-J.Kowalski-Glikman-L.Smolin).

Relational Quantum Theory

In 1994, C.Rovelli elaborated **relational quantum mechanics** as an attempt to radically solve the interpretational problems of quantum theory. This approach is based on two assumptions:

- All physical systems should be treated in the same way: there is no difference between observed systems and observers.
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Analysis of the third observer problem (Schrödinger cat) entails:

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Strict Higher Categories

A **strict globular *n*-category** $(\mathcal{C}, \circ_0, \cdots \circ_{n-1})$ is a set $\mathcal{C}$ equipped with a family of partially defined binary compositions $\circ_p$, for $p := 0, \dots, n-1$, that satisfy the following list of axioms:

- for all $p = 0, \dots, n-1$, $(\mathcal{C}, \circ_p)$ is a partial 1-monoid, whose partial identities are denoted by $\mathcal{C}^p$,
- for all $p, q = 0, \dots n-1$, with $q < p$, the $\circ_q$-composition of $\circ_p$-identities, whenever exists, is a $\circ_p$-identity: $\mathcal{C}^p \circ_q \mathcal{C}^p \subset \mathcal{C}^p$,
- for all $q < p$, a $\circ_q$-identity is also a $\circ_p$-identity: $\mathcal{C}^q \subset \mathcal{C}^p$,
- the exchange property holds for all $q < p$: whenever $(x \circ_p y) \circ_q (w \circ_p z)$ exists also $(x \circ_q w) \circ_p (y \circ_p z)$ exists and they coincide.[2]

[2] By symmetry, the exchange property automatically holds for all $q \neq p$.

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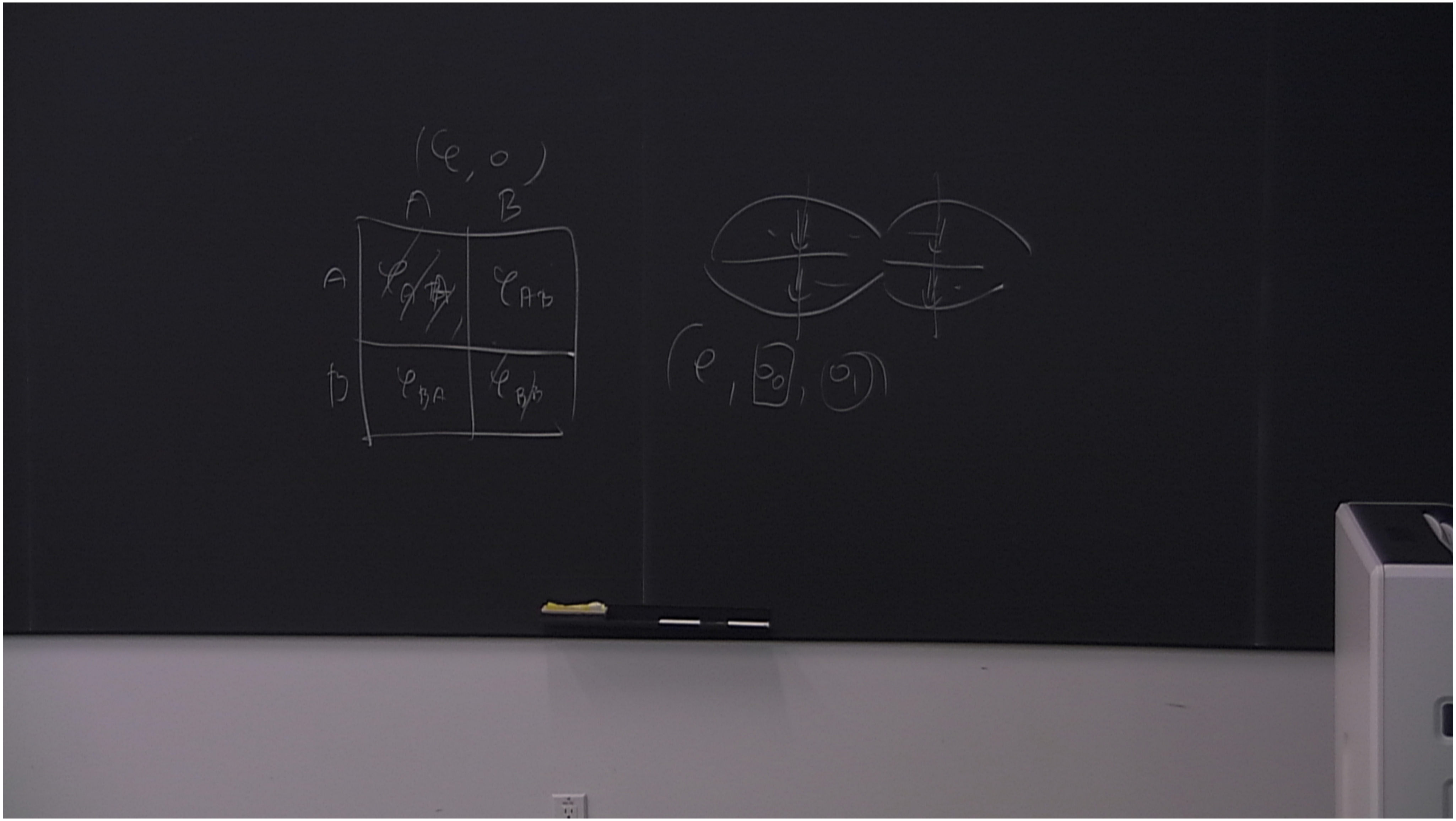
Non-commutative Exchange / Non-globular Categories

- the exchange property (Eckmann-Hilton argument) forces a collapse of the structure: for all $e \in \mathcal{C}^q$ with $q < n-1$, $\circ_q = \cdots = \circ_{n-1}$ and $(\mathcal{C}_{ee}, \circ_q)$ is Abelian.
- In order to accommodate non-commutative fibers we proposed a relaxed **non-commutative exchange** property: for all $\circ_p$-identities x, for all $q < p$, the partially defined maps $x \circ_q - : (\mathcal{C}, \circ_p) \to (\mathcal{C}, \circ_p)$ and $- \circ_q x : (\mathcal{C}, \circ_p) \to (\mathcal{C}, \circ_p)$ are functorial (homomorphisms of partial 1-monoids).
- It is also possible to consider n-categories and n-C*-categories that are not based on globular or cubical n-quivers.
- We can produce "iterated" n-C*-categories with separate norms and linear structures for each level $1, \ldots, n$.



Strict Higher C*-Categories

A **fully involutive strict** n**-C*-category** $(\mathcal{C}, \circ_0, \dots, \circ_{n-1}, *_0, \dots, *_{n-1}, +, \cdot, \| \cdot \|)$ is a fully involutive strict n-category such that:

- for all $a, b \in \mathcal{C}^{n-1}$, the fiber $\mathcal{C}_{ab}$ is Banach with norm $\| \cdot \|$,[3]
- for all p, $\circ_p$ is fiberwise bilinear and $*_p$ is conjugate-linear,
- for all $\circ_p$, $\|x \circ_p y\| \leq \|x\| \cdot \|y\|$, whenever $x \circ_p y$ exists,
- for all p, $\|x^{*_p} \circ_p x\| = \|x\|^2$, for all $x \in \mathcal{C}$,
- for all p, $x^{*_p} \circ_p x$ is positive in the C*-algebra envelope of $\mathcal{C}_{ee}$ $(\mathcal{E}(\mathcal{C}_{ee}), \circ_p, *_p, +, \cdot, \| \cdot \|)$, where e is the p-source of x.

A **partially involutive strict** n**-C*-category** will be equipped with only a subfamily of the previous involutions and will satisfy only those properties that can be formalized using the given involutions.

[3] By definition $\mathcal{C}_{ab} := \{x \in \mathcal{C} \mid b \circ_{n-1} x, \quad x \circ_{n-1} a \quad \text{both exist}\}$.

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Hypermatrices 2

- there are 2^n involutions taking the conjugate of all the entries and, at every level, either the transposed or the identity: $[x^{i_1 \dots i_k \dots i_n}_{j_1 \dots j_k \dots j_n}]^{\star_\gamma} := [\overline{x}^{i_1 \dots j_{k_1} \dots j_{k_m} \dots i_n}_{j_1 \dots i_{k_1} \dots i_{k_m} \dots j_n}]$, for all $\gamma := \{k_1, \dots, k_m\} \subset \{1, \dots, n\}$.
- there are 2^n C*-norms taking either the operator norm or the maximum norm at every level: using the natural isomorphism $\mathbb{M}_{N_1^2 \dots N_n^2}(\mathbb{C}) \simeq \mathbb{M}_{N_1^2}(\mathbb{C}) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{M}_{N_n^2}(\mathbb{C})$, $\forall \gamma \subset \{1, \dots, n\}$, $\|[x^{i_1}_{j_1}] \otimes \cdots \otimes [x^{i_n}_{j_n}]\|_\gamma := \prod_{k \in \gamma} \|[x^{i_k}_{j_k}]\| \cdot \prod_{k' \notin \gamma} \|[x^{i_{k'}}_{j_{k'}}]\|_\infty$, where $\|[x^{i_k}_{j_k}]\|$ is the C*-norm on $\mathbb{M}_{N_k}(\mathbb{C})$ and $\|[x^{i_k}_{j_k}]\|_\infty := \max_{i,j} |x^i_j|$.

$(\mathbb{M}_{N_1^2 \dots N_n^2}(\mathbb{C}), \bullet_\gamma, \star_\gamma, \| \cdot \|_\gamma, \gamma \subset \{1, \dots, n\})$ is a hyper C*-algebra.

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Symmetries as Twisted Bimodules

In algebraic quantum theory (following Wigner), symmetries are described by linear isomorphisms (or conjugate-linear anti-isomorphisms) $\phi : \mathcal{A} \to \mathcal{B}$ between two C*-algebras of observables.

To every such symmetry ϕ, there is a naturally associated adjoint pair of $\mathcal{A}$-$\mathcal{B}$ bimodules ${}_\phi\mathcal{B}$ and $\mathcal{B}_\phi$ obtained by left or right ϕ-twisting of the product in $\mathcal{B}$:

$$a \cdot x \cdot b := \phi(a)xb, \quad \forall a \in \mathcal{A},\ b \in \mathcal{B},\ x \in {}_\phi\mathcal{B},$$
$$b \cdot x \cdot a := bx\phi(a), \quad \forall a \in \mathcal{A},\ b \in \mathcal{B},\ x \in \mathcal{B}_\phi,$$

Composition of symmetries functorially corresponds to the internal tensor product of bimodules:

$$\mathcal{A} \xrightarrow{\phi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \quad \mapsto \quad \mathcal{C}_{\psi\circ\phi} \simeq \mathcal{C}_\psi \otimes_\mathcal{B} \mathcal{B}_\phi.$$

Physical Systems = Categories of Correlations 1

Different observers are now mutually related by a family of quantum correlation channels, some of them describing symmetries, others quantum interactions.

Each observer is still equipped with a family of potential states, but now states of different observers can be compared via the family of binary correlations so far introduced.

The dynamic of the quantum theory has been totally codified via correlations and the potentially huge Cartesian product of state-spaces of the observers is now collapsed to a much more manageable set of states that are compatible under the given correlations.

From Empirical Practice to Observables and the Action Principle

Bruno Hartmann

Loops13 26.07.2013

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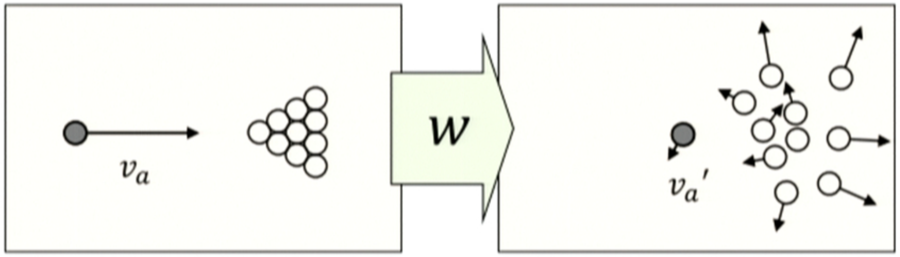
Action Functional

$$S_{\text{Ham}}[\gamma] := \int dt \left(\frac{1}{2} \cdot m_I \cdot \boldsymbol{v}_I{}^2 - V_{\text{pot}}\right)$$

Action Functional

$$S_{\text{Ham}}[\gamma] := \int dt \left(\frac{1}{2} \cdot m_I \cdot \boldsymbol{v}_I{}^2 - V_{\text{pot}}\right)$$

Physical Interaction



Physical Operation

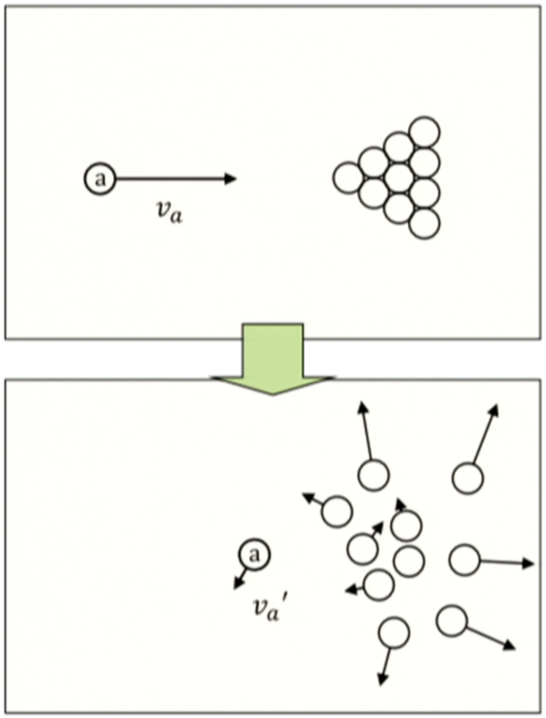
1kg '+' 1kg   1sec '+' 1sec   1m '+' 1m

$$\boldsymbol{F} = \frac{\boldsymbol{p}}{t}$$

$$\boldsymbol{p} = m \text{ `$\cdot$' } \boldsymbol{v}$$

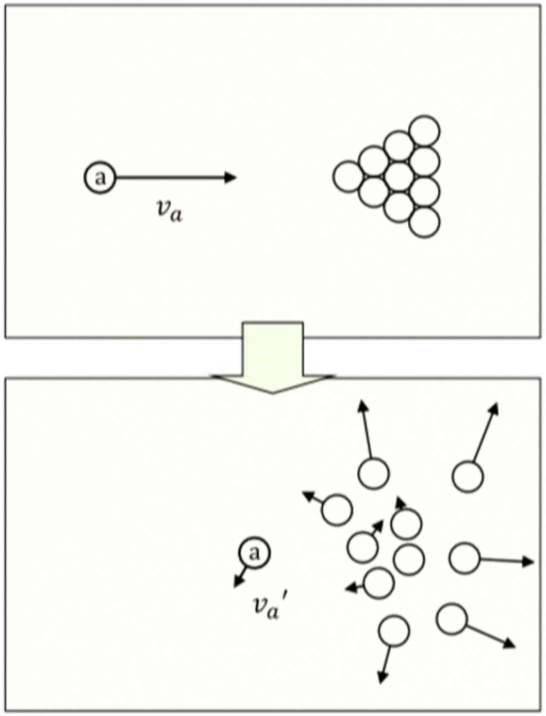
etc.

generic Billiard collision

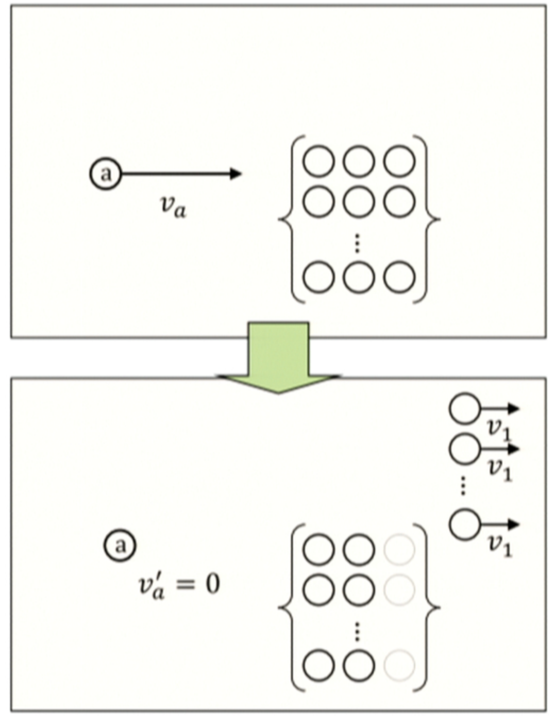


- ‘potential to cause action’
- ‘striking power’ or ‘impulse’

generic Billiard collision

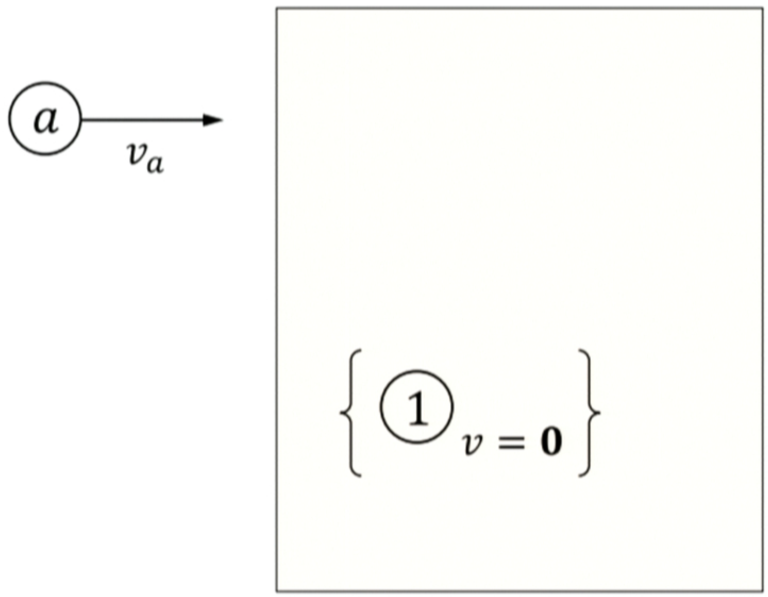


controlled replacement process



- ‘potential to cause action’
- ‘striking power’ or ‘impulse’

Calorimeter Model



Assemble Calorimeter

elementary standard interaction

w_1

Assemble Calorimeter

elementary standard interaction

w_1

elastic collision

${w_1}^{-1} * w_1$

Assemble Calorimeter

elementary standard interaction

$$w_1$$

elastic collision

$$w_1{}^{-1} * w_1$$

elastic transversal collision

$$w_T := (w_1{}^{-1} * w_1)^{(B)}$$

Assemble Calorimeter

elementary standard interaction

$$w_1$$

elastic collision

$$w_1^{-1} * w_1$$

elastic transversal collision

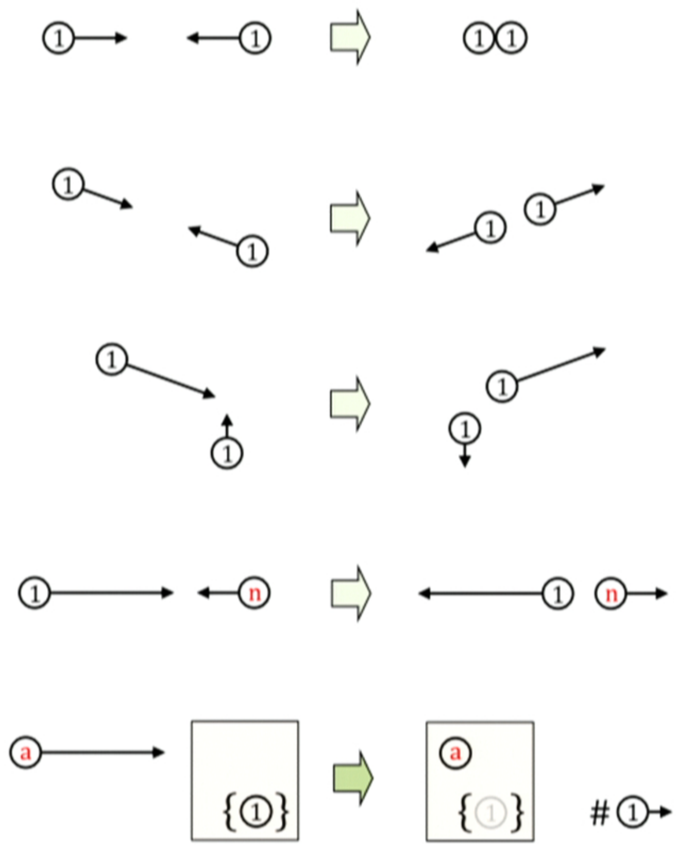
$$w_T := (w_1^{-1} * w_1)^{(B)}$$

elastic longitudinal collision

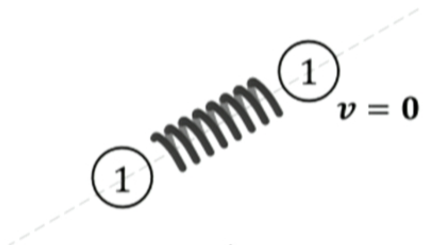
$$w_L := w_T * \cdots * w_T$$

absorption in calorimeter

$$W_{\text{cal}} := w_L^{(A)} * w_L^{(B)} * \cdots$$



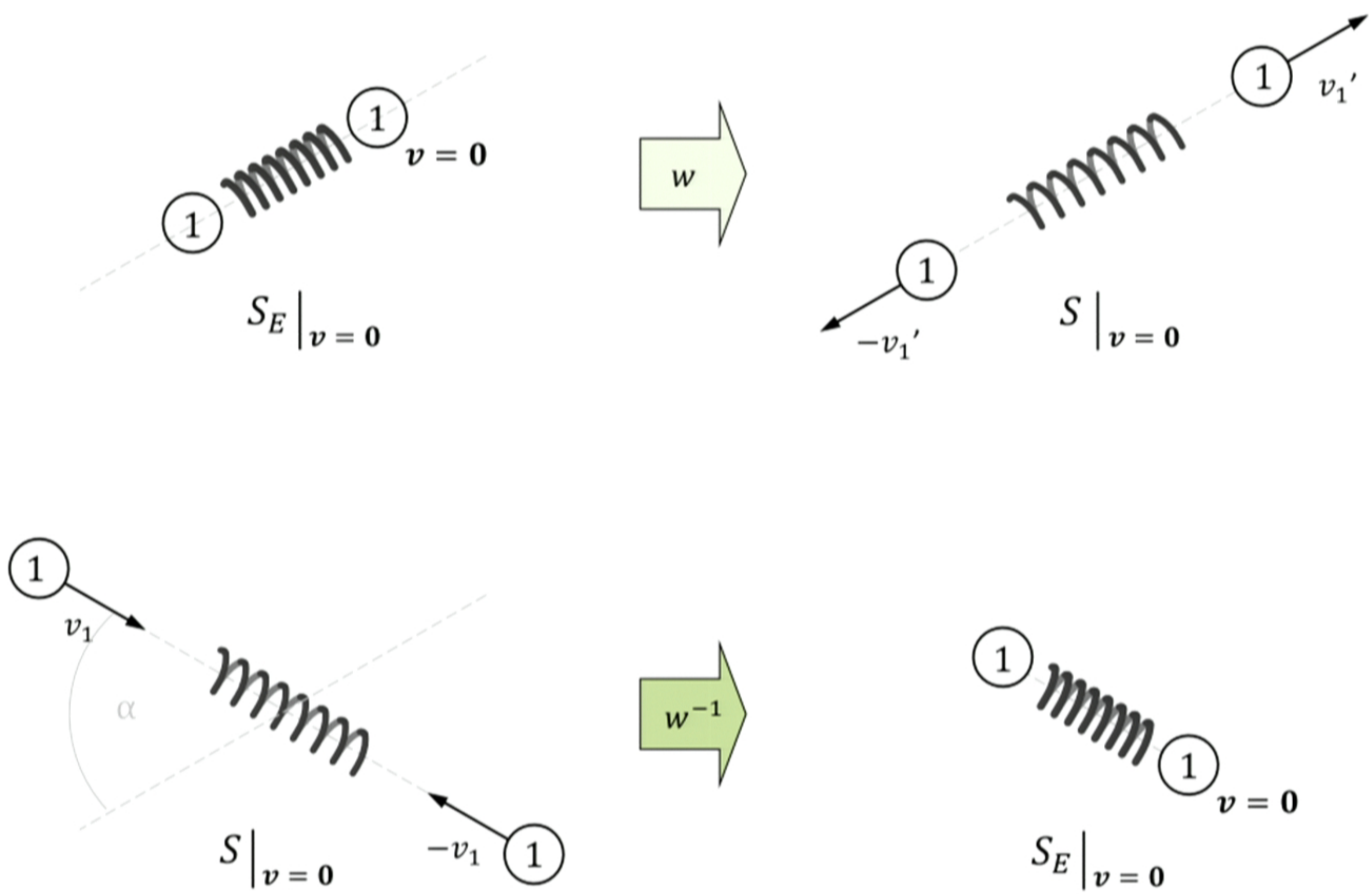
Standard Interaction (unit action $w_{\mathbf{1}}$)



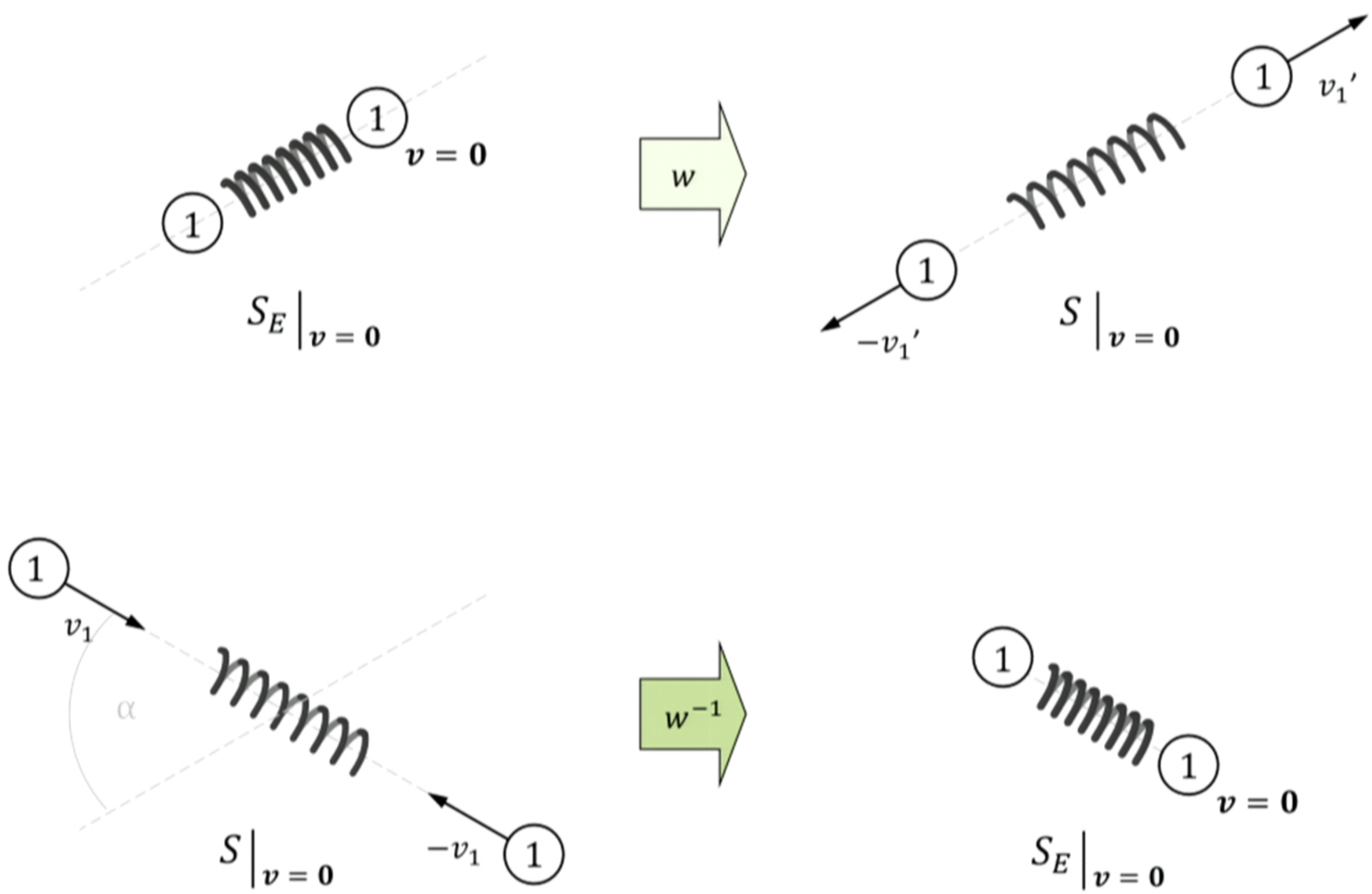
$v = \mathbf{0}$

$S_E \big|_{v=\mathbf{0}}$

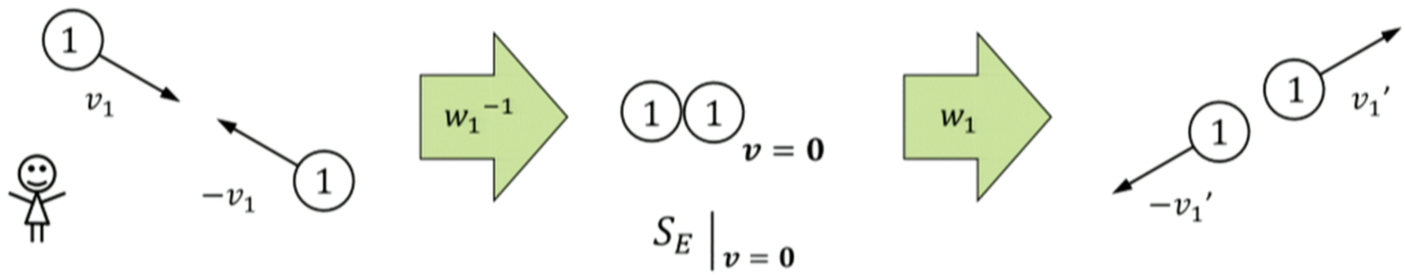
Standard Interaction (unit action w_1)



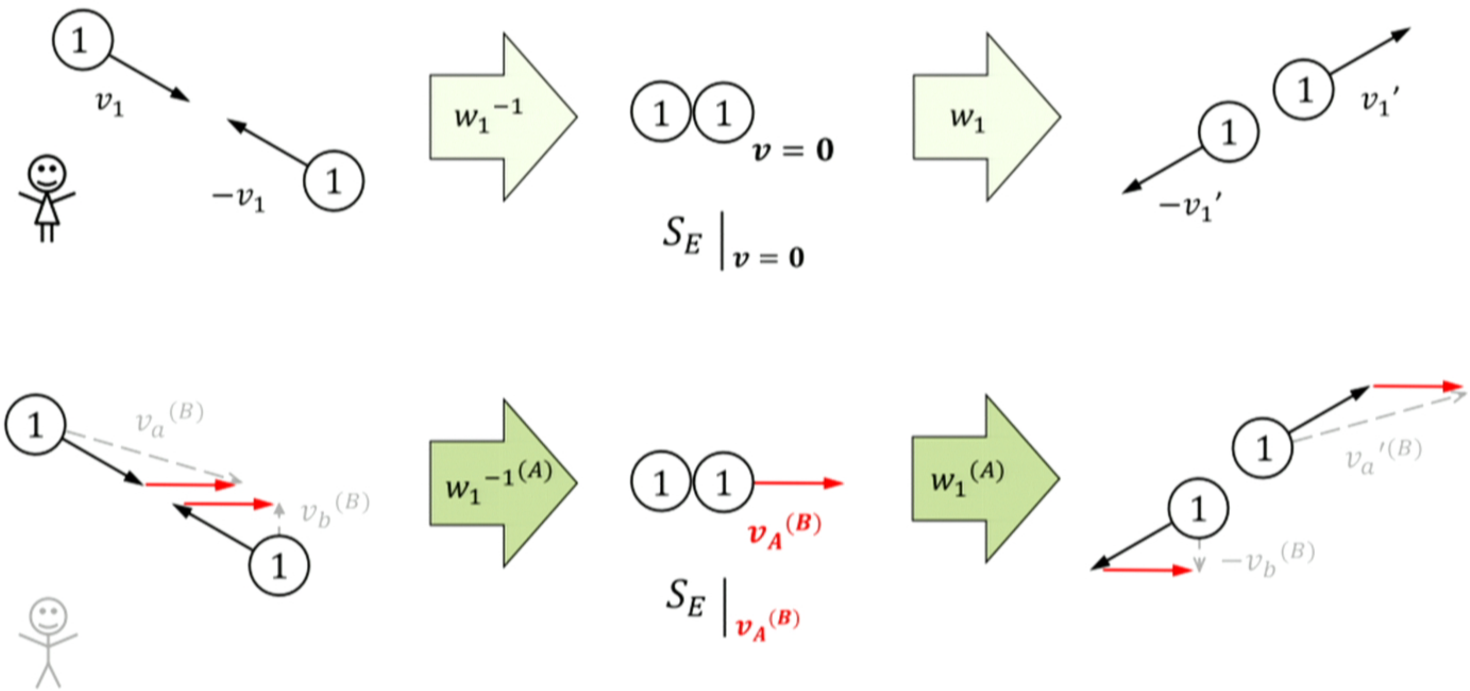
Standard Interaction (unit action $w_{\mathbf{1}}$)



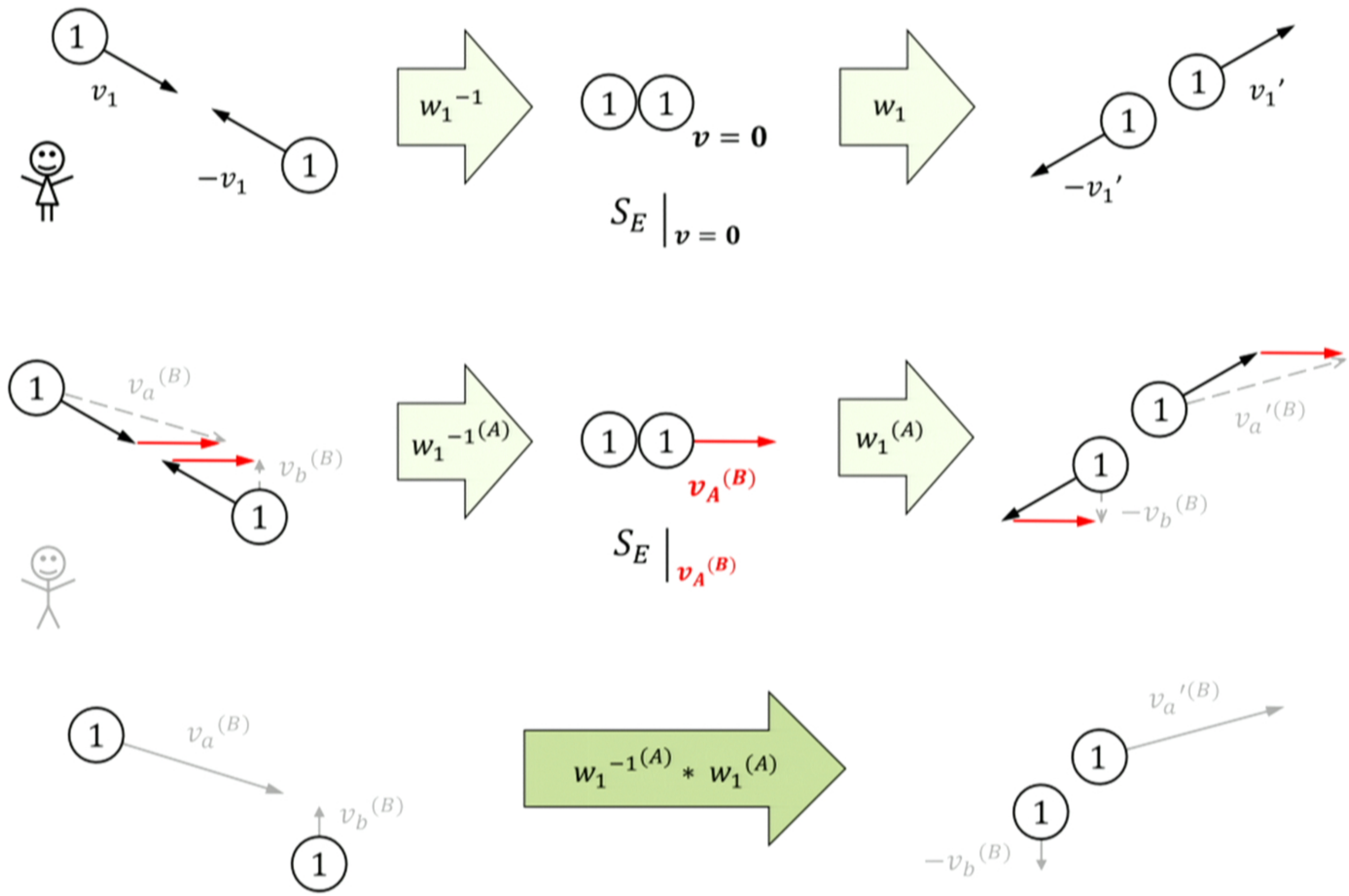
Consecutive Association (concatenation $*$)



Consecutive Association (concatenation $*$)

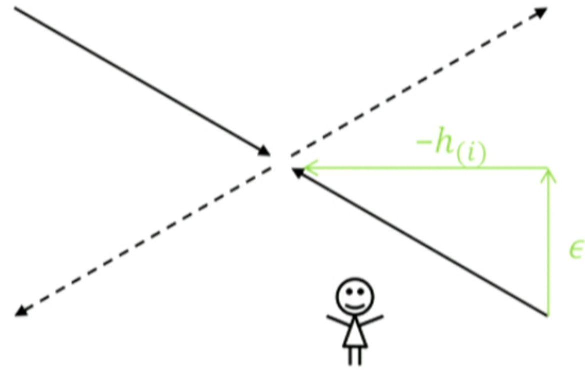


Consecutive Association (concatenation ∗)



Elastic Transversal Collision

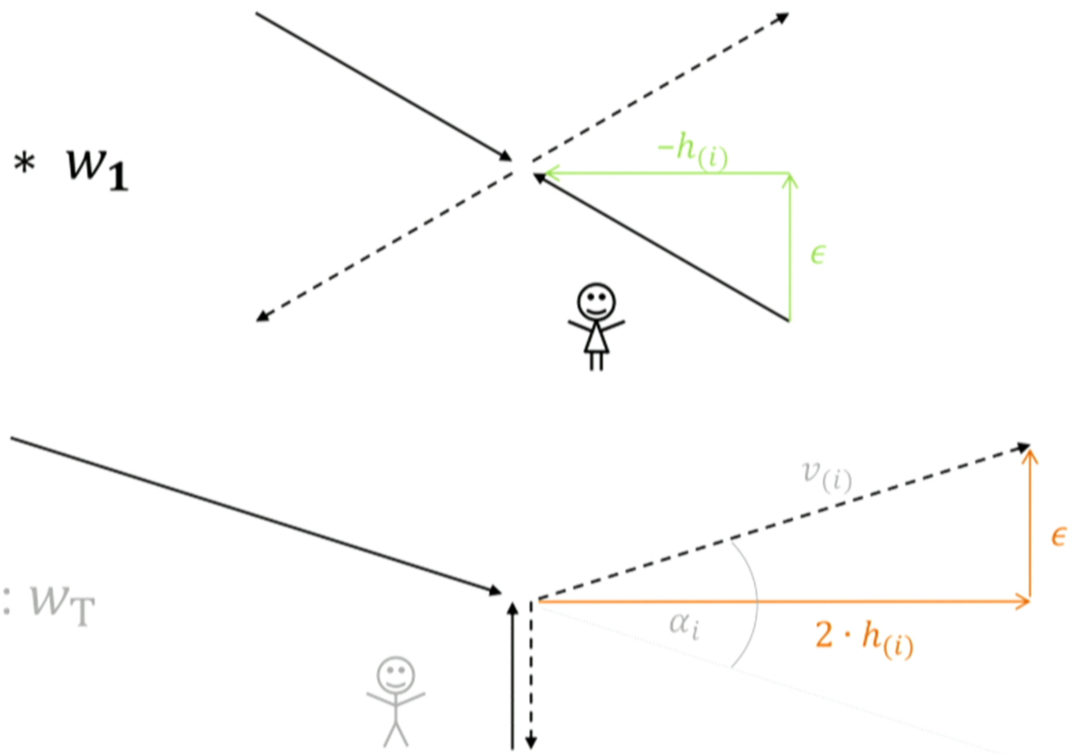
$$w_{\mathbf{1}}{}^{-1} * w_{\mathbf{1}}$$



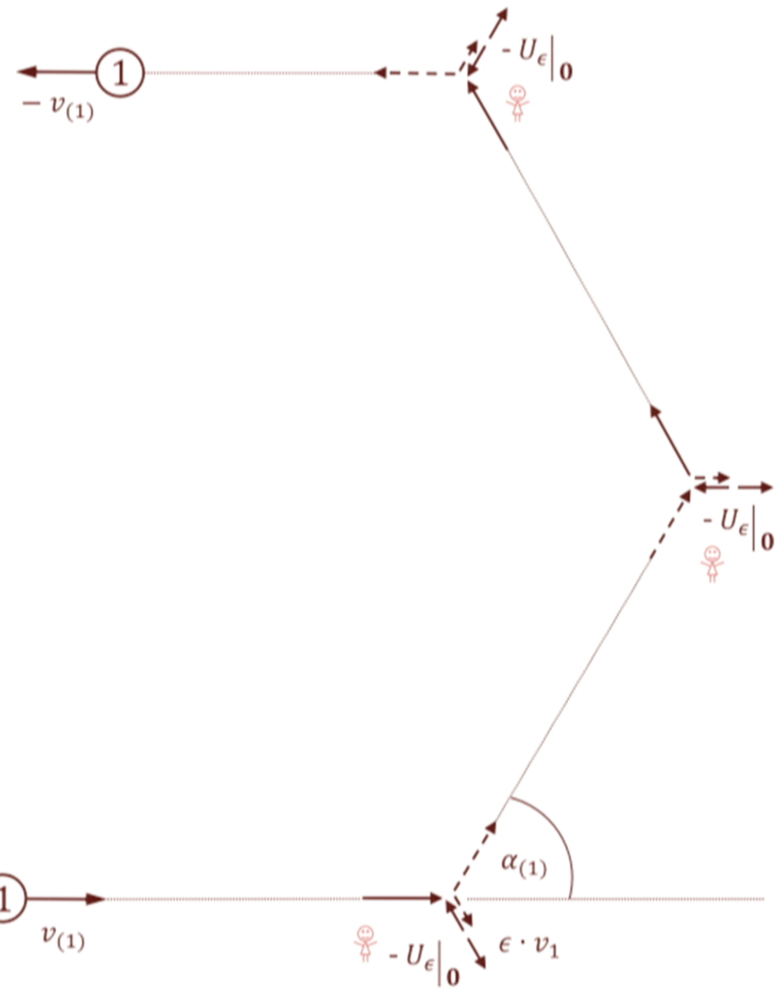
Elastic Transversal Collision

$w_1^{-1} * w_1$

$=: w_T$

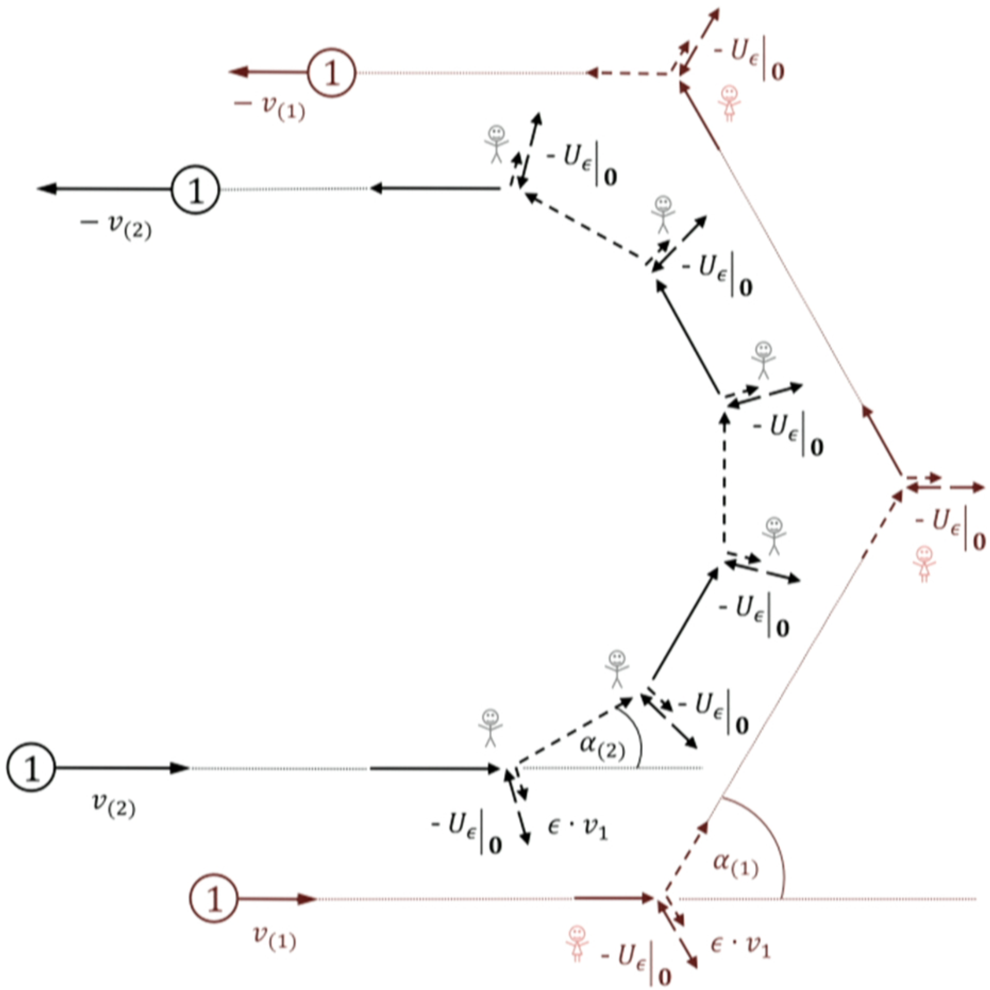


Impulse Reversion Process

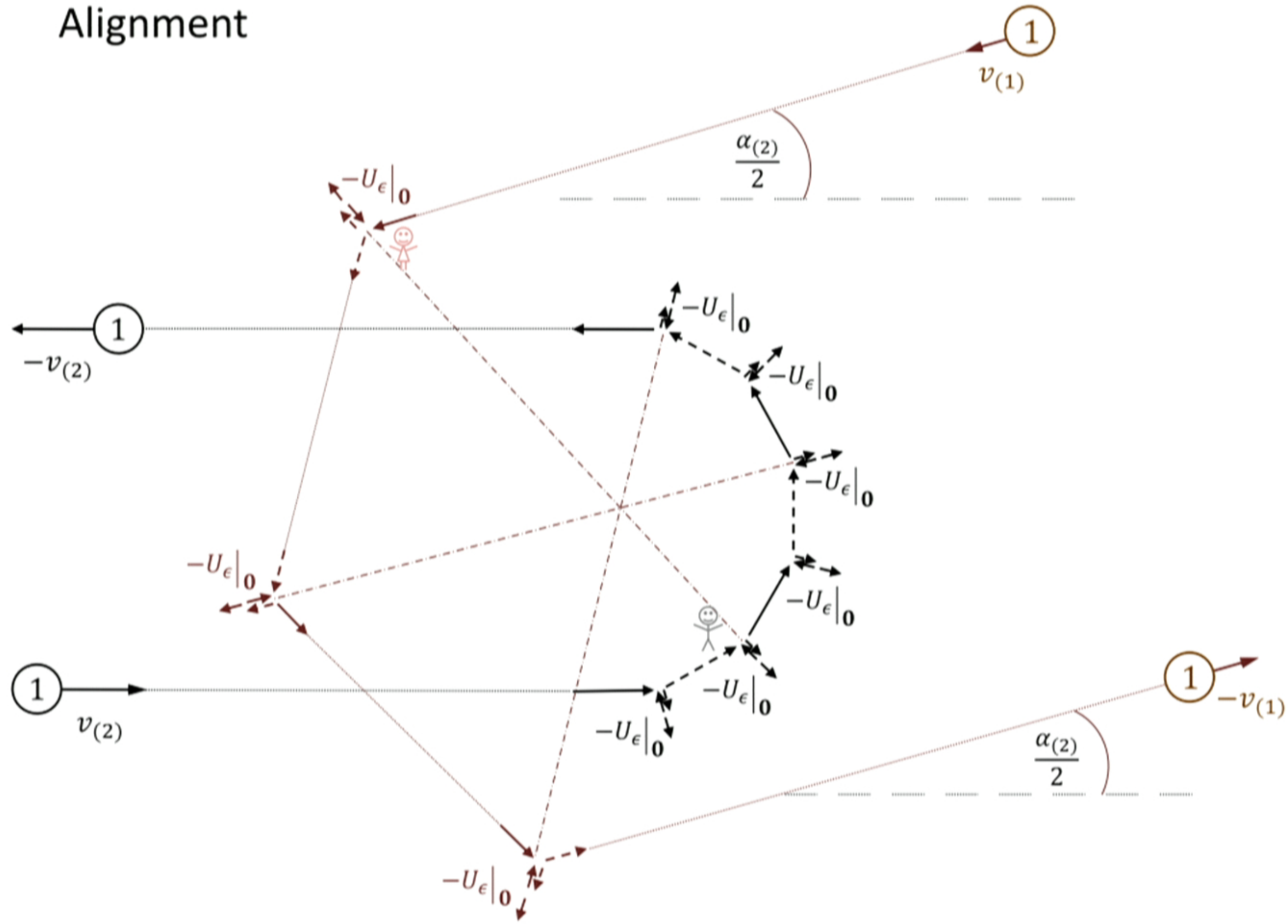


$$w_T * w_T * w_T$$

Impulse Reversion Process

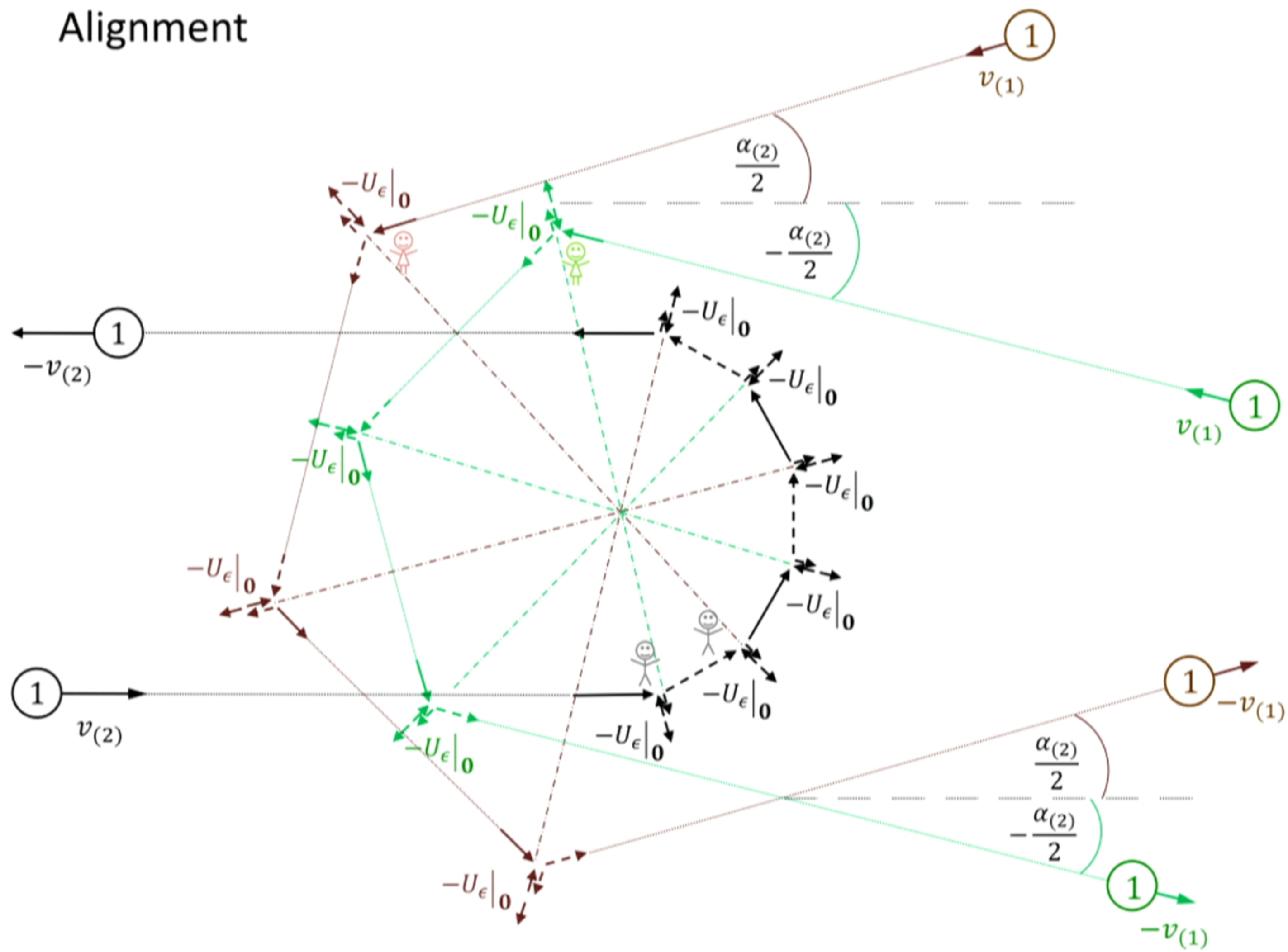


Alignment

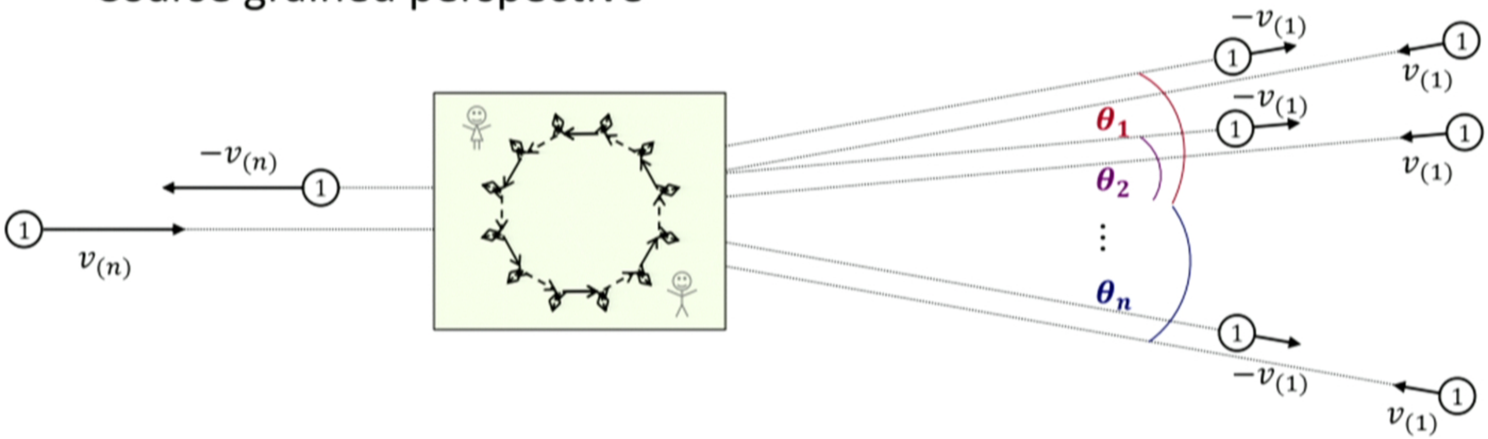


1
$v_{(1)}$
$\frac{\alpha_{(2)}}{2}$
$-U_\epsilon|_0$
1
$-v_{(2)}$
$-U_\epsilon|_0$
$-U_\epsilon|_0$
$-U_\epsilon|_0$
$-U_\epsilon|_0$
$-U_\epsilon|_0$
1
$v_{(2)}$
$-U_\epsilon|_0$
$-U_\epsilon|_0$
1
$-v_{(1)}$
$\frac{\alpha_{(2)}}{2}$
$-U_\epsilon|_0$

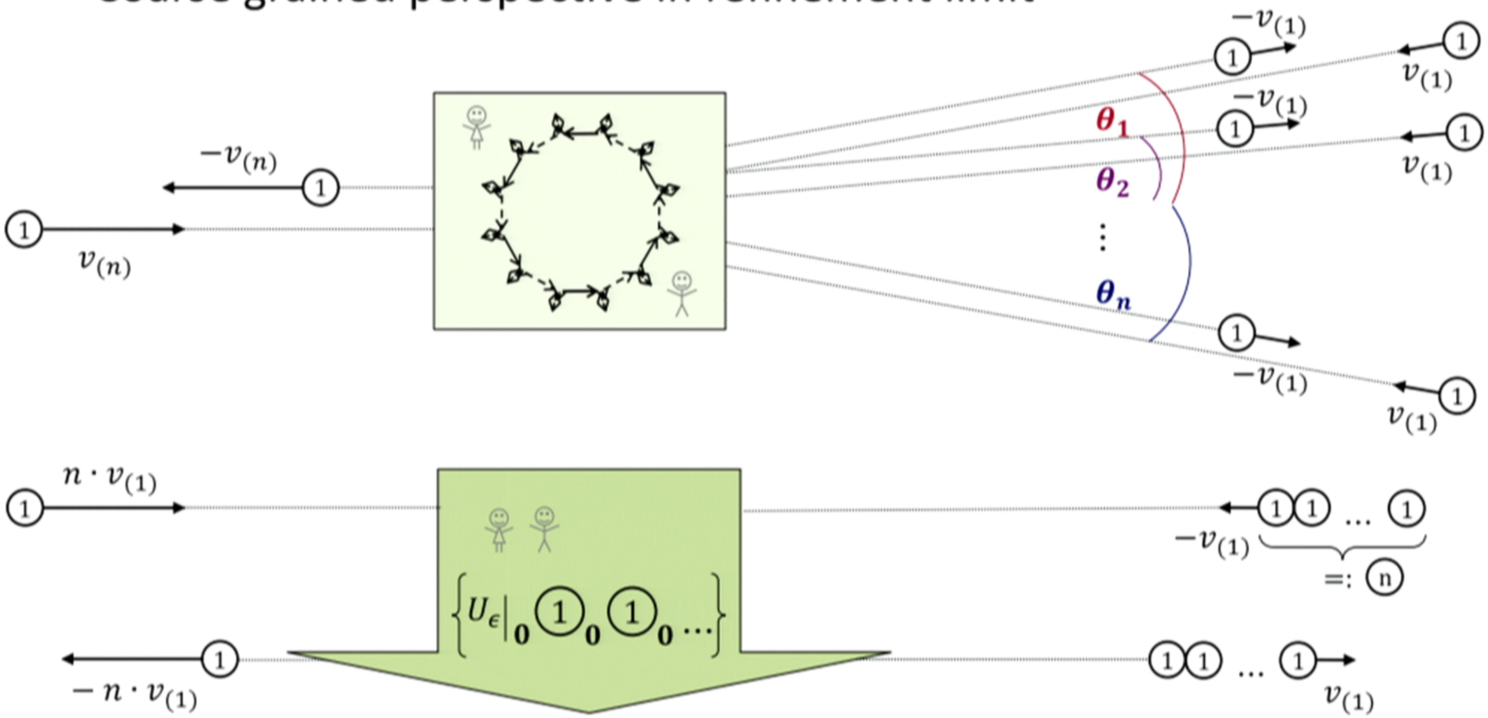
Alignment



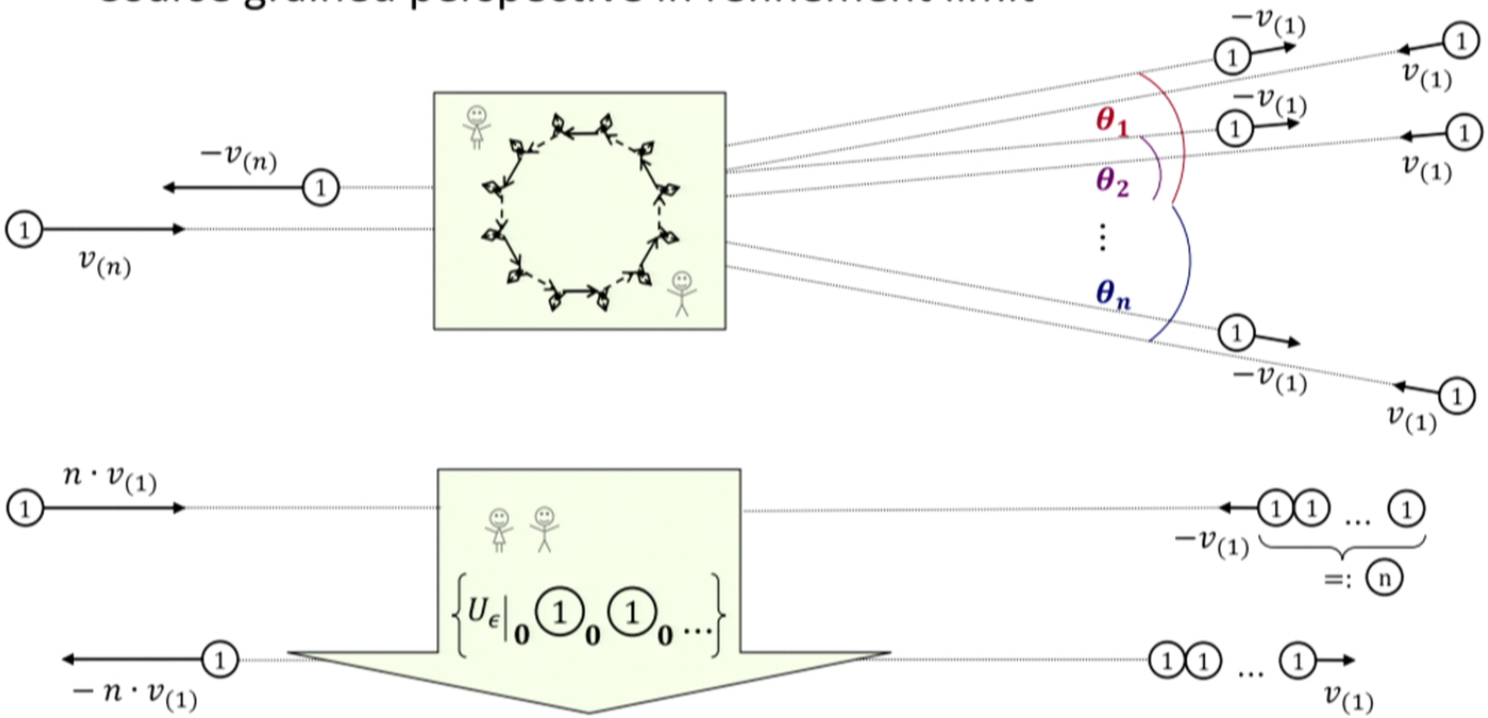
Coarse grained perspective



Coarse grained perspective in refinement limit



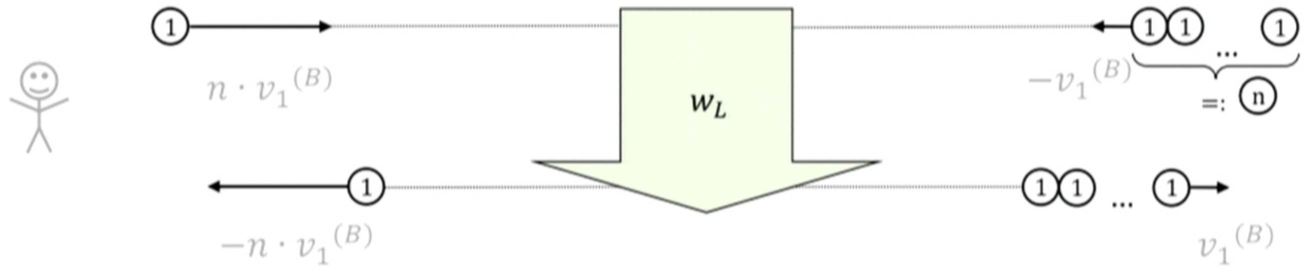
Coarse grained perspective in refinement limit



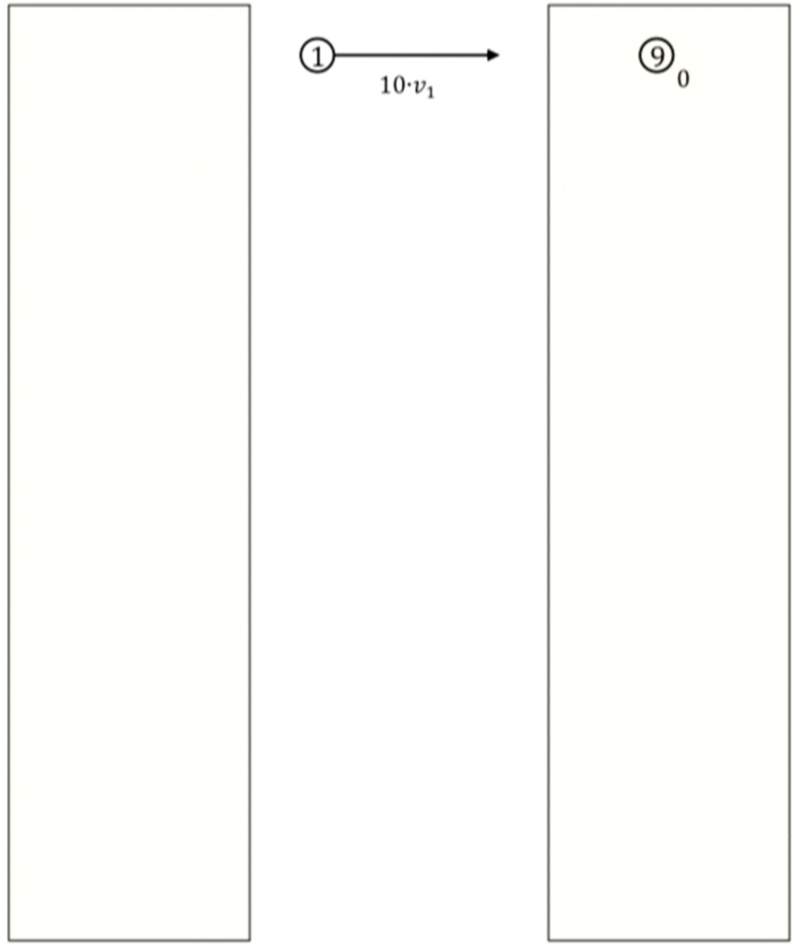
Quantification of Elastic Collision

$$w : \textcircled{n}_{1\cdot\mathbf{v}}, \textcircled{1}_{-n\cdot\mathbf{v}} \Longrightarrow \textcircled{n}_{-1\cdot\mathbf{v}}, \textcircled{1}_{n\cdot\mathbf{v}}$$

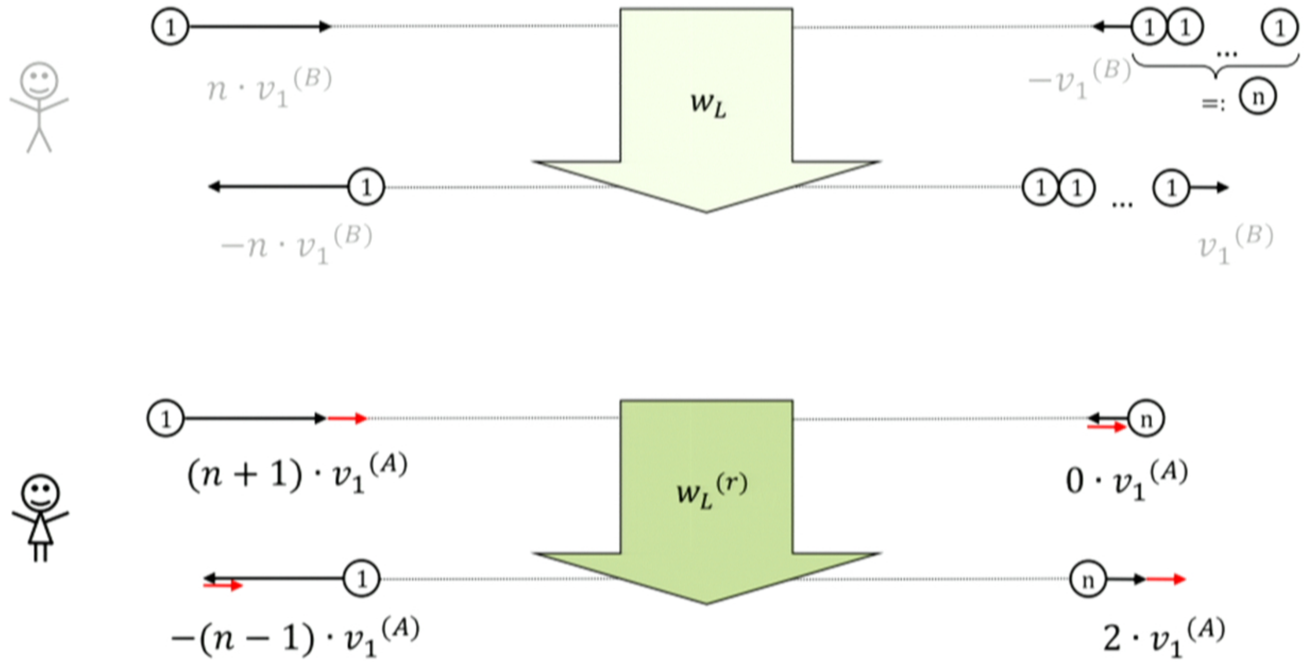
Elastic Longitudinal Collision



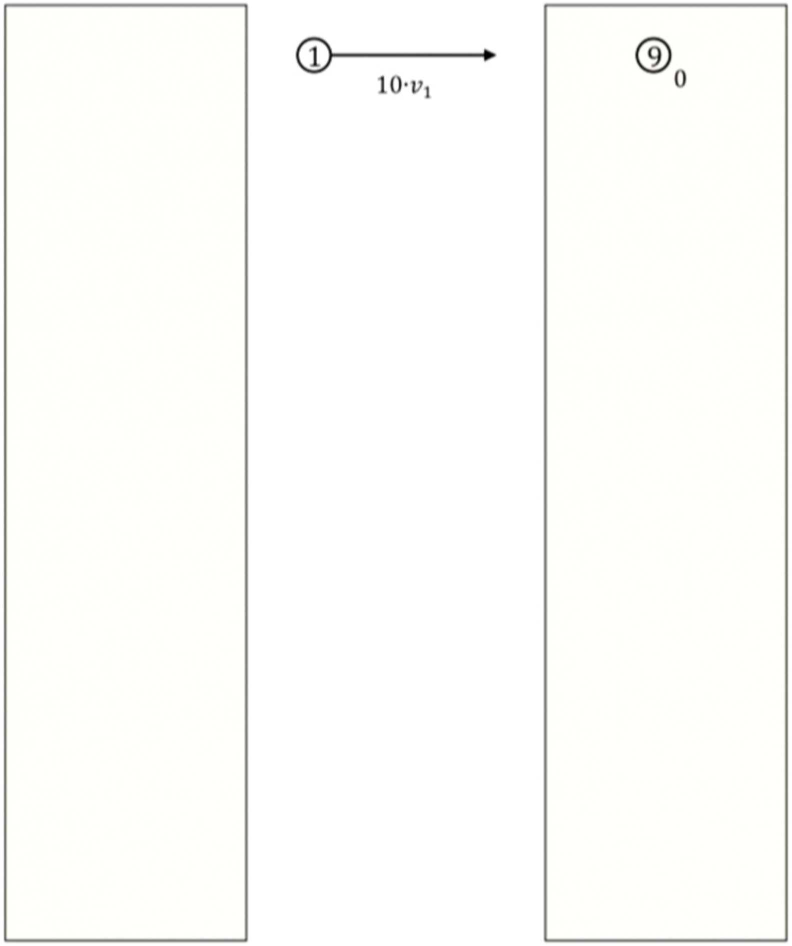
Absorption Action



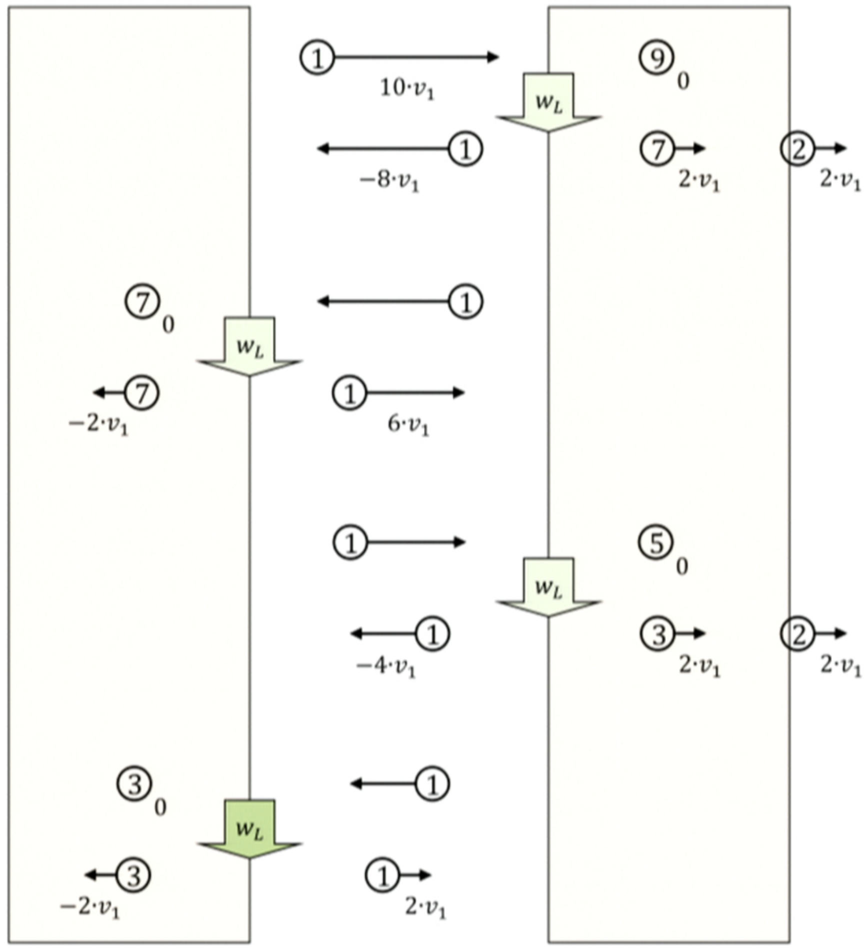
Elastic Longitudinal Collision

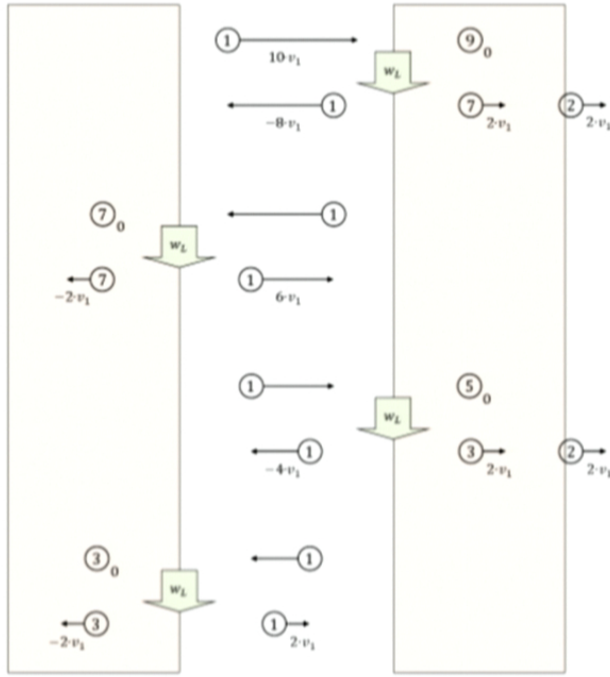


Absorption Action



Absorption Action





Quantification of Absorption Action

$$W_{\text{cal}} : \text{①}_{10\cdot\mathbf{v}},\ 25\cdot\text{①}_{\mathbf{0}} \Rightarrow \text{①}_{\mathbf{0}},\ \underbrace{10\cdot\{\text{①}_{2\cdot\mathbf{v}},\text{①}_{-2\cdot\mathbf{v}}\},\ 5\cdot\text{①}_{2\cdot\mathbf{v}}}_{\text{Calorimeter Extract}}$$

Quantification

Momentum:

$\sim_{\boldsymbol{p}}$ $\quad \mathbf{1}_{\boldsymbol{p}} := ①_{v_1} \quad *$

$ⓐ_{v_a} \sim_{\boldsymbol{p}} \underbrace{①_{v_1} * \cdots * ①_{v_1}}_{\text{Calorimeter Extract}}$

$\boldsymbol{p}\,[\,ⓐ_{v_a}\,] = \boldsymbol{p}_a{}^{(A)} \cdot \boldsymbol{p}\,[\,①_{v_1}\,]$

Quantification

quantified (physical) measure

(basic) physical quantities

$$E_a = E_a^{(A)} \cdot E_{\mathbf{1}(A)} \qquad \boldsymbol{p}_a = \boldsymbol{p}_a^{(A)} \cdot \boldsymbol{p}_{\mathbf{1}(A)}$$

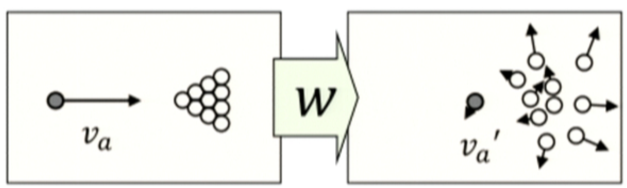
$$w \sim_{E,\boldsymbol{p}} W_{\text{cal}} := w_{\mathbf{1}} * \ldots * w_{\mathbf{1}}$$

observable/ physical measure

$$E_{\textcircled{a}_{v_a}} \qquad \boldsymbol{p}_{\textcircled{a} \cup \textcircled{b}_{v_l}}$$

$$\sim_E \qquad \sim_{\boldsymbol{p}}$$

empirical basis



Physical Principles

Principle of Causality

Principle of Inertia

Impossibility of a Perpetuum Mobile

Principle of Sufficient Reason

Equivalence Principle

Superposition Principle

Methodical Principles

Basic measurement: as doubling of physical measures

Congruence Principle: for reliable quantification

Equipollence Principle: of measuring the cause of potential action by its (kinetic) effect

Fundamental Equations

count equivalent elements in calorimeter Model W_{cal}

$$\#\{S_1|_0\} \qquad \#\{①_{v_1}\} \qquad \#\{①\} \qquad \#\{\boldsymbol{v_1}\}$$

when built in Galilei-Kinematics

$$E[ⓐ_{v_a}] = \left\{\frac{1}{2} \cdot m_a{}^{(A)} \cdot v_a{}^{(A)^2}\right\} \cdot E[\, S_1|_0 \,]$$

$$\boldsymbol{p}[ⓐ_{v_a}] = \left\{m_a{}^{(A)} \cdot \boldsymbol{v}_a{}^{(A)}\right\} \cdot \boldsymbol{p}[①_{v_1}]$$

Fundamental Equations

count equivalent elements in calorimeter Model W_{cal}

$$\#\{S_1|_0\} \qquad \#\{①_{v_1}\} \qquad \#\{①\} \qquad \#\{\boldsymbol{v_1}\}$$

when built in Poincare-Kinematics

$$E[ⓐ_{v_a}] = \{m_a \cdot (\gamma - 1) \cdot c^2\} \cdot E[\,S_1|_0\,]$$

$$\boldsymbol{p}[ⓐ_{v_a}] = \{m_a \cdot \gamma \cdot \boldsymbol{v}_a\} \cdot \boldsymbol{p}[①_{v_1}]$$

Fundamental Equations

count equivalent elements in calorimeter Model W_{cal}

$$\#\{S_1|_0\} \qquad \#\{\textcircled{1}_{v_1}\} \qquad \#\{\textcircled{1}\} \qquad \#\{\boldsymbol{v}_1\}$$

when built in Galilei-Kinematics

$$E[\textcircled{a}_{v_a}] = \left\{\frac{1}{2}\cdot m_a{}^{(A)}\cdot v_a{}^{(A)^2}\right\}\cdot E[\,S_1|_0\,]$$

$$\boldsymbol{p}[\textcircled{a}_{v_a}] = \left\{m_a{}^{(A)}\cdot \boldsymbol{v}_a{}^{(A)}\right\}\cdot \boldsymbol{p}[\textcircled{1}_{v_1}]$$

(tailored) quantitative equations

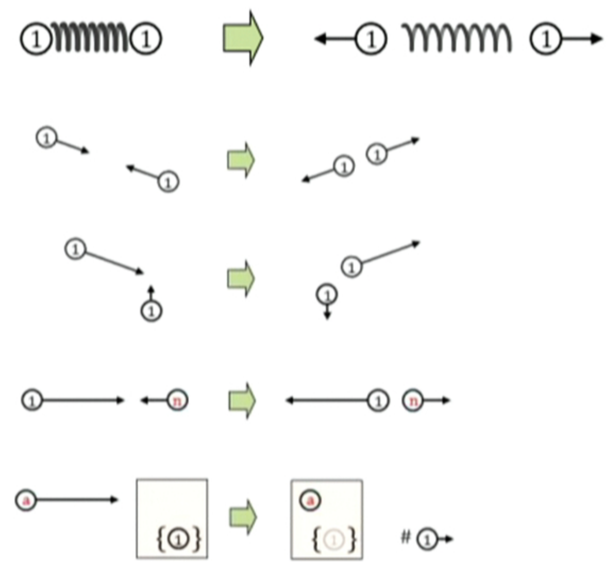
$$E_a{}^{(A)} = \frac{1}{2}\cdot m_a{}^{(A)}\cdot \boldsymbol{v}_a{}^{(A)^2} \qquad \boldsymbol{p}_a{}^{(A)} = m_a{}^{(A)}\cdot \boldsymbol{v}_a{}^{(A)}$$

numerical values in the form

$$E_a{}^{(A)} = \frac{E\,[\textcircled{a}_{v_a}]}{E_{1(A)}} \qquad \boldsymbol{p}_a{}^{(A)} = \frac{\boldsymbol{p}\,[\textcircled{a}_{v_a}]}{\boldsymbol{p}_{1(A)}} \qquad m_a{}^{(A)} = \frac{m\,[\textcircled{a}_{v_a}]}{m_{1(A)}} \qquad \boldsymbol{v}_a{}^{(A)} = \frac{\boldsymbol{v}\,[\textcircled{a}_{v_a}]}{\boldsymbol{v}_{1(A)}}$$

Measurement and Steering *Tool*

elementary standard interaction w_1 :



elastic collision

$w_1^{-1} * w_1$

elastic transversal collision

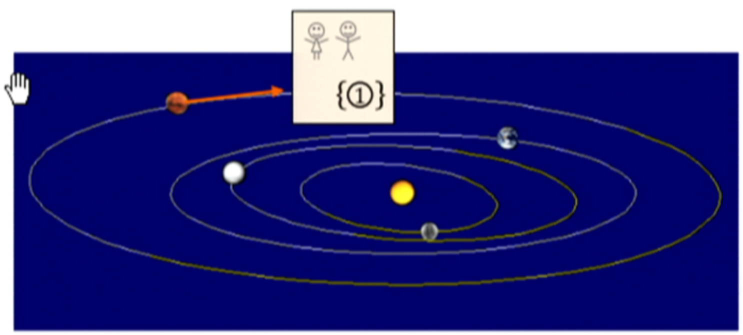
$w_T := (w_1^{-1} * w_1)^{(B)}$

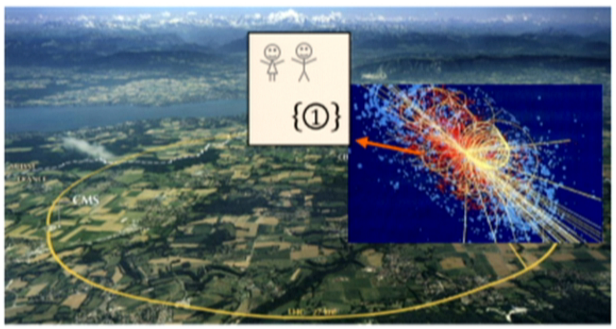
elastic longitudinal collision

$w_L := w_T * \cdots * w_T$

absorption in calorimeter

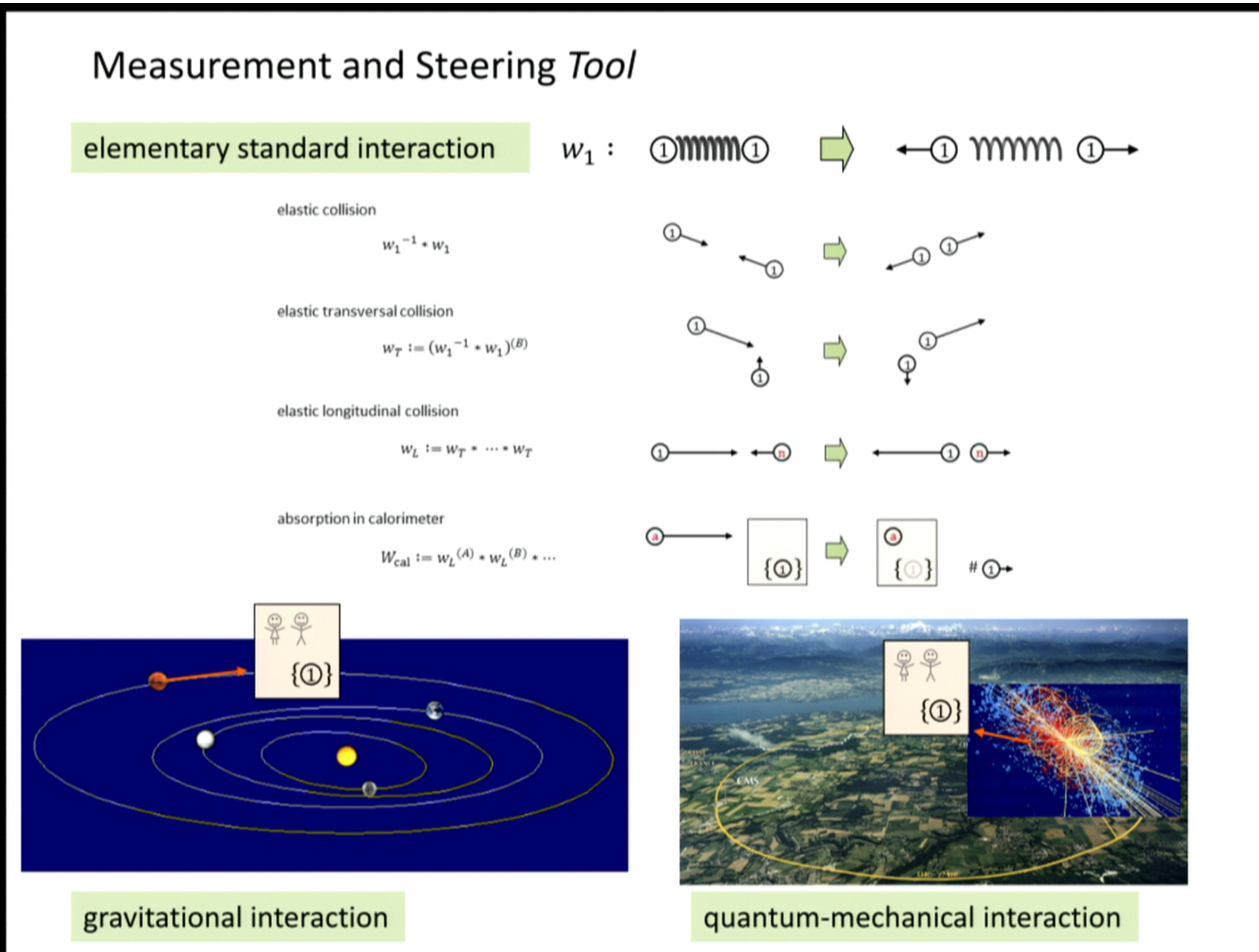
$W_{\text{cal}} := w_L^{(A)} * w_L^{(B)} * \cdots$





gravitational interaction

quantum-mechanical interaction


Measurement and Steering Tool
elementary standard interaction
w_1 :
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elastic transversal collision
$w_T := (w_1^{-1} * w_1)^{(B)}$
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$w_L := w_T * \cdots * w_T$
absorption in calorimeter
$W_{\text{cal}} := w_L^{(A)} * w_L^{(B)} * \cdots$
gravitational interaction
quantum-mechanical interaction

$0 < \delta S_{\mathrm{Ham}}[\gamma]$

Principle of Least Action (external steering effort)

steer Hamilton type variation $\delta\gamma^{(\mathrm{Ham})}$ of free course γ of intrinsic action w

meaningful derived physical quantities & EOM

$V_{\mathrm{pot}}[\boldsymbol{x}_I \to \boldsymbol{x}'_I] := V_{\mathrm{pot}}[\gamma] \big/_{\mathrm{mod}\ \gamma}$ $\quad \boldsymbol{F}_a := \frac{\Delta \boldsymbol{p}_a}{\Delta t_a}\left[w|_{x_I,v_I}\right] \big/_{\mathrm{mod}\ \boldsymbol{v}_I}$ $\quad m_i \cdot \frac{\mathrm{d}^2 \boldsymbol{s}_i}{\mathrm{d}t^2}\left[w|_{x_I,v_I}\right] = -\nabla^{(i)} V_{\mathrm{pot}}$ $\quad \forall\, i \in I$

analyse course of intrinsic action w by external steering action $\mathrm{RB}^{(i)}$

$E_a{}^{(A)} = \frac{1}{2} \cdot m_a{}^{(A)} \cdot \boldsymbol{v}_a{}^{(A)2}$ $\quad \boldsymbol{p}_a{}^{(A)} = m_a{}^{(A)} \cdot \boldsymbol{v}_a{}^{(A)}$

quantitative equations

$\#\{\mathbf{1}_E|_0\}$ $\quad \#\{\mathbf{1}_p\}$ $\quad \#\{①\}$

equivalent elements in calorimeter model W_{cal}

quantified (physical) measure $\quad E_a = E_a{}^{(A)} \cdot E_{\mathbf{1}(A)}$ $\quad \boldsymbol{p}_a = \boldsymbol{p}_a{}^{(A)} \cdot \boldsymbol{p}_{\mathbf{1}(A)}$

(basic) physical quantities

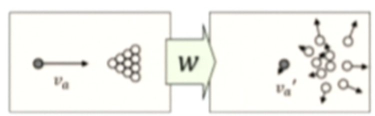
basic physical measurement/ quantification

$w \sim_{E,p} W_{\mathrm{cal}} := w_1 * \ldots * w_1$

observable/ physical measure $\quad E_{ⓐ_{v_a}}$ $\quad \boldsymbol{p}_{ⓐ \cup ⓑ_{v_I}}$

Abstraction: $\sim_E$ $\quad \sim_p$

empirical basis




0 < δS_Ham[γ]
Principle of Least Action (external steering effort)
steer Hamilton type variation δγ^(Ham) of free course γ of intrinsic action w
meaningful derived physical quantities & EOM
V_pot[x_I → x'_I] := V_pot[γ] / mod γ
F_a := Δp_a/Δt_a [w|x_I,v_I] / mod v_I
m_i · d²s_i/dt² [w|x_I,v_I] = −∇^(i) V_pot
∀ i ∈ I
analyse course of intrinsic action w by external steering action RB^(i)
E_a^(A) = 1/2 · m_a^(A) · v_a^(A)2
p_a^(A) = m_a^(A) · v_a^(A)
quantitative equations
#{1_E|0} #{1_p} #{①}
equivalent elements in calorimeter model W_cal
quantified (physical) measure
E_a = E_a^(A) · E_1(A)
p_a = p_a^(A) · p_1(A)
(basic) physical quantities
basic physical measurement/ quantification
w ∼_E,p W_cal := w_1 * … * w_1
observable/ physical measure
E_ⓐ v_a
p_ⓐ∪ⓑ v_I
Abstraction: ∼_E ∼_p
empirical basis
v_a
w
v_a'