Title: Canonical Quantum Gravity and Spin Foams

Date: Jul 26, 2013 02:30 PM

URL: http://pirsa.org/13070086

Abstract:

On the relation between covariant and canonical Quantum Gravity

[arXiv:gr-qc/1307.5885]

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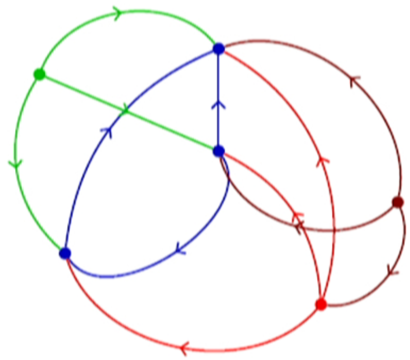
Motivation

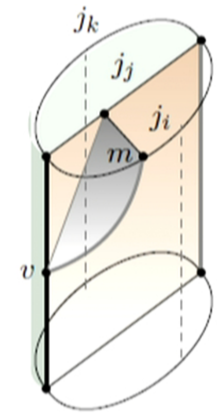
Canonical LQG



$$T_s(A) = \mathrm{Tr}[\prod_l h^j_l(A) \prod_n \iota_n]$$

$$\mathcal{H}_{kin} = \bigoplus_{\gamma \in \Sigma} \mathcal{H}_{kin,\gamma}$$

Covariant LQG



$$Z[\kappa] = \sum_c \prod_v \mathcal{A}_v \prod_f \mathcal{A}_f \times \mathcal{B}$$

$$\mathcal{H}_{\partial\kappa} \simeq \mathcal{H}_{kin,\gamma}$$

Is it possible to merge covariant and canonical LQG?

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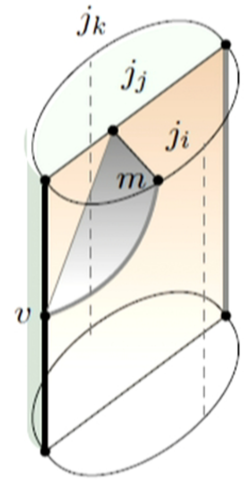
A "rigging" map for LQG



$$\langle T_s | Z[\kappa] | T_{s'} \rangle$$
$$\partial\kappa = \gamma_s \cup \gamma_{s'}$$

[Engle, Pereira, Rovelli, Livine]/[Freidel, Krasnov]
[Kaminski, Kisielowski, Lewandowski]

Heuristic Idea

$$\delta(H) = \int \exp(itH)\,\mathrm{d}t \leftrightsquigarrow \sum_\kappa \underbrace{Z[\kappa]}_{\text{Feynman diagrams}}$$

[Reisenberger, Rovelli]

"Physical Scalar Product"

$$\eta[T_s](T_{s'}) := \sum_{\kappa:\gamma_{s'}\to\gamma_s} \langle T_s | Z[\kappa] | T_{s'} \rangle$$

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Spin foam operator

We need:

$$Z[\kappa] : \mathcal{H}_{kin,\gamma} \to \mathcal{H}_{kin,\gamma'}$$

EPRL-FK Model (Eucl.)

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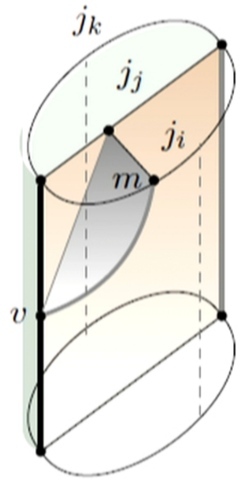

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triang. $\leftrightarrow$ topology $\leftrightarrow$ geometry

But ...

Need n-valent vertices
3-valent v generated by H [Thiemann]

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... then suppose

Abstract foam: $\kappa = \{f, e, v\}$

Boundary graph:
$\gamma = \{(e, v)|e \text{ in only one } f\}$

Technical restriction:
κ p.l.-homeomorphic to convex 2-complex

Induced boundary space: $\mathcal{H}_{\partial\kappa}$

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KKL-Model

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$\mathcal{H}_{\partial\kappa} = \mathcal{H}_{kin,\partial\kappa}$ $\surd$

Barbero-Imirzi Parameter: $\beta \in 2\mathbb{N} + 1$

[Kaminski, Kisielowski, Lewandowski]

Spin foam operator

We almost have:

$$Z[\kappa] : \mathcal{H}_{kin,\gamma} \to \mathcal{H}_{kin,\gamma'}$$

What is missing?

A better Rigging Map

$$\eta[T_s](T_{s'}) = \sum_{[s']_A \in N_A} \eta_{[s]_A,[s']_A} L_{[s']_A} \quad \text{with} \quad \eta_{[s]_A,[s']_A} = \sum_{\kappa_A : s'_A \to s_A} Z[\kappa]$$

$$\text{and} \quad L_{[s']_A} = \eta_{[s']_A} \sum_{\hat{s} \in [s']_A} \langle T_{\hat{s}}, \cdot \rangle$$

$[s]_A$ abstract equivalence class, in the following: $\eta_{[s]_A} = 1$

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Properties of $Z[\kappa]$

$$\eta_{[s]_A,[s']_A}(\kappa) := \langle T_s | Z[\kappa] | T_{s'} \rangle =$$

$$\sum_{j_f,\iota_e} \prod_f \mathcal{A}_f \prod_{e \in \kappa_{int}} Q_e \prod_{v \in \kappa_{int}} \mathcal{A}_v(j_f, \iota_e) \prod_{l \in \partial\kappa} \delta_{j_l, j_{f_l}} \prod_{n \in \partial\kappa} \delta_{\iota_n, \iota_{e_n}}$$

[Kaminski, Kisielowski, Lewandowski], [Ding, Han, Rovelli],[Bahr, Hellmann, Kaminski, Kisielowski, Lewandowski]

Can be defined s.t. ...

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Can be defined s.t. . . .

- The map $Z[\kappa]$ is cylindrically consistent
- $\forall\, \gamma\ \exists\ \kappa^0_\gamma$ s.t. $Z[\kappa^0_\gamma] : \mathcal{H}_{kin,\gamma} \to \mathcal{H}_{kin,\gamma}$ with $Z[\kappa^0_\gamma] = \mathbb{1}_\gamma$

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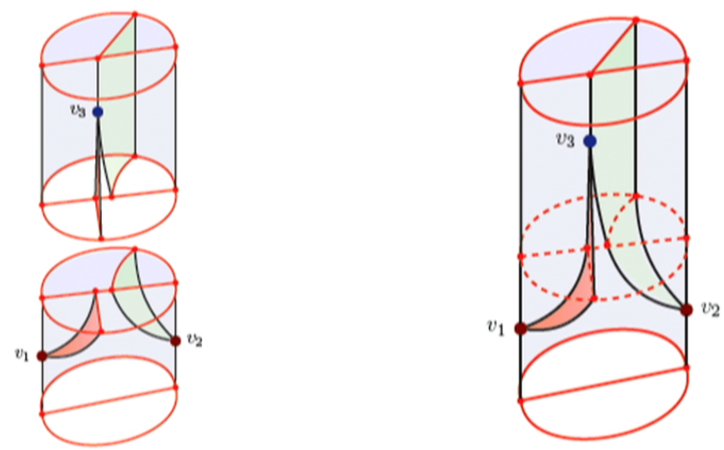
Gluing



Suppose $\kappa_1 \cap \kappa_2 = \partial\kappa_1 \cap \partial\kappa_2 = \tilde{\gamma}$ then

$$\sum_{\tilde{s}(\tilde{\gamma})} \langle T_s | Z[\kappa_1] | T_{\tilde{s}} \rangle \langle T_{\tilde{s}} | Z[\kappa_2] | T_{s'} \rangle = \langle T_s | Z[\kappa_1 \sharp \kappa_2] | T_{s'} \rangle$$

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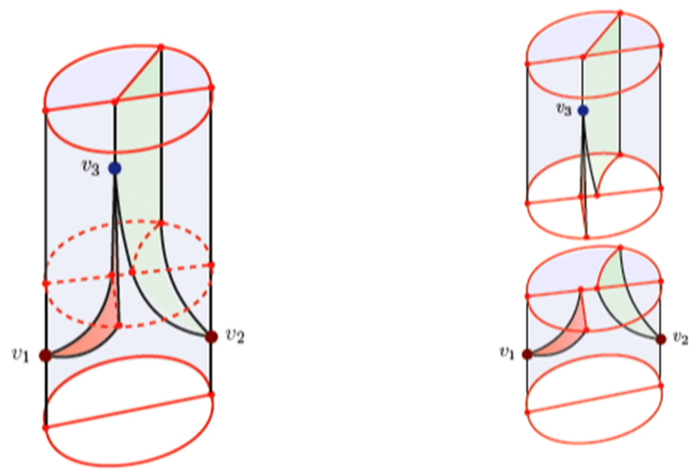
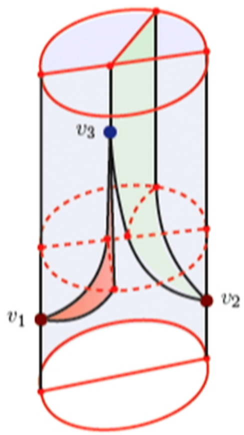
Spliting



Idea: Use this to split big complexes to get a better control on

$$\eta[T_s](T_{s'}) := \sum_{\kappa:\gamma_{s'}\to\gamma_s} \langle T_s|Z[\kappa]|T_{s'}\rangle$$

Time ordering



Definition

- $v \in \kappa_{int}$ s.t. $\exists n \in \gamma_i$ and $\exists e \in \kappa_{int}$ with $s(e) = n$ and $t(e) = v$
  $\rightsquigarrow$ vertex of first generation
- Inductively: Vertex of nth generation
- Only final graph $\Rightarrow$ count backwards
- $\partial\kappa = \emptyset$ all $v \in \kappa$ of first generation

Theorem

$(\kappa, \{j_f\}, \{Q_e\})$ can be uniquely split into $(\kappa_i\{, j_{f_i}\}, \{Q_{e_i}\})$ containing only vertices of ith generation with respect to the original foam

$$\kappa = \kappa_1 \sharp \cdots \sharp \kappa_n$$ *where n is the maximal generation of κ.*

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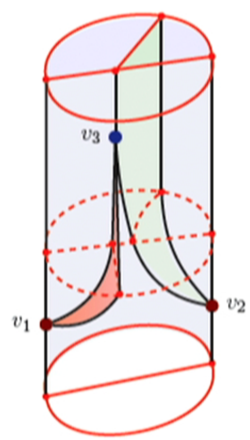

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Factorization of the Rigging map

The Rigging map

$$\eta[T_{s_f}](T_{s_i}) = \sum_{\kappa \in K_{\gamma(s_i),\gamma(s_f)}} \langle T_{s_f},\ Z(\kappa)\ T_{s_i} \rangle$$

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$$\eta[T_{s_f}](T_{s_i}) = \sum_{\kappa \in K_{\gamma(s_i),\gamma(s_f)}} \langle T_{s_f},\ Z(\kappa_1 \sharp \cdots \sharp \kappa_N)\ T_{s_i} \rangle$$

Spin foam transfer matrix

$\hat{K}_{\gamma,\gamma'}$ set of "one time step" foam and $P_\gamma : \mathcal{H}_{kin} \to \mathcal{H}_{kin,\gamma}$

$$\hat{Z} := \sum_{\gamma,\gamma'} P_{\gamma'} \,[\sum_{\hat{\kappa} \in \hat{K}_{\gamma,\gamma'}} Z(\hat{\kappa})]\, P_\gamma$$

Note: $\hat{Z}$ is symmetric

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$$\eta[T_{s_f}](T_{s_i}) = \sum_{N=0}^{\infty} \langle T_{s_f}, \hat{Z}^N T_{s_i} \rangle$$

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Is η a Rigging map?

Issue I: η and $\hat{Z}$ very likely diverging

Cut-Off: $\hat{K}_{\gamma,\gamma'}$ finite

Weight: ω s.t. $\omega(\kappa\sharp\kappa') = \omega(\kappa)\omega(\kappa')$:

$$||\hat{Z}' T_s||^2 = \sum_{\gamma'} \sum_{j',\iota'} | \sum_{\hat{\kappa}\in\hat{K}_{\gamma,\gamma'}} w(\hat{\kappa}) \langle T_{\gamma',j',\iota'}, Z(\hat{\kappa}) T_{\gamma,j,\iota}\rangle|^2 < \infty$$

Issue II: Not a projector

$$A := \sum_{N=0}^{\infty} Z^N \Rightarrow A = \mathbb{1} + ZA$$

Formal identity; but holds rigorously on suitable subspace if exist semi analytic extension

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What is going wrong?

Option 0: Z itself a projector, e.g. BF-theory
[Dittrich, Hellmann, Kaminski],[Alesci, Thiemann, A.Z.]

Option I: Wrong assumption on the weight!

Option II: Vertex amplitude too local

Option III: Necessity to restrict to one Plebanski sector?

What is going wrong?

Option III: Necessity to restrict to one Plebanski sector?

$$2\pi\delta(C) = \lim_{T\to\infty} \int_{-T}^{T} e^{itC} = \lim_{T\to\infty} \int_{0}^{T} [e^{itC} + e^{-itC}]$$

$$= \lim_{T\to\infty} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{T}{n} [\{e^{iCT/n}]^k + [e^{-iCT/n}]^k\}$$

But: $\hat{Z}$ is symmetric $\rightsquigarrow$ $\hat{Z} = U + U^\dagger$

$$\Rightarrow \sum_{k\in\mathbb{N}} Z^k = \sum_{k\in\mathbb{N}} (U + U^\dagger)^k \neq \delta(H)$$

Solution: Proper vertex [Engle]?

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Option III: Necessity to restrict to one Plebanski sector?

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Summary

Technical Aspects

- Single time step $\rightsquigarrow$ better control on the sum
- Formalism to design single time step foams;
  [Kisielowski, Lewandowski, Puchta]

Regularization

- η very likely divergent (infinite sum)
- Regularize $\hat{Z}$
- Connection to GFT, coarse graining?

Is η a Rigging map? $\rightarrow$ No!

Possible Causes: Weight? Too local amplitude? Plebanski sectors?

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Thank you for your attention!

Contents

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Melonic phase transition in group field theory

Aristide Baratin

Max-Planck-Institut
für Gravitationsphysik
(Albert-Einstein-Institut)

Loops' 13 – Perimeter Institute

based on arXiv:1307.5026
with S. Carrozza, D. Oriti, J. P. Ryan, M. Smerlark.

Context

Group field theory formulation of spin foam models:

Boulatov, Ooguri (1993), Di-Pietri, Freidel, Krasnov, Rovelli (1999)

spin foams $\Longleftrightarrow$ Feynman diagrams

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spin foams $\Longleftrightarrow$ Feynman diagrams

Possibility to define the continuum limit in a way analogous to matrix models

Key analytical tool: 1/N expansion [Gurau '11]

- Intensely exploited for tensor models:
  see talks by Rivasseau, Gurau, Bonzom, Ryan, Dartois
  - Critical behaviour at leading order giving continuum polymer phase
  - Progresses towards the definition of double scaling limits

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Our results

arXiv:1307.5026

Context: Boulatov-Ooguri models

1. Combinatorial formula for the (melonic) amplitudes
   in terms of a two-dimensional analogue of the Symanzik graph polynomials

2. Bounds on the amplitudes and existence of a critical point

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Colored Boulatov-Ooguri models

Boulatov, Ooguri '93, Gurau '10

- Variable: collection of $D+1$ fields on D-copies of a Lie group G:

$$\varphi_\ell : G^{\times D} \to \mathbb{C}$$

with shift invariance:

$$\varphi_\ell(hg_1, \cdots hg_D) = \varphi_\ell(g_1, \cdots g_D) \quad \forall h \in G.$$

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$$\varphi_\ell(hg_1, \cdots hg_D) = \varphi_\ell(g_1, \cdots g_D) \quad \forall h \in G.$$

- Interaction with a simplicial combinatorial structure:

$$S_{\text{int}} = \lambda \int \prod_{i<j} \mathrm{d}g_{ij} \prod_{\ell=0}^{D} \varphi_\ell(g_\ell) + \text{c.c.}$$

and "trivial" kinetic term $\sum_\ell |\varphi_\ell|^2$.

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Colored Boulatov-Ooguri models

Boulatov, Ooguri '93, Gurau '10

- Variable: collection of $D+1$ fields on D-copies of a Lie group G:

$$\varphi_\ell \,:\, G^{\times D} \to \mathbb{C}$$

with shift invariance:

$$\varphi_\ell(hg_1, \cdots hg_D) = \varphi_\ell(g_1, \cdots g_D) \quad \forall h \in G.$$

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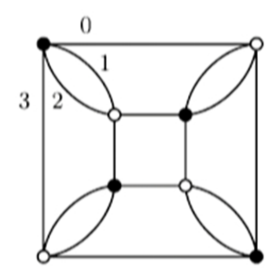
- Feynman expansion:



Regularization

Heat kernel regularization:

$$\delta(g) \longrightarrow K_\tau(g) \qquad \begin{aligned} (\partial_\tau - \Delta)K_\tau &= 0 \\ \lim_{\tau \longrightarrow 0} K_\tau(g) &= \delta(g) \end{aligned}$$

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Feynman expansion of the free energy

$$F_{\tau,\lambda\overline{\lambda}} = \sum_{\mathcal{G}} \frac{(\lambda\overline{\lambda})^p}{\mathrm{SYM}(\mathcal{G})} A_\tau(\mathcal{G})$$

with graph amplitudes:

$$A_\tau(\mathcal{G}) \propto \int \prod_{e \in \mathcal{E}\setminus\mathcal{T}} \mathrm{d}h_e \prod_{f \in \mathcal{F}} K_{m_f \tau}\left(\overrightarrow{\prod_{e \in \partial f}} h_e{}^{\epsilon_{fe}} \right)$$

$\epsilon_{fe} = \pm 1$ the face-edge adjacency matrix
m_f = number edges of the face f.

The limit $\tau \longrightarrow 0$

Gurau 1/N-expansion result: upon rescaling the coupling constant

$$\lambda \quad \longrightarrow \quad \lambda / N_{\tau}^{(\dim G)\frac{(D-2)(D-1)}{4}} \qquad N_{\tau} = (4\pi\tau)^{-\frac{1}{2}}$$

the expansion of the free energy can be organized as:

$$F_{\tau,\lambda\overline{\lambda}} \quad = \quad N_{\tau}^{(\dim G)(D-1)}\, F_{\lambda\overline{\lambda}}^{(0)} \quad + \quad \mathcal{O}(N_{\tau}^{\dim G})$$

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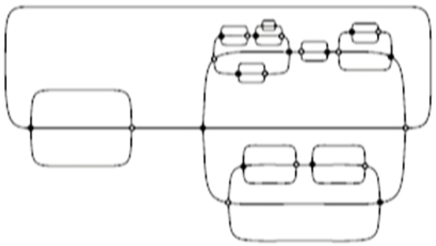
The dominant contribution comes from melonic graphs:

- maximizes the number of faces
- dual to triangulated spheres



Amplitudes and Laplacian matrix

As a result: $F^{(0)}_{\lambda\bar{\lambda}} = \sum_{p\in\mathbb{N}} \frac{(\lambda\bar{\lambda})^p}{p} \sum_{\mathcal{G}\in M_p} a(\mathcal{G})$

where:

$$a(\mathcal{G}) = \lim_{\tau\to 0} N_\tau^{-(\dim G)(D-1)} A_\tau(\mathcal{G}) < +\infty \, .$$

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Melonic amplitudes can be evaluated in terms of a Laplacian matrix:

$$a(\mathcal{G}) = \left[\det(\widetilde{L}) \prod_f m_f\right]^{-\frac{\dim G}{2}}$$

$$L_{e,e'} = \sum_f \frac{1}{m_f} (\partial_2)_{ef} (\partial_2)^{\intercal}_{fe'}$$

$\widetilde{L}$ = submatrix of L with rows and columns indexed by $\mathcal{E}\backslash\mathcal{T}$.

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A tree-matrix theorem

1st Result: *The melonic amplitudes* $a(\mathcal{G})$ *may be expressed as weighted sums over their spanning 2–trees:*

$$a(\mathcal{G}) = \left[\sum_{\mathrm{T} \in \mathcal{T}_2(\mathcal{G})} |H_1(\mathrm{T}, \mathbb{Z})|^2 \prod_{f \notin \mathrm{T}} m_f \right]^{-\frac{\dim G}{2}} .$$

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- Homological definition: 2-subcomplexes $\mathrm{T} \subset \mathcal{G}$ containing all edges and all vertices of $\mathcal{G}$ and such that: Adin, 92; Duval, Klivans, Martin '09; Petersson '09

$$H_2(\mathrm{T},\mathbb{Z}) = 0 \quad \text{and} \quad |H_1(\mathrm{T},\mathbb{Z})| < \infty$$

- T has enough faces to avoid "holes", but not too many, so that they do not form higher dimensional cycles.

The proof relies on Cauchy-Binet formula and homological computations.

The $D = 3$ case

Two simplifying facts:

1. 1-1 correspondence between spanning 2-trees of $\mathcal{G}$ and spanning trees of the dual triangulation $\Delta_{\mathcal{G}}$
2. Trivial homological factor $H_1(\mathrm{T}, \mathbb{Z}) = 0$ for 2-trees in 3D melons.

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Melonic amplitudes = Kirchhoff polynomials:

$$a_{3\mathrm{D}}(\mathcal{G}) = \left[\sum_{\mathcal{T}^* \in \mathcal{T}_1(\Delta_{\mathcal{G}})} \prod_{l \in \mathcal{T}^*} m_{f(l)} \right]^{-\frac{3}{2}}$$

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Critical behaviour

Bounds on $a(\mathcal{G}_p)$

Calling $\mathcal{N}(\mathcal{G}_p)$ the number of 2–trees in $\mathcal{G}_p$, one has:

$$(k_1 {c_1}^p \mathcal{N}(\mathcal{G}_p))^{\frac{-D}{2}} \leq a(\mathcal{G}_p) \leq \mathcal{N}(\mathcal{G}_p)^{-\frac{D}{2}}$$

for some $k_1 > 0$ and $c_1 > 0$.

Bounds on the series

The melonic contribution to the free energy satisfies the bounds:

$$k \sum_{p \in \mathbb{N}} F_p c^p (\lambda\bar{\lambda})^p \leq F^{(0)}_{\lambda\bar{\lambda}} \leq \sum_{p \in \mathbb{N}} F_p (\lambda\bar{\lambda})^p \,.$$

for some $k > 0$ and $c > 0$, where $F_p = \frac{1}{(D+1)p+1}\binom{(D+1)p+1}{p}$ is an exact counting of the number of melonic graphs with $2p$ vertices.

Critical behaviour

2nd result: *At leading order in the $1/N$–expansion, the free energy of topological group field theories possesses critical behaviour.*

Outlook

- To analyze the properties of this graph polynomial: new computational tool for GFT amplitudes   Toy models: abelian group field theories
- To extend the analysis to quantum gavity models
- To go beyond the melonic sector

Null Twisted Geometry

Mingyi ZHANG

Centre de Physique Théorique

LOOPS 13, Perimeter Institute

Based on work with Simone SPEZIALE

July 26, 2013

Motivation I

Loop gravity and twisted geometry

Loop gravity with space-like hypersurface boundary gives a very clear discrete picture of space-time

- **Canonical loop quantum gravity**: spin-network states correspond to twisted geometry (polyhedra without shape matching) L.Freidel, S.Speziale (2010) PRD82
- **Spinfoam gravity**: at large-j limit the simplicial spinfoam gravity becomes quantum Regge gravity (Regge action emerges from spinfoam amplitude) F.Conrady, L.Freidel (2008) PRD78, J.Barrett, R.Dowdall, W.Fairbairn, F.Hellmann, R.Pereira (2010) CQG27, M.Han, M.Z (2011) CQG30

Motivation II

How about null(light-like) hypersurface boundary?

It is important and interesting both in physics and mathematics

- Physically we cannot avoid considering the null hypersurface boundary. Null hypersurface plays an important role in General Relativity, for instance in black hole physics. The black hole horizon is exactly a null hypersurface.
- It is well known that the initial data problem of General Relativity on piecewise null hypersurfaces is simpler than the space-like case. (Initial data on null hypersurface is constraint free) Penrose, Bondi, Newman, Sachs, *et al.* 1960s

Can we use it in loop quantum gravity? It is an open question.

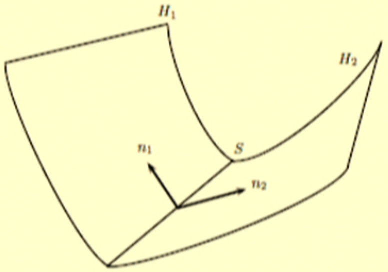
- The canonical analysis on a null hypersurface is missing works by M.Reisenberger (2008) PRL101 define a Poisson bracket on null initial data
- The "time evolution" problem with null initial data is still not exactly understood.

Outline

We use the recent developed tool (twistorial parametrization) for loop gravity to explore a possible and natural path in the direction of understanding null hypersurface boundary.

Main results

- The linear simplicity constraints with the normal n^I to be null, $n \cdot n = 0$. They are all first class .
- The reduced phase space from the twistorial phase space of loop gravity has a 2-D geometrical interpretation.



2-D point-wise conical structure .

- Quantization and kinematic Hilbert space: the system can be quantized and the result are U(1) spin-networks.

Constraints are simpler when $n \cdot n = 0$

Simplicity constraints and closure constraints

From BF-theory to General Relativity, **the linear simplicity constraint** is needed.

$$B^{IJ}n_J = 0 \tag{1}$$

B is an anti-symmetric Lie-algebra-valued 2-form. Simplicity constraints guarantee $B = e \wedge e$ simple

- In the situation n is null. Pick a fixed gauge $n^I = (1, 0, 0, 1)$, simplicity constraints still make B simple.

The closure constraints are the discrete version of Gauss law in loop gravity.

$$\sum_{l \in n} B_l = 0 \tag{2}$$

l are the links attached to node n. B_l is the 2-D smearing of the B field.

- The Minkowski theorem is still true. The closure constraints guarantee the existence of the null polyhedra. the degrees of freedom of a l-face null polyhedron is $3l - 7$. $3l$ d.o.f. of $A_l b_l$ module 3 closure constraints, 3 null rotation and 1 rescaling transformation on the null direction. d.o.f $= 3l - 4 - 3 = 3l - 7$. Exactly the same d.o.f.

Constraints are simpler

- With non-zero Immirzi parameter γ, the linear simplicity constraints become

$$K_3 + \gamma L_3 = 0, \quad P_1 = P_2 = 0 \tag{3}$$

$P_1 = L_1 - K_2$ and $P_2 = L_2 + K_1$ are the translation generators of the null rotation that keeps n invariant.

- Define $\star B_l \equiv A_l b_l \wedge n$, then the closure constraints become

$$\sum_{l \in n} A_l b_l \propto n \quad \Rightarrow \quad \sum_{l \in n} \varepsilon_l A_l = 0 \tag{4}$$

Bs are space-like and are embedded in null hypersurface. b is a null vector with $n \cdot b = \varepsilon = \pm 1$. The area closure constraint is the result of the degeneracy of the null hypersurface.

- The constraints are all first class constraints!

Twistorial parametrization

The twistorial parametrization of the holonomy-flux phase space of loop gravity $(h, \Pi) \in T^*\mathrm{SL}(2,\mathbb{C})$ is well studied. M.Dupuis, L.Freidel, E.Livine, S.Speziale, J.Tambornino, W.Wieland, *et al.* (2008-now)

$$\Pi^{AB} = \frac{1}{2}\omega^{(A}\pi^{B)}, \quad h^A{}_B = \frac{\tilde{\omega}^A\omega_B + \tilde{\pi}^A\pi_B}{\sqrt{\tilde{\omega}_C\tilde{\pi}^C}\sqrt{\omega_D\pi^D}}, \quad \{\pi_A, \omega^B\} = \delta_A^B \tag{5}$$

area matching condition $C = \pi_A\omega^A - \tilde{\pi}_A\tilde{\omega}^A = 0$ is needed to get $T^*\mathrm{SL}(2,\mathbb{C})$

- In terms of spinors the simplicity constraints read

$$F_1 = \mathrm{Re}(\pi_A\omega^A) + \gamma\mathrm{Im}(\pi_A\omega^A) = 0, \quad F_2 = \delta^o_{A\dot{A}}\omega^A\bar{\pi}^{\dot{A}} = \omega^1\bar{\pi}^1 = 0, \tag{6}$$

F_2 is new and is first class, $\{F_2, \bar{F}_2\} = 0$. $\delta^o_{A\dot{A}} = o_A\bar{o}_{\dot{A}}$ is the spinorial version of null normal n.

- They can be packaged as the incidence relation:

$$\pi^A = -\frac{1}{r}e^{i\frac{\theta}{2}}\delta^{oA\dot{A}}\bar{\omega}_{\dot{A}} \tag{7}$$

since $\delta^{oA\dot{A}}$ is not invertible, the solutions actually contain the gauge d.o.f. generated by F_2.

Geometric Interpretation of the Symplectic Reduced Phase Space

Symplectic reduction

Consider the phase space of one link $(\omega, \pi, \tilde{\omega}, \tilde{\pi})$. The reduced symplectic potential is defined by the pull-back of the embedding map $\Psi : \mathbb{C}^2 \to \mathbb{T}^2$.

$$\Psi^*\Theta = \Psi^*\left(\pi_A \mathrm{d}\omega^A + \tilde{\pi}_A \mathrm{d}\tilde{\omega}^A + cc.\right) = \frac{\mathrm{i}}{2}\left(z\mathrm{d}\bar{z} + \tilde{z}\mathrm{d}\bar{\tilde{z}} - cc.\right) \tag{8}$$

Ψ embeds one orbit of simplicity constraint surface to twistor space. If we also impose the area matching condition and module its orbits, then the symplectic potential actually just

$$\Theta_{\mathbb{C}} = J\mathrm{d}\varphi \tag{9}$$

$J \equiv z\bar{z}$ and $\varphi \equiv \arg z + \arg \tilde{z}$. Both of them are gauge invariant variables. We can directly read the Poisson bracket $\{J, \varphi\} = 1$.

$$\begin{array}{ccccc}
\mathbb{T}\times\mathbb{T} & \xrightarrow{F_1,\ \tilde{F}_1,\ \text{orbits}} & \mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2 & \xrightarrow{F_2,\ \tilde{F}_2,\ \text{orbits}} & \mathbb{C}\times\mathbb{C}(z\otimes\tilde{z}) \\
\downarrow{\scriptstyle C,\ \bar{C}\ \text{orbit}} & & \downarrow{\scriptstyle C_{\text{red}},\ \text{orbit}} & & \downarrow{\scriptstyle C_{\text{red}},\ \text{orbit}} \\
T^*\mathrm{SL}(2,\mathbb{C}) & \xrightarrow{F_{1,\text{red}},\ \text{orbit}} & \mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C} & \xrightarrow{F_2,\ \tilde{F}_2,\ \text{orbits}} & \mathbb{C}(J,\varphi)
\end{array}$$

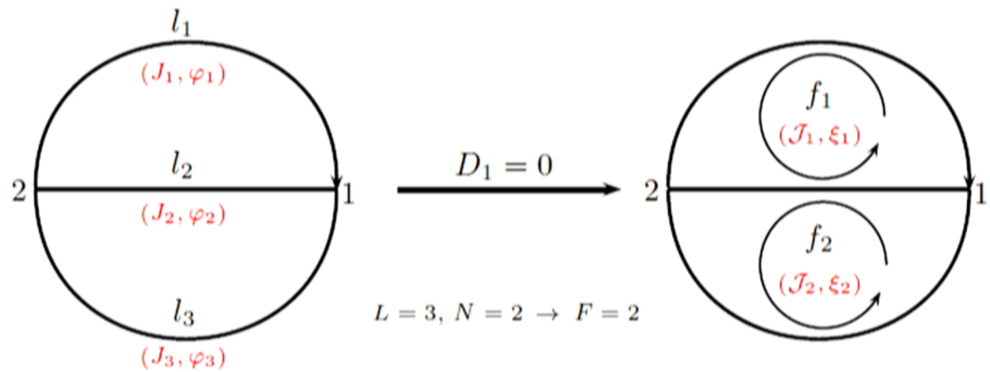
Geometric interpretation I

- We have simplicity constraints, closure constraints and area matching conditions. They are all first class. On the constraint surface, there is a way to construct null polyhedra thanks to the Minkowski theorem.
- The reduced phase space is a much smaller space since all constraints are first class. The geometry is encoding in the independent loops of a graph. given a certain closed graph Γ with L links N nodes and F dual faces (loops), the d.o.f. of reduced phase space is $2(L-N+1)=2(F-\chi+1)$, χ is the Euler characteristic of Γ.
- The gauge-invariance based on loops are

$$\mathcal{J}_f = \frac{1}{N}\sum_{i=1}^{N}\sigma_i J_i, \quad \xi_f = \sum_{i=1}^{N}\sigma_i\varphi_i, \quad \{\mathcal{J}_f, \xi_{f'}\} = \delta_{ff'} \qquad (10)$$

$\sigma_i = 1$ when the loop orientation and link orientation are consistent, otherwise -1. J_i and φ_i are already parallel transformed to the center of the loops.

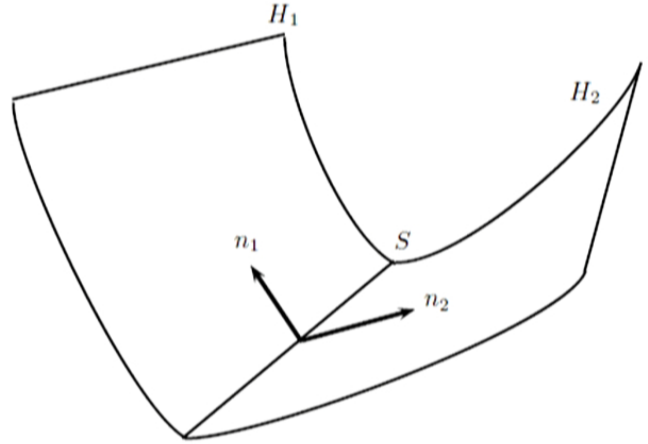
Geometric interpretation II



A graph with 3 links and 2 nodes, which is isomorphic to a disk ($\chi = 1$), has only 2 independent dual faces. Each face is parametrized by two gauge-invariant variables $(\mathcal{J}_f, \xi_f)$.

Geometric interpretation III

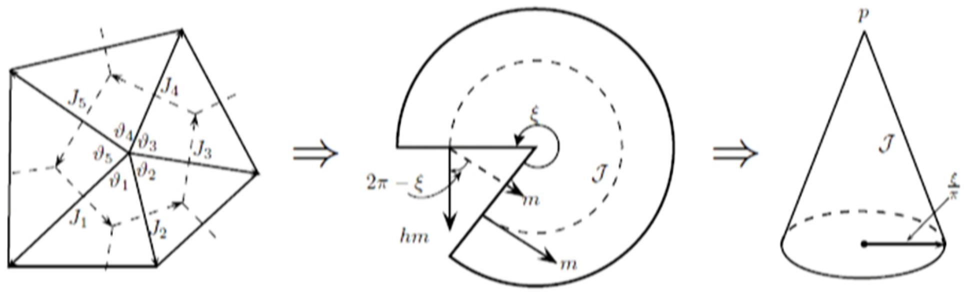
The flux $\Pi \propto J\sigma^3$ on the reduced phase space implies that the bivector B is not only orthogonal to $n_1 = n$ but also $n_2 = \mathcal{P}n$. It means the reduced phase space is actually gives a 2-D plane which is the intersect of two null hypersurface with normals are n_1 and n_2 respectively.

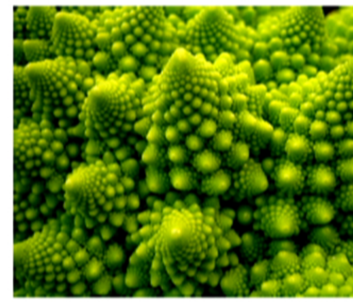


it can also be understood as that from the reduced phase space the two intersecting null hypersurfaces cannot really be distinguished. If we use F_1 and F_2 to change the orbit on the constraint surface, we will recover the geometry of null polyhedrons.

Geometric interpretation IV

- The geometry of the 2-plane is a discrete point-wise manifold in which for each point dual to loop f there is a cone with the peak angle ξ_f and scale $\mathcal{J}_f$. We call it conical singular structure.





- ξ_f encodes the intrinsic curvature of the 2-plane. There is a natural point-wise metric equipped with this conical manifold. $\mathrm{d}s^2 = |z|^{(\frac{\xi}{\pi}-2)}|\mathrm{d}z|^2$. J.Kazdan, F.Warner (1974) AM99.

Quantization

Schrödinger quantization

- Canonical Poisson algebra

$$[\hat{\pi}_A, \hat{\omega}^B] = -\mathrm{i}\hbar\delta_A^B \quad \Rightarrow \quad \hat{\pi}_A = -\mathrm{i}\hbar\frac{\partial}{\partial\omega^A} \tag{11}$$

- Simplicity constraints

$$\hat{F}_1 = \frac{\hbar}{\gamma^2+1}\left((\gamma+\mathrm{i})\bar{\omega}^{\dot{A}}\frac{\partial}{\partial\bar{\omega}^{\dot{A}}} - (\gamma-\mathrm{i})\omega^A\frac{\partial}{\partial\omega^A} + 2\mathrm{i}\right), \quad \hat{F}_2 = -\mathrm{i}\hbar\frac{\partial}{\partial\omega^0} \tag{12}$$

where we use the normal order

$: \hat{\pi}_A\hat{\omega}^A :\equiv (\hat{\pi}_A\hat{\omega}^A + \hat{\omega}^A\hat{\pi}_A)/2 = -\mathrm{i}\hbar\left(\omega^A\partial_{\omega^A} + 1\right)/2.$

- Area operator

$$\hat{A}^2 \equiv \frac{-\gamma^2\hbar^2}{2(\gamma^2+1)^2}\left[(\gamma-\mathrm{i})^2\left(\omega^A\frac{\partial}{\partial\omega^A}+1\right)^2 + (\gamma+\mathrm{i})^2\left(\bar{\omega}^{\dot{A}}\frac{\partial}{\partial\bar{\omega}^{\dot{A}}}+1\right)^2\right] \tag{13}$$

Kinematic Hilbert space and spin-network state I

1. The Hilbert space $H^{(k,\rho)}$ of homogeneous function realizes a unitary irreducible representation of $\mathrm{SL}(2,\mathbb{C})$. The basis in the Hilbert space is ISO(2) basis.

$$f_p^{(k,\rho)}(\omega^A) = \frac{1}{2\pi}(\omega^1)^{k-1+\mathrm{i}\rho}(\bar{\omega}^{\dot{1}})^{-k-1+\mathrm{i}\rho}\exp\left[\frac{\mathrm{i}}{2}\left(\frac{\bar{\omega}^{\dot{0}}}{\bar{\omega}^{\dot{1}}}p + \frac{\omega^0}{\omega^1}\bar{p}\right)\right] \quad (14)$$

the basis is labelled by the eigenvalues of P_1 and P_2. $p = -p_2 + \mathrm{i}p_1$

2. Imposing simplicity constraints strongly on these states. $H^{(k,\rho)}$ becomes $H^{(k)} \equiv H^{(k,-\gamma k)}$. Only one state survives.

$$\hat{F}_1 f_p^{(k,\rho)} = 0 \to \gamma k = -\rho, \quad \hat{F}_2 f_p^{(k,\rho)} = 0 \to p = 0 \quad (15)$$

$$f_k(\omega^A) \equiv f_0^{(k,-\gamma k)}(\omega^A) = \frac{1}{2\pi}(\omega^1)^{-1+(1-\mathrm{i}\gamma)k}(\bar{\omega}^{\dot{1}})^{-1-(1+\mathrm{i}\gamma)k} \quad (16)$$

3. Hilbert space $H \equiv \bigoplus_k H^{(k)}$ is the space of the U(1) unitary irreducible representation with eigenvalue $2k$. Coincide with the fact that one particle state of massless particle is only labelled by helicity. Weinberg's QFT book

Kinematic Hilbert space and spin-network state II

Following the Dirac procedure, we should solve all the constraints on the Hilbert space $\bigotimes_{l\in\Gamma}[H_l^{(k)} \otimes H_l^{(\tilde{k})}]$ for a given graph Γ

1. Area matching condition gives

$$k_l = -\tilde{k}_l \qquad (17)$$

2. Closure constraints on the reduced phase space just the area closure constraint

$$K = \sum_{l \text{ incoming} \in n} k_l - \sum_{l \text{ outgoing} \in n} k_l = 0 \qquad (18)$$

   This restriction on the Hilbert space picks out the U(1) gauge-invariant part of the Hilbert space, U(1) intertwiner: δ_{K0}. The Hilbert space on node n becomes

$$H_n \equiv \text{Inv}\left(\bigotimes_{l \text{ outgoing}} H^{(k_l)} \otimes \bigotimes_{l \text{ ingoing}} \tilde{H}^{(k_l)}\right) \qquad (19)$$

Kinematic Hilbert space and spin-network state III

- The Hilbert space H_Γ of a closed connected oriented graph Γ

$$H_\Gamma \equiv \left[\bigoplus_{k_l} \bigotimes_{l}^{L} \left(H^{(k_l)} \times \tilde{H}^{(k_l)}\right)\right] / \mathrm{U}(1)^{N-1} = \left(\bigoplus_{k_l} \bigotimes_{n}^{N} H_n\right) / \mathrm{U}(1)^{N-1} \tag{20}$$

- Spin-network state

$$|S\rangle \equiv \sum_{k_1, \cdots, k_{N-1}} \prod_{n=1}^{N} \delta_{K_n 0} |\Gamma, k_l\rangle \tag{21}$$

As a function of spinors, the spin-network state is

$$\Psi_S(\omega, \tilde{\omega}) = \langle \omega\tilde{\omega}|S\rangle \equiv \left(\bigotimes_{n}^{N-1} i_{K_n}\right) \cdot \left[\bigotimes_{l}^{L} \left(f_{k_l}(\omega^A) \otimes \tilde{f}_{k_l}(\tilde{\omega}^A)\right)\right] \tag{22}$$

- Area expectation value

$$\hat{A}_l^2 \Psi_S(\omega, \tilde{\omega}) = \gamma^2 \hbar^2 k_l^2 \Psi_S(\omega, \tilde{\omega}) \tag{23}$$

Conclusions

Definite answers

1. The constraint algebra is much simpler: they are all first class, especially the simplicity constraints, in the quantum level, they should all be imposed strongly.
2. At the same time, the reduced phase space is quite small: each link is only parametrized by two real gauge-invariances (J, φ), in the quantum theory the state ends up with U(1) spin-network state.
3. The geometric interpretation of the reduced phase space coincides with the double null hypersurface formalism: the phase space corresponds to a 2-D conical singular structure, which can be interpreted as the intersection of two null hypersurfaces.

Outlooks

Open questions

- Can we give the dynamics with this formulation? There are several attempts.
  - Boost Hamiltonian: change the peak angle ξ_f of the cone while keep $\mathcal{J}_f$ invariant. But if we sum all boost Hamiltonians of all nodes in the same weight, it seems that the reduced phase space is invariant.
  - Spinfoam: 4-simplex doesn't work if we want the boundary to be null. A 4-simplex can have at most 4 null tetrahedra. Somehow we should jump out of thinking 4-simplices and try a new boundary.
- The application on black hole. (entropy, horizon dynamics?)
- Revisit of the free initial data problem, probably the canonical formulation on null hypersurface?

Thank you!

Purely geometric path integral for spin foams

Atousa Ch. Shirazi
Florida Atlantic University
LOOPS 13, Perimeter Institute, July 2013

Friday, July 26, 13

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Motivation

- Need to use **Liouville measure** on the reduced phase space which preserves the phase space volume

$$\int \mathcal{D}q \exp iS \to \int \mathcal{D}q\mathcal{D}p \exp i \int d^4x(p\dot{q} - \mathcal{H})$$

- **The phase part**

J. Barrett, et al, J.Math.Phys. 50 (2009) 112504, J. Barrett, et al., Class.Quant.Grav. 27 (2010) 165009

J. Engle, Phys. Rev. D, vol. 87, p. 084048, 2013, J. Engle, arXiv:1201.2187

- **The measure factor:** One way to fix it is with equivalence to canonical theory!

- **Plebanski-Holst formulation:**

Has desired variables $(\omega_\mu^{IJ}, X_{\mu\nu}^{IJ})$

J. Engle, M. Han and T. Thiemann, 2010 Class. Quantum Grav. 27 245014

However, in spin foam sums over spin and intertwiners which label eigenstates of $X_{\mu\nu}^{IJ}$
We need to integrate out the connection.

We call this purely geometric path integral!

Outline

- Reduced phase space path integral approach for a general Hamiltonian system with constraints
- Applying that to Plebanski Holst formulation
- Integrating out the connection
- ADM path integral
- Conclusion

Friday, July 26, 13

Reduced phase space path integral

Path integral on the reduced phase space: $\int \mathcal{D}q^\star \mathcal{D}p^\star \exp i(p\dot{q} - \mathcal{H})$

In terms of the whole phase space variables:
(Quantization of Gauge Systems, Henneaux and Teitelboim)

$$\mathcal{Z} := \int \mathcal{D}q\mathcal{D}p\sqrt{\det(\{S,S\})}|\det(\{F,\xi\})|\delta[S]\delta[F]\delta[\xi]\exp i\int dx^4(p\dot{q} - \mathcal{H})$$

Faddeev Popov term: $\Delta_{FP} = |\det(\{F,\xi\})|$

Expectation values:

$$\mathcal{Z}(\mathcal{O}) := \int \mathcal{D}q\mathcal{D}p\sqrt{\det(\{S,S\})}|\det(\{F,\xi\})|\mathcal{O}\delta[S]\delta[F]\delta[\xi]\exp iS$$

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Plebanski Holst formulation

- The action:

$$S_{BF} = \int \overset{(\gamma)}{X}_{IJ} \wedge F^{IJ}$$

$$\overset{(\gamma)}{X}{}^{IJ} = (X - \frac{1}{\gamma}{}^{\star}X)^{IJ}$$

$$F^{IJ} = d\omega^{IJ} + \omega^{I}{}_{K} \wedge \omega^{KJ}$$

- Conjugate variables: $(\omega_\mu^{IJ}, X_{\mu\nu}^{IJ})$
- Simplicity constraint:

$$C_{\mu\nu\rho\sigma} := \epsilon_{IJKL} X^{IJ}_{\mu\nu} X^{KL}_{\rho\sigma} - \frac{s}{4!}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{IJKL} X^{IJ}_{\alpha\beta} X^{KL}_{\gamma\delta} \approx 0$$

$$X^{IJ} = \begin{cases} \pm\frac{1}{\kappa} e^I \wedge e^J & (I\pm) \\ \pm\frac{1}{2\kappa}\epsilon^{IJ}{}_{KL} e^K \wedge e^L & (II\pm) \end{cases}$$

- Reduced phase space integral:

J. Engle, M. Han and T. Thiemann, 2010 Class. Quantum Grav. 27 245014

$$\mathcal{Z} = \int_{(II_\pm)} \mathcal{D}\omega_\mu^{IJ} \mathcal{D}X^{IJ}_{\mu\nu} \delta(C) \mathcal{V}^9 V_s \exp i \int \overset{(\gamma)}{X}_{IJ} \wedge F^{IJ}$$

- Appearance of 4-volume $\mathcal{V} = g^{\frac{1}{2}}$ and 3- volume $V_s = h^{\frac{1}{2}}$
- Not Gauge fixed!

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Integrating out the connection

$$S[X,\omega] = \int a[^{\mu}_{IJ}][^{\nu}_{KL}]\omega_{\mu}^{IJ}\omega_{\nu}^{KL} + b^{\mu}_{IJ}\omega_{\mu}^{IJ}$$

Gaussian integral:

$$I(X) = \int \mathcal{D}\omega \exp i \int (a\omega^2 + b\omega)$$

$$\hookrightarrow \quad I(X) \hat{=} (\det a)^{-\frac{1}{2}} \exp \frac{-i}{4}(b, a^{-1}b)$$

The measure: $\det a = \mathcal{V}^{12}$

The exponential is the BF action

$$\mathcal{Z} = \int_{(II_{\pm})} \mathcal{D}\omega_{\mu}^{IJ}\mathcal{D}X_{\mu\nu}^{IJ}\delta(C)\mathcal{V}^{9}V_s \exp iS_{BF}$$

$$\hookrightarrow \quad \mathcal{Z} \hat{=} \int_{(II_{\pm})} \mathcal{D}X_{\mu\nu}^{IJ}\delta(C)\mathcal{V}^{3}V_s \exp iS_{BF}$$

ADM Formulation

- Canonical variables: (h_{ab}, π^{ab}) $\qquad \pi^{ab} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ab}}$

- First class constraints: Hamiltonian H_0 and 3-diff constraints H_a

$$H_0 = -\frac{h^{-\frac{1}{2}}}{2}[h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd}]\pi^{ab}\pi^{cd} - {}^{(3)}R(h)h^{1/2}$$
$$H_a = h^{-\frac{1}{2}}h_{ab}D_c\pi^{bc}$$

- No second class constraints

$$\mathcal{Z} := \int \mathcal{D}q\mathcal{D}p\sqrt{\det(\{S,S\})}|\det(\{F,\xi\})|\delta[S]\delta[F]\delta[\xi]\exp i\int dx^4(p\dot{q} - \mathcal{H})$$

$$\hookrightarrow \mathcal{Z}_{ADM} = \int \mathcal{D}h_{ab}\mathcal{D}\pi^{ab}\mathcal{D}N\mathcal{D}N^a \exp i\int d^4x(\pi^{ab}\dot{h}_{ab} - \mathcal{H}_G(h_{ab}, \pi^{ab}, N, N^a))$$

Using: $\{\int \mathcal{D}N \exp -iNH = \delta(H), \qquad \int \mathcal{D}N^a \exp -iN^a H_a = \delta(H_a)\}$

- N and N^a are Lagrange multipliers
- No gauge fixing no Fadeev-Popov term!

Comparison

(1) $\mathcal{Z}_{ADM} \hat{=} \int N^{-3} h^{-\frac{1}{2}} \mathcal{D}h_{ab} \mathcal{D}N \mathcal{D}N^a \exp iS_G$

(2) $\mathcal{Z}_{PH} \hat{=} \int \mathcal{D}X^{IJ}_{\mu\nu} \delta(C) \mathcal{V}^3 V_s \exp iS_G$

Change of variables $\quad (h_{ab}, N, N^a) \to (g_{\mu\nu})$

Change of measure $\quad \mathcal{D}g_{\mu\nu} = \mathcal{D}g_{ab}\mathcal{D}g_{00}\mathcal{D}g_{0a} = \det J \mathcal{D}h_{ab}\mathcal{D}N\mathcal{D}N^a$

$$\begin{aligned} g_{ab} &= h_{ab} \\ g_{00} &= -N^2 + h_{ab}N^aN^b \\ g_{0a} &= h_{ab}N^b \end{aligned} \qquad \det J = \left| \frac{\partial(g_{ab}, g_{00}, g_{0a})}{\partial(h_{cd}, N, N^c)} \right| = -2hN$$

$$\begin{aligned} \mathcal{Z}_{ADM} &\hat{=} \int N^{-3} h^{-\frac{1}{2}} \mathcal{D}h_{ab} \mathcal{D}N \mathcal{D}N^a \exp iS_G \\ &\hat{=} \int N^{-4} h^{-\frac{3}{2}} \mathcal{D}g_{\mu\nu} \exp iS_G \end{aligned}$$

Other change of variables: $\quad X^{IJ}_{\mu\nu} \to (e^I_\mu, C)^{(1)} \qquad \mathcal{D}X^{IJ}_{\mu\nu} = \mathcal{V}^{-6}\mathcal{D}e^I_\mu \mathcal{D}C$

$(e^I_\mu) \to (g_{\mu\nu}, \Lambda_I{}^J)^{(2)} \qquad \mathcal{D}g_{\mu\nu}\mathcal{D}\Lambda_I{}^J = \sqrt{g}\mathcal{D}e^I_\mu$

1) Canonical path integral measures for Holst and Plebanski gravity: I. Reduced phase space derivation, J. Engle, M. Han and T. Thiemann, 2010 Class. Quantum Grav. 27 245014
2) Path integral measure for first-order and metric gravities, R. Aros, M. Contreras and J. Zanelli, 2003 *Class. Quantum Grav.* **20** 2937

Conclusion

- Appearance of some powers of the 4-vol and 3-vol.
- Breaking the manifest general covariance because of the appearance of 3-volume.
- **Gauge fixing** $|\det(\{F,\xi\})|\delta\xi$
- Equivalence of gauge fixed and non gauge fixed path integral is still an open question in case of gravity since the structure functions appear and the gauge group is not a Lie group!
- **Continuum path integral**

Next step: Discretizing and quantizing the measure and importing to spin foam models.

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Twisted Geometries and Secondary Constraints

Fabio Anzà

work with Simone Speziale

Università di Pisa & CPT Marseille

July 26, 2013

Purpose and Program

Twisted Geometries: The Main Problem

Twisted geometries arise as a generalization of the Regge's geometries in a smeared version of GR motivated by LQG. Is there a consistent dynamics for these objects?

S. Speziale's review

J. Hnybida's talk → arXiv:1305.3326

W. Wieland's talk → arXiv:1301.5859

E. Livine's plenary talk

Purpose and Program

Twisted Geometries: The Main Problem

Twisted geometries arise as a generalization of the Regge's geometries in a smeared version of GR motivated by LQG. Is there a consistent dynamics for these objects?

Addressing some dynamics' aspects

- It has been argued that the dynamics naturally selects the Regge subcase. We study a simplified hamiltonian dynamics and show that this is indeed the case.
- If true in general there will be important consequences for the spin-foam

Program

1. LQG: basic aspects of the twistorial structure
2. Twistor networks and *"twisted"* geometries - basic ideas
3. Toy-model for the study of the secondary constraints, geometrical interpretation

Phase space of the smeared Loop Gravity

Canonical Analysis

- Thanks to the Dirac-Bergmann formalism, we can treat GR as an Hamiltonian constrained theory, usually starting from the Holst's action
- The arising structure leads to $SL(2,\mathbb{C})$ variables of the (Covariant) Loop Gravity

$$\left\{\Pi^a_i(p), A^j_b(q)\right\} = \left\{\bar{\Pi}^a_i(p), \bar{A}^j_b(q)\right\} = \delta^a_b \delta^j_i \delta(p,q)$$

Smeared variables: HF Algebra on each link

$h[l] = \mathrm{Pexp}\left[-\int_l A\right] \in SL(2,\mathbb{C})$
$\Pi[l] = \int_{q\in l} h_{q\to p}\Pi_q h^{-1}_{q\to p} \in \mathfrak{sl}(2,\mathbb{C})$
$\Pi[l^{-1}] = -h[l]\Pi[l]h[l]^{-1} \equiv \underset{\sim}{\Pi}[l] \in \mathfrak{sl}(2,\mathbb{C})$

Loop Gravity's Phase Space on each link

$SL(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \cong T^* SL(2,\mathbb{C})$

Phase space of the smeared Loop Gravity

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Loop Gravity's Phase Space on each link

$SL(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \cong T^* SL(2,\mathbb{C})$

Fixed Graph Truncation - Physical Meaning

On a fixed graph Γ the phase space of the Covariant Loop Gravity is $T^* SL(2,\mathbb{C})^L$
Rovelli and Speziale showed that fixed graph smearing is a truncation of the full GR to a finite number of degrees of freedom - PRD **82** 044018 (2010)

Twistors and $T^*SL(2,\mathbb{C})$

Definition: a couple of spinors

- $\mathbb{T} := \mathbb{C}^2 \oplus \bar{\mathbb{C}}^{2*}$
- $Z \in \mathbb{T} \;:\; Z = (\omega^A, \bar{\pi}_{\dot{A}})$

$SL(2,\mathbb{C})$ - invariant symplectic structure

$$\{\pi_A, \omega^B\} = \delta_A^B = \{\underset{\sim}{\omega}_A, \underset{\sim}{\pi}^B\}$$

$\mathbb{T}^2$ carries a $T^*SL(2,\mathbb{C})$ representation - Area-Matching symplectic reduction

$$C \equiv \pi_A\omega^A - \underset{\sim}{\pi}_B\underset{\sim}{\omega}^B \overset{!}{\approx} 0 \qquad \Rightarrow \qquad \mathbb{T}^2/\!/C \cong T^*SL(2,\mathbb{C})$$

Finally: the twistorial representation of the HF Algebra on $T^*SL(2,\mathbb{C})$

$$\Pi^{AB} = -\frac{1}{2}\pi^{(A}\omega^{B)} \qquad \underset{\sim}{\Pi}^{AB} = \frac{1}{2}\underset{\sim}{\pi}^{(A}\underset{\sim}{\omega}^{B)} \qquad h_B^A = \frac{\underset{\sim}{\omega}^A\pi_B - \underset{\sim}{\pi}^A\omega_B}{\sqrt{\pi\omega}\sqrt{\underset{\sim}{\pi}\underset{\sim}{\omega}}}$$

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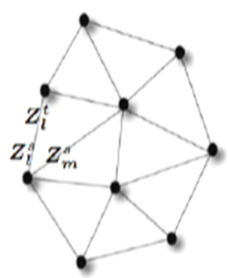
The unfolding picture: Covariant Twisted Geometries

The picture

- So far, we have a graph where we attached a $\mathbb{T}^2$ on each link. These objects are called *Twistor Networks*. Imposing the area-matching we reach the phase space of the covariant loop gravity.



Covariant Loop Gravity's phase space

$$\mathbb{T}^2 /\!/ C \simeq T^* SL(2,\mathbb{C})^L$$

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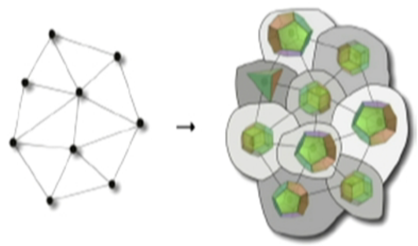
- So far, we have a graph where we attached a $\mathbb{T}^2$ on each link. These objects are called *Twistor Networks*. Imposing the area-matching we reach the phase space of the covariant loop gravity.
- Imposing the Gauss constraint allows to bring in the geometrical interpretation as collection of polyhedra, locally flat. Through simplicity constraints one reach the gauge invariant space



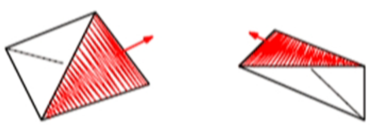
Simplicity constraints: gauge invariant phase space

$$\vec{K}_t + \gamma \vec{L}_t = 0 \; ; \; T^* SL(2, \mathbb{C})^L /\!/ F_t^L \simeq T^* SU(2)^L$$

The unfolding picture: Covariant Twisted Geometries

The picture

- So far, we have a graph where we attached a $\mathbb{T}^2$ on each link. These objects are called *Twistor Networks*. Imposing the area-matching we reach the phase space of the covariant loop gravity.
- Imposing the Gauss constraint allows to bring in the geometrical interpretation as collection of polyhedra, locally flat. Through simplicity constraints one reach the gauge invariant space
- The geometries arising from this picture are quite different from the Regge geometries: lack of the gluing conditions, discontinuous metric



Lack of gluing conditions

Triangle's area matches but they might have different shape

The point: twistor networks and covariant twisted geometries

1. (Covariant) **twisted geometries** fully represent the phase-space of a finite d.o.f. truncation of covariant loop gravity
2. The shape of the triangle shared by two tetrahedra, depend on the frame used to compute the edge lengths
3. They can be thought as a generalization of Regge's geometries but with the lack of the so-called *gluing conditions*

Twistors and classical loop gravity

Twistors' space

$$\mathbb{T}^2 \downarrow \mathbb{C} - \textit{Area Matching} \downarrow T^*SL(2,\mathbb{C})$$

Loop Gravity's phase-space

$SL(2,\mathbb{C})$	$\rightarrow$	*Constraints*	$\rightarrow$	*AB variables*
$\Downarrow$		*Smearing*		$\Downarrow$
$T^*SL(2,\mathbb{C})$	$\rightarrow$	*Constraints*	$\rightarrow$	$T^*SU(2)$

Constraints in the continuum

- Uniqueness of the metric structure, simple bi-vectors
- Torsionless constraint providing the embedding in the covariant space, $\Gamma = \Gamma(g)$

Smeared theory - opening the problem

- Primary: *"simple"* twistors, unique locally flat metric: (twisted) geometries
- Consistency conditions are an open question: discrete torsion? embedding of $T^*SU(2)$ in $T^*SL(2,\mathbb{C})$? discrete $\Gamma = \Gamma(E)$?

ArXiv:1207.6348 - Wieland, Speziale 2012 *Class. Quantum Grav.* **29** - Wieland

Continuum VS Discrete

Continuum picture

$$\begin{array}{lll} \textit{Primary} & \vec{K}+\gamma\vec{L} & \textit{First class} \\ \textit{Secondary} & \Gamma=\Gamma(E) & \textit{Second class} \end{array}$$

What happens in the discrete?

Discrete picture

$$\textit{Primary} \; : \; \vec{K}_l+\gamma\vec{L}_l \longrightarrow \left\{ \begin{array}{ll} D_l & \textit{Lorentz invariant} - 1^{st} \textit{ class} \\ F_l & \textit{Second class} \end{array} \right\} \quad \boxed{\left\{\Pi^i,\Pi^j\right\} = \varepsilon^{ij}_{k}\Pi^k}$$

Mechanism proposed: dynamics

- Secondary constraints make also D_l second-class
- They can be interpreted as the gauge fixing of its orbits

What should be done?

- Identify the orbits
- How the dynamics gauge-fix them?

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An issue: torsionless condition and secondary constraints in the discrete

The main question

Covariant twisted geometries represent the phase space of a truncation of LG:

1. Is there a consistent dynamics for these objects?
2. What is its relation with the Regge case? Role of the "mismatch"?

The idea by Dittrich and Ryan

- Matching conditions as secondary constraints. Mismatch could encode torsion and dynamics is Regge-type. *ArXiv:* 1209.4892 - Dittrich, Ryan
- They derive them through the discretization of the continuum theory, rather then from the study of a discrete Hamiltonian

A counterargument from Marseille

- The torsionless equation is about the connection, which in principle has nothing to do with the geometry or with the matching conditions
- Mismatch $\neq$ Torsion: *twisted* Levi-Civita connection
  PRD **87** (2013) Haggard, Rovelli, Wieland, Vidotto

An issue: torsionless condition and secondary constraints in the discrete

A conundrum arise

Do the twisted geometries have a consistent dynamics, or it is just a "kinematical" parametrization and the dynamics deal just with Regge geometries?

Is it so hard to solve it?

- *Pseudo*-constraints arise after the smearing of the theory
  Bahr and Dittrich (2009)
- Only the dynamics will have the last word

Our strategy

- Even in the discrete, if there is no curvature, the evolution is given by a constraint
- Search for secondary constraints in a toy-model imposing flatness

The model: ingredients

The model - Smearing over a graph with triangular faces

$$\mathcal{H} = \underbrace{\sum_l a_l C_l}_{\text{Area Matching}} + \underbrace{\sum_l \lambda_l D_l + b_l F_l^{(2)} + \underset{\sim}{b}_l \underset{\sim}{F}_l^{(2)}}_{\text{Simplicity}} + \underbrace{\sum_k g_k \vec{\mathcal{G}}_k}_{\text{Gauss}} + \underbrace{\sum_f N_f H_f}_{\text{Hamiltonian}}$$

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Primary Constraints

- Area-Matching

Physical meaning

- $\mathbb{T}^2 \to T^* SL(2, \mathbb{C})$

The model: ingredients

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- Simplicity Constraints

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- $\mathbb{T}^2 \to T^*SL(2,\mathbb{C})$
- "Simple" Twistors - Bivectors

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Primary Constraints

- Area-Matching
- Simplicity Constraints
- Gauss Law - Closure

Physical meaning

- $\mathbb{T}^2 \to T^* SL(2, \mathbb{C})$
- "Simple" Twistors - Bivectors
- Polyhedra - Gauge Invariance

The "toy" part: scalar constraint

$$H_f = \Re \left[\mathrm{Tr} \left\{ h_f - \mathbb{I} \right\} \right]$$

The model: ingredients

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The "toy" part: scalar constraint

$$H_f = \Re\left[\mathrm{Tr}\left\{h_f - \mathbb{I}\right\}\right]$$

We ask for zero (discrete) scalar curvature

$$h_f = h_{\alpha_{ab}} \approx \mathbb{I} + \tfrac{1}{2}\epsilon^2 F^i_{ab}\tau_i + \mathcal{O}(\epsilon^4)$$

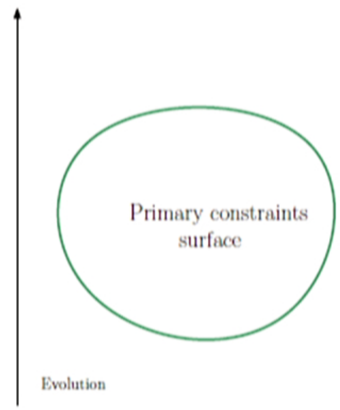
Canonical Analysis - Secondary constraints

Secondary constraints and simplicity constraints

Secondary constraints arise from the diagonal part of the simplicity constraints $\dot{D}_I \stackrel{!}{\approx} 0$

Secondary constraints: the standard guess

Often they are overlooked. One hope that imposing the primary constraints in some consistent way will assure they are preserved through the evolution.



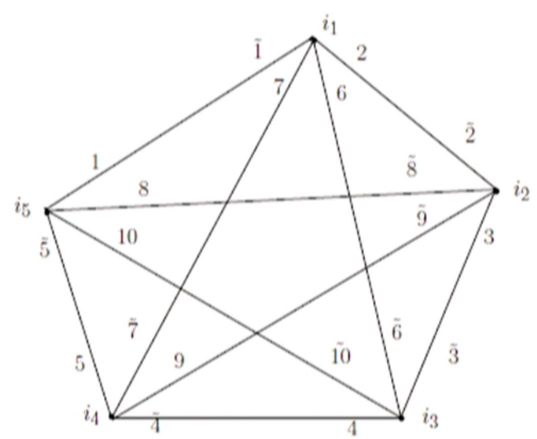
Second-class Poisson algebra - Consistency conditions

Some constraints are second class and they may not be preserved under the evolution

Secondary Constraints - Solution

Strategy

1. Fix the graph for the smearing. We picked up the simplest: a 4-Simplex



Secondary Constraints - Solution

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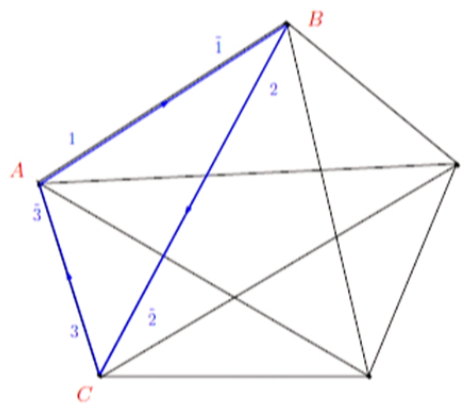

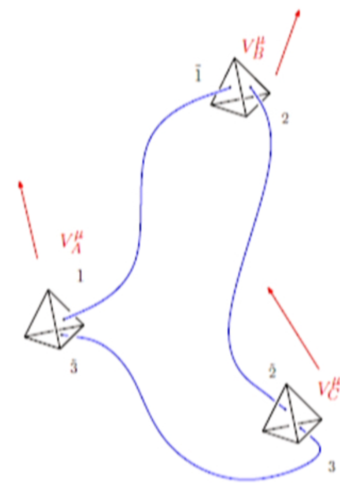

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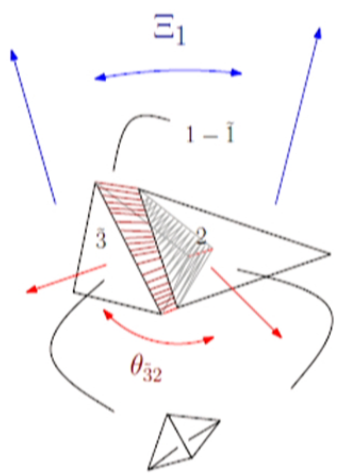

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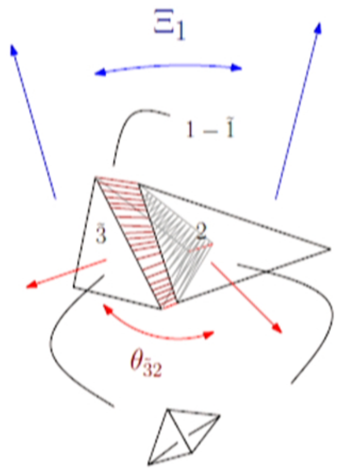

Secondary Constraints - Solution

Click on Sign to add text and place signature on a PDF File.

Strategy

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Here the solution for Ξ_1, arising from the secondary constraints on the face $1 - 2 - 3$

$$\cosh \Xi_1 = \frac{\cos \theta_{\underset{\sim}{23}} + \cos \theta_{\underset{\sim}{31}} \cos \theta_{\underset{\sim}{12}}}{\sin \theta_{\underset{\sim}{31}} \sin \theta_{\underset{\sim}{12}}}$$

"Reconstruction formula"

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Each link l is in the boundary of tree independent faces

$$\Xi_l^{(A)} = \Xi_l^{(B)} = \Xi_l^{(C)} \quad \Rightarrow \quad \frac{\cos\theta_{ij,k} + \cos\theta_{ik,j}\cos\theta_{jk,i}}{\sin\theta_{ik,j}\sin\theta_{jk,i}} \stackrel{!}{=} \frac{\cos\theta_{ij,h} + \cos\theta_{ih,j}\cos\theta_{jh,i}}{\sin\theta_{ih,j}\sin\theta_{jh,i}}$$

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Dittrich & Speziale
New J. Phys. (2008)

Shape – matching conditions

Final remarks

What do we learn?

1. Twistorial formalism is a powerful framework which perfectly suit LQG
2. Gauge inv. phase space is **NOT** Regge: Twisted geometries
3. Piecewise-flat and discontinuous $3D$ geometries

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In a flatness toy-model, the twisted geometries correctly parametrize LQG phase-space **BUT** the dynamics select the Regge solutions through the secondary constraints

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In a flatness toy-model, the twisted geometries correctly parametrize LQG phase-space **BUT** the dynamics select the Regge solutions through the secondary constraints

- Orbits of the simplicity constraints are $4D$ dihedral angles
- Secondary constraints perform a gauge-fixing imposing shape-matching

Thanks

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Thank you!