Title: Discrete Approaches and Mixed Session

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Abstract:

COUPLING GENERAL COVARIANT SYSTEMS

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July 26

LOOPS 13 PERIMETER PI INSTITUTE FOR THEORETICAL PHYSICS

w/ H.Haggard & C. Rovelli work in progress...

introduction

HOW TO ATTACK THE PROBLEM ?

EQUILIBRIUM

⇕

THERMALIZING INTERACTION

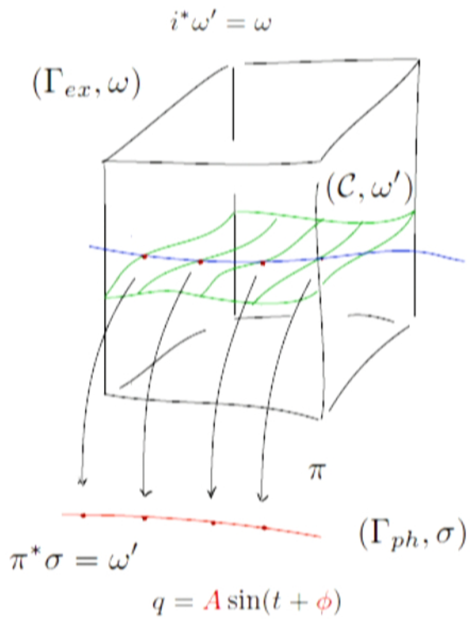
⇕

DYNAMICAL COUPLING => how to couple parametrized systems in a consistent pre-symplectic description

what is dynamics for this systems?

pre-symplectic formulation

- A general covariant system is defined by a Lagrangian that leads to a vanishing canonical Hamiltonian. The Legendre transform of the Lagrangian defines a phase space with constraints: the dynamics is coded in the constraints



- The constraints generate Hamiltonian vector fields X_{dC} , which are tangent vectors to the constraint surface, given by

$$X_{dC} = -\frac{\partial C}{\partial p}\partial_q + \frac{\partial C}{\partial q}\partial_p$$

and $\omega'(X_{dC}) = 0$

- the integral curves of these Hamiltonian vectors fields constitute the gauge submanifold or the orbits of the constraint surface, and the dynamics of the system with respect to τ is the unfolding of this gauge symmetry, i.e., dynamics is gauge.

- Each point in Γ_{ph} is a motion, and establishes a system of relations (defined by its orbit) among the functions on Γ_{ex} . These correlations are defined without specifying one of these as the independent time variable.

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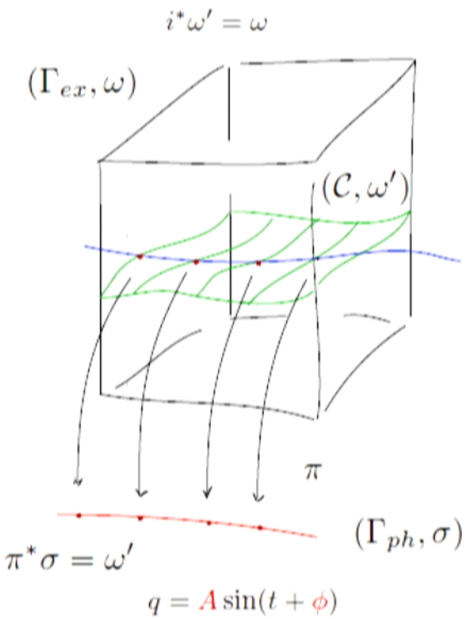

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C - coupling general covariant systems

Take two non-relativistic systems S_1 and S_2, with phase spaces Γ_1 and Γ_2 and Hamiltonians H_1 and H_2.

S1 + S2 => S

We can write a unified description of the two by considering the coupled phase space $\Gamma = \Gamma_1 \times \Gamma_2$, with Hamiltonian $H = H_1 + H_2$. This kinematical coupling allows a dynamical coupling between the systems, for instance by adding a term to H.

recipe:

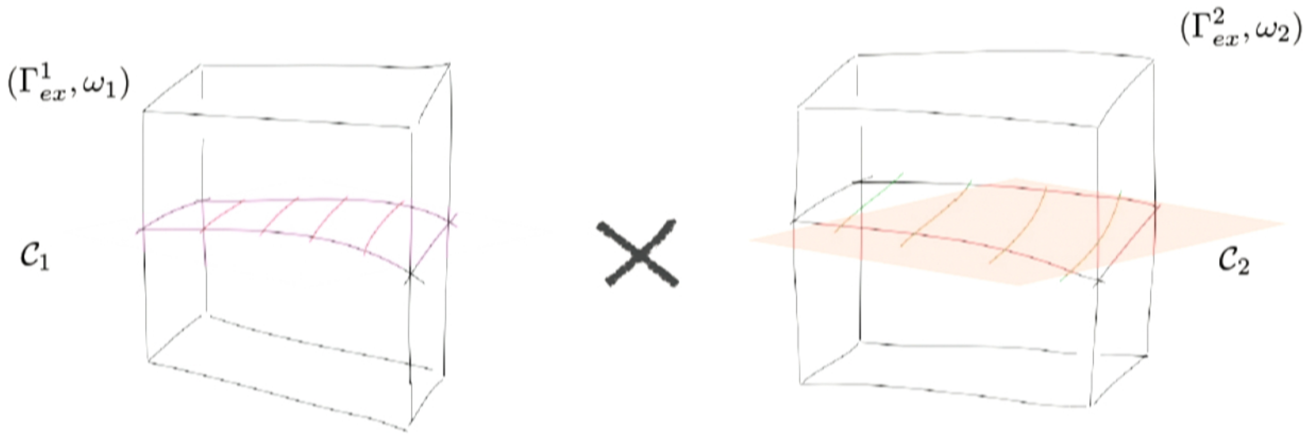
- SET KINEMATIC COUPLING
- DYNAMICAL COUPLING FOLLOWS

CAN THE SAME BE DONE FOR GENERAL COVARIANT SYSTEMS ?

kinematical level : cartesian product

- Consider two general covariant systems S_1 and S_2, with (extended) phase spaces $\Gamma^1{}_{ex}$ and $\Gamma^2{}_{ex}$ with Hamiltonian constraints $C_1 = 0$ and $C_2 = 0$. How do we couple them?

- The answer is to consider the (extended) phase space $\Gamma_{ex} = \Gamma^1{}_{ex} \times \Gamma^2{}_{ex}$, with symplectic form $\omega = \omega_1 + \omega_2$ and the two constraints $C_1 = 0$ and $C_2 = 0$.



single out a time variable, deparametrize

- Let $\Gamma^1{}_{ex}$ admit canonical coordinates $(q_1, t_1, p_1, -E_1)$ and $C_1 = E_1 - H_1(p_1, q_1)$. This is the general covariant form of a system with one degree of freedom and Hamiltonian $H(q,p)$. Similarly for $(q_2, t_2, p_2, -E_2)$ and $C_2 = E_2 - H_2(p_2, q_2)$.

- The surface where the constraints are satisfied admits coordinates $(q_1, p_1, t_1, q_2, p_2, t_2)$.

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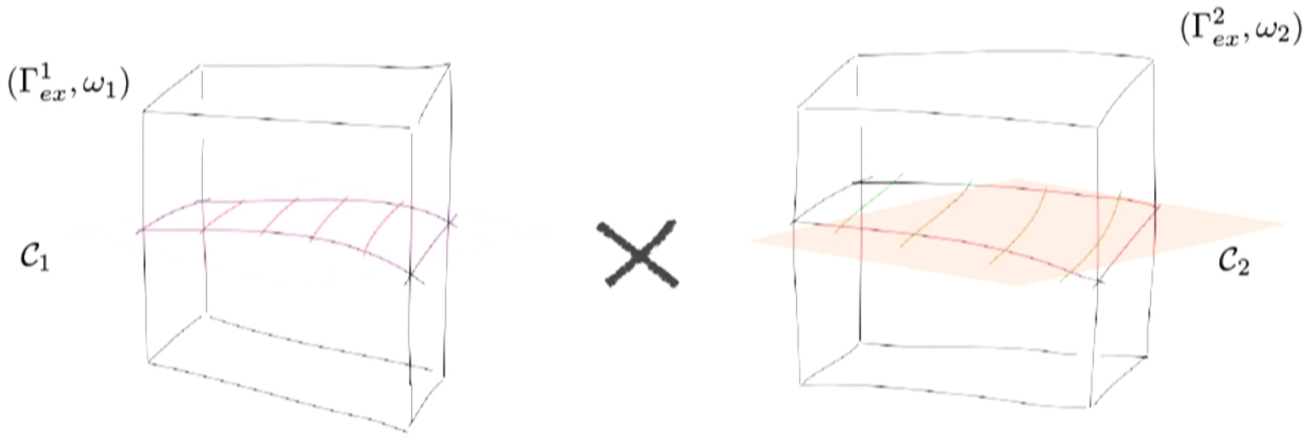

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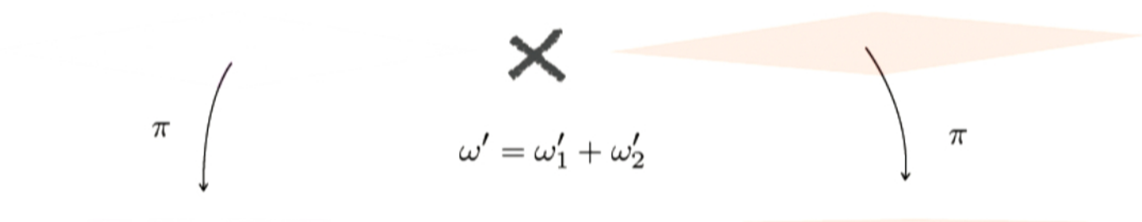
choosing among two times

DINAMICAL LEVEL $(\mathcal{C}, \omega')$

$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$



=> $\Gamma_{ph} = \Gamma^1_{ph} \times \Gamma^2_{ph}$ (no interaction!)

- The constraint orbits are two-dimensional, and (t_1, t_2) can be taken as parameters along each orbit. The physical phase space (the space of the orbits) can be coordinatized by the values of (q_1,p_1,q_2,p_2) at $t_1 = t_2 = 0$.

PROBLEM 1 dynamics is characterized by two times. we would like one reference observable.

When coupling two covariant systems there is a certain redundancy in the partial observable description. A dynamical system is defined by stating which are its partial observables, which represent the quantities to which we have access.

UNIFIED DESCRIPTION => CHOICE

hamiltonian constraint and gauge?

REDUCTION

- assume that the partial observable are, say, (q_1, p_1, q_2, p_2) and $t = 1/2\,(t_1 + t_2)$, while the difference $\chi = 1/2\,(t_1 - t_2)$ is not observable

$C = C_1 + C_2$	must be interpreted as the physical time evolution in t. hamiltonian constraint
$\Delta = C_1 - C_2$	generates an evolution in χ at fixed t which we must interpret as a gauge. In particular, without loss of generality, we can restrict the description to the surface $\chi = 0$, which is to say $t_1 = t_2$. gauge constraint

- the relation between the coupled systems and a system with a single time variable is obtained by interpreting one combination of constraints as gauge, and the other as the Hamiltonian constraint.

PROBLEM 2: what is hamiltonian and what is gauge is arbitrary here. Is there any physical interpretation for this choice?

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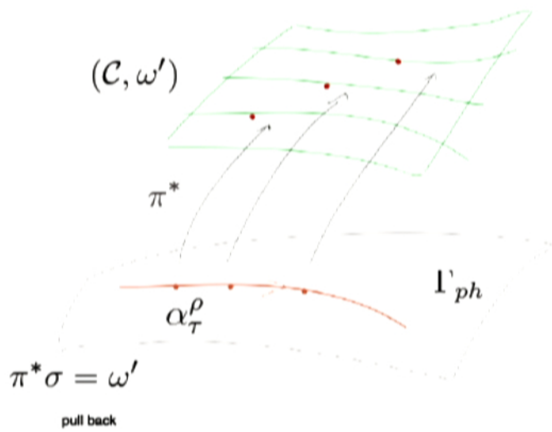
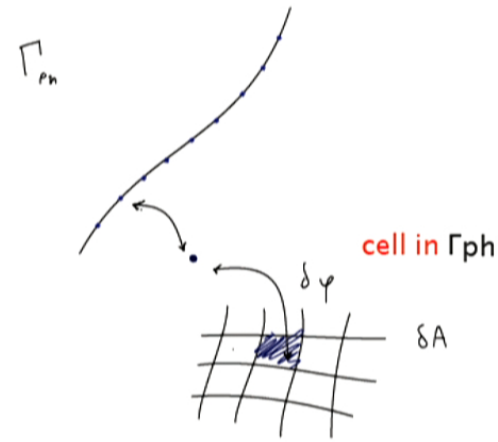
the thermal time criterium

In classical systems, the information on the time flow is coded into the Gibbs states $\rho = \exp(-\beta H)$ as well as in the hamiltonian, namely the Gibbs state is dual to the time flow Xt in the sense that $\beta\rho\,\omega(X_t) = -d\rho$.

- Define a statistical state ρ as a positive function $\rho : \Gamma_{ph} \to R^+$, normalized with respect to the Liouville measure . Any statistical state defines a "hamiltonian" vector field X_ρ by $\rho\,\sigma(X_\rho) = -d\rho$





- The field X_ρ generates a flow $\alpha_\tau\rho$ on Γ_{ph} called thermal flow; its generator $h = -\ln\rho$ is called the thermal hamiltonian, while the flow parameter τ is called thermal time of ρ.

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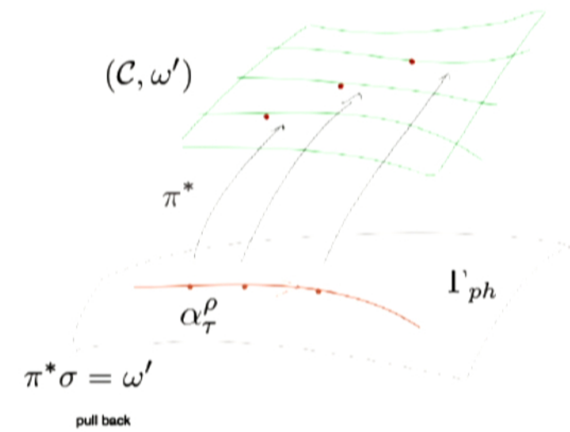

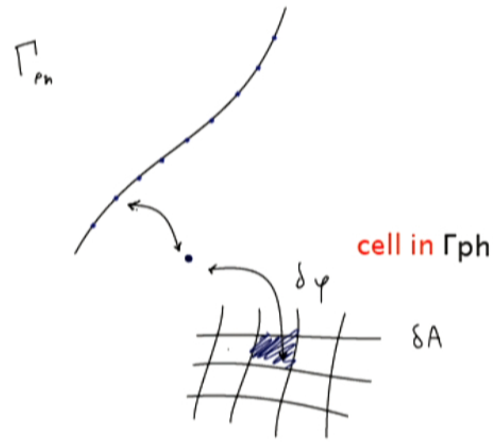

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factorizing the statistical state

This thermal flow determines a time flow T on the constraint surface by $X_\rho = \pi^* T$. In turn, X is a time flow in Γ_{ex} if $X = i^* T$. In this way, the statistical state select a time variable t on the extended phase space.

- A generic statistical state of the combined system is defined by a probability distribution ρ on the physical phase space of the coupled system, $\rho : \Gamma_{ph}{}^1 \times \Gamma_{ph}{}^2 \to R^+$

- We assume ρ to be an equilibrium state, so that we can factorize it in terms of two states, ρ_1 and ρ_2 defined on the respective physical phase space,

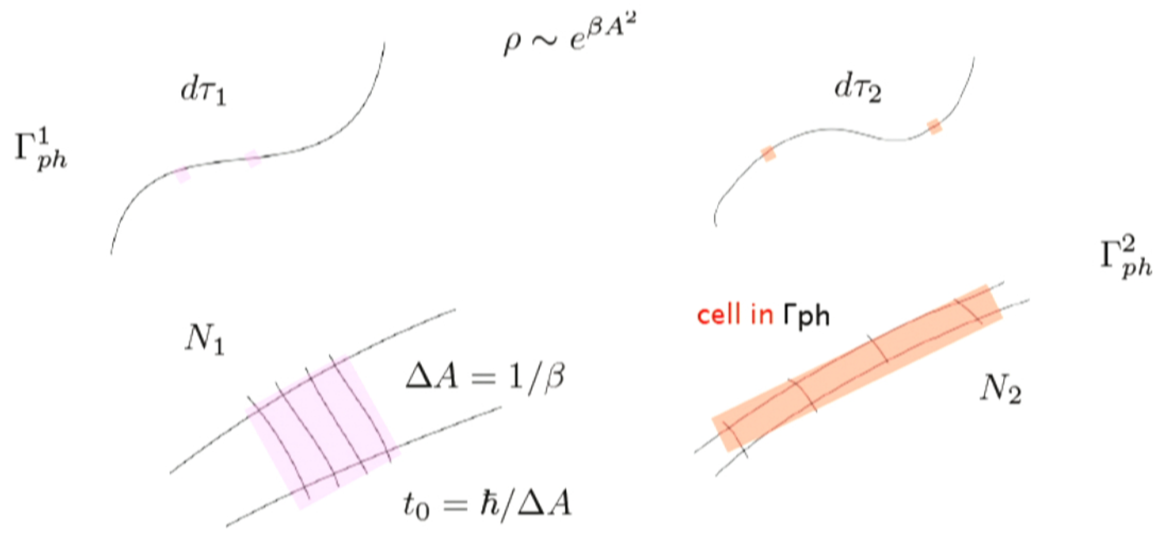
$$\rho(q_1, p_1, q_2, p_2) = \rho_1(q_1, p_1) \cdot \rho_2(q_2, p_2)$$

- Thus, given an arbitrary separation of the system into two macroscopic subsystem, the time vector field defined by the equilibrium state has two independent components associated to the two thermal hamiltonians, $\sigma(X_\rho) = -d\ln\rho = - dh_1 - dh_2$

In this case, the thermal hamiltonian of the coupled system is a sum of two terms, each one defining a flow in its own physical phase space. However, at same time, the statistical state ρ naturally defines a common flow, a single time for the two.

equilibrium, information & gauge

- The thermal time τ parametrizes a curve defined by a sequence of physical states on Γph. Along this history of states, one can think thermal time τ = t/t0 as time measured in number of elementary "time steps", where a step is the characteristic time taken to move to a distinguishable cell in the phase space at a given temperature.



- consider two systems are coupled via some interaction during a certain interval. During the interaction interval the first system transits N_1 states, and the second N_2.

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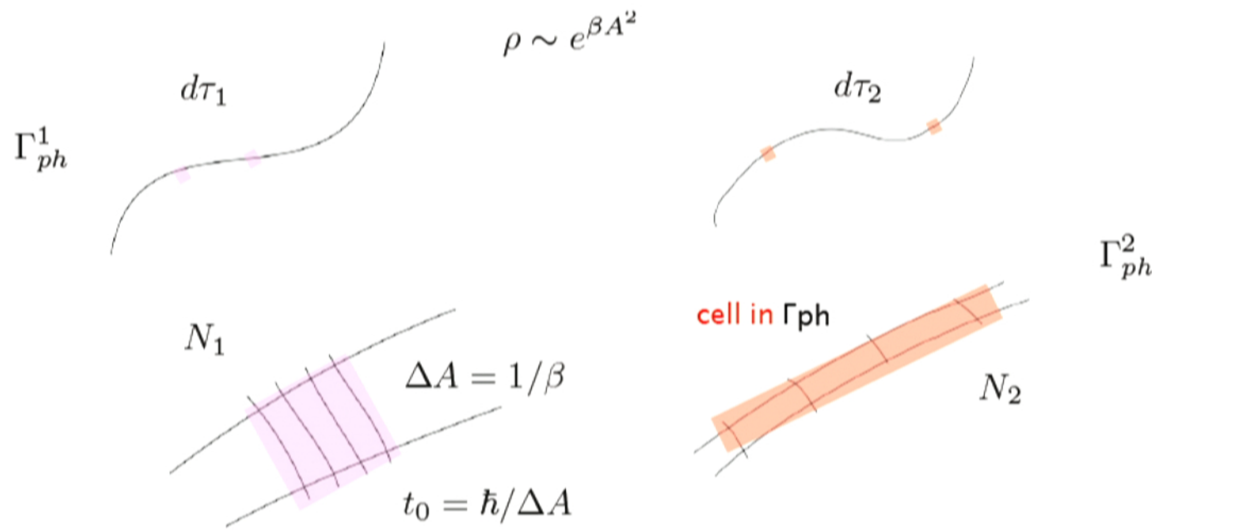

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equilibrium, information & gauge

- If system 2 has access to an amount of information I_1 = $\log N_1$ about system 1, and system 1 has access to an amount of information I_2 = $\log N_2$ about system 2, then the net flow of information can be defined as δI = $I_2 - I_1$.

POSTULATE

At equilibrium, one can postulate that, as any other flow, also information flow δI must vanish.

$$\delta I = 0 \quad \Rightarrow \quad N_1 = N_2$$

Rovelli, Haggard

- In particular, since the rate that states are transited is given by τ and we assume a fixed interaction interval, the equilibrium conditions also reads

$$\tau_1 = \tau_2$$

! The gauge fixing in is equivalent to the equilibrium condition. the gauge is implicitly fixed where the equilibrium is assumed. **!**

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DISCRETE GRAVITY

Application of Discrete Differential Forms

Eugene Kur

With
Robert Littlejohn

University of California, Berkeley
July 26, 2013

LOOPS 2013

WHY DISCRETE DIFFERENTIAL FORMS?

- Numerical schemes require discretization
  - Symplectic integrators for ODE's have greater numerical stability
  - Discrete forms implement symplectic integration for field theories
  - Conservation laws and constraints preserved exactly
- Simplifies classical limit of some quantum gravity theories
  - Regge action central to classical limit of loop quantum gravity
  - Discrete forms simplify Regge calculus
  - Greater flexibility in discrete theories, smoother transition to continuum GR

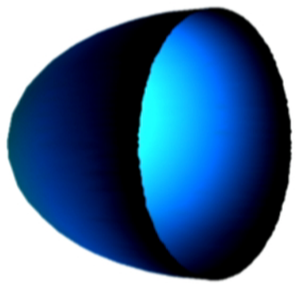
DISCRETE DIFFERENTIAL FORMS

- Natural approach to discretizing actions
- Preserve multisymplectic structure
- Exact conservation laws; constraint preservation
- Numerical Stability
- Relatively new in GR*

*Stern 2009; McDonald, Miller, et. al. 2012; Frauendiener 2006

DISCRETE DIFFERENTIAL FORMS*

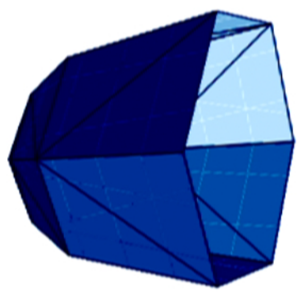
Manifold



α

Differential k-form

Triangulation



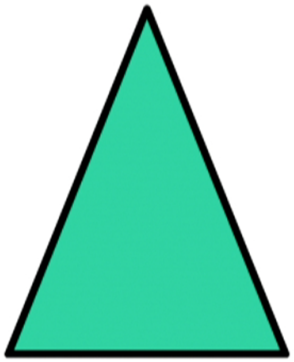
$$\int_{\sigma^k} \alpha \equiv \langle \alpha, \sigma^k \rangle$$

Discrete k-form

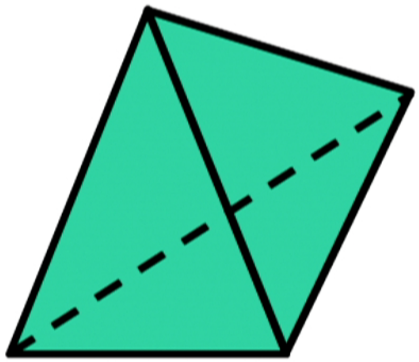
* Hirani 2003; Hirani, Marsden, et. al. 2005; Bossavit 2001-2012; Whitney 1957

DISCRETE EXTERIOR DERIVATIVE

$$\langle \alpha, \sigma^k \rangle$$

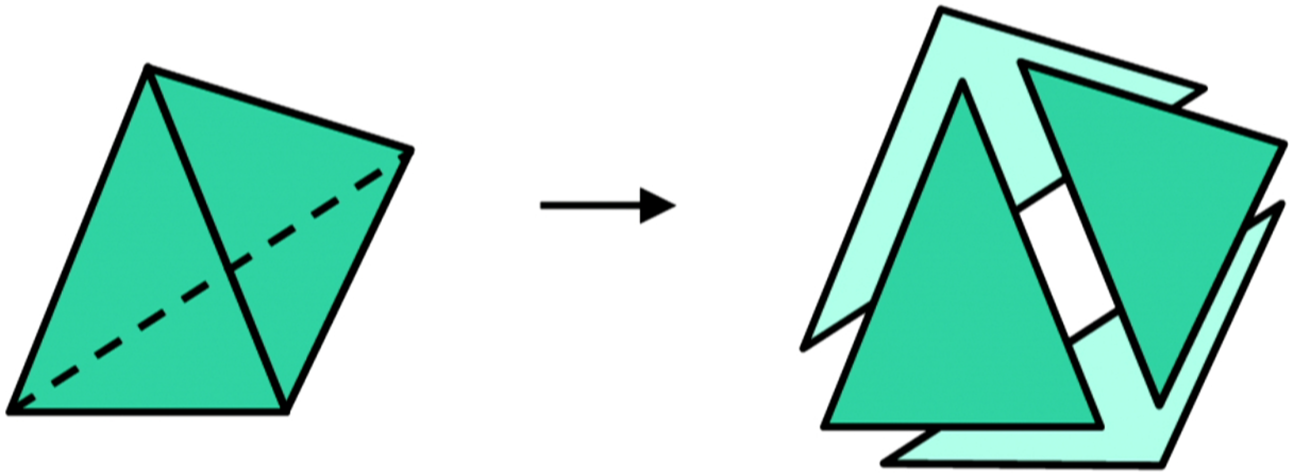


DISCRETE EXTERIOR DERIVATIVE



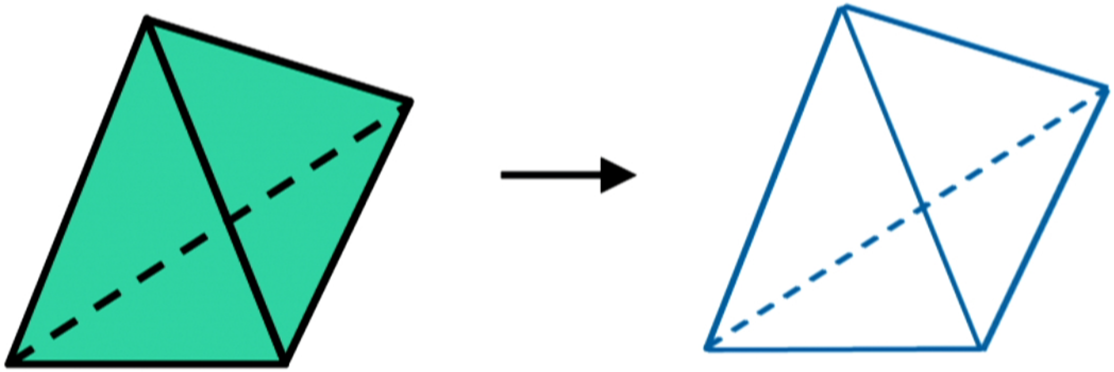
$$\langle \mathrm{d}\alpha, \sigma^{k+1} \rangle$$

DISCRETE EXTERIOR DERIVATIVE



$$\langle \mathrm{d}\alpha, \sigma^{k+1} \rangle = \langle \alpha, \partial\sigma^{k+1} \rangle$$

DISCRETE WEDGE PRODUCT



$$\langle \alpha \wedge \beta, \sigma^{r+s} \rangle \;=$$

$$\frac{1}{N+1} \sum_{\mu} (-1)^{\mu} \frac{r!s!}{N!} \sum_{P \in \mathcal{S}_N} (-1)^{P} \langle \alpha, \mu\mu_1 \ldots \mu_r \rangle \langle \beta, \mu\nu_1 \ldots \nu_s \rangle$$

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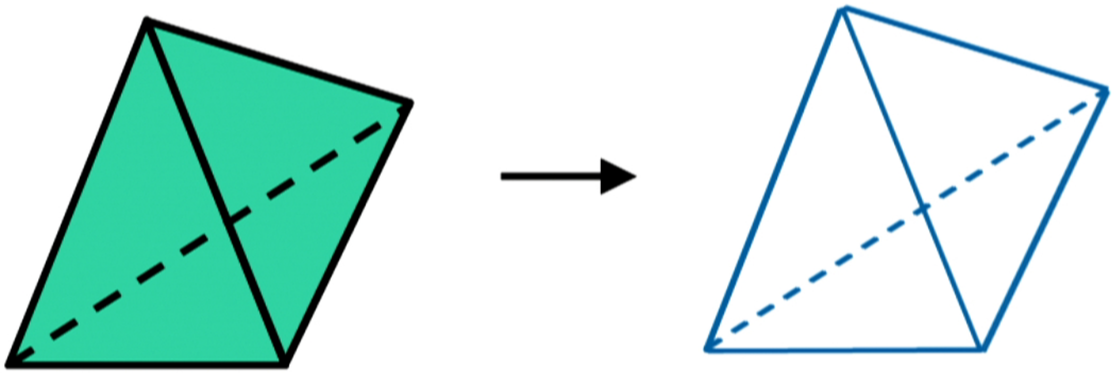
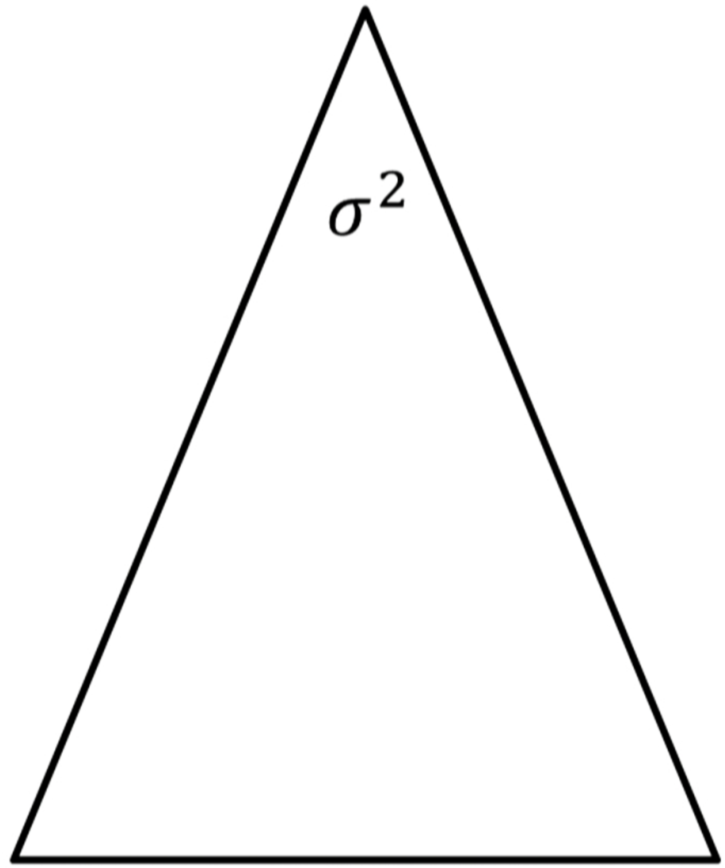


$$\langle \alpha \wedge \beta, \sigma^{r+s} \rangle \;=\;$$

$$\sum_{Q \in \mathcal{S}_{N+1}} \langle \alpha, \mu_0 \ldots \mu_r \rangle \sum_{\nu} (-1)^Q \frac{r!s!}{(N+1)!} \langle \beta, \nu\nu_1 \ldots \nu_s \rangle$$

DUAL CELL



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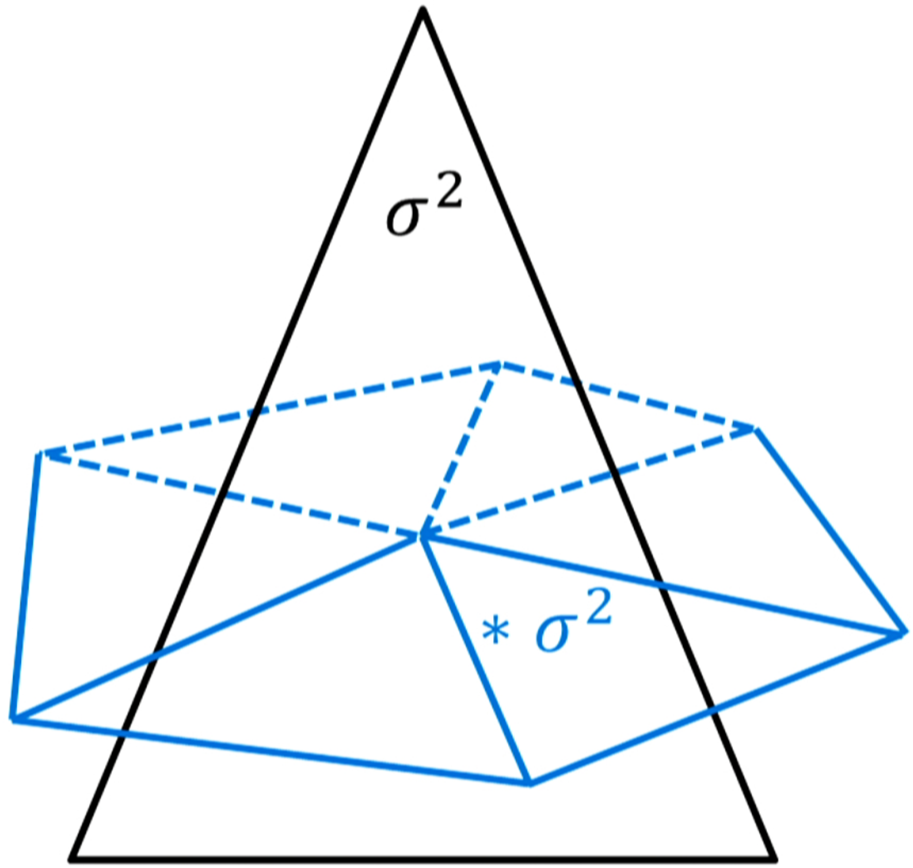

DISCRETE GRAVITATIONAL ACTION

- GR action as integral of 4-form*: $S = \frac{1}{16\pi G}\int_M \frac{1}{2}\epsilon_{IJKL}\, e^I \wedge e^J \wedge R^{KL}$

- Discretize, then wedge product: $S = \frac{1}{16\pi G}\sum_{\sigma^2}\left\langle \frac{1}{2}\epsilon_{IJKL}\, e^I \wedge e^J, \sigma^2 \right\rangle \left\langle R^{KL}, *\sigma^2 \right\rangle$

* Cartan, 1929

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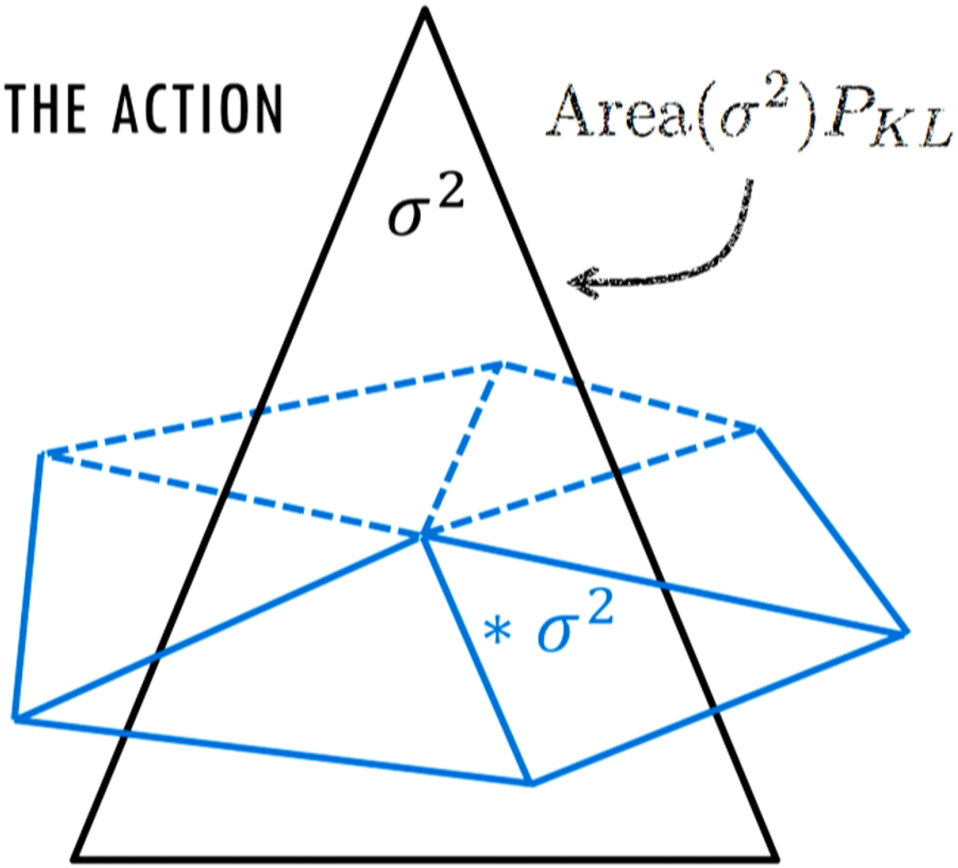
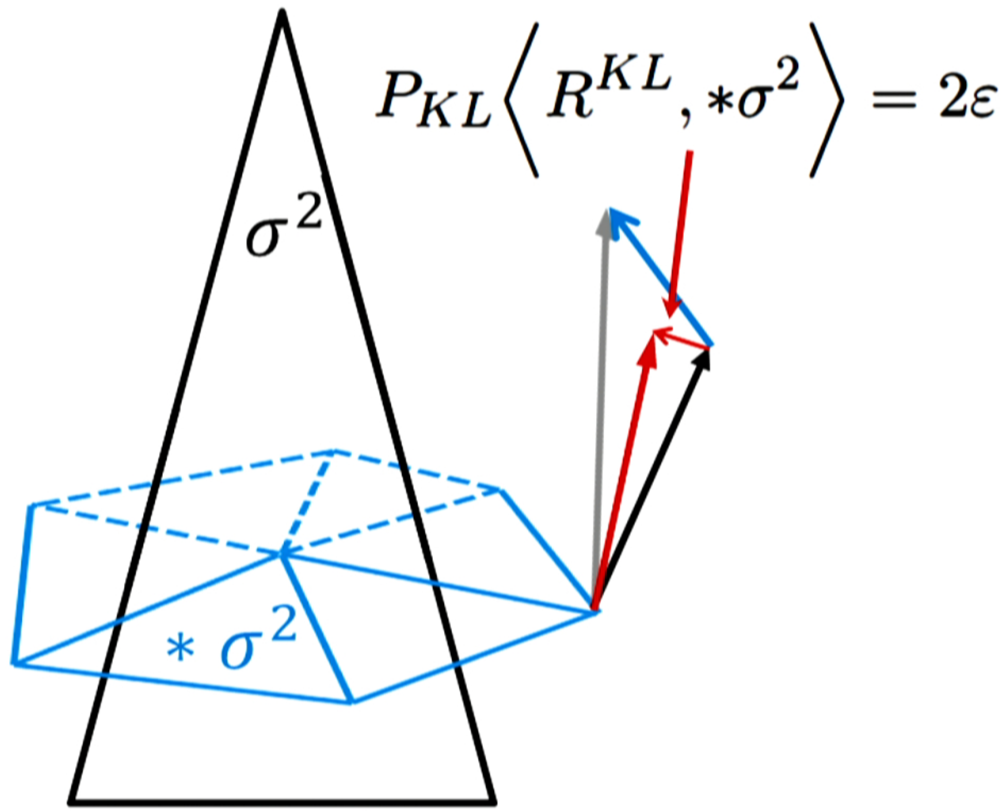
VISUALIZING THE ACTION



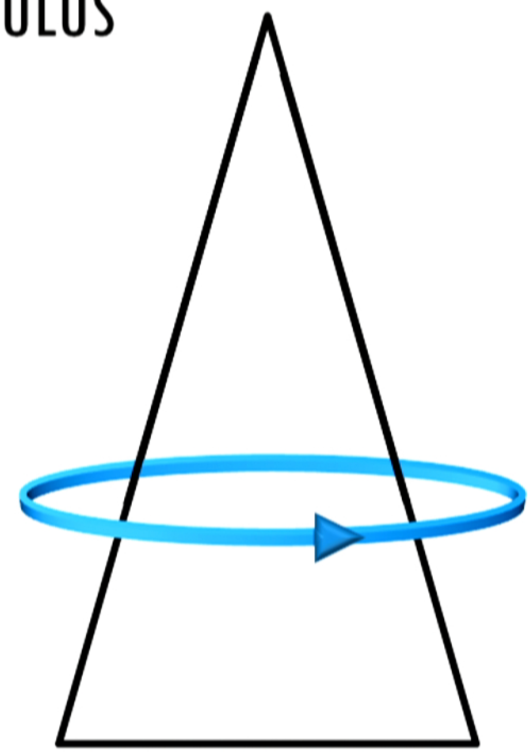
VISUALIZING THE ACTION

- Curvature generates transformation of parallel transport
- Full curvature two-form, not just one angle



CONNECTION WITH REGGE CALCULUS*

- Triangulate manifold
- Require 4- and 3- simplexes to be flat (curvature concentrated on 2-simplexes or "bones")
- Curvature produces rotation in plane orthogonal to bone



* Tulio Regge, 1960

VARIATONAL PRINCIPLE

$$S = \frac{1}{16\pi G} \sum_{\sigma^2} \left\langle \frac{1}{2} \epsilon_{IJKL} \, e^I \wedge e^J, \sigma^2 \right\rangle \left\langle R^{KL}, *\sigma^2 \right\rangle$$

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$$\langle \epsilon_{IJKL}\, e^J \wedge R^{KL}, *\sigma^1 \rangle = 0$$

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A NEW APPROACH TO DISCRETE GRAVITY

$$S = \frac{1}{16\pi G} \sum_{\sigma^2} \left\langle \frac{1}{2} \epsilon_{IJKL}\, e^I \wedge e^J, \sigma^2 \right\rangle \left\langle R^{KL}, *\sigma^2 \right\rangle$$

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A NEW APPROACH TO DISCRETE GRAVITY

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$$\downarrow \qquad\qquad \downarrow$$

$$\left\langle \epsilon_{IJKL}\, e^J \wedge R^{KL}, *\sigma^1 \right\rangle = 0 \qquad\qquad \sum_{\sigma^2 \succ \sigma^1} \varepsilon(\sigma^2)\cot\left[\theta_{\sigma^1}(\sigma^2)\right] = 0$$

TIME EVOLUTION

$$S_{\mathrm{ADM}} = \int \mathcal{L} \, \mathrm{d}t$$

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FUTURE DIRECTIONS

- Numerical simulations
- Understand conservation laws and constraint preservation
- Study the (multi)symplectic structure
- Use approaches closer to loop quantum gravity (Ashtekar connection, Plebanski area forms, etc.)
- Consider quantum aspects (path integral, semiclassics, etc.)

SUMMARY

Discrete differential forms:

- New approach to discrete gravity
- Suited for symplectic time evolution
- Generalize Regge calculus
- Lead to Regge action
- General approach to discrete theories

Studying Topology Change with Topspin Networks

LOOPS 13, Perimeter Institute

Christopher Duston[1]
with Matilde Marcolli[2]

[1]Physics Department, Stony Brook University

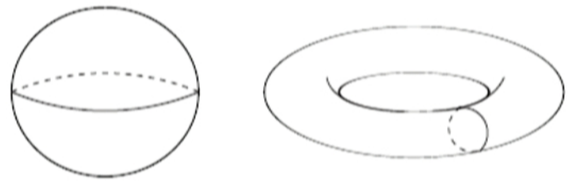
[2]Mathematics Department, Caltech

July 2013

Outline of Talk

- Topology and Loop Quantum Gravity
- Topspin Networks: Representations of spatial sections as a branched covering space
- Examples: The topology of simple topspin networks and topspin foams.
- (Model building)

Topology in Quantum Gravity: Why?



- By "Topology" we mean things measured by topological invariants, the fundamental group, homology groups, *etc*...
- Unlike geometry, topology is not part of the background independence of GR; we must specify a topology *a priori*.
  - A similar thing is true of dimension and differential structure...
  - "GR is not a purely relational theory"[1]
- "Where does this topology come from?" A nice answer would be: the quantum nature of gravity.

[1] Smolin 2005, The Case for Background Independence

Topology in Loop Quantum Gravity

- At first glace, the canonical LQG approach to QG destroys all topological knowledge.
- Restricting from a 3-sphere to an embedded graph trades smooth information for discrete information; one can embed a given graph (spin network) in a large number of topologically inequivalent 3-manifolds:

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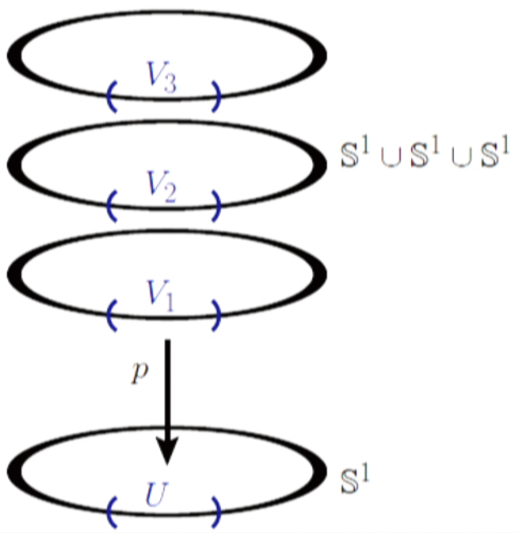
Question:

Can one modify LQG in such a way that includes topological information and also preserves the geometrical structure?

Representation of Space as a Branched Covering

- A **covering space** M is a space and a surjective map $p: M \to B$ to a base space B. The inverse image of any set $U \subset B$ is a disjoint product of sets in M:

$$p^{-1}(U) = V_1 \cup V_2 \cup \ldots \cup V_n.$$



Topspin Networks[2]

Based on the following observation:

Alexander's Theorem (1920)

Any compact oriented 3-manifold can be described as a branched covering of $\mathbb{S}^3$, branched along a graph.

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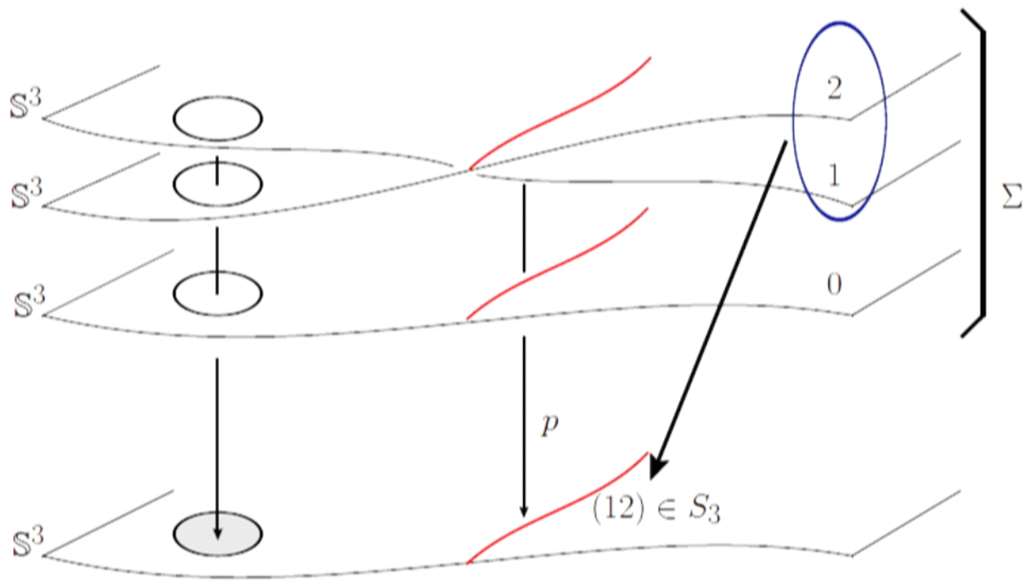

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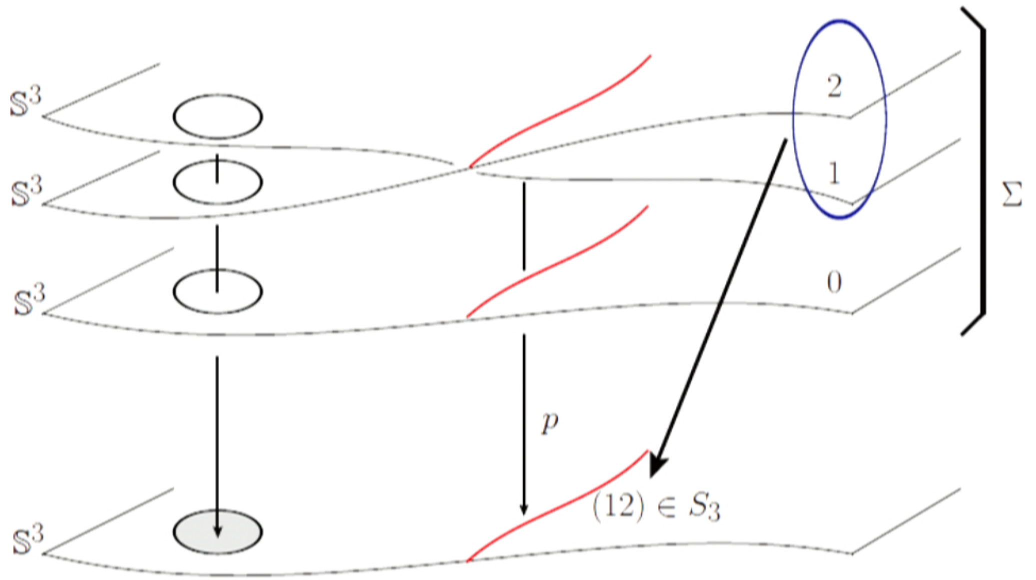

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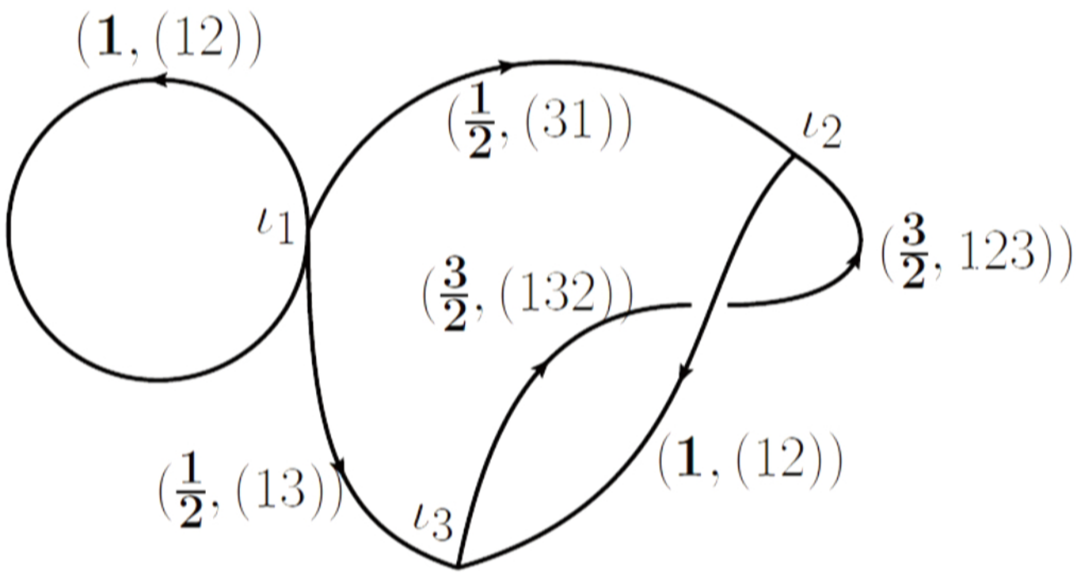
The Point:

Σ is a spatial section! **Topspin networks** track both topology and geometry.



[a] CD 2012 Class. Quantum Grav. 29 205015
[b] CD 2011, IJGMMP (08), Asselmeyer-Maluga 2010, CQG (27)
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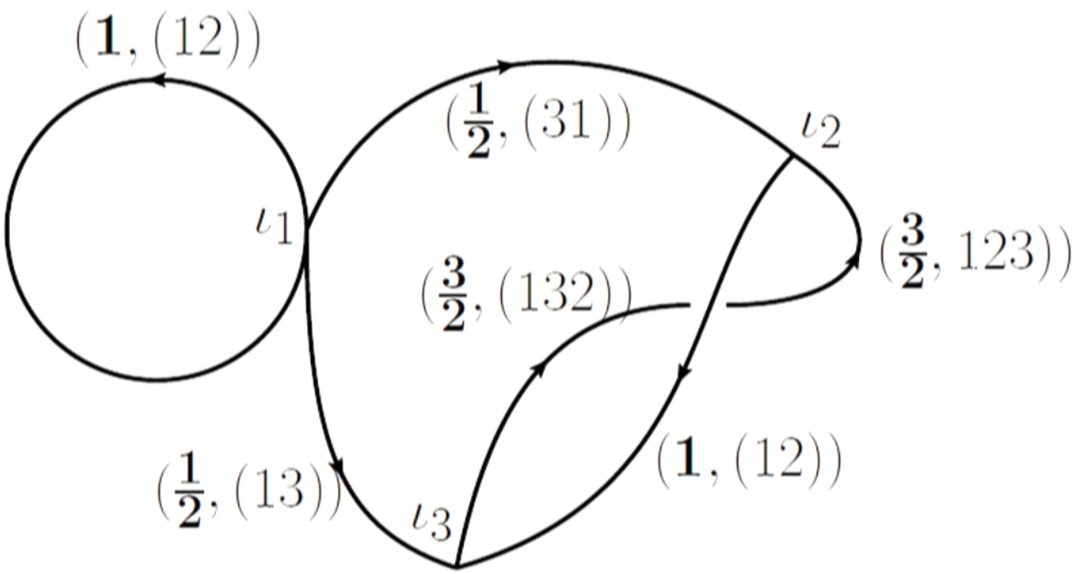

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- Completely compatible with usual spin networks, and allow for arbitrary spatial sections.
- Hilbert space and operator structure compatible with LQG[a].
- Slight Aside:
  - Alexander's theorem applies dimension $d = 4$ as well...
  - Provides a complete specification of the geometry and topology (no exotic smoothness![b]).
  - Provides a reparametrization of the gravitational field in terms of surfaces and topological labels[c].

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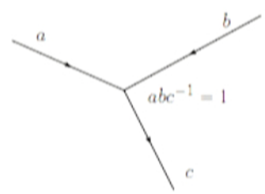
The Fundamental Group of the Spatial Section

In the case of a connected spatial section represented as a g-fold cover, the fundamental group can be found by the following algorithm[3]:

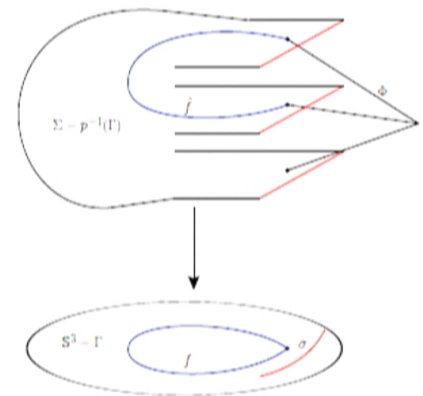
- $G = \pi_1(\mathbb{S}^3 - \Gamma)$ is the *graph group* generated by edges $a, b, c, \ldots$ under relations $r_1, r_2, r_3, \ldots$.



- Map the elements of G to the cover under a specific homeomorphism ϕ which respects the permutation elements:

$$G = \langle a, b, c, \ldots | r_1, r_2, \ldots \rangle$$

$$\phi(G) = \langle a_{\sigma(0)}, a_{\sigma(1)}, \ldots | r_{1\sigma(0)}, r_{2\sigma(1)}, \ldots \rangle$$



- There will be elements in this group which connect all the sheets together, and also elements belonging to the branch locus - these are trivial and should be removed.

[3] Adapted from Fox R 1961 *A Quick Trip Through Knot Theory*.

Example: 2 Vertex Σ_2

2 vertices, 3 edges, g=3.

$$G = \pi_1(\mathbb{S}^3 - \Gamma) = \langle a, b, c | c = ab \rangle.$$

$$\pi_1(\Sigma_2 - p^{-1}(\Gamma)) * F_2 = \langle a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2 | c_0 = a_0 b_1, c_1 = a_1 b_2, c_2 = a_2 b_0 \rangle.$$

$$F_3 = \langle a_0, a_1 \rangle.$$

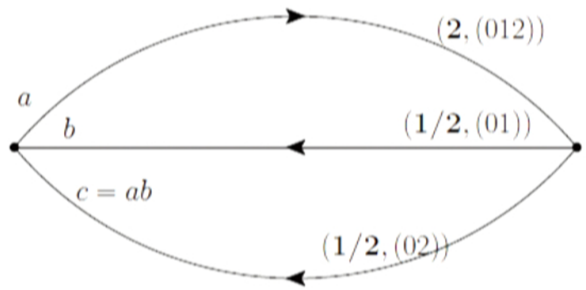
$$\phi^0(a)\phi^1(a)\phi^2(a) = a_0 a_1 a_2 = 1,$$

$$\phi^0(b)\phi^1(b) = b_0 b_1 = 1, \ \phi^2(b) = b_2 = 1,$$

$$\phi^2(c)\phi^0(c) = \phi^2(ab)\phi^0(ab) = a_2 b_0 = 1,$$

$$\phi^1(c) = \phi^1(ab) = a_1 b_2 = 1.$$

$$\boxed{\pi_1(\Sigma_2) = 0}$$



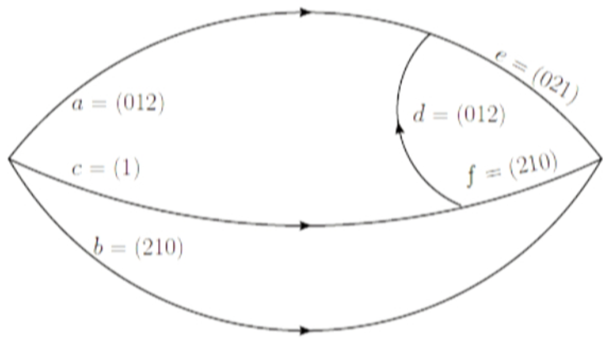
Example: Adding extraordinary vertex Σ_4

$$H := \pi_1(\Sigma_4 - p^{-1}(\Gamma))$$

$$H * F_2 = \left(\begin{array}{c|c} a_0, b_0, c_0, d_0, e_0, f_0 & c_0 = b_0 a_2, e_0 = d_0 a_1, f_0 = a_2^{-1} d_1^{-1} b_2^{-1} \\ a_1, b_1, c_1, d_1, e_1, f_1 & c_1 = b_1 a_0, e_1 = d_1 a_2, f_1 = a_0^{-1} d_2^{-1} b_0^{-1} \\ a_2, b_2, c_2, d_2, e_2, f_2 & c_2 = b_2 a_1, e_2 = d_2 a_0, f_2 = a_1^{-1} d_0^{-1} b_1^{-1} \end{array} \right)$$

After relations:

$$\pi_1(\Sigma_4) = \langle d_0, d_1, d_2 | d_2 = d_1^{-1} d_0^{-1} \rangle \simeq \mathbb{Z} * \mathbb{Z}$$



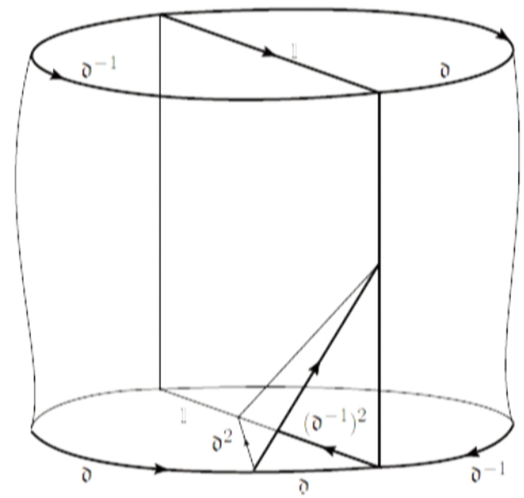
$$H_1(\Sigma_4) = H^2(\Sigma_4) \simeq \mathbb{Z} \oplus \mathbb{Z}$$

Dynamics of Topology Change: Topspin Foams

Dynamics underlying LQG still unknown, but we have **spinfoams**. With topological labels these are **topspin foams**.

- Example: set $\eth \in S_3$ to be any cycle of maximum length ((012),(201), *etc.*)
- The boundaries of this topspin foam are connected by induced action of the (graph-changing) Hamiltonian.
- By ensuring that the topological labels are consistent we can be sure the foam represents a smooth 4-manifold M with

$$\partial M = \Sigma_2 \cup \Sigma_4.$$



The above topspin foam represents a transition between trivial spatial topology and $\pi_1(\Sigma_4) = \mathbb{Z} * \mathbb{Z}$.

Summary

- Look for topology change in LQG.
- Include topological data into the spin networks to allow for arbitrary spatial sections $\longrightarrow$ **topspin networks**
- The (modified) Fox algorithm can be used to identify the topological class of the spatial section.
- Examples: Any 2 vertex topspin network is a sphere!
- Examples: Topspin foam representing topological transitions.
- Future:
  - Continue with Topspin foams
  - S_n-GFT *a la* Ben Geloun & Ramgoolam (1307.6490)
  - Modeling building to study topology change: Schwinger representations and coherent states -

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I, \qquad \mathcal{H}_I = \sum v_i(\ell\ell')(\mathbf{a}_i(\ell)\mathbf{a}_i^\dagger(\ell') + \mathbf{b}_i(\ell)\mathbf{b}_i^\dagger(\ell')).$$

Spectral dimension: an observable of quantum geometry

Motivation

Characterize geometric meaning of quantum gravity states/quantum histories in LQG/SF/GFT

To this end, the spectral dimension d_s is an example much discussed:

[Ambjorn *et al* 2005], [Lauscher, Reuter 2005], [Horava 2009], [Benedetti 2009], [Modesto *et al* 2008/09]

$$\langle P(\tau)\rangle_{QG} = \langle \mathrm{Tr} K(x,y;\tau)\rangle_{QG} = \left\langle \mathrm{Tr} e^{\tau\Delta}\right\rangle_{QG} \propto \tau^{-\frac{d_s}{2}}$$

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Definition for discrete geometries

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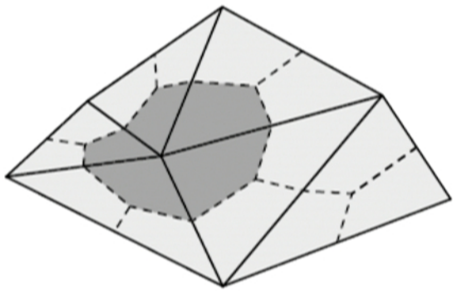
- So far either purely combinatorial (CDT) or smooth setting (AS, HL, NCFT,...)
- LQG/SF/GFT and Regge calculus built on discrete geometries
  → Use discrete (exterior) calculus (DEC) [Desbrun *et al* 2005]:

Definition of $\Delta = \mathbf{d}\delta + \delta\mathbf{d}$ acting on p-forms on abstract simplicial (or polyhedral) d-complexes with geometric interpretation (assignment of volumes to simplices).

On dual scalar fields ϕ:

$$(\Delta\phi)_\sigma = \frac{1}{V_\sigma} \sum_{\sigma' \sim \sigma} \frac{V_{\sigma\cap\sigma'}}{V_{\star(\sigma\cap\sigma')}} (\phi_\sigma - \phi_{\sigma'})$$



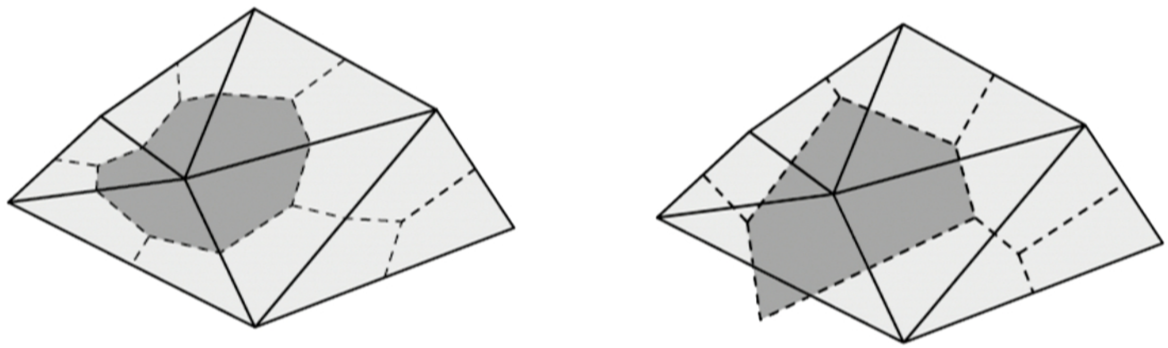
Bra-ket formulation conceptually unifying the necessary duality between chains/cochains and between combinatorially primal/dual complex

Geometric data

Volumes can be defined (motivated by simplicial setting) as functions of

- edge lengths
- $(d-1)$-face normals
- face bivectors/fluxes or area-angle variables (in $4d$)

Barycentric vs. circumcentric dual volumes:



Positivity of Laplacian on generic geometries → barycentric dual preferred

Topology and geometry

Before analyzing d_S of LQG states/SF histories:

- Understand classical features of underlying complexes!

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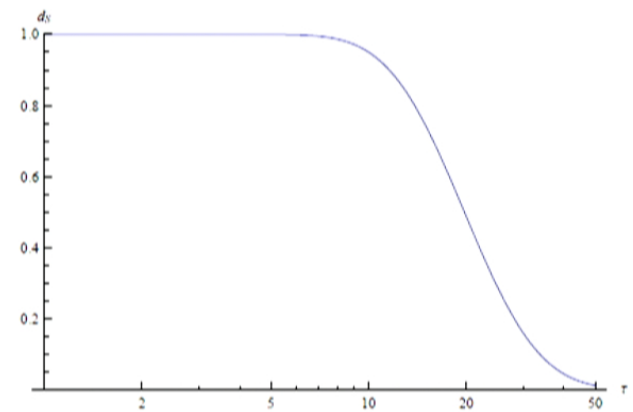
Finite closed complexes $\rightarrow$ compact topology of a certain kind

Circle S^1



$$P_{S^1}(\tau) = \frac{1}{R}\sum_{k\in\mathbb{Z}} e^{-(\frac{k}{R})^2\tau} = \frac{1}{R}\theta_3\left(0, e^{-(\frac{1}{R})^2\tau}\right) = \frac{1}{R}\theta_3\left(0|(\frac{1}{R})^2\frac{i\tau}{\pi}\right)$$

Curvature R just shifts the plot ($\equiv$ rescaling of τ)

Topology and geometry

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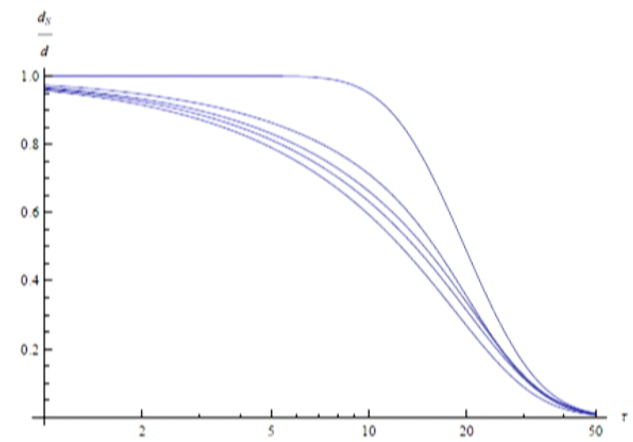
Spheres S^d, $d = 1, 2, 3, 4, 5$

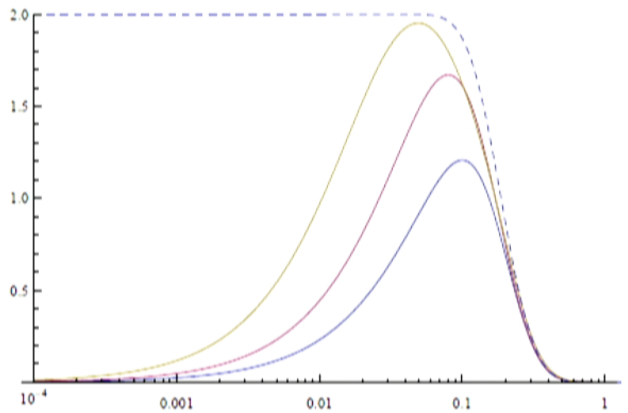


$$P_{S^d}(\tau) = \frac{1}{R^d}\left(1 + \sum_{j=1}^{\infty}\left[\binom{d+j}{d} - \binom{d+j-2}{d}\right] e^{-\frac{j(j+d-1)}{R^2}\tau}\right) \quad (0.1)$$

Discreteness effect

Minimal distance effects small τ behavior

Refined triangulations of the sphere

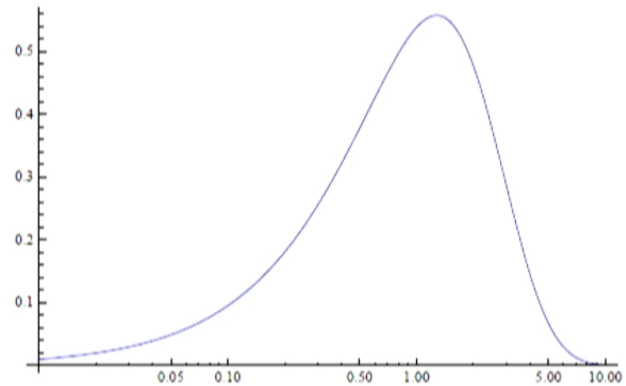


Only boundary of platonic solids are equilateral:
Plots for tetrahedron, octahedron, icosahedron
Complexes are to small to even peak on the value of topological dimension!

Discreteness effect

Minimal distance effects small τ behavior

The dipole triangulation of S^d



d-independent analytic solution of d_S (maximum at ≈ 0.557)

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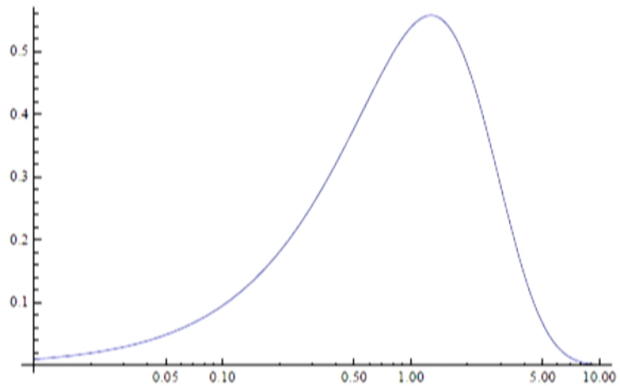

d-independent analytic solution of d_S (maximum at ≈ 0.557)

Lesson: For a concept of d_S as dimension at all, complexes must be large enough, i.e. regime between discreteness and topological effect is needed.

For these reasons, toroidal complexes are used in the following.

LQG states in 2+1 dimensions

$d = 2 + 1$: LQG states are geometric, direct observation of quantum effects

$$\left\langle \widehat{P(\tau)} \right\rangle_\psi \propto \left\langle \psi | \mathrm{Tr} e^{\tau \widehat{\Delta}} | \psi \right\rangle = \sum_s |\psi(s)|^2 \left\langle s | \mathrm{Tr} e^{\tau \widehat{\Delta}} | s \right\rangle = \sum_s |\psi(s)|^2 \, \mathrm{Tr} e^{\tau \langle s | \widehat{\Delta} | s \rangle}$$

Edge length Laplacian on SN states $\widehat{\Delta} = \widehat{\Delta(l^2)} = \Delta(\widehat{l^2}) = \Delta(j)$

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- peaked on equilateral ($l = l(J)$) intrinsic geometry, spread σ
- semiclassical approximation $\rightarrow$ Gaussian sum
- implementation: approximation by sum over some Gaussian samples

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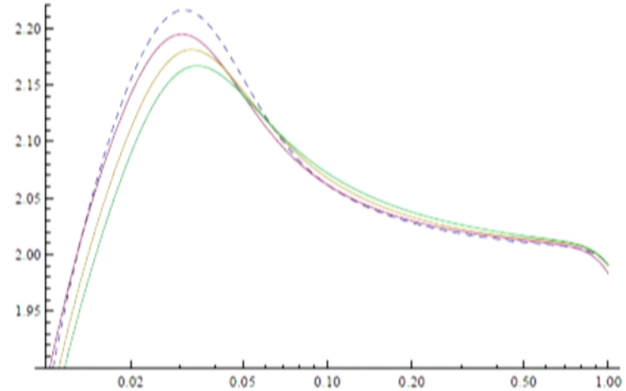

$J = 10$, $\sigma = 1.5, 2, 2.5$ on $N_2 = 1152$ torus

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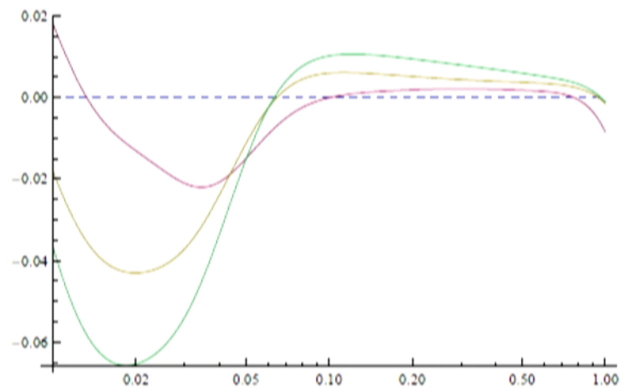

$J = 10$, $\sigma = 1.5, 2, 2.5$ on $N_2 = 1152$ torus

Only small ($\mathcal{O}(10^{-2})$) "quantum corrections" even for small J, large σ

LQG states in 2+1 dimensions

$d = 2+1$: LQG states are geometric, direct observation of quantum effects

$$\left\langle \widehat{P(\tau)} \right\rangle_\psi \propto \left\langle \psi | \mathrm{Tr} e^{\tau \widehat{\Delta}} | \psi \right\rangle = \sum_s |\psi(s)|^2 \left\langle s | \mathrm{Tr} e^{\tau \widehat{\Delta}} | s \right\rangle = \sum_s |\psi(s)|^2 \, \mathrm{Tr} e^{\tau \langle s | \widehat{\Delta} | s \rangle}$$

Edge length Laplacian on SN states $\widehat{\Delta} = \widehat{\Delta(l^2)} = \Delta(\widehat{l^2}) = \Delta(j)$

Coherent states

- peaked on equilateral ($l = l(J)$) intrinsic geometry, spread σ
- semiclassical approximation $\rightarrow$ Gaussian sum
- implementation: approximation by sum over some Gaussian samples

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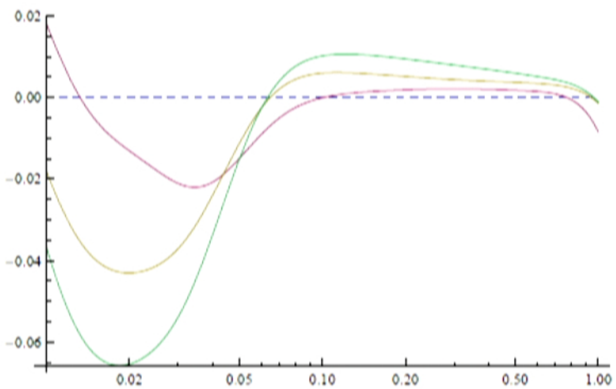

$J = 10$, $\sigma = 1.5, 2, 2.5$ on $N_2 = 1152$ torus

Conclusion

- for discrete geometries new definition of Laplacian is needed $\rightarrow$ extension of discrete exterior calculus to QG

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- for discrete geometries new definition of Laplacian is needed $\rightarrow$ extension of discrete exterior calculus to QG
- concept of dimension only for states on large complexes
- semiclassical states provide good approximation to classical case
- stronger quantum effects (e.g. dimensional reduction) don't show up
- superpositions of states peaked on different scales and states on complexes of different size under investigation (compare CDT)

Outlook: d_S of spin foams

Challenges

- $P(\tau)$ doesn't factorize into ultralocal parts $\to$ results on expectations values [Livine, Ryan 2009]

$$\int \mathbf{d}B\; B^{2n} e^{i\,\mathrm{Tr} BG} = \langle l^{2n} \rangle\, d_j \chi_j(G)$$

can't be used for

$$\langle P(\tau) \rangle_{SF} = \int [\mathbf{d}B_f][\mathbf{d}g_e]\; P(B_f) e^{i \sum \mathrm{Tr} B_f G_f(g_e)}$$

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- degenerate configurations $\rightarrow$ imaginary contribution to $P(\tau)$
- divergences and regularization: gauge fixing on maximal tree not applicable

Thank you for your attention!