Title: Diffeomorphism Invariant Gauge Theories

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Abstract: I will describe a very large class of gauge theories that do not use any external structure such as e.g. a spacetime metric in their construction. When the gauge group is taken to be SL(2) these theories describe interacting gravitons, with GR being just a particular member of a whole family of gravity theories. Taking larger gauge groups one obtains gravity coupled to various matter systems. In particular, I will show how gravity together with Yang-Mills gauge fields arise from one and the same diffeomorphism invariant gauge theory Lagrangian. Finally, I will describe what is known about these theories quantum mechanically.

Diffeomorphism Invariant Gauge Theories

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Past work:

70's Penrose, Plebanski, Atiyah-Hitchin-Singer

80's Gindikin, Ashtekar, Jacobson-Smolin

90's Capovilla, Dell, Jacobson, Bengtsson,
Peldan, Chakraborty

Recent work:

KK, Shtanov, Speziale, Ishibashi, Smolin, Lisi, Freidel, Beke, Palmisano, Fine, Torres-Gomez, Scarinci, Delfino, Espin, Groh, Steinwachs, Alexander, Marciano

This talk: top-to-bottom approach

Forget everything you know about gravity

Civilization with a thick impenetrable to sight atmosphere that did not discover Newtonian gravity

But did discover laws of mechanics, electromagnetism, then quantum mechanics and quantum field theory

They know the importance of gauge theories

(forces between particles)

They also know about topological quantum field theory

(because of applications to CFT, condensed matter)

Diffeomorphism invariant gauge theories

(in 4 spacetime dimensions)

Dynamically non-trivial theories of gauge fields that use no external structure (metric) in their construction

"TQFT's" with local degrees of freedom

Let A be a G-connection

$$F = dA + (1/2)[A, A]$$

curvature 2-form

Can define a gauge and diffeomorphism invariant action

$$S[A] = \int f(F \wedge F)$$

no dimensionful coupling constants!

Functions of the curvature

Let f be a function on $\mathfrak{g} \otimes_S \mathfrak{g}$

satisfying

$\mathfrak{g}$ - Lie algebra of G

$$f : X \to \mathbb{R}(\mathbb{C})$$

$$X \in \mathfrak{g} \otimes_S \mathfrak{g}$$

defining function

1) $f(\alpha X) = \alpha f(X)$ homogeneous degree 1

2) $f(\mathrm{Ad}_g X) = f(X), \quad \forall g \in G$ gauge-invariant

Then $f(F \wedge F)$ is a well-defined 4-form (gauge-invariant)

In practice:

define $\tilde{X}^{IJ} := \frac{1}{4} \tilde{\epsilon}^{\mu\nu\rho\sigma} F^I_{\mu\nu} F^J_{\rho\sigma}$ so that

$$F^I \wedge F^J = \tilde{X}^{IJ} d^4x$$

then $f(F^I \wedge F^J) = f(\tilde{X}^{IJ}) d^4x$

Functions of the curvature

$\mathfrak{g}$ - Lie algebra of G

Let f be a function on $\mathfrak{g} \otimes_S \mathfrak{g}$ satisfying

$$f : X \to \mathbb{R}(\mathbb{C})$$

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Field equations: $d_A B = 0$

where $B = \frac{\partial f}{\partial X} F$ and $X = F \wedge F$

Second-order (non-linear) PDE's

compare Yang-Mills equations: $d_A B = 0$

where $B = {}^* F$

* - encodes the metric

$n = \dim(G)$

Dynamically non-trivial theory with 2n-4 propagating DOF

apart from the single point $f_{top} = \mathrm{Tr}(F \wedge F)$

Gauge symmetries:

$\delta_\phi A = d_A \phi$ gauge rotations

$\delta_\xi A = \iota_\xi F$ diffeomorphisms

Remark: first order formulation also available

$$S[B,A] = \int B^I \wedge F^I - V(B^I \wedge B^J)$$

B^I Lie algebra valued 2-form

$V : \mathfrak{g} \otimes_S \mathfrak{g} \to \mathbb{R}(\mathbb{C})$

gauge invariant, homogenous degree one function

integrating out B get a second order "pure connection" formulation

Are there any such theories?

G=U(1) $F \wedge F$ is just a 4-form

$$\Rightarrow \quad f(F \wedge F) = F \wedge F$$ trivial dynamics

G=SU(2)~SO(3) $F^i \wedge F^j$ is a 3 x 3 symmetric matrix (times a 4-form)

$$X^{ij} = O \,\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) O^T$$

$$\Rightarrow \quad f(X^{ij}) = f(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 \chi(\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1})$$

homogeneity degree one
function of 3 variables
invariant under $\lambda_1 \leftrightarrow \lambda_2$ etc.

no dimensionful couplings

Even for SU(2) the class of such theories is infinitely large

What are these theories about?

Let us take a closer look at SU(2)

Given a theory, i.e. a function f(X), and a connection A^i one can define a spacetime metric

This metric owes its existence to the isomorphism

$$\mathrm{SO}(6,\mathbb{C}) \sim \mathrm{SL}(4,\mathbb{C})$$

Very important for twistor theory

Dynkin diagrams

$\mathfrak{sl}(n+1)$ $\quad$ $\mathfrak{so}(2n)$

Proof: Consider the 6-dimensional space Λ^2 of 2-forms in $\mathbb{R}$

The wedge product makes Λ^2 into a metric space

$$\Lambda^2 \ni U, V \rightarrow (U, V) = U \wedge V / d^4x \in \mathbb{R}$$

metric of signature (3,3) if over $\mathbb{R}$

$\mathrm{SL}(4,\mathbb{R})$ acts on Λ^2 $\qquad G_\mu^\nu \in \mathrm{SL}(4,\mathbb{R})$

$$^G U_{\mu\nu} = G_\mu^\alpha G_\nu^\beta U_{\alpha\beta}$$

the wedge product metric is preserved

$$\Rightarrow \quad \mathrm{SL}(4,\mathbb{R}) \sim \mathrm{SO}(3,3)$$

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The isomorphism implies

$\mathrm{SL}(4)/\mathrm{SO}(4)$		$\mathrm{SO}(3,3)/\mathrm{SO}(3)\times\mathrm{SO}(3)$
	$\Leftrightarrow$	
conformal metrics on M		Grassmanian of 3-planes in Λ^2

Conformal metrics can be encoded into the knowledge of which 2-forms are self-dual

<u>Explicitly:</u> a triple of linearly independent 2-forms $B^i_{\mu\nu}$

$$\Rightarrow \quad g_{\mu\nu} \sim \tilde{\epsilon}^{\alpha\beta\gamma\delta}\epsilon^{ijk}B^i_{\mu\alpha}B^j_{\nu\beta}B^k_{\gamma\delta}$$

Urbantke formula

2-forms $B^i_{\mu\nu}$ are self-dual with respect to this metric

The isomorphism implies

SL(4)/SO(4)	$\Leftrightarrow$	SO(3,3)/SO(3) × SO(3)
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Urbantke formula

2-forms $B^i_{\mu\nu}$ are self-dual with respect to this metric

Definition of the metric:

Let A^i be an SU(2) connection

(SL(2,C) connection for Lorentzian signature)

$$F^i = dA^i + (1/2)[A, A]^i$$

$$F^i \wedge (F^j)^* = 0$$

reality conditions

declare F^i to be self-dual 2-forms $\Rightarrow$ conformal metric

To complete the definition of the metric need to specify the volume form

$$(\mathrm{vol}) := \frac{1}{\Lambda^2} f(F \wedge F)$$

$$\Lambda \sim 1/L^2$$

dimensionful parameter

$$S[A] = \Lambda^2 \int_M (\mathrm{vol})$$

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Field equations:

(*) $$d_A \left(\mathrm{Tr}\sqrt{X} (X^{-1/2})^{ij} F^j \right) = 0$$

second-order PDE's for the connection

Theorem:

For connections A^i satisfying (*)
the metric $g(A)$ is Einstein with non-zero scalar curvature Λ

In the opposite direction, the self-dual part of the Levi-Civita connection for an Einstein metric satisfies (*)

Caveat: only metrics with $\Lambda/3 + W^+$ invertible almost everywhere covered

examples not covered
$S^2 \times S^2$
Kahler metrics

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What do other choices of f(X) correspond to?

Can understand via perturbation theory

Need a non-zero connection to expand about

Homogeneous isotropic connection $A^i = a(t) dx^i$

$$F^i = a' dt \wedge dx^i + \frac{a^2}{2} \epsilon^{ijk} dx^j \wedge dx^k$$

$$\Rightarrow \quad F^i \wedge F^j = 2a' a^2 \delta^{ij} d^4 x$$

Such a connection is a solution for any f(X) for any $a(t)$

The corresponding metric is de Sitter of cosmological constant Λ (in flat slicing)

$$F^i = -\frac{\Lambda}{3} \Sigma^i$$

where Σ^i is a basis of self-dual 2-forms

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One gets the following linearised Lagrangian

$$L^{(2)} \sim \frac{\partial^2 f}{\partial X^{ij} X^{kl}} (\Sigma^{i\mu\nu} d_\mu a^j_\nu)(\Sigma^{k\rho\sigma} d_\rho a^l_\sigma)$$

Easy to show that for any f(X)

a^i_μ connection perturbation

d_μ de Sitter covariant derivative

$$\frac{\partial^2 f}{\partial X^{ij} X^{kl}}\Big|_{X=\mathrm{Id}} \sim P^{(2)}_{ijkl} := \delta_{i(k}\delta_{l)j} - \frac{1}{3}\delta_{ij}\delta_{kl}$$

Linearized Lagrangian is the same for any f(X)

point f(X)=Tr(X) is singular

Easy to show that describes spin 2 particles on de Sitter space

Any of SU(2) theories is a gravity theory!

Summary of the pure connection formulation of GR

- Uses connections instead of metrics to describe gravitons

  At the linearized level $S_+^3 \otimes S_-$ instead of $S_+^2 \otimes S_-^2$

parity invariance
non-manifest!

- Simpler than the metric-based GR in many aspects

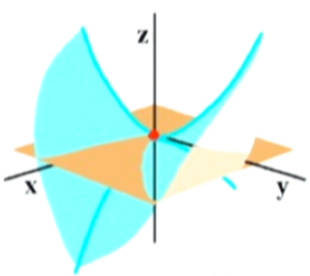
hermiticity
non-manifest!

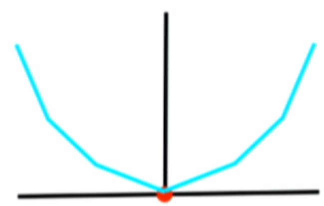
vertices are much simpler in this formulation

functional in the space of conformal classes rather than full metrics

convex action functional

$S_{\rm EH}[g]$





$S[A]$

space of me

space of conformal metrics =
2) connections/gauge

Gravity-Yang-Mills Unification

Consider a larger gauge group $G \supset \mathrm{SU}(2)$

No longer can define any natural metric

Can still understand the theory via linearization

Take a homogeneous isotropic connection to expand about

A class of backgrounds is classified by how SU(2) embeds into G

$$A^I = e_i^I a(t) dx^i$$

e_i^I an embedding of SO(3)~SU(2) into G

$$\epsilon^{ij}{}_k e_i^I e_j^J = f^{IJ}{}_K e_k^K$$

Solution of the field equations for any f(X)

Generically, there is a part of G that commutes with the SU(2)

E.g. $\mathrm{SU}(3) \ni \begin{pmatrix} \mathrm{SU}(2) & * \\ * & * \end{pmatrix}$

The SU(2) part of the connection continues to describe gravitons

The part that commutes with SU(2) describes YM gauge fields

Linearized Lagrangian

$$L^{(2)} \sim \frac{\partial^2 f}{\partial X^{i\alpha} X^{k\beta}} (\Sigma^{i\mu\nu} \partial_\mu a^\alpha_\nu)(\Sigma^{k\rho\sigma} \partial_\rho a^\beta_\sigma) \sim (\partial_{[\mu} a^\alpha_{\nu]})^2$$

linearized Yang-Mills

because $$\frac{\partial^2 f}{\partial X^{i\alpha} X^{k\beta}} \sim \delta_{ij} g_{\alpha\beta}$$

The part that does not commute with SU(2) describes somewhat exotic massive fields

Thus, depending on the background, different interpretation of the excitations of the gauge field

Generally: gravity + Yang-Mills gauge fields + massive fields

E.g. can have massive scalar field with SU(2)xSU(2) diagonal embedding

Other backgrounds are possible, but do not seem to change the qualitative picture

(Smolin, Speziale, Lizi, Alexander)

Gravity-Yang-Mills unification by "internal" Kaluza-Klein mechanism

simplest known unification mechanism; compare to usual KK

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Quantization: Very little is known

Can study perturbative quantization around any fixed background

Well under way, with some one-loop results already available

Hard to go further:

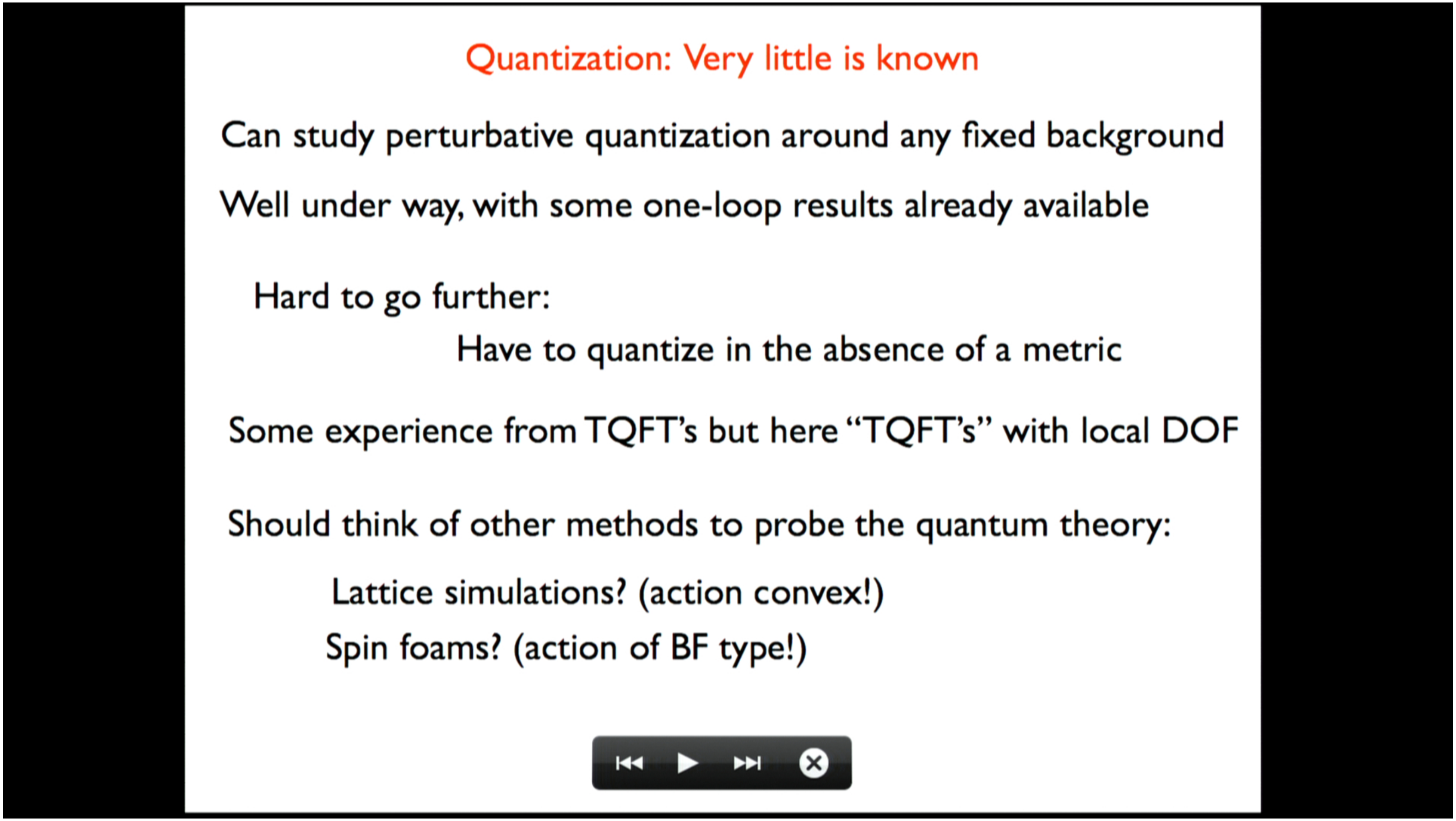
Have to quantize in the absence of a metric

Some experience from TQFT's but here "TQFT's" with local DOF

Should think of other methods to probe the quantum theory:

Lattice simulations? (action convex!)

Spin foams? (action of BF type!)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Spin foams? (action of BF type!)

Quantum Theory Hopes

Remark: no dimensionful coupling constants in any of these gravitational theories

(negative) dimension coupling constant comes when expanded around a background

Non-renormalizable in the usual sense

Hope: the class of theories {all possible f()} is large enough to be closed under renormalization

$$\frac{\partial f(F \wedge F)}{\partial \log \mu} = \beta_f(F \wedge F)$$

I.e. physics at higher energies continues to be described by theories from the same family

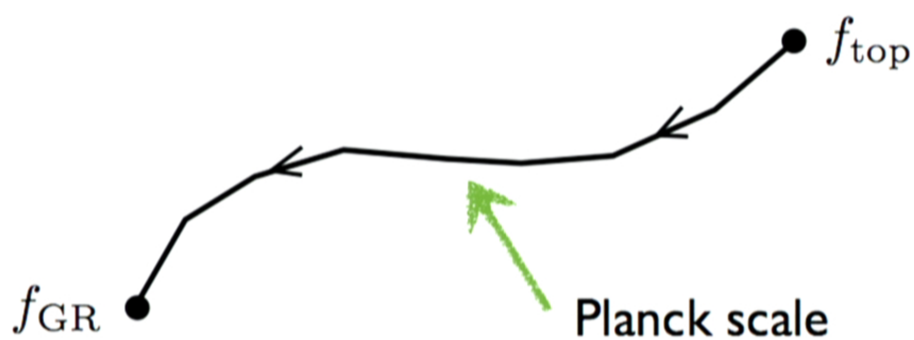
= no new DOF appear at Planck scale, just the dynamics changes

The speculative RG flow: topological theory ?

$$f_{\rm top}(F \wedge F) = {\rm Tr}(F \wedge F)$$

necessarily a fixed point of the RG flow

corresponds to a topological theory
(no propagating DOF)



flow from very steep in IR towards very flat in UV potential

Summary:

- Dynamically non-trivial diffeomorphism invariant gauge theories
- The simplest non-trivial such theory G=SU(2) - gravity

Diffeomorphism invariance leads to spin 2 particles!

- GR can be described in this language:

bounded from below Euclidean action

- Computationally efficient alternative to the usual description (no propagating conformal mode even off-shell)
- Can unify gravity with Yang-Mills in this framework:

"internal" Kaluza-Klein

- If this class of theories is closed under renormalization

$\Rightarrow$ understanding of the gravitational RG flow
description of the Planck scale physics

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Open problems

- Chiral, thus complex description. Unitarity?
- Coupling to matter?

  Enlarging the gauge group - rather general types of matter coupled to gravity can be obtained. Fermions?
- Closedness under renormalization?

Are these theories Yang-Mills theories 20 years before Veltman and ‘t Hooft?

Thank you