Title: Canonical Quantum Gravity - 4

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Abstract:

An Alternative State Space for Quantum Gravity

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Why?

- LQG treatment of holonomies / flux is very unbalanced
  $\rightarrow$ serious issue when looking for well-behaved coherent states
- working with a stack of small theories is technically comfortable until we try to go beyond fixed graph
  $\rightarrow$ 'cylindrical consistency' is hard to get, going to the dual space has its own drawbacks
- physical interpretation as specializing into specific d.o.f.'s of the continuous theory: why $\oplus$? it should be $\otimes$!
  [see also: Thiemann & Winkler 01]

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  [see also: Thiemann & Winkler 01]

How?

- usual construction relies on writing the **configuration** space as a projective limit $\rightarrow$ let's write the **phase** space as a projective limit... [see also: Thiemann 01]
- transcription at the quantum level $\rightarrow$ projective families of density matrices, the projections are given by appropriate partial traces [Kijowski 76, Okołów 09 & 13]
- physical insight $\rightarrow$ a given experiment only measures a finite number of observables

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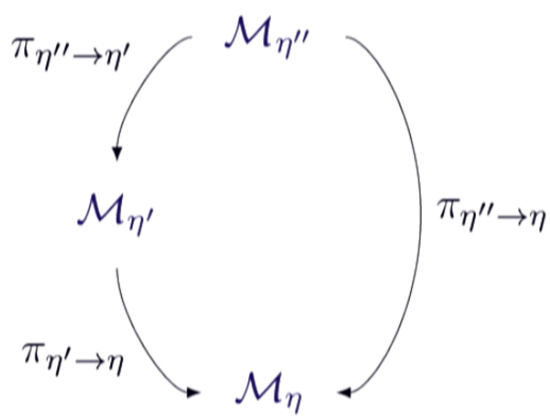
Contents

Projective Systems of Phase Spaces



$$\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$$

Collection of partial theories:

- label set $\mathcal{L}$, $\preccurlyeq$
- $\eta \in \mathcal{L}$ = a selection of d.o.f.'s
- 'small' symplectic manifolds $\mathcal{M}_\eta$

Ensuring consistency:

- projections $\pi_{\eta'\to\eta}$ for $\eta \preccurlyeq \eta'$
- compatible with symplectic structures
- 3-spaces-consistency $\to$ projective system

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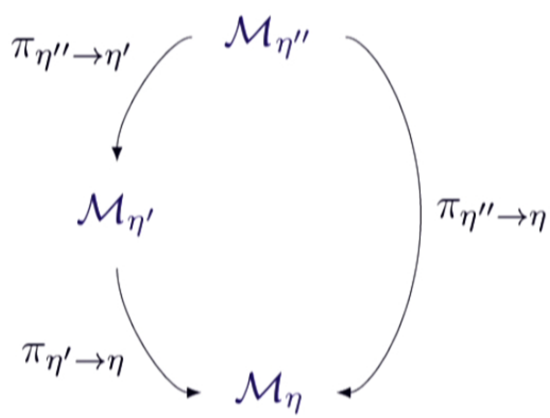

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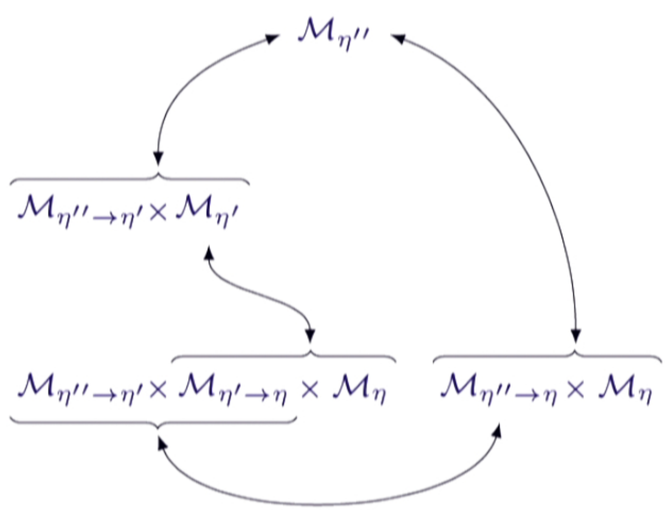
Projective Systems of Quantum State Spaces



$$\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$$

Modeled on special case:

- classical factorizations
  $\mathcal{M}_{\eta'} \approx \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_{\eta}$
- 3-spaces consistency
  $\mathcal{M}_{\eta''\to\eta} \approx \mathcal{M}_{\eta''\to\eta'} \times \mathcal{M}_{\eta'\to\eta}$
- quantum equivalent
  $\to \otimes$-factorizations

Projective families $(\rho_\eta)_{\eta\in\mathcal{L}}$:

- ρ_η density matrix on $\mathcal{H}_\eta$
- $\mathrm{Tr}_{\mathcal{H}_{\eta'\to\eta}}\, \rho_{\eta'} = \rho_\eta$

Holonomy-Flux Algebra

The label set

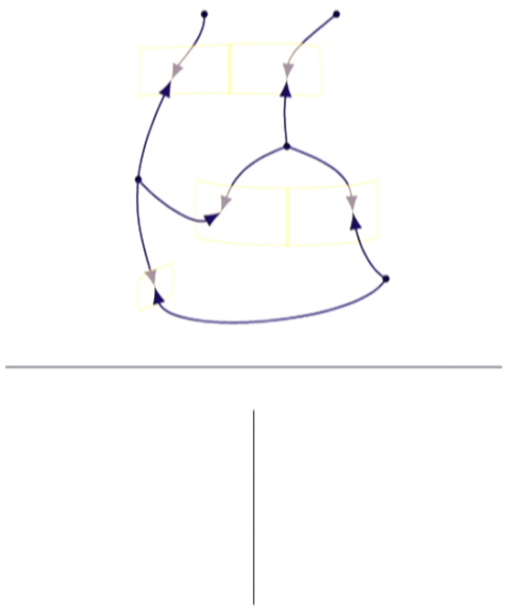
The label set:

- a graph = a choice of configuration variables
- a set of flux for this graph = a choice of conjugate momentum variables
- the label set must be directed (any two labels η, η' have a common finer label $\eta'' \succcurlyeq \eta, \eta'$)

[Holonomy-flux algebra: Ashtekar, Isham, Rovelli, Smolin, Lewandowski, Pullin, Gambini,...]

Holonomy-Flux Algebra

The factorizations



The state spaces:

- $L_2\left(G^n,\, d\mu_{\mathrm{Haar}}\right)$
- one group variable per edge

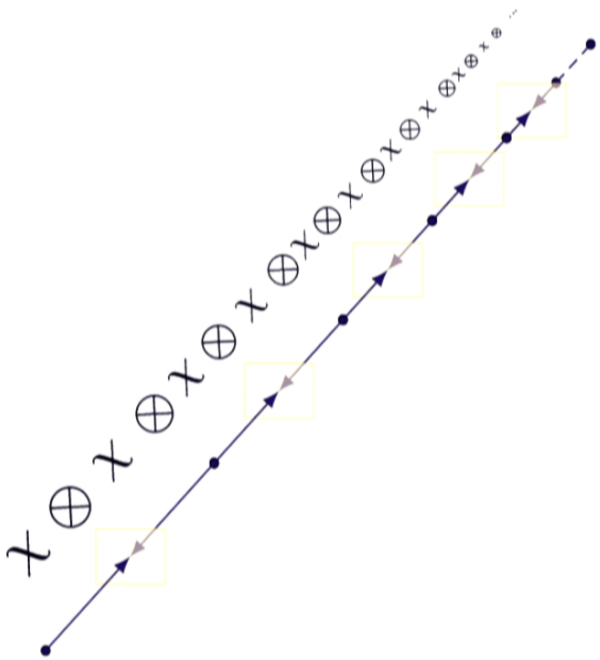
The factorizations:

- $G^n \approx G^m \times G^{n-m}$
- selecting specific edges $\rightarrow$ prescribes the factor G^m
- selecting specific flux $\rightarrow$ prescribes the complementary factor G^{n-m}

[Holonomy-flux algebra: Ashtekar, Isham, Rovelli, Smolin, Lewandowski, Pullin, Gambini,...]

Holonomy-Flux Algebra

Relation to the usual LQG Hilbert space



There is an **injective** map embedding the space of density matrices on $\mathcal{H}_{\text{LQG}}$ into the projective state space.

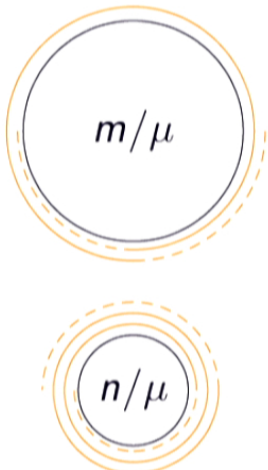
This map is **not surjective**.

We have states with narrow distribution for infinitely many holonomies:

- first step toward satisfactory coherent states
- but there remain deeper problems...

[LQG Hilbert space: Isham, Ashtekar, Lewandowski,...]

Loop Quantum Cosmology



$L_2(\mathcal{U}_1)$

↕

$L_2(\mathcal{U}_1) \otimes \mathbb{C}^k$

$$n = m/k$$

$$m,\ n,\ k \in \mathbb{N}$$

Label set $\{n \in \mathbb{N}\}$:

- with order $n \mid m$
- less observables than on $\mathcal{H}_{\text{LQC}}$

The projective structure:

- Hilbert spaces $L_2(\mathcal{U}_1)$
- $L_2(\mathcal{U}_1) \approx L_2(\mathcal{U}_1) \otimes \mathbb{C}^k$

[LQC: Bojowald, Ashtekar, Pawlowski, Singh, Lewandowski,...]

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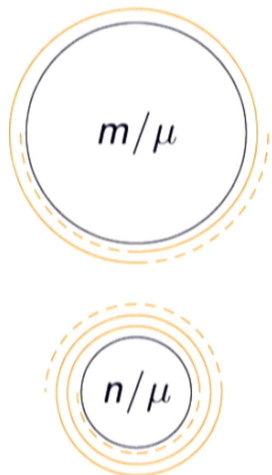

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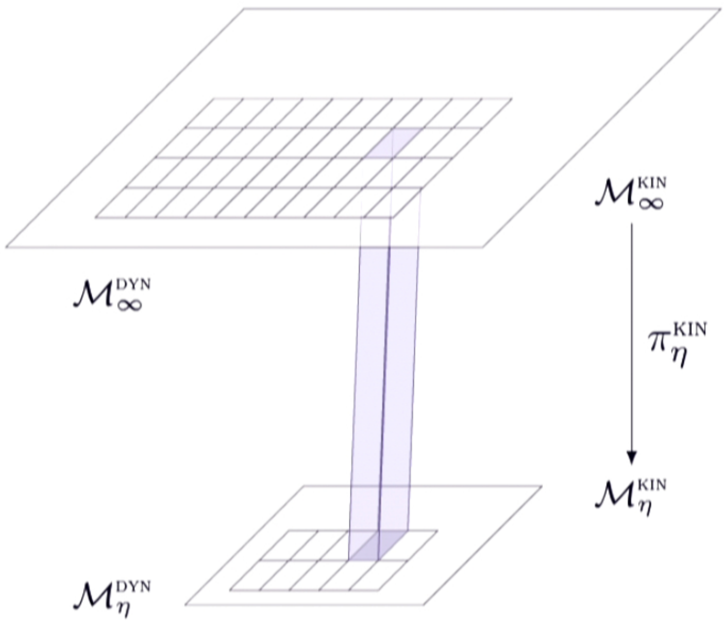
Nice Constraints



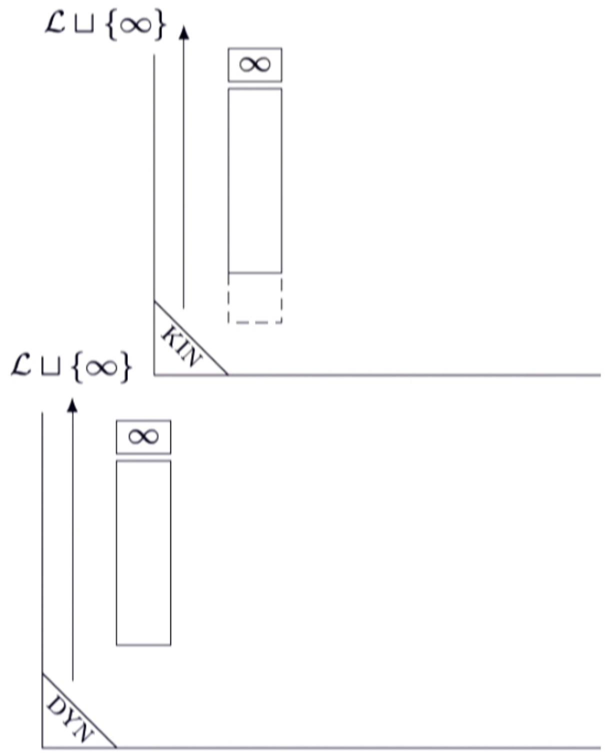
Restrictive requirements:

- orbits are projected on orbits $\rightarrow \pi^{\text{DYN}}_{\eta}$ between reduced phase spaces
- compatible with symplect. structures

Dynamical projective system & transport maps:

- states to projective families of orbits
- observables

Unfitting Constraints



Successive approximations:

- labeled by $\varepsilon \in \mathcal{E}$
- nice on smaller and smaller cofinal parts of $\mathcal{L}$
- dynamical projective system on a subset of $\mathcal{E} \times \mathcal{L}$
- convergence for a subset of the dynamical projective state space

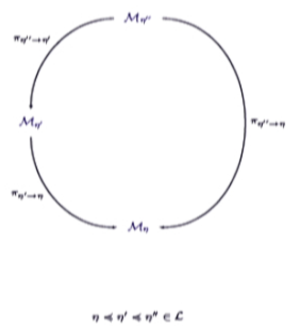
Summary

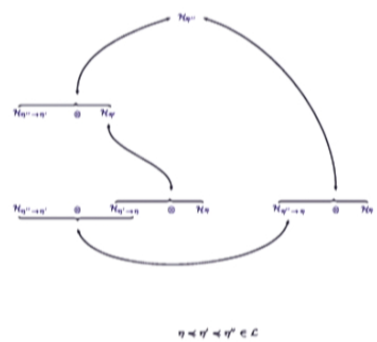
- we can construct projective state spaces for LQG and LQC
- results obtained in fixed graph can be directly imported
- assembling is done with a different interpretation $\rightarrow \eta$ selects **observables**, not **states**
- immediate payoff $\rightarrow$ states that were not constructible on $\mathcal{H}_{\mathsf{LQG}}$ can be designed
- needed input for dealing with constraints $\rightarrow$ regularizing scheme + projections between the approximated theories

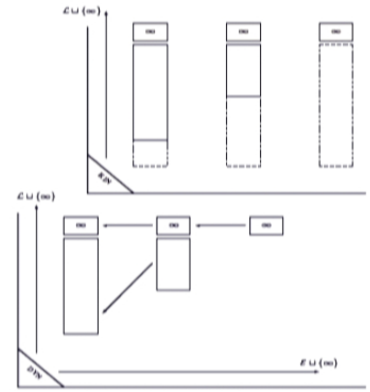
What next?

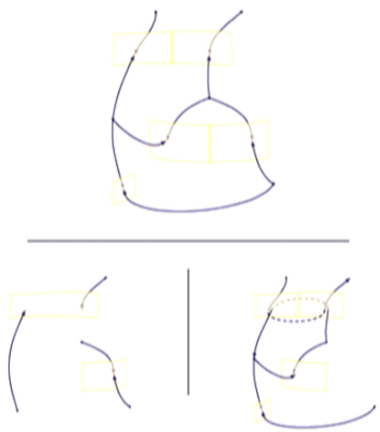
- good coherent states: there are deeper problems (related to the structure of the algebra itself) $\rightarrow$ drastically cut down the label set?
- link between LQG and LQC $\rightarrow$ partly depends on progress in the previous point
- solving Gauss and diffeo constraints, ultimately even Hamiltonian constraint
- application to QFT $\rightarrow$ relation between regularization schemes and renormalization techniques?

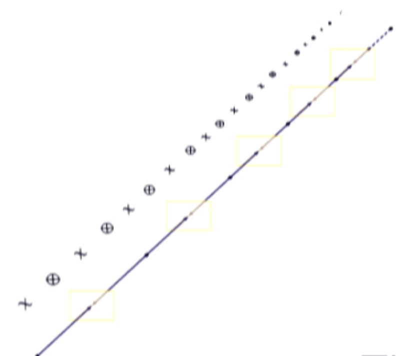
Projective State Spaces for LQG / LQC











Thank you!

Classical and Distributional Symmetric Connections in LQG

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Emmy Noether-Programm
Deutsche Forschungsgemeinschaft DFG

July 25. 2013

General Setting: Symmetries

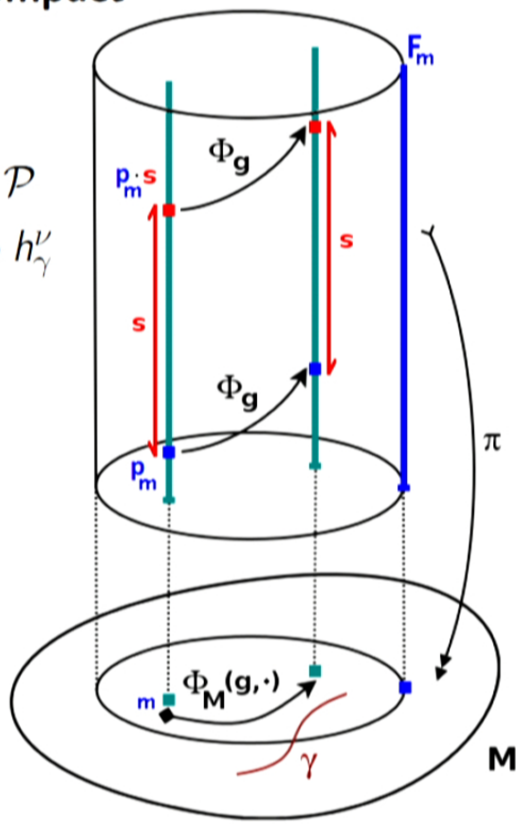
Principal fibre bundle (P, π, M, S) **with** S **compact**

- $\mathcal{A}$ - smooth connections on P
- $\mathcal{P}$ - some smooth curves in M
- $\mathrm{Cyl}(\mathcal{P})$ - cylindrical functions on $\mathcal{A}$ w.r.t. $\mathcal{P}$

  $\rightarrow$ unital C^*-subalgebra generated by $\rho_{ij} \circ h^{\nu}_{\gamma}$

Quantum Configuration Space

$\overline{\mathcal{A}} = \mathrm{Spec}(\mathfrak{A})$ for $\mathfrak{A} = \mathrm{Cyl}(\mathcal{P})$



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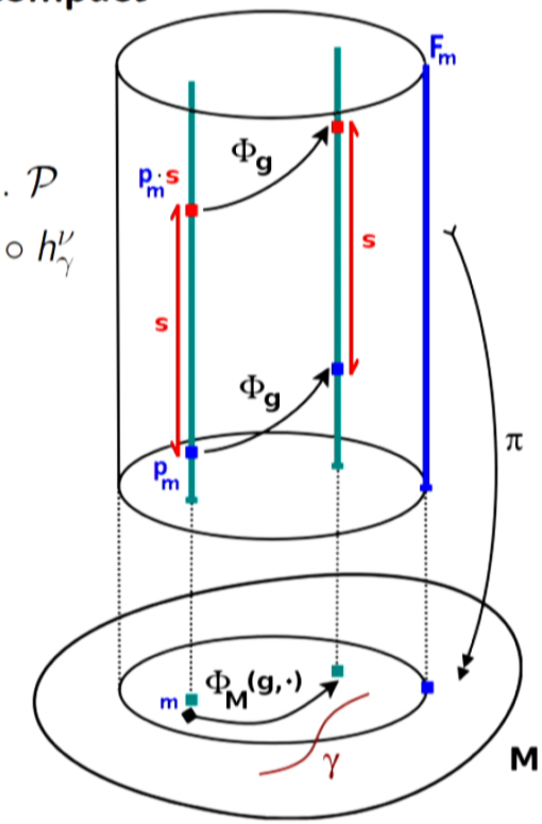

Quantization of Reduced Classical Configuration Space

Symmetry (G, Φ) on P

Choose $\mathcal{P}$

$\overline{\mathcal{A}} = \mathrm{Spec}(\mathrm{Cyl}(\mathcal{P}))$

$\iota \colon \mathcal{A} \to \overline{\mathcal{A}}$

$\omega \mapsto [f \mapsto f(\omega)]$

Reduction

Compute G-invariant connections

$\mathcal{A}_G = \{\omega \in \mathcal{A} \mid \Phi_g^* \omega = \omega \text{ for all } g \in G\}$

$\Rightarrow \omega$ invariant iff $\mathrm{Stab}_\phi(\omega) = G$

Compute $\overline{\iota(\mathcal{A}_G)}$

$\widehat{\mathcal{A}_G} := \overline{\iota(\mathcal{A}_G)} \cong \mathrm{Spec}\left(\overline{\mathrm{Cyl}(\mathcal{P})|_{\mathcal{A}_G}}\right)$

Two Cosmological Quantum Configuration Spaces

- $\mathfrak{B} := \overline{\mathrm{Cyl}(\mathcal{P})|_{\mathcal{A}_G}} \longrightarrow \widehat{\mathcal{A}_G} \cong \mathrm{Spec}(\mathfrak{B})$
- $i\colon \mathcal{A}_G \to \mathcal{A}$ inclusion map

	ABL [2003]	Fleischhack [2010]
$\mathcal{P}$	linear curves	embedded analytic curves
$\mathfrak{B}$	$C_{\mathrm{AP}}(\mathbb{R})$	$C_0(\mathbb{R}) \oplus C_{\mathrm{AP}}(\mathbb{R})$
$\widehat{\mathcal{A}_G}$	$\mathbb{R}_{\mathrm{Bohr}}$	$\mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$
$+$	μ_{Bohr}	i extends to embedding $\overline{i}\colon \mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}} \to \overline{\mathcal{A}}_\omega$
$-$	no canonical extension of i *a.)*	no Haar measure *b.)*

a.) Brunnemann, Fleischhack 2009

b.) **Theorem** (*MH: math-ph arXiv:1307.5303v1*)

There is no continuous group structure on $\mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$.

Cylindrical measures on $\overline{\mathbb{R}} := \mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}}$

However, topology on $\overline{\mathbb{R}}$ is such that each normalized Radon measure μ on $\mathfrak{B}(\overline{\mathbb{R}})$ is of form

$$\mu(A) = t \cdot \mu_1(A \cap \mathbb{R}) + (1-t) \cdot \mu_2(A \cap \mathbb{R}_{\mathrm{Bohr}}) \qquad \forall\, A \in \mathfrak{B}(\overline{\mathbb{R}})$$

for some $0 \leq t \leq 1$ with n.R.m.'s μ_1 on $\mathfrak{B}(\mathbb{R})$, μ_2 on $\mathfrak{B}(\mathbb{R}_{\mathrm{Bohr}})$.

Motivation to look for alternative reduction concept

Non-transitive situations for $P = \mathbb{R}^3 \times SU(2)$

(Semi-)homogeneous case:

$V \subseteq \mathbb{R}^3$ linear subspace and $\Phi(v, (x, s)) = (v + x, s)$

Isotropic case:

$SU(2)$ linear subspace and $\Phi(\sigma, (x, s)) = (\sigma(x), \sigma s)$

Symmetry	$\mathcal{A}_G$	Methods $\widehat{\mathcal{A}_G}$
homogeneous	$\cong \mathbb{R}^9$	$\mathrm{Spec}(\mathfrak{B})$ (hard)
semi-homogeneous	par. by functions	other ?
Isotropic	par. by functions	other ?

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(P, π, M, S) with S compact

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Symmetry (G, Φ) on P

↓

$\phi\colon G \times \mathcal{A} \to \mathcal{A}$

General Concept

(P, π, M, S) with S compact

MH: math-ph
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Symmetry (G, Φ) on P

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$\overline{\mathcal{A}} = \mathrm{Spec}(\mathrm{Cyl}(\mathcal{P}))$

$\phi\colon G \times \mathcal{A} \to \mathcal{A}$

Lift

$\theta\colon G \times \overline{\mathcal{A}} \to \mathcal{A}$

$\mathcal{A}_G = \{\omega \in \mathcal{A} \mid \mathrm{Stab}_\phi(\omega) = G\}$

compact spaces

$\overline{\mathcal{A}}_G := \{\overline{\omega} \in \overline{\mathcal{A}} \mid \mathrm{Stab}_\theta(\overline{\omega}) = G\}$ $\supseteq$ $\widehat{\mathcal{A}_G} = \overline{\iota(\mathcal{A}_G)} \subseteq \overline{\mathcal{A}}$

$P = \mathbb{R}^3 \times SU(2)$

Φ	isotropic	(semi-)homogeneous	homogeneous isotropic
Φ_M	rotations	translations	euclidean group

$\Rightarrow \mathcal{P}_1,\ \mathcal{P}_\omega$ Φ-invariant

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- Bijection $\kappa : \overline{\mathcal{A}}_\alpha \to \mathrm{Hom}(\mathcal{P}, SU(2))$ for $\alpha = 1, \omega$
- $\mathrm{Hom}_G(\mathcal{P}, SU(2)) := \kappa(\overline{\mathcal{A}}_G)$ (G-inv. homomorphisms)

Loop Quantum Cosmology

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- Characterized by simple algebraic relations: $\phi \in \mathrm{Hom}_G(\mathcal{P}, SU(2))$ iff

$$\begin{array}{llcll} \mathrm{Euclidean:} & \phi(\vec{v} + \delta(\sigma)(\gamma)) & = & \alpha_\sigma \circ \phi(\gamma) & \forall\, (\vec{v}, \sigma) \in E,\ \gamma \in \mathcal{P}, \\ \mathrm{Rotations:} & \phi(\delta(\sigma)(\gamma)) & = & \alpha_\sigma \circ \phi(\gamma) & \forall\, \sigma \in SU(2),\ \gamma \in \mathcal{P}, \\ V \subseteq \mathbb{R}^3: & \phi(\vec{v} + \gamma) & = & \phi(\gamma) & \forall\, \vec{v} \in V,\ \gamma \in \mathcal{P}. \end{array}$$

Consequences:

- (Homogeneous) isotropic case: $\mu_{AL}(\widehat{\mathcal{A}_G}) \leq \mu_{AL}(\overline{\mathcal{A}}_G) = 0$ for $\alpha = 1, \omega$

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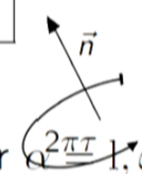

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-

$\mathcal{P}$	G = euclidean group
linear	$\mathbb{R}_{\mathrm{Bohr}} \cong \widehat{\mathcal{A}_G} = \overline{\mathcal{A}}_G$
embedded analytic	$\mathbb{R} \sqcup \mathbb{R}_{\mathrm{Bohr}} \cong \widehat{\mathcal{A}_G} \subsetneq \overline{\mathcal{A}}_G$

New elements e.g.:
$\phi(\gamma_c) = \exp(2\pi\tau\mu(\vec{n}))$
$\phi(\gamma) = e$ if γ not circular

Conclusion

G-invariant $\mathcal{P}$ allows for symmetry reduction on quantum level:

- Reduced space embedded in full theory since subset
- Det. by algebraic relations $\Rightarrow \mu_{\mathrm{AL}}(\mathcal{C}^{\omega}_{\mathrm{red,quant}}) = 0$ for $G = E, R$ (? V)
- $\mathcal{C}_{\mathrm{quant,red}} \leq \mathcal{C}_{\mathrm{red,quant}}$
- $$\left.\begin{array}{l} \overline{\mathcal{A}}_{\omega} \cong \mathrm{Hom}(\mathcal{P}_{\omega}, SU(2)) \longrightarrow \mathcal{C}^{\omega}_{\mathrm{quant,red}} \lneq \mathcal{C}^{\omega}_{\mathrm{red,quant}} \\ \overline{\mathcal{A}}_{\mathrm{l}} \cong \mathrm{Hom}(\mathcal{P}_{\mathrm{l}}, SU(2)) \longrightarrow \mathcal{C}^{\mathrm{l}}_{\mathrm{quant,red}} = \mathcal{C}^{\mathrm{l}}_{\mathrm{red,quant}} \end{array}\right| \quad ? \text{ for } G = R, V$$

On the Koslowski-Sahlmann representation of LQG

Miguel Campiglia

Raman Research Institute

PART I (based on work in collaboration with M. Varadarajan)

- How to describe non-compact, asymptotically flat geometries in LQG?
- Not clear how to impose condition

$$E^a \xrightarrow{r \to \infty} \mathring{E}^a$$

in terms of spin networks

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- How to describe non-compact, asymptotically flat geometries in LQG?
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$$E^a \xrightarrow{r\to\infty} \mathring{E}^a$$

  in terms of spin networks
- Idea: Use representation introduced by Koslowski and Sahlmann (KS)

KS representation

Koslowski (2007), Sahlmann (2010): Diffeo-covariant rep of holonomy-flux algebra that supports smooth geometries:

- Hilbert space basis

$$|s, \bar{E}\rangle = |s\rangle \otimes |\bar{E}\rangle \quad \text{with} \quad \langle s', \bar{E}' | s, \bar{E}\rangle = \langle s'|s\rangle \delta_{\bar{E}'\bar{E}},$$

s = spin network; $\bar{E}^a$ = $su(2)$-valued densitized vector field

- Action of holonomies, fluxes

$$\boxed{h_\gamma[A] = \mathcal{P}e^{\int_\gamma A}} \qquad \boxed{F_{S,f}[E] = \int_S dS_a \mathrm{Tr}[f\, E^a]}$$

$$\begin{aligned} \widehat{h_\gamma}|s, \bar{E}\rangle &:= \hat{h}^{\mathrm{LQG}}_\gamma |s\rangle \otimes |\bar{E}\rangle \\ \widehat{F_{S,f}}|s, \bar{E}\rangle &:= \hat{F}^{\mathrm{LQG}}_{S,f}|s\rangle \otimes |\bar{E}\rangle + F_{S,f}[\bar{E}]|s, \bar{E}\rangle \end{aligned}$$

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$$|s, \bar{E}\rangle = |s\rangle \otimes |\bar{E}\rangle \quad \text{with} \quad \langle s', \bar{E}'|s, \bar{E}\rangle = \langle s'|s\rangle \delta_{\bar{E}'\bar{E}},$$

s = spin network; $\bar{E}^a = su(2)$-valued densitized vector field

- Action of holonomies, fluxes and background exponentials

$$h_\gamma[A] = \mathcal{P}e^{\int_\gamma A} \qquad F_{S,f}[E] = \int_S dS_a \mathrm{Tr}[f\, E^a] \qquad \beta_{\bar{E}}[A] = e^{i\int_\Sigma \mathrm{Tr}[\bar{E}^a A_a]}$$

Varadarajan (2013)

$$\begin{aligned}
\widehat{h_\gamma}|s, \bar{E}\rangle &:= \hat{h}_\gamma^{\mathrm{LQG}}|s\rangle \otimes |\bar{E}\rangle \\
\widehat{F_{S,f}}|s, \bar{E}\rangle &:= \hat{F}_{S,f}^{\mathrm{LQG}}|s\rangle \otimes |\bar{E}\rangle + F_{S,f}[\bar{E}]|s, \bar{E}\rangle \\
\widehat{\beta_{\bar{E}'}}|s, \bar{E}\rangle &:= |s, \bar{E}' + \bar{E}\rangle
\end{aligned}$$

KS representation

Koslowski (2007), Sahlmann (2010): Diffeo-covariant rep of holonomy-flux algebra that supports smooth geometries:

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\end{aligned}$$

Phases in gauge transformations

- Under gauge transformations $g \in \mathcal{G}$, background exponentials transform as:

$$g \cdot \beta_{\bar{E}}[A] := \beta_{\bar{E}}[g^{-1} \cdot A] = e^{i\alpha(g,\bar{E})} \beta_{g\bar{E}g^{-1}}[A] \qquad (\star)$$

$\alpha(g,\bar{E}) = \int_{\Sigma} \mathrm{Tr}[\bar{E}^a g^{-1} \partial_a g]$

Example: $U(1)$ gauge averaging in Abelian theory

Abelian $U(1)$ theory:

- $\bar{E}^a$=densitized vector field,

$$g = e^{i\theta(x)}, \quad U(g)|\bar{E}\rangle = e^{-i\int_\Sigma \theta\partial_a \bar{E}^a}|\bar{E}\rangle$$

- $\mathcal{G}$ group averaging:

$$\eta(|\bar{E}\rangle) = \sum_{[g]\in\mathcal{G}/\mathrm{Sym}_{|\bar{E}\rangle}} U(g)|\bar{E}\rangle \qquad (*)$$

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- If $\partial_a \bar{E}^a = 0, \qquad \mathcal{G}/\mathrm{Sym}_{|\bar{E}\rangle} = 1 \qquad \Rightarrow \eta(|\bar{E}\rangle) = |\bar{E}\rangle$
- If $\partial_a \bar{E}^a \neq 0, \qquad \mathcal{G}/\mathrm{Sym}_{|\bar{E}\rangle} \approx U(1) \qquad \Rightarrow \eta(|\bar{E}\rangle) := 0$
  - $(*)$ Ill-defined: Infinite sum of phases
  - But sum takes the form: $\sum_{u\in U(1)} u$

$SU(2)$ case

- In $SU(2)$ theory: Group averaging sum potentially ill-defined if $\exists\ g$ such that

$$U(g)|\bar{E}\rangle \quad = \quad e^{i\alpha(g,\bar{E})}|\bar{E}\rangle \quad \neq |\bar{E}\rangle$$
$$\Updownarrow$$
$$\bar{E}^a = X^a \hat{n} \quad \text{and} \quad g = e^{\theta \hat{n}}$$

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- $\alpha(e^{\theta\hat{n}}, X^a\hat{n}) = -\int_\Sigma \theta\partial_a X^a$

- Situation analogous to Abelian case:

$$\partial_a X^a \neq 0 \quad \rightarrow \sum_{u\in U(1)} u \Rightarrow \quad \eta(|X^a\hat{n}\rangle) := 0$$

- All potentially ill-defined cases are taken care of and one obtains a well-defined $\mathcal{G}$ group averaging map
- We have also implemented $\mathcal{G} \rtimes \text{Diff}$ group averaging and included spin networks. But due to time constraints I will now move on to Part II

PART II

- KS states $|s, \bar{E}\rangle$ have two type of labels
- Can there be a unified description of s and $\bar{E}$?
- Evidence (for the case of a Wilson loop) that at gauge invariant level: "$s =$ distributional $\bar{E}$"
- The following argument is in connection representation so "$\beta_{\bar{E}}[A] = \langle A|\bar{E}\rangle$"

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Wilson loops and distributional $\bar{E}^a$

- $U(1)$ Abelian case: $\bar{E}^a(x) := \int dt\, \dot{\gamma}^a(t)\delta(x, \gamma(t))$

$$\Rightarrow \beta_{\bar{E}}[A] = e^{i\int_\Sigma \bar{E}^a A_a} = e^{i\int_\gamma A} = W_\gamma[A]$$

Wilson loops and distributional $\bar{E}^a$

- Idea: write $W_\gamma[A] = \mathrm{Tr}[h_\gamma[A]]$ as composition of N holonomies and insert resolution of identity $\mathbf{1} = \int_{S^2} |\hat{n}\rangle\langle\hat{n}|$ in coherent state basis:

$$
\begin{aligned}
W_\gamma &= \mathrm{Tr}[h_{\gamma(\epsilon)} h_{\gamma(2\epsilon)} \ldots] \\
&= \int_{(S^2)^N} \langle \hat{n}_0 | h_{\gamma(\epsilon)} | \hat{n}_1 \rangle \langle \hat{n}_1 | h_{\gamma(2\epsilon)} \ldots | \hat{n}_0 \rangle
\end{aligned}
$$

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\end{aligned}
$$

- In the limit $N \to \infty$ one obtains (Diakonov and Petrov 1989):

$$W_\gamma[A] = \int \mathcal{D}\hat{n}(t)\, e^{i\Phi[\hat{n}]} e^{i\int_\gamma \mathrm{Tr}[A\hat{n}]} \qquad (\star)$$

$\Phi[\hat{n}]$ = area enclosed by $\hat{n}(t) \in S^2$

- $g\cdot\; e^{i\Phi[\hat{n}]} e^{i\int_\gamma \mathrm{Tr}[A\hat{n}]} = e^{i\Phi[g\hat{n}g^{-1}]} e^{i\int_\gamma \mathrm{Tr}[Ag\hat{n}g^{-1}]}$
- $(\star)$ Represents an $SU(2)$ gauge group averaging

Summary/Outlook

- Improved KS treatment
  - Unnoticed subtlety in group averaging due to presence of phases
  - Identified potentially 'problematic' configurations (those admitting symmetries with non-trivial phases)
  - Well defined $\mathcal{G} \rtimes \text{Diff}$ group averaging with the corrected action (Here only discussed $\mathcal{G}$ group averaging of 'pure backgrounds' in Abelian case)
- Importance of phases
  - To recover correct answer in Abelian theory
  - Relation with Wilson loop in non-Abelian theory
- In progress:
  - Use KS to address asymptotically flat spaces (analogue problem in PFT addressed by Sengupta)
  - Characterization of KS quantum configuration space $\overline{\mathcal{A}}$
  - Gauge invariant spin networks from gauge averaging of distributional backgrounds

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Outline

1. **Motivation and Introduction**
2. **Connection Dynamics of Scalar-Tensor Theories**
3. **Loop Quantum Scalar-Tensor Theories of Gravity**
4. **Loop Quantum Brans-Dicke Cosmology**
5. **Summary and Outlook**

Success of LQG and Its Scope

⋆ It is remarkable that, as a non-renormalizable theory, GR can be non-perturbatively quantized by the loop quantization procedure. What is the applicable scope of loop quantum gravity?

- LQG can be extended to $f(R)$ theories of gravity [Zhang, YM, 2011].
- LQG is applicable to GR in arbitrary dimensions [Bodendorfer, Thiemann, Thurn, 2011].

Success of LQG and Its Scope

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  - LQG can be extended to $f(R)$ theories of gravity [Zhang, YM, 2011].
  - LQG is applicable to GR in arbitrary dimensions [Bodendorfer, Thiemann, Thurn, 2011].
- To explain the accelerated expansion of the universe, as well as dark matter, from fundamental physics is now a great challenge.
- A large variety of models of $f(\mathcal{R})$ modified gravity have been proposed to account for the dark energy and the dark matter problems. [Sotiriou and Faraoni 2010]

Gravity as Geometry

- $f(\mathcal{R})$ modified gravity is also implied by some other approaches to quantum gravity [e.g. Asymptotically safe gravity: talk by Saueressig].
- If some modified gravity theory becomes fundamental rather than GR, one has to consider its quantization as well.

Gravity as Geometry

- $f(\mathcal{R})$ modified gravity is also implied by some other approaches to quantum gravity [e.g. Asymptotically safe gravity: talk by Saueressig].
- If some modified gravity theory becomes fundamental rather than GR, one has to consider its quantization as well.
- Besides GR, scalar-tensor theories belong to metric theories of gravity.
- For metric theories, gravity is still geometry with diffeomorphism invariance as in GR.

  The differences between them are just reflected in dynamical equations and additional variables.

  Hence, a background-independent and non-perturbative quantization for metric theories of gravity is preferable.
- Since scalar-tensor theories (STT) include $f(\mathcal{R})$ theories as special case and have received increased attention due to motivations coming form cosmology and astrophysics, we will take them as examples to carry out the extension of LQG to metric theories.

Action Principle of STT

- The most general action of STT reads

$$S[g,\phi]=\int_M d^4x\sqrt{-g}[\frac{1}{2}(\phi\mathcal{R}-\frac{\omega(\phi)}{\phi}(\partial_\mu\phi)\partial^\mu\phi)-\xi(\phi)] \qquad (1)$$

where we set $8\pi G=1$, $\mathcal{R}$ denotes the scalar curvature of spacetime metric $g_{\mu\nu}$, the coupling parameter $\omega(\phi)$ and potential $\xi(\phi)$ can be arbitrary functions of scalar field ϕ.

First-order Action for STT

- A first-order action for STT, which is equivalent to action (1) but can lead to a Hamiltonian connection formalism, reads [Zhou, Guo, Han, YM, 2013]

$$S[e,\omega,\phi] = \int_M \frac{e}{2}\Big(\phi e^a_I e^b_J \bar{\Omega}_{ab}{}^{IJ} - 2e^a_I e^b_J \bar{\omega}^{IJ}_a \bar{\partial}_b\phi + e_I^{[a} e_J^{b]} \bar{\partial}_a (e^I_b e^{cJ} \bar{\partial}_c \phi) + \frac{\omega(\phi)}{\phi}(\bar{\partial}_a\phi)\bar{\partial}^a\phi - 2V(\phi) + e^a_I e^b_J \frac{1}{\gamma} {}^\star\bar{\Omega}_{ab}{}^{IJ}\Big) d^4x \,, \quad (2)$$

where $e = det(e^I_a)$ is the determinant of the right-handed cotetrad e^I_a, $\bar{\Omega}_{ab}{}^{IJ} = \bar{\partial}_{[a}\bar{\omega}^{IJ}_{b]} + \bar{\omega}^{IK}_{[a}\bar{\omega}_{b]K}{}^J$ is the curvature of the $SL(2,\mathbb{C})$ spin connection $\bar{\omega}^{IJ}_a$, * denotes the Hodge dual of a differential form, and γ is an arbitrary real number.

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- Another first-order action was proposed by [Cianfrani, Montani, 2009], which can give the Hamiltonian connection formalism of STT in Einstein frame.

Connection Dynamics of STT

- The detailed Hamiltonian analysis of action (2) leads to the connection dynamics of STT in two sectors marked respectively by $\omega(\phi) \neq -\frac{3}{2}$ and $\omega(\phi) = -\frac{3}{2}$, coinciding with the results of canonical transformation from geometrical dynamics [Zhang, YM, 2011].
- The basic conjugate pairs consist of a $SU(2)$ connection A_a^i and the densitized triad E_j^b, together with the scalar ϕ and its momentum π.

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Polymer-like Representation

- The quantum kinematics of LQG can be straightforwardly extended to STT.
- For the geometry sector, we have the unique diffeomorphism and internal gauge invariant representation for the quantum holonomy-flux algebra [LOST 2005] .
- There is also a unique diffeomorphism invariant measure $d\mu$ on the space $\bar{\mathcal{U}}$ of polymer scalar fields [Ashtekar, Lewandowski, Sahlmann, 2002; Kaminski, Lewandowski, Bobienski, 2006].

Quantum Dynamics

- In the sector of $\omega(\phi) = -3/2$, we promote the conformal constraint $S(\lambda)$ as a well-defined operator by acting on a given basis vector $T_{\alpha,X} \in \mathcal{H}_{\rm kin}$ as

$$\hat{S}(\lambda) \cdot T_{\alpha,X} = \Big(\sum_{v \in V(\alpha)} \frac{\lambda(v)}{\gamma^{3/2}(i\hbar)} [\hat{H}^E(1), \hat{V}_v] - \sum_{x \in X} \lambda(x)\hat{\phi}(x)\hat{\pi}(x) \Big) \cdot T_{\alpha,X}.$$

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- By the regularization techniques developed for the Hamiltonian constraint operators of LQG [Thiemann, 1996], Polymer scalar field [Han, YM, 2006] and loop quantum $f(\mathcal{R})$ gravity [Zhang, YM, 2011], the Hamiltonian constraints in both sectors can be quantized as operators acting on cylindrical functions in $\mathcal{H}_{kin}$ in state-dependent ways.

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Symmetric Reduction

- Example: Spatially flat FRW universe
- The connections and the density weighted triads is reduced to [Ashtekar, Bojowald, Lewandowski, 2003]:
  $A_a^i = \tilde{c}\, V_o^{-(1/3)}\ {}^o\omega_a^i$ and $E_i^a = p\, V_o^{-(2/3)} \sqrt{{}^oq}\ {}^oe_i^a$.
- In the cosmological model of scalar-tensor theory, the fundamental Poisson brackets are given by:
  $\{\tilde{c},\, p\} = \gamma/3, \quad \{\phi,\, \pi\} = 1.$

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- For $\omega \neq -3/2$ sector of Brans-Dicke theory, the gravitational Hamiltonian reads

$$H = -\frac{3\tilde{c}^2\sqrt{p}}{\gamma^2\phi} + \frac{1}{(3+2\omega)\phi|p|^{\frac{3}{2}}}\left(\frac{3\tilde{c}p}{\gamma} + \pi\phi\right)^2. \qquad (3)$$

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- By employing (3) one can show that ϕ is monotonous wrt the cosmic time and hence can be viewed as an internal time.

Quantization Scheme

- To quantize the model, in the geometrical sector, we employ the polymer-like representation of connection $\tilde{c}$ in $\mathcal{H}_{\rm kin}^{\rm grav} = L^2(\mathbb{R}_{\rm Bohr}, \mathrm{d}\mu_{\rm Bohr})$.

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- To quantize the model, in the geometrical sector, we employ the polymer-like representation of connection $\tilde{c}$ in $\mathcal{H}_{\rm kin}^{\rm grav} = L^2(\mathbb{R}_{\rm Bohr}, {\rm d}\mu_{\rm Bohr})$.
- For a neat formulation of quantum dynamics, it is convenient to introduce new conjugate variables in the geometrical sector by a canonical transformation:

$$b := \frac{\sqrt{\Delta}}{2}\frac{\tilde{c}}{\sqrt{|p|}}, \qquad \nu := \frac{4}{3\sqrt{\Delta}} sgn(p)|p|^{\frac{3}{2}},$$

where Δ ($\sim 4\sqrt{3}\pi\gamma\, l_p^2$) is the smallest non-zero eigenvalue of area operator in full LQG.

- In the kinematical Hilbert space $\mathcal{H}_{\rm kin}^{\rm grav}$, eigenstates of $\hat{\nu}$, which are labeled by real numbers v, constitute an orthonormal basis as: $\langle v_1|v_2\rangle = \delta_{v_1,v_2}$.

Polymer-like representations

- For the quantization of the scalar field, we have two schemes: (i) Polymer-like representation; (ii) Schrodinger representation.
- Polymer-like representation:
  For the convenience of constructing a Hamiltonian constraint operator, we employ the polymer-like representation of the momentum π of ϕ.
- One parameter ambiguity: Some small number λ_0 has to be introduced in order to define the operator $\hat{\pi} \equiv \frac{\sin(\lambda_0 \pi)}{\lambda_0}$.
- In the kinematical Hilbert space $\mathcal{H}_{\rm kin}^{\rm scalar}$, eigenstates of $\hat{\phi}$, which are labeled by real numbers λ, constitute an orthonormal basis.
- The operator corresponding to ϕ^{-1} can be defined as

$$\begin{aligned}\widehat{\phi}^{-1}\,|\lambda\rangle &= \frac{4\mathrm{sgn}(\phi)}{(\lambda_0)^2\hbar}\Big||\lambda+\lambda_0|^{1/2}-|\lambda|^{1/2}\Big|^2\,|\lambda\rangle \\ &=: D(\lambda)\,|\lambda\rangle.\end{aligned}$$

The Dynamical Setting

- The Hamiltonian constraint for the $\omega \neq -3/2$ sector in the full theory of Brans-Dicke gravity reduces to 5 terms $H = \sum_{i=1}^{5} H_i$ for the spatially flat cosmology.
  They all can be quantized as following well-defined operators.

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- The Hamiltonian constraint for the $\omega \neq -3/2$ sector in the full theory of Brans-Dicke gravity reduces to 5 terms $H = \sum_{i=1}^{5} H_i$ for the spatially flat cosmology.
  They all can be quantized as following well-defined operators.
- The actions of the corresponding terms of the Hamiltonian operator on a basis vector $|\lambda, v\rangle \equiv |\lambda\rangle \otimes |v\rangle$ read respectively as

$$
\begin{aligned}
&(\hat{H}_1 + \hat{H}_2)|\lambda, v\rangle \\
= \quad & \frac{D(\lambda)}{2}\left(f_+(v)|\lambda, v+4\rangle + f_0(v)|\lambda, v\rangle + f_-(v)|\lambda, v-4\rangle\right),
\end{aligned}
$$

$$
\begin{aligned}
&\hat{H}_3|\lambda, v\rangle \\
= \quad & -\frac{4\gamma\beta D(\lambda)}{\hbar^2}\left(\tilde{f}_+(v)|\lambda, v+8\rangle - \tilde{f}_0(v)|\lambda, v\rangle + \tilde{f}_-(v)|\lambda, v-8\rangle\right),
\end{aligned}
$$

Quantum Dynamics

$$
\begin{aligned}
& \hat{H}_4|\lambda, v\rangle \\
= & -\frac{2\gamma^{1/2}\beta}{\lambda_0\hbar}[(B(v+4)+B(v))\times \\
& (f_+(v)|\lambda-\lambda_0, v+4\rangle + f_+(v)|\lambda+\lambda_0, v+4\rangle) \\
- & (B(v)+B(v-4))(f_-(v)|\lambda-\lambda_0, v-4\rangle - f_-(v)|\lambda+\lambda_0, v-4\rangle)],
\end{aligned}
$$

$$
\begin{aligned}
& \hat{H}_5|\lambda, v\rangle \\
= & -\frac{\beta\hbar B(v)}{4\lambda_0^2}\left((\lambda-\lambda_0)|\lambda-2\lambda_0, v\rangle - 2\lambda|\lambda, v\rangle + (\lambda+\lambda_0)|\lambda+2\lambda_0, v\rangle\right),
\end{aligned}
$$

where $\beta \equiv 3 + 2\omega$.

Polymer-like plus Schrodinger representations

- If we employ the Schrodinger representation of the scalar field, the Hamiltonian operator for Brans-Dicke cosmology can also be well defined on the coupled Hilbert space.
- In this scheme, we can obtain the effective Hamiltonian constraint as

$$H_F = -\frac{\sqrt{3\Delta}}{2\gamma^2\kappa\phi}|v|\sin^2 b + \frac{2\sqrt{3}\kappa}{\beta\Delta^{3/2}|v|\phi}\left(\frac{3\hbar}{4}\sin(b)v + \pi\phi\right)^2 + \frac{\Delta^{3/2}|v|}{2\sqrt{3}}\rho,$$

where ρ is the energy density of minimally coupled matter field.
- This effective Hamiltonian constraint of loop quantum Brans-Dicke cosmology gives the modified evolution equation:

$$\left(\frac{\dot{a}}{a} + \frac{\dot{\phi}}{2\phi}\right)^2 = \left(\frac{1}{\phi}\sqrt{\frac{\kappa}{3}\rho_e(1-\frac{\rho_e}{\rho_c})} + \frac{\dot{\phi}}{2\phi}(1-\sqrt{1-\frac{\rho_e}{\rho_c}})\right)^2, \quad (4)$$

where $\rho_e \equiv \frac{\beta\dot{\phi}^2}{4\kappa} + \phi\rho$ and $\rho_c \equiv \frac{3}{\gamma^2\Delta\kappa}$.

Quantum Bounce of Loop Quantum Brans-Dicke Cosmology

[Zhang, Artymowski, YM, 2013]

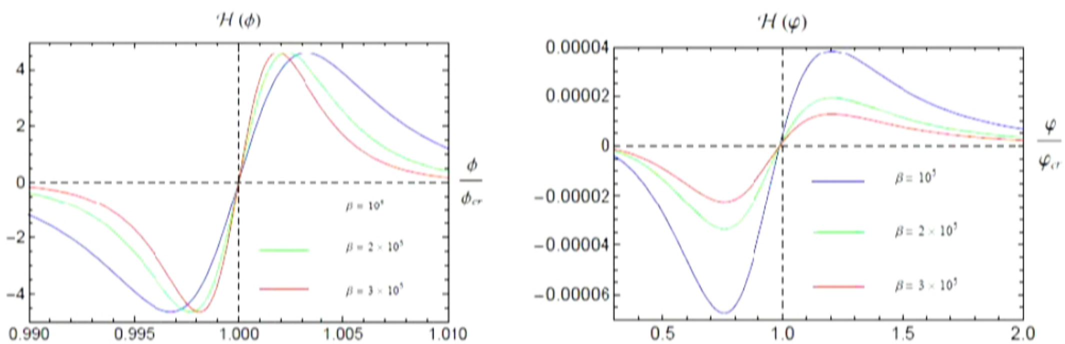

FIG. 1. *Left and right panels present the evolution of the Hubble parameter in the Planck units as a function of the Brans-Dicke field (left panel, vacuum solution) or the massless scalar field (right panel, massless scalar field domination) for realistic values of* β. *Initial conditions are chosen to be:* $\phi_{cr} = 1$ *(left panel) and* $\frac{\dot{\phi}_{cr}}{\dot{\varphi}_{cr}} = -\frac{\varphi_{cr}}{\beta}$, $\phi_{cr} - \frac{\varphi_{cr}^2}{\beta} - \frac{\dot{\phi}_{cr}}{\dot{\varphi}_{cr}}\varphi_{cr} = M_{pl}$ *(right panel)*

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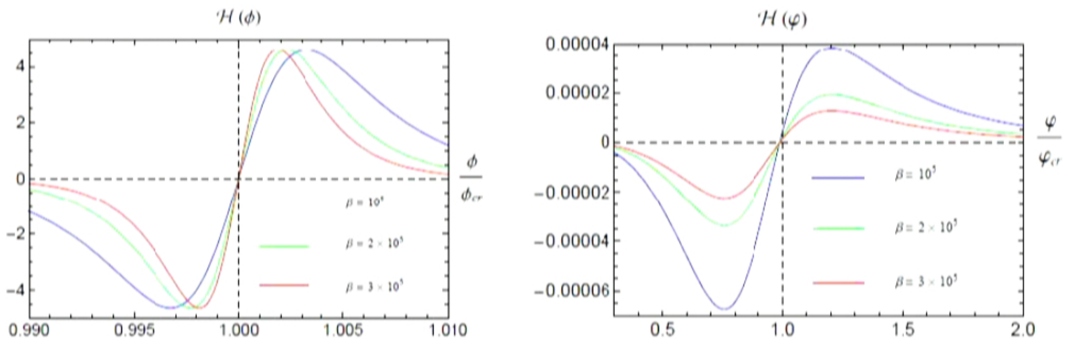

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Conclusions

- The Hamiltonian connection formulation of STT of gravity has been derived from their Lagrangian formulation.
  Two sectors of STT are marked off by the coupling parameter $\omega(\phi)$.
- In the sector of $\omega(\phi) = -3/2$, the feasible theories are restricted and a new primary constraint generating conformal transformations of spacetime is obtained.

Conclusions

- Scheme (i): polymer-like representations
  - The polymer representation of the momentum π of the scalar field is more convenient for the purpose of constructing manageable Hamiltonian constraint operator.
  - An one-parameter ambiguity appears in this construction for the scalar sector, which is also the feature of the full theory.
  - The Hamiltonian constraint operator gives rise to a difference equation representing a discrete evolution of the universe.
  - In contrast to the old treatment of LQC, both space and (internal) time are "discrete" in this treatment of loop quantum BD cosmology.

Outlook

- The method proposed by [Bodendorfer, Thiemann, Thurn, 2011] for the loop quantization of higher dimensional GR can also be extended to higher(> 4) dimensional scalar-tensor theories.
  [Han, YM, Zhang, 2013]

Outlook

- The method proposed by [Bodendorfer, Thiemann, Thurn, 2011] for the loop quantization of higher dimensional GR can also be extended to higher(>4) dimensional scalar-tensor theories.
[Han, YM, Zhang, 2013]
- Applications to black holes are desirable
[Guo, Zhang, YM...; related work see e.g. talk by Bodendorfer].
- Inflation of loop quantum Brans-Dicke cosmology:
[Zhang, Artymowski, YM...]
- It is also desirable to quantize metric theories of gravity by the covariant spin foam approach [Zhou, Zhang, YM...].

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!Thanks!