Title: Phenomenology - 4

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Abstract:

Exploring Cartan gravity with dynamical symmetry breaking

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- "Gravity, Cartan geometry, and idealized waywisers", arXiv:1203.5709
- "Cartan gravity, matter fields, and the gauge principle", arXiv:1209.5358
- "Gravity from dynamical symmetry breaking", arXiv:1302.1103

Is gravity just another Yang-Mills field?

"Electromagnetism is, as we have seen, a gauge field. That gravitation is a gauge field is universally accepted, although exactly how it is a gauge field is a matter still to be clarified."
Shiing-Shen Chern

Similarity with electroweak theory

- Cartan gravity provides a precise answer to Chern's question: gravity in the mathematical formulation of Cartan is a symmetry broken $SO(1,4)$ Yang-Mills gauge theory.
- As such Cartan gravity mirrors the structure of the electroweak theory of the standard model.
- In the electroweak theory we have:
  - a Yang-Mills field valued in the Lie-algebra of $SU(2) \times U(1)$
  - a Higgs field Φ that breaks the $SU(2) \times U(1)$ gauge symmetry down to $U(1)$, i.e. the gauge group of electromagnetism.
  - The electromagnetic gauge group is the subgroup of $SU(2) \times U(1)$ transformations that leaves the Higgs field Φ invariant.
- In Cartan gravity we have:
  - a Yang-Mills gauge field A^{AB} valued in the Lie albebra of $SO(1,4)$
  - a Higgs field V^A that breaks the $SO(1,4)$ gauge symmetry down to $SO(1,3)$, i.e. the gauge group of General Relativity in the tetrad formulation.
  - The local Lorentz group is the subgroup of $SO(1,4)$ transformations that leave the Higgs field V^A invariant.

Motivation and basic idea

- A glaring difference between these two theories is that, while the Higgs field Φ of the electroweak theory is treated as a genuine dynamical degree of freedom, the Higgs field in Cartan gravity V^A is not treated as a dynamical field satisfying non-trivial field equations. Instead, V^A is typically required to satisfy $V^2 = V^A V^B \eta_{AB} = const.$
- The idea here is to generalize Cartan gravity in a straightforward way and propose a simple way to provide dynamics to the Higgs field V^A. (Stelle&West 1980, Randono 2010).
- In doing so we introduce an additional scalar degree of freedom $|V|$ into gravity.
- At the same time we know that cosmologists frequently introduce, by hand, such scalar fields in order to model quintessence, dark matter, and inflation.
- In this generalized dynamical Cartan gravity such a scalar field comes as a part of the Cartan geometric package and it is then reasonable to expect it to have direct applications for cosmology.

Results

Indeed, by applying Cartan gravity with dynamical symmetry breaking to FRW cosmology we will see that that this theory yields:

1. The extensively studied Peebles-Ratra rolling quintessence model for a very simple action.
2. Signature change for the MacDowell-Mansouri action: the universe starts out in a timeless 4D Euclidean state, then undergoes a smooth transition from Euclidean to Lorentzian signature and time emerges. This is similar to Hartle-Hawking no-boundary proposal.
3. Recovery of General Relativity at late times with positive non-zero cosmological constant by adding a 'kinetic term' for $|V|$ to the MacDowell-Mansouri action.

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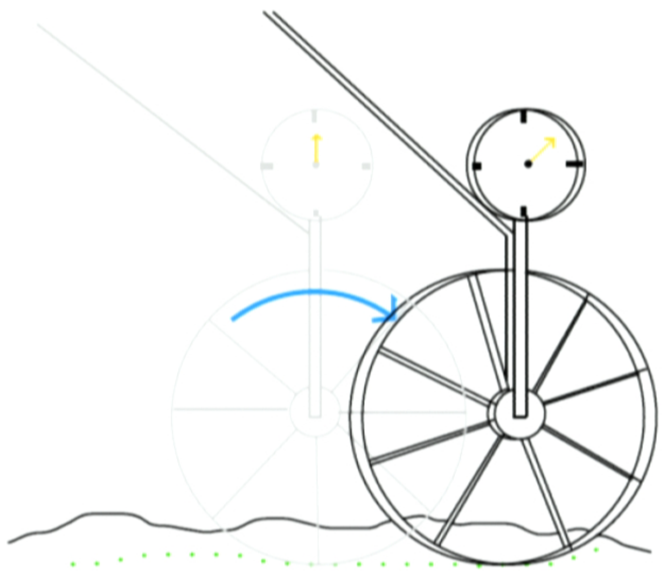
An introduction to Cartan geometry

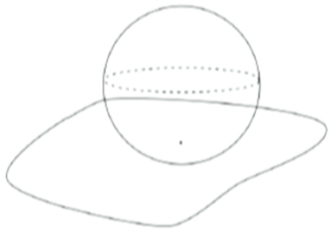
Basic idea of Cartan geometry



A waywiser measures the traversed distance by keeping track of how much the wheel has rotated.

Physical waywisers vs idealized waywisers

- Idealized waywisers, on the other hand, are Platonic creations of the mind where all irrelevant features, inherent in their material incarnations, have been stripped and abstracted away.
- The 'wheel' of an idealized waywiser we take to be the two-sphere. One point on that sphere will be singled out as the point of contact between the the sphere itself and the manifold.



- In addition to a contact point we need also a prescription for how the idealized waywiser rotates when rolled along some path.

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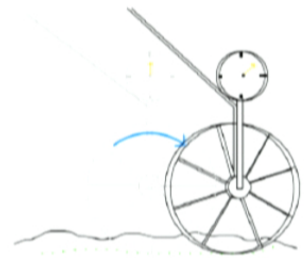

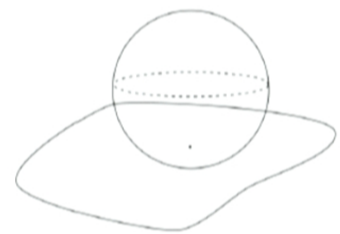

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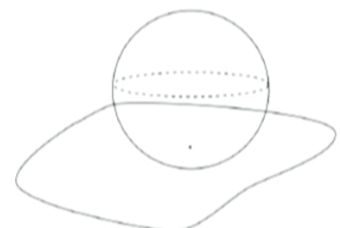

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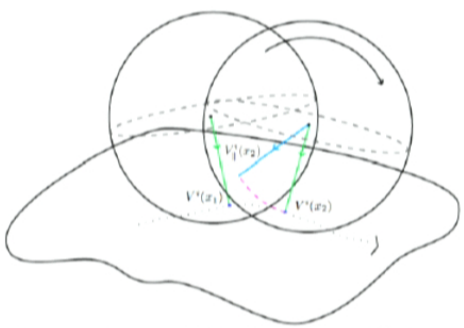
Mathematical representation of the contact point

- The wheel of the idealized waywiser is taken to be a sphere of some radius ℓ: $\mathcal{S}^2 = \{X^i \in \mathbb{R}^3 | X^i X^j \delta_{ij} = \ell^2\}$, $i = 1, 2, 3$.
- Since the contact point is also a point on the sphere we can mathematically represent it by a **contact vector** V^i satisfying $V^i V^j \delta_{ij} = \ell^2$. This contact vector breaks the $SO(3)$ symmetry to that of the tangent space $SO(2)$.
- We note that the contact vector $V^i(x)$ at some location x, points in the same direction (it is normal to the surface) regardless of how the waywiser ended up at that point. Thus, we can specify without loss of generality a *field* of contact vectors $V^i(x)$, with x^a $(a = 1, 2)$ coordinatizing the two-dimensional embedded surface.

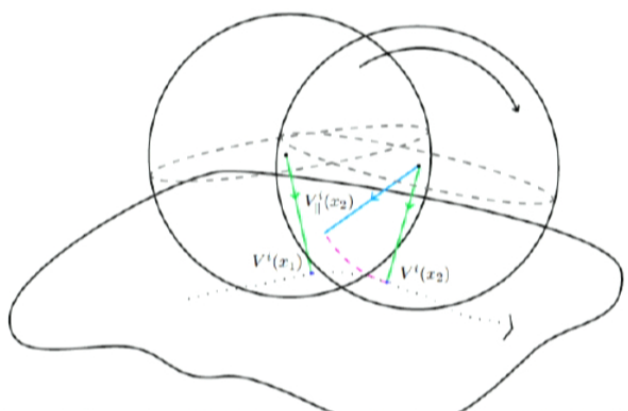


Mathematics of rolling without slipping

- When we roll the idealized waywiser along some path on the manifold it undergoes a succession of infinitesimal rotations and the point of contact changes.
- Since we are dealing with a two-sphere these rotations belong to $SO(3)$.
- An infinitesimal $SO(3)$ rotation associated with moving from point x_1 to x_2 on the manifold is given by

$$\delta\Omega^i{}_j = \delta^i_j - dx^a A_a{}^i{}_j$$

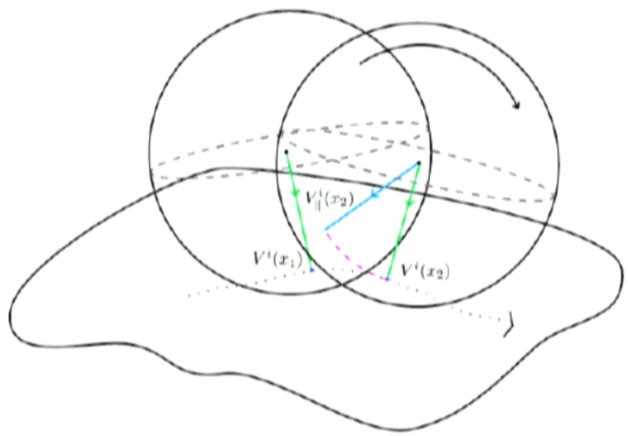
where $dx^a = x_2^a - x_1^a$ and $A_a{}^i{}_j$ is an $\mathfrak{so}(3)$-valued connection.



Calculating the change in contact point

- Let us now mathematically quantify the change in contact point when the idealized waywiser is rolled.
- To do this we denote $V^i(x_1)$ the contact vector at x_1 and similarly for $V^i(x_2)$.
- In order to compare these two contact vectors we roll the contact vector $V^i(x_1)$ from the point x_1 to x_2. This yields the vector $V_{\|}^i(x_2)$ given by:

$$V_{\|}^i(x_2) = \delta\Omega^i_j V_{\|}^j(x_1) = (\delta^i_j - dx^a A_a{}^i{}_j) V_{\|}^j(x_1) = V^i(x_1) - dx^a A_a{}^i{}_j V^j(x_1)$$



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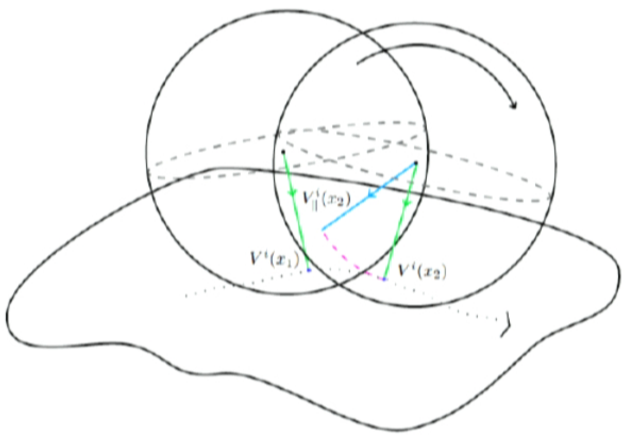


The change in contact point and the metric tensor

- The change of contact point is then represented by the object δV^i given by

$$\begin{aligned}\delta V^i &\equiv V^i(x_2) - V^i_{\|}(x_2) = V^i(x_2) - (V^i(x_1) - dx^a A_a{}^i{}_j V^j(x_1)) \\ &= dx^a(\partial_a V^i + A_a{}^i{}_j V^j) \equiv dx^a D_a V^i.\end{aligned}$$

- An ideal waywiser deduces the traversed distance ds by quantifying the change of contact point. Thus we have:

$$ds^2 = \delta_{ij}\delta V^i \delta V^j = dx^a dx^b \delta_{ij} D_a V^i D_b V^j.$$

- We can now identify the metric tensor as

$$g_{ab} = \delta_{ij} D_a V^i D_b V^j.$$

and $e^i = D_a V^i$ as the (co)-zwei-bein since $\delta_{ij} V^i e^j_a = 0$ which notably no longer appears as a fundamental variable.

Reconstructing the affine connection

- What about the *rate* of change of the contact point?

$$D_a D_b V^i = \partial_a D_b V^i + A_a{}^i{}_j D_b V^j$$

- The component which is normal to V^i is again just the metric. But from the orthogonal component we can read off the affine connection:

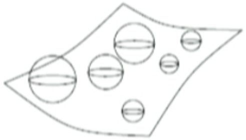
$$P^i{}_j (D_a D_b V^j - \Gamma^c_{ab} D_c V^i) = 0$$

where and $P^i{}_j \equiv \delta^i_j - \frac{1}{\ell^2} V^i V_j$ is a projector.

- We note that the affine connection is not a fundamental object but something we can construct from the basic variables $\{V^i, A^{ij}\}$ if we need it.
- We have now showed how to mathematically characterize geometry using the variables (V^i, A^{ij}).

Cartan geometry with arbitrary size of the spheres

- The glaring difference with the electroweak theory suggests a straightforward generalization of Cartan geometry: why not allow for the spheres to have different size at different points on the manifold?



- From a geometrical point of view it seems like an unwanted new degree of freedom, the norm $|V|$, has been introduced.
- However, within cosmology researchers frequently introduce additional scalar fields to model dark matter, quintessence and inflation. Instead of being a nuisance, such a scalar degree of freedom comes with the Cartan geometry package and does not have to be introduced in addition to gravity.
- We shall see below how the well-known Peebles-Ratra quintessence model pops up naturally, as well as a signature change resembling that of the Hartle-Hawking no-boundary proposal.

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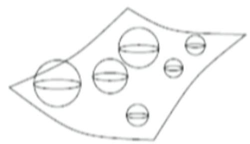

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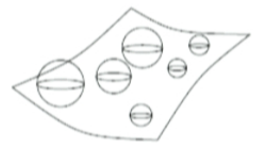

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Relativistic Cartan geometry and polynomial actions for gravity

Relativistic waywisers

- To accommodate spacetime geometries and relativistic theories we must adapt the above waywiser formalism accordingly.
- From a mathematical point of view the obvious change to make is to make use of symmetric **spacetimes**, rather than spaces, as idealized waywiser wheels.
- In the literature the symmetric spacetimes, representing relativistic waywiser wheels, go by the name model spaces or model spacetimes.
- Although we can choose both a flat Minkowski spacetime or an anti-DeSitter spacetime as model spacetimes we will consider here only the De Sitter spacetime which exhibits the symmetry group $SO(1,4)$

$$-T^2 + X^2 + Y^2 + Z^2 + W^2 = \ell^2 \quad V^A V^B \eta_{AB} = \pm\phi^2$$

- If the contact vector V^A is spacelike $V^2 > 0$ the subgroup of transformations that leaves the contact vector V^A invariant is then by construction the Lorentz group $SO(1,3)$.
- Similarly, if the contact vector V^A is timelike $V^2 < 0$ the remnant symmetry group is $SO(4)$ and we are dealing with 4D Euclidean manifolds.

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How to translation to more standard notation

- In this talk we treat the waywiser pair $\{V^A, A^{AB}\}$ as the basic variables and think of all other variables as constructed out of these basic variables. This difference will be important when we consider action principles.
- It is anyhow helpful to establish the relation with the more standard notation used in Einstein-Cartan theory. This is achieved by fixing the gauge so that $V^A \stackrel{*}{=} \phi\delta^A_4 = (0, 0, 0, 0, \phi)$

$$
\begin{aligned}
DV^A &\stackrel{*}{=} dV^A + A^A{}_B V^B \stackrel{*}{=} (e^I, d\phi) \\
A^{AB} &\stackrel{*}{=} \begin{pmatrix} \omega^{IJ} & \frac{e^I}{\phi} \\ -\frac{e^J}{\phi} & 0 \end{pmatrix} \\
F^{AB} &\stackrel{*}{=} \begin{pmatrix} R^{IJ} - \frac{1}{\ell^2} e^I \wedge e^J & \frac{1}{\phi}\left(T^I - \frac{d\phi}{\phi} e^I\right) \\ -\frac{1}{\phi}\left(T^J - \frac{d\phi}{\phi} e^J\right) & 0 \end{pmatrix}
\end{aligned}
$$

- Note that $V^2 = \pm\phi^2$ is not assumed to be constant here.

A class of polynomial actions for gravity

- The basic variables of Cartan waywiser geometry are $\{V^A, A^{AB}\}$.
- The list of possible actions is, of course, infinite, but if we restrict ourselves to actions which are polynomial in the variables $\{V^A, A^{AB}\}$ and their gauge covariant derivatives the list is rather short.

$$\mathcal{S} = \int a_{ABCD}F^{AB}F^{CD} + b_{ABCD}DV^A DV^B F^{CD} + c_{ABCD}DV^A DV^B DV^C DV^D$$

where the wedge product between forms is implicitly understood and

$$\begin{aligned} a_{ABCD} &= a_1\epsilon_{ABCDE}V^E + a_2 V_A V_C \eta_{BD} + a_3\eta_{AC}\eta_{BD} \\ b_{ABCD} &= b_1\epsilon_{ABCDE}V^E + b_2 V_A V_C \eta_{BD} + b_3\eta_{AC}\eta_{BD} \\ c_{ABCD} &= c_1\epsilon_{ABCDE}V^E \end{aligned}$$

- The equations of motion are first order partial differential equations, something which follows from the fact that the action is polynomial in all variables and gauge invariant.
- If a_i, b_i, c_1 do not depend on V^2 then a_3 term is topological and the a_2 and b_3 terms are topologically equivalent. Thus, we have five topologically different possible terms.

MacDowell-Mansouri action and propagating torsion

- By writing out the MacDowell-Mansouri action, the a_1 term, in the gauge $V^A \stackrel{*}{=} \phi\delta^A_4$

$$S_{MM} = \int a_1 \phi \epsilon_{IJKL} \left(R^{IJ} R^{KL} - 2\frac{a_1}{\phi^2} e^I e^J R^{KL} + \frac{a_1}{\phi^4} e^I e^J e^K e^L \right)$$

  we see that the contorsion one-form C^{IJ} defined by $T^I = D^\omega e^I = C^{IJ} e_J$ does not appear algebraically the the first term. Specifically, the contorsion one-form appears with a derivative which can be removed by partial integration only if $\phi = const$.
- This means that a non-constant V^2 awakens new dynamical degrees of freedom that are dormant in Einstein-Cartan theory or general relativity.
- This might lead to new interesting phenomenology.

The General Relativity limit: $V^2 \to const.$

- Any new theory must yield old tested physics in some well-defined limit. In fact, the polynomial action yields General Relativity for spacelike V^A in the limit $V^2 \to const.$
- This can be seen by assuming $V^2 = const. > 0$ and rewriting the polynomial action in the gauge $V^A \stackrel{*}{=} \phi_0 \delta^A_4$ yielding

$$S \to \int \left(b_1 \phi_0 - \frac{2a_1}{\phi_0} \right) \epsilon_{IJKL} e^I e^J R^{KL} + \left(\frac{a_1}{\phi_0^3} - \frac{b_1}{\phi_0} + c_1 \phi_0 \right) \epsilon_{IJKL} e^I e^J e^K e^L$$
$$+ (a_2 + b_3) \phi_0^2 e_I e_J R^{IJ}$$

  where boundary terms have been discarded.
- Here we recognize an Einstein-Hilbert-Palatini term, a cosmological constant term, and the Holst term.
- This understood let us now explore the applications of Cartan gravity with a dynamical symmetry breaking field V^A to cosmology.

Peebles-Ratra type inflation

- Since Cartan gravity comes naturally equipped with such a scalar field $V^2(x)$ it is reasonable to ask whether well-known scalar tensor theories of cosmology are hidden in the polynomial action.
- Indeed, the extensively studied Peebles-Ratra quintessence model is in fact just the $b_1 - b_2$ action

$$S_{PR} = \int (b_1 \epsilon_{ABCDE} V^E + b_2 V_A V_C \eta_{BD}) DV^A DV^B F^{CD}$$

  but here disguised in 1st order language.
- To see this we make use of the fact that contorsion one-form C^{IJ} appears algebraically in the action which allows us to solve for it

$$C_{IJ} = e_{[I} e^{\mu}_{J]} \partial_\mu \log \phi + \frac{b_2}{4b_1} \epsilon_{IJKL} e^{K\mu} \partial_\mu \log \phi e^L$$

- Then we plug the solution back into the action and after a Weyl rescaling yields a Peebles-Ratra action of the form

$$S_{PR} = \int d^4x \sqrt{-g} \left(\kappa_1 R - g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\kappa_2}{\phi^3} \right)$$

The MacDowell-Mansouri action and signature change

- Let us now explore the MacDowell-Mansouri action corresponding to the a_1 term

$$S_{MM} = \int a_1 \epsilon_{ABCDE} V^E F^{AB} F^{CD}$$

- In order to explore its consequences for cosmology we impose *FRW* symmetry, i.e. homogeneity and isotropy of space.

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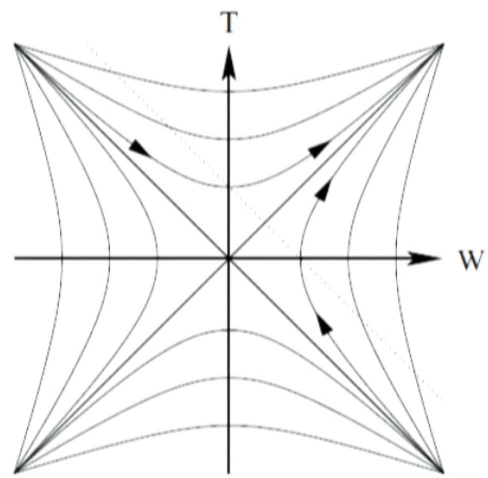
Signature change

- If the Higgs field V^A is spacelike then the $SO(1,4)$ symmetry is broken down to $SO(1,3)$.
- However, it V^A is timelike the remnant symmetry is $SO(4)$, i.e. we are describing a 4D geometry with Euclidean signature.
- Since we have now allowed V^A to be a dynamical variable we simply have to be open to the possibility that the dynamics implies a transition from spacelike to timelike Higgs field $V^A = (T, 0, 0, 0, W)$.



FRW reduced form of V^A and A^{AB}

- In this better suited gauge the contact point V^A and connection A^{AB} takes the form

$$A^{AB} \stackrel{*}{=} \begin{pmatrix} 0 & BE^j & nE^0 \\ -BE^i & \omega^{ij} & AE^i \\ -nE^0 & -AE^j & 0 \end{pmatrix}$$

$$V^A \stackrel{*}{=} (\psi, 0, 0, 0, \phi)$$

with

$$\omega^{0i} = BE^i \quad \omega^{12} = -\frac{K(r)}{r}E^2 - CE^3 \quad \omega^{13} = -\frac{K(r)}{r}E^3 + CE^2$$

$$\omega^{23} = -\frac{\cot\theta}{r}E^3 - CE^1$$

where $K(r) = \sqrt{1 - kr^2}$ and (A, B, C, ϕ, ψ, n) only depending on t and

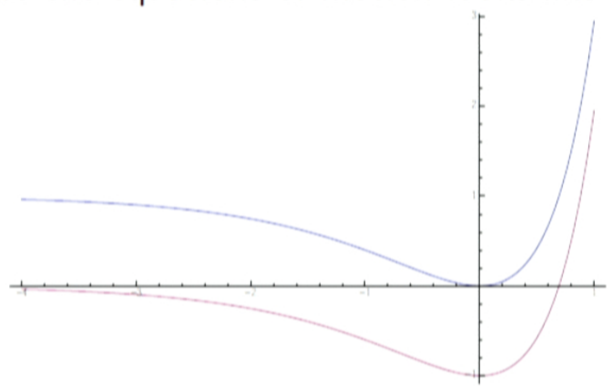
$$E^0 = dt \quad E^1 = \frac{dr}{K(r)} \quad E^2 = rd\theta \quad E^3 = r\sin\theta d\varphi$$

The FRW-reduced MacDowell-Mansouri action

- The FRW reduced MacDowell-Mansouri action becomes

$$L_{FRWMM} = (\dot{A}\psi + \dot{B}\phi)(k + B^2 - A^2 - C^2) - \dot{C}C(\phi B + \psi A) - n(\psi B + \phi A)(k + B^2 - A^2 - C^2)$$

- A typical solution to the equations of motion looks like



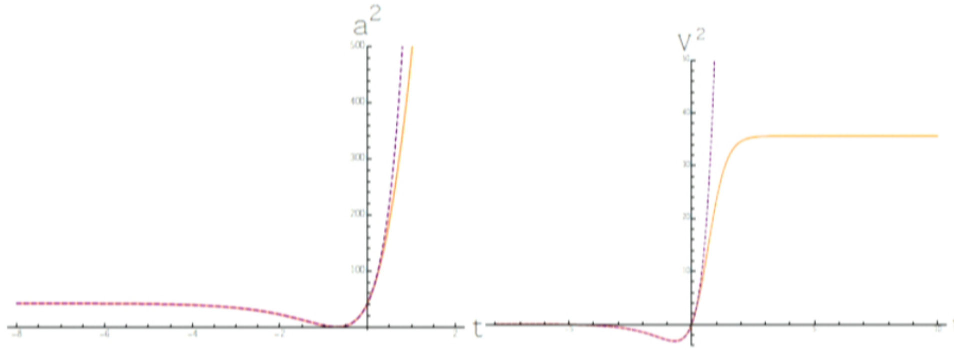
- Here we see a perfectly smooth well-behaved process of signature change. The purple line is V^2 which changes sign. The scale factor $a^2 \equiv (A\phi + B\psi)^2$ exhibits expontial acceleration. Importantly we do not see V^2 settling to a constant value and this means that we do not recover General Relativity at late times.

Adding the 'kinetic' term b_2

- Adding the b_2 term to the MacDowell-Mansouri action

$$S_{MM+} = \int a_1 \epsilon_{ABCDE} V^E F^{AB} F^{CD} + b_2 V_A V_C \eta_{BD} DV^A DV^B F^{CD}$$

yields the typical FRW solutions



- The signature change survives this modification but more important is that V^2 stabilizes to a constant value at late times so that General Relativity is recovered.
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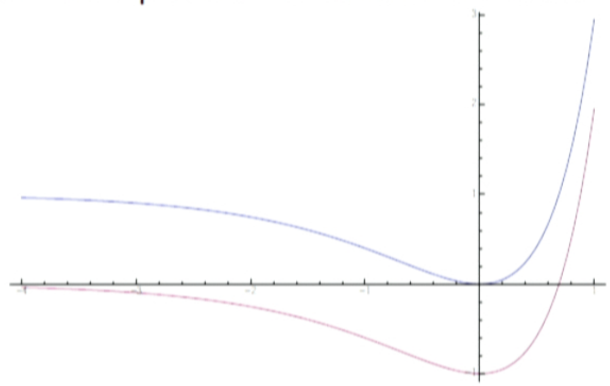
The FRW-reduced MacDowell-Mansouri action

- The FRW reduced MacDowell-Mansouri action becomes

$$L_{FRWMM} = (\dot{A}\psi + \dot{B}\phi)(k + B^2 - A^2 - C^2) - \dot{C}C(\phi B + \psi A) - n(\psi B + \phi A)(k + B^2 - A^2 - C^2)$$

- A typical solution to the equations of motion looks like



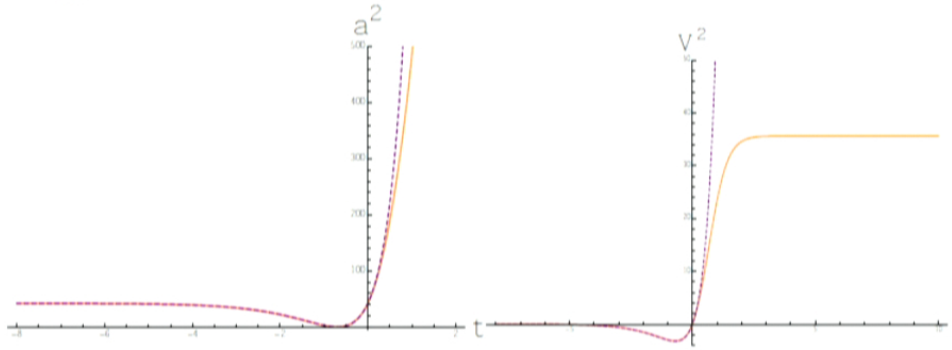
- Here we see a perfectly smooth well-behaved process of signature change. The purple line is V^2 which changes sign. The scale factor $a^2 \equiv (A\phi + B\psi)^2$ exhibits expontial acceleration. Importantly we do not see V^2 settling to a constant value and this means that we do not recover General Relativity at late times.

Adding the 'kinetic' term b_2

- Adding the b_2 term to the MacDowell-Mansouri action

$$S_{MM+} = \int a_1 \epsilon_{ABCDE} V^E F^{AB} F^{CD} + b_2 V_A V_C \eta_{BD} DV^A DV^B F^{CD}$$

yields the typical FRW solutions



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Conclusions & Outlook

Conclusions and Outlook

- We saw how the extensively studied Peebles-Ratra quintessence model was hidden in first order form within the simple polynomial class of actions.
- We also saw how the MacDowell-Mansouri action leads to signature change instead of a big bang singularity, resembling the Hartle-Hawking no-boundary proposal.
- Finally, we saw how we can recover General Relativity at late times by adding a 'kinetic term' to the MacDowell-Mansouri action.
- These are qualitative results and it remains to determine if this Cartan geometric cosmology can be made in quantitative agreement with cosmological data.
- It is not unreasonable to expect that the singularity inside black holes could also be replaced by a signature change, but this remains to be investigated.
- Cartan geometric matter actions are polynomial in all variables and are therefore robust against degenerate geometries such as signature changing ones. In fact, all field equations, both matter and gravity, are in a Cartan geometric formulation first order partial differential equations rather than of second order.

UNIVERSIDAD DE BURGOS
DEPARTAMENTO DE FÍSICA

Drinfel'd doubles for (2+1) gravity

A. Ballesteros (U. Burgos)
F.J. Herranz (U. Burgos), C. Meusburger (FAU Erlangen)

LOOPS'13 PERIMETER INSTITUTE, JULY 2013

Preface: Quantum $sl(2)$ in a nutshell

The **standard or Drinfel'd–Jimbo** quantum deformation of $U(sl(2,\mathbb{R}))$:

- **Deformed commutation rules**:

$$[J_3, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = \frac{\sinh(2\,z\,J_3)}{z} \qquad z \in \mathbb{R}$$

- **Deformed coproduct** compatible with them:

$$\Delta(J_3) = J_3 \otimes 1 + 1 \otimes J_3 \qquad \Delta(J_\pm) = J_\pm \otimes e^{zJ_3} + e^{-zJ_3} \otimes J_\pm .$$

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- The coproduct map Δ can be **expanded as a formal power series in z**

$$\Delta = \sum_{k=0}^{\infty} \Delta_{(k)} = \sum_{k=0}^{\infty} z^k \delta_{(k)} = \Delta_0 + z\, \delta_{(1)} + o[z^2]$$

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  This Lie bialgebra is generated by a **classical r-matrix** on $sl(2) \otimes sl(2)$:

$$r = J_+ \wedge J_- = J_+ \otimes J_- - J_- \otimes J_+$$

  since

$$\delta_1(X) = [1 \otimes X + X \otimes 1, r]$$

Preface: The associated Poisson-Lie group

Each Lie bialgebra has a **unique Poisson-Lie structure** associated to it:

- Take the ***r*-matrix** that generates the Lie bialgebra: $r = J_+ \wedge J_-$.
- Compute the **Sklyanin bracket**:

$$\{f, g\} = r^{\alpha\beta}\left(X^L_\alpha f \, X^L_\beta g - X^R_\alpha f \, X^R_\beta g\right) \qquad f, g \in C^\infty(SL(2)).$$

- If we consider $D(SL(2)) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we get the **PL brackets**

$$\{b, a\} = a\,b \qquad \{c, a\} = a\,c \qquad \{c, b\} = 0$$
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- The group multiplication induces a natural **coproduct** on $C^\infty(SL(2))$:

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c \\ \Delta(b) &= a \otimes b + b \otimes d \\ \Delta(c) &= c \otimes a + d \otimes c \\ \Delta(d) &= c \otimes b + d \otimes d \end{aligned}$$

which, by construction, is a **Poisson map** for the PL structure.

Quantum groups in (2+1) gravity with cosmological constant

For a given Lie algebra/group, there are **many possible quantum deformations** (for (2+1) (A)dS see [1]).

[1] A.B., F.J. Herranz, F. Musso, arXiv:1302.0684
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- Poisson-Lie (PL) structures on the isometry groups of (2+1) spaces with constant curvature play a relevant role as **phase spaces when (2+1) gravity is considered as a Chern-Simons gauge theory.**[2 3 4]

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- These PL structures are given by certain **classical r-matrices** that **have to be 'compatible' with the CS formalism**.

  Explicitly, the **symmetric component of** r has to be directly related with the Ad-invariant symmetric bilinear form in the CS action.

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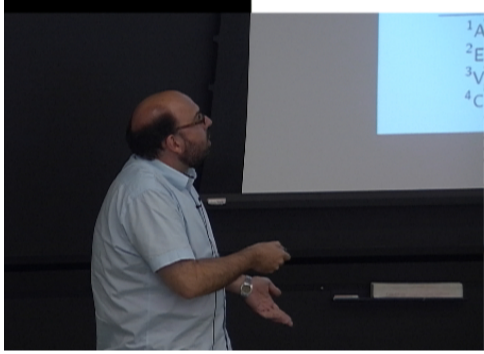

Therefore, it seems natural that the appropriate **quantum groups** in (2+1) should be the **quantizations of these admissible PL symmetries**.

B., F.J. Herranz, F. Musso, arXiv:1302.0684

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Main results

We present the **(2+1) gravity $\longleftrightarrow$ Drinfel'd double** relationship: [5] [6]

MAIN FACT: **All the classical r-matrices coming from a Drinfel'd double structure of the isometry group -(A)dS and Poincaré- fulfill the Fock-Rosly condition and are compatible with the CS formalism**. Thus:

- We give the **4 possible DD structures for the de Sitter Lie algebra** $so(3,1)$ and the **3 possible ones for the Anti de Sitter** one $so(2,2)$.
- We obtain (4+3) candidates for quantum deformations of the (A)dS symmetries that would be appropriate in a (2+1) setting.

---

[5] A.B., F.J. Herranz, C. Meusburger, Phys. Lett. B 687 (2010) 375

[6] A.B., F.J. Herranz, C. Meusburger, Class. Quantum Grav. 30 (2013) 155012

[7] A.B., F.J. Herranz, M.A. del Olmo, M. Santander, J. Phys. A **27** (1994) 1283

[8] G. Amelino-Camelia, L. Smolin, A. Starodubtsev, Class. Quantum Grav. 21 (2004) 3095

1. Spacetimes and symmetries of (2+1) gravity

Spacetimes of (2+1) gravity

Any solution of the 3d vacuum Einstein equations is of **constant curvature** which **cosmological constant** Λ and is locally isometric to one of the **six standard spacetimes**:

- **Euclidean signature**: the **three-sphere** $\mathbf{S}^3$ ($\Lambda > 0$), 3d **hyperbolic** space $\mathbf{H}^3$ ($\Lambda < 0$) and 3d **Euclidean** space $\mathbf{E}^3$ ($\Lambda = 0$).
- **Lorentzian signature**: the 3d **de Sitter** space $\mathbf{dS}^{2+1}$ ($\Lambda > 0$), **Anti-de Sitter** space $\mathbf{AdS}^{2+1}$ ($\Lambda < 0$) and **Minkowski** space $\mathbf{M}^{2+1}$ ($\Lambda = 0$).

All of them are **homogeneous spaces** given as quotients of their isometry group by either $SO(3)$ (Euclidean) or $SO(2,1)$ (Lorentzian).

$\Lambda > 0$	$\Lambda = 0$	$\Lambda < 0$
$\mathbf{dS}^{2+1} = SO(3,1)/SO(2,1)$ $\mathrm{Isom}(\mathbf{dS}^{2+1}) = SO(3,1)$	$\mathbf{M}^{2+1} = ISO(2,1)/SO(2,1)$ $\mathrm{Isom}(\mathbf{M}^{2+1}) = ISO(2,1)$	$\mathbf{AdS}^{2+1} = SO(2,2)/SO(2,1)$ $\mathrm{Isom}(\mathbf{AdS}^{2+1}) = SO(2,2)$
$\mathbf{S}^3 = SO(4)/SO(3)$ $\mathrm{Isom}(\mathbf{S}^3) = SO(4)$	$\mathbf{E}^3 = ISO(3)/SO(3)$ $\mathrm{Isom}(\mathbf{E}^3) = ISO(3)$	$\mathbf{H}^3 = SO(3,1)/SO(3)$ $\mathrm{Isom}(\mathbf{H}^3) = SO(3,1)$

Lie algebras of (2+1) gravity

- The **Lie algebras** of these isometry groups can be written in a common **kinematical basis** in terms of generators $J_a, P_a,\ a = 0, 1, 2$.
- In this basis the cosmological constant Λ and the signature of the metric arise as **parameters** in the Lie bracket: [9] [10]

$$[J_a, J_b] = \epsilon_{abc} J^c \qquad [J_a, P_b] = \epsilon_{abc} P^c \qquad [P_a, P_b] = \chi\, \epsilon_{abc} J^c$$

$$\text{where } \chi = \begin{cases} \Lambda & \text{for Euclidean signature;} \\ -\Lambda & \text{for Lorentzian signature.} \end{cases}$$

- If $g = \mathrm{diag}(\alpha, 1, 1)$ with $\alpha = \pm 1$ denotes the Euclidean / Minkowski metric and $\Lambda = \alpha\chi$, we have

$$\begin{array}{lll}
[J_0, J_1] = J_2, & [J_0, J_2] = -J_1, & [J_1, J_2] = \alpha\, J_0, \\
[J_0, P_0] = 0, & [J_0, P_1] = P_2, & [J_0, P_2] = -P_1, \\
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[9] E. Witten, Nucl. Phys. B 311 (1988) 46

[10] A. Achucarro, P.K. Townsend, Phys. Lett. B 180 (1986) 89

Lie algebras of (2+1) gravity

The basis $\{J_a, P_a\}_{a=0,1,2}$ have a **direct geometrical interpretation**

- J_a are the infinitesimal generators of **boosts / rotations**.
- P_a generate **translations**, which commute if $\Lambda = 0 = \chi$.

For all values of the parameters α, χ we have **two quadratic Casimir elements**

$$C_1 = \alpha\, P_0^2 + P_1^2 + P_2^2 + \chi\,(\alpha\, J_0^2 + J_1^2 + J_2^2),$$
$$C_2 = \tfrac{1}{2}\,(\alpha\,(J_0\,P_0 + P_0\,J_0) + J_1\,P_1 + P_1\,J_1 + J_2\,P_2 + P_2\,J_2)\,.$$

and the space of Ad-invariant symmetric bilinear forms is two-dimensional.

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and the space of Ad-invariant symmetric bilinear forms is two-dimensional.

If the duals of J_a and P_a are identified with, respectively, P_a and J_a, the **symmetric bilinear forms** associated to C_1 and C_2 are

$$\langle J_a, P_b\rangle_s = 0, \qquad \langle J_a, J_b\rangle_s = g_{ab}, \qquad \langle P_a, P_b\rangle_s = \chi\, g_{ab}.$$
$$\langle J_a, P_b\rangle_t = g_{ab}, \qquad \langle J_a, J_b\rangle_t = 0, \qquad \langle P_a, P_b\rangle_t = 0,$$

with $g = \mathrm{diag}(\alpha, 1, 1)$.

Two symmetric bilinear forms

The two symmetric Ad-invariant bilinear forms can be used in (2+1)-gravity: [11]

- **The one associated with C_2**

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$$\langle J_a, P_b \rangle_s = 0 \qquad \langle J_a, J_b \rangle_s = g_{ab} \qquad \langle P_a, P_b \rangle_s = \chi\, g_{ab}$$

  yields a CS theory with the **same equations of motion but with a different symplectic structure**. [12] [13] [14]

  Note that this form is the only one that can be **generalized to (3+1)D.**

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Admissible r-matrices

- The **Poisson structure on the phase space of (2+1) gravity** has a natural description in terms of Poisson–Lie structures on the corresponding isometry group. [15]
- The **admissible classical r-matrices** defining such Poisson-Lie groups are such that their **symmetric component coincides with a tensorized Casimir element**. For instance,

$$t = P_a \otimes J^a + J_a \otimes P^a$$

that corresponds to the Casimir

$$C_2 = \tfrac{1}{2}\left(\alpha\left(J_0\, P_0 + P_0\, J_0\right) + J_1\, P_1 + P_1\, J_1 + J_2\, P_2 + P_2\, J_2\right)$$

of the symmetry algebra of the Lorentzian spacetimes.

[15] A.Y. Alekseev, A.Z. Malkin, Commun. Math. Phys. 169 (1995) 99
V.V. Fock, A.A. Rosly, ITEP-72-92 (1992); Am. Math. Soc. Transl. 191 (1999) 67

Drinfel'd doubles

A $2d$-dimensional Lie algebra $\mathfrak{a}$ has the structure of a (classical) Drinfel'd double [16] if there exists a basis $\{X_1, \ldots, X_d, x^1, \ldots, x^d\}$ of $\mathfrak{a}$ in which the Lie bracket takes the form

$$[X_i, X_j] = c_{ij}^k X_k \qquad [x^i, x^j] = f_k^{ij} x^k \qquad [x^i, X_j] = c_{jk}^i x^k - f_j^{ik} X_k .$$

[16] V.G. Drinfel'd, *Proc. Int. Congress of Math.*, *AMS* (1987) 798

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- This implies that the two sets of generators $\{X_1, \ldots, X_d\}$ and $\{x^1, \ldots, x^d\}$ form **two Lie subalgebras** with structure constants c_{ij}^k and f_k^{ij}, respectively.
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$$\langle X_i, X_j \rangle = 0 \qquad \langle x^i, x^j \rangle = 0 \qquad \langle x^i, X_j \rangle = \delta_j^i \qquad \forall i, j .$$

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The DD canonical r-matrix

Moreover, if $\mathfrak{a}$ is a DD Lie algebra, **its corresponding Lie group can be always endowed with a PL structure** generated by the **canonical classical r-matrix**

$$r = \sum_i x^i \otimes X_i$$

which is a (constant) solution of the Classical Yang-Baxter equation $[[r, r]] = 0$.

- The **skew-symmetric component** of the r-matrix is

$$r' = \tfrac{1}{2} \sum_i x^i \wedge X_i .$$

- And the **symmetric component** Ω coincides with the **tensorized form of the canonical quadratic Casimir element in $\mathfrak{a}$**

$$\Omega = r - r' = \tfrac{1}{2} \sum_i (x^i \otimes X_i + X_i \otimes x^i),$$

which is just the **Fock-Rosly condition**.

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3. (A)dS Lie algebras in (2+1) as Drinfel'd doubles

$so(3,1)$ and $so(2,2)$ as Drinfel'd double Lie algebras

The **complete classification** of the six-dimensional DD Lie algebras is known [17] and is equivalent to the classification of three-dimensional real Lie bialgebras. [18]

The **de Sitter Lie algebra** $so(3,1)$ admits four families of DD structures $(c^i_{jk}|f^{ij}_k|\eta)$: $[X_i, X_j] = c^k_{ij} X_k$ $[x^i, x^j] = f^{ij}_k x^k$ $[x^i, X_j] = c^i_{jk} x^k - f^{ik}_j X_k$.

- A: $(8|5.ii|\eta) \equiv (so(2,1)|\mathfrak{an}(2)''|\eta)$
- B: $(9|5|\eta) \equiv (so(3)|\mathfrak{an}(2)|\eta)$
- C: $(7_0|5.ii|\eta) \equiv (iso(2)|\mathfrak{an}(2)''|\eta)$
- D: $(7_\mu|7_{1/\mu}|\eta)$

While the **Anti de Sitter Lie algebra** $so(2,2)$ admits three:

- E: $(8|5.i|\eta) \equiv (so(2,1)|\mathfrak{an}(2)'|\eta)$
- F: $(6_0|5.iii|\eta) \equiv (iso(1,1)|\mathfrak{an}(2)'''|\eta)$
- G: $(6_a|6_{1/a}.i|\eta)$

[17] L. Snobl and L. Hlavaty, *Int. J. Mod. Phys. A* 17 (2002) 4043
[18] X. Gomez, *J. Math. Phys.* 41 (2000) 4939

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Case A: the $(8|5.ii|\eta)$ de Sitter DD

- **Lie brackets** for $(8|5.ii|\eta)$:

$$8 \simeq so(2,1): \quad [X_0, X_1] = X_2 \qquad [X_0, X_2] = -X_1 \qquad [X_1, X_2] = -X_0$$

$$5.ii \simeq \mathfrak{an}(2)'': \quad [x^0, x^1] = -\eta\, x^1 \qquad [x^0, x^2] = -\eta\, x^2 \qquad [x^1, x^2] = 0$$

[19] A.B., F.J. Herranz, C. Meusburger, Class. Quantum Grav. 30 (2013) 155012

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together with the **crossed relations** $[x^i, X_j] = c^i_{jk} x^k - f_j^{ik} X_k$:

$$[x^0, X_0] = 0 \qquad [x^0, X_1] = -x^2 + \eta\, X_1 \qquad [x^0, X_2] = x^1 + \eta\, X_2$$
$$[x^1, X_0] = -x^2 \qquad [x^1, X_1] = -\eta\, X_0 \qquad [x^1, X_2] = x^0$$
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$$[x^1, X_0] = -x^2 \quad [x^1, X_1] = -\eta\, X_0 \quad [x^1, X_2] = x^0$$
$$[x^2, X_0] = x^1 \quad [x^2, X_1] = -x^0 \quad [x^2, X_2] = -\eta\, X_0.$$

- The **change to the kinematical basis** is found to be: [19]

$$J_0 = X_0 \quad J_1 = \tfrac{1}{\sqrt{2}}(X_1 - X_2) \quad J_2 = \tfrac{1}{\sqrt{2}}(X_1 + X_2)$$
$$P_0 = -x^0 \quad P_1 = \tfrac{1}{\sqrt{2}}\left(\eta(X_1 + X_2) + x^1 - x^2\right) \quad P_2 = \tfrac{1}{\sqrt{2}}\left(\eta(X_2 - X_1) + x^1 + x^2\right).$$

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Case A: the $(8|5.ii|\eta)$ de Sitter DD

And we get

$$
\begin{array}{lll}
[J_0, J_1] = J_2, & [J_0, J_2] = -J_1, & [J_1, J_2] = -J_0, \\
[J_0, P_0] = 0, & [J_0, P_1] = P_2, & [J_0, P_2] = -P_1, \\
[J_1, P_0] = -P_2, & [J_1, P_1] = 0, & [J_1, P_2] = -P_0, \\
[J_2, P_0] = P_1, & [J_2, P_1] = P_0, & [J_2, P_2] = 0, \\
[P_0, P_1] = -\eta^2 J_2, & [P_0, P_2] = -\eta^2 J_1, & [P_1, P_2] = \eta^2 J_0.
\end{array}
$$

Summary: DD r-matrices for $so(3,1)$

#	Metric	Λ	Pairing	Skew-symmetric r-matrix	Space
A	$(-1,1,1)$	η^2	$\langle\ ,\ \rangle_t$	$r'_{\rm A} = \eta J_1 \wedge J_2 + \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	$\mathbf{dS}^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm A} = \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	$\mathbf{M}^{2+1}$
B	$(1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\rm B} = -\eta J_1 \wedge J_2 + \frac{1}{2}(P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	$\mathbf{H}^3$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm B} = \frac{1}{2}(P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	$\mathbf{E}^3$
C	$(-1,1,1)$	η^2	$\langle\ ,\ \rangle_t$	$r'_{\rm C} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	$\mathbf{dS}^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm C} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	$\mathbf{M}^{2+1}$
D	$(1,1,1)$	$-\eta^2$	$\frac{\mu(\mu^2-1)}{(1+\mu^2)^2}\langle\ ,\ \rangle_t - \frac{2\mu^2}{\eta(1+\mu^2)^2}\langle\ ,\ \rangle_s$	$r'_{\rm D} = J_0 \wedge P_1 - J_1 \wedge P_0 + \frac{(1+\mu^2)}{2\mu} P_2 \wedge J_2 + \frac{(\mu^2-1)}{2\eta\mu}(\eta^2 J_0 \wedge J_1 - P_0 \wedge P_1)$	$\mathbf{H}^3$
		0	None	$r'_{\rm D} = J_0 \wedge P_1 - J_1 \wedge P_0 + P_2 \wedge J_2 \quad (\mu = 1)$	$\mathbf{E}^3$

Summary: DD *r*-matrices for $so(3,1)$

#	Metric	Λ	Pairing	Skew-symmetric *r*-matrix	Space
A	$(-1,1,1)$	η^2	$\langle\,,\,\rangle_t$	$r'_{\mathrm{A}} = \eta J_1\wedge J_2 + \frac{1}{2}(-P_0\wedge J_0 + P_1\wedge J_1 + P_2\wedge J_2)$	$\mathbf{dS}^{2+1}$
		0	$\langle\,,\,\rangle_t$	$r'_{\mathrm{A}} = \frac{1}{2}(-P_0\wedge J_0 + P_1\wedge J_1 + P_2\wedge J_2)$	$\mathbf{M}^{2+1}$
B	$(1,1,1)$	$-\eta^2$	$\langle\,,\,\rangle_t$	$r'_{\mathrm{B}} = -\eta J_1\wedge J_2 + \frac{1}{2}(P_0\wedge J_0 + P_1\wedge J_1 + P_2\wedge J_2)$	$\mathbf{H}^3$
		0	$\langle\,,\,\rangle_t$	$r'_{\mathrm{B}} = \frac{1}{2}(P_0\wedge J_0 + P_1\wedge J_1 + P_2\wedge J_2)$	$\mathbf{E}^3$
C	$(-1,1,1)$	η^2	$\langle\,,\,\rangle_t$	$r'_{\mathrm{C}} = \frac{1}{2}(J_1\wedge P_0 - J_0\wedge P_1 + J_2\wedge P_2)$	$\mathbf{dS}^{2+1}$
		0	$\langle\,,\,\rangle_t$	$r'_{\mathrm{C}} = \frac{1}{2}(J_1\wedge P_0 - J_0\wedge P_1 + J_2\wedge P_2)$	$\mathbf{M}^{2+1}$
D	$(1,1,1)$	$-\eta^2$	$\frac{\mu(\mu^2-1)}{(1+\mu^2)^2}\langle\,,\,\rangle_t - \frac{2\mu^2}{\eta(1+\mu^2)^2}\langle\,,\,\rangle_s$	$r'_{\mathrm{D}} = J_0\wedge P_1 - J_1\wedge P_0 + \frac{(1+\mu^2)}{2\mu}P_2\wedge J_2 + \frac{(\mu^2-1)}{2\eta\mu}(\eta^2 J_0\wedge J_1 - P_0\wedge P_1)$	$\mathbf{H}^3$
		0	None	$r'_{\mathrm{D}} = J_0\wedge P_1 - J_1\wedge P_0 + P_2\wedge J_2 \quad (\mu=1)$	$\mathbf{E}^3$

Summary: DD r-matrices for $so(3,1)$

#	Metric	Λ	Pairing	Skew-symmetric r-matrix	Space
A	$(-1,1,1)$	η^2	$\langle\ ,\ \rangle_t$	$r'_{\rm A} = \eta J_1\wedge J_2 + \frac{1}{2}(-P_0\wedge J_0 + P_1\wedge J_1 + P_2\wedge J_2)$	$\mathbf{dS}^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm A} = \frac{1}{2}(-P_0\wedge J_0 + P_1\wedge J_1 + P_2\wedge J_2)$	$\mathbf{M}^{2+1}$
B	$(1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\rm B} = -\eta J_1\wedge J_2 + \frac{1}{2}(P_0\wedge J_0 + P_1\wedge J_1 + P_2\wedge J_2)$	$\mathbf{H}^3$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm B} = \frac{1}{2}(P_0\wedge J_0 + P_1\wedge J_1 + P_2\wedge J_2)$	$\mathbf{E}^3$
C	$(-1,1,1)$	η^2	$\langle\ ,\ \rangle_t$	$r'_{\rm C} = \frac{1}{2}(J_1\wedge P_0 - J_0\wedge P_1 + J_2\wedge P_2)$	$\mathbf{dS}^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm C} = \frac{1}{2}(J_1\wedge P_0 - J_0\wedge P_1 + J_2\wedge P_2)$	$\mathbf{M}^{2+1}$
D	$(1,1,1)$	$-\eta^2$	$\frac{\mu(\mu^2-1)}{(1+\mu^2)^2}\langle\ ,\ \rangle_t - \frac{2\mu^2}{\eta(1+\mu^2)^2}\langle\ ,\ \rangle_s$	$r'_{\rm D} = J_0\wedge P_1 - J_1\wedge P_0 + \frac{(1+\mu^2)}{2\mu}P_2\wedge J_2 + \frac{(\mu^2-1)}{2\eta\mu}(\eta^2 J_0\wedge J_1 - P_0\wedge P_1)$	$\mathbf{H}^3$
		0	None	$r'_{\rm D} = J_0\wedge P_1 - J_1\wedge P_0 + P_2\wedge J_2 \quad (\mu=1)$	$\mathbf{E}^3$

- The **κ-deformation is given by** $J_0\wedge P_1 - J_1\wedge P_0$ and it does not appear by itself.
- Case D is associated to a **superposition of both bilinear forms**.
- The **bilinear form** $\langle\ ,\ \rangle_s$ is obtained in case D when $\mu = 1$ (self-duality).

Summary: DD r-matrices for $so(2,2)$

#	Metric	Λ	Pairing	Skew-symmetric r-matrix	Space
E	$(-1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\mathrm{E}} = \eta J_0 \wedge J_2 + \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	**AdS**$^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\mathrm{E}} = \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	**M**$^{2+1}$
F	$(-1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\mathrm{F}} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	**AdS**$^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\mathrm{F}} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	**M**$^{2+1}$
G	$(-1,1,1)$	$-\eta^2$	$\frac{(1+\rho^2)}{2\rho^2}\langle\cdot,\cdot\rangle_t + \frac{(1-\rho^2)}{2\eta\rho^2}\langle\cdot,\cdot\rangle_s$	$r'_{\mathrm{G}} = \frac{(1+\rho^2)}{4}(J_1 \wedge P_0 - J_0 \wedge P_1) + \frac{\rho}{2} J_2 \wedge P_2 + \frac{(1-\rho^2)}{4\eta}(\eta^2 J_0 \wedge J_1 + P_0 \wedge P_1)$	**AdS**$^{2+1}$
		0	None	None	**M**$^{2+1}$

Summary: DD r-matrices for $so(2,2)$

#	Metric	Λ	Pairing	Skew-symmetric r-matrix	Space
E	$(-1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\rm E} = \eta J_0 \wedge J_2 + \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	$\mathbf{AdS}^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm E} = \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	$\mathbf{M}^{2+1}$
F	$(-1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\rm F} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	$\mathbf{AdS}^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm F} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	$\mathbf{M}^{2+1}$
G	$(-1,1,1)$	$-\eta^2$	$\frac{(1+\rho^2)}{2\rho^2}\langle\cdot,\cdot\rangle_t + \frac{(1-\rho^2)}{2\eta\rho^2}\langle\cdot,\cdot\rangle_s$	$r'_{\rm G} = \frac{(1+\rho^2)}{4}(J_1 \wedge P_0 - J_0 \wedge P_1) + \frac{\rho}{2} J_2 \wedge P_2 + \frac{(1-\rho^2)}{4\eta}(\eta^2 J_0 \wedge J_1 + P_0 \wedge P_1)$	$\mathbf{AdS}^{2+1}$
		0	None	None	$\mathbf{M}^{2+1}$

Summary: DD r-matrices for $so(2,2)$

#	Metric	Λ	Pairing	Skew-symmetric r-matrix	Space
E	$(-1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\rm E} = \eta J_0 \wedge J_2 + \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	$\mathbf{AdS}^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm E} = \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	$\mathbf{M}^{2+1}$
F	$(-1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\rm F} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	$\mathbf{AdS}^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm F} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	$\mathbf{M}^{2+1}$
G	$(-1,1,1)$	$-\eta^2$	$\frac{(1+\rho^2)}{2\rho^2}\langle\cdot,\cdot\rangle_t + \frac{(1-\rho^2)}{2\eta\rho^2}\langle\cdot,\cdot\rangle_s$	$r'_{\rm G} = \frac{(1+\rho^2)}{4}(J_1 \wedge P_0 - J_0 \wedge P_1) + \frac{\rho}{2} J_2 \wedge P_2 + \frac{(1-\rho^2)}{4\eta}(\eta^2 J_0 \wedge J_1 + P_0 \wedge P_1)$	$\mathbf{AdS}^{2+1}$
		0	None	None	$\mathbf{M}^{2+1}$

- Again, the **κ-deformation** $J_0 \wedge P_1 - J_1 \wedge P_0$ appears when combined with a twist.
- Case G is associated to a **superposition of both bilinear forms**.
- However, the bilinear form $\langle\ ,\ \rangle_s$ **cannot be obtained** since $-1 < \rho < 1$.

Summary: DD r-matrices for $so(2,2)$

#	Metric	Λ	Pairing	Skew-symmetric r-matrix	Space
E	$(-1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\mathrm{E}} = \eta J_0 \wedge J_2 + \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	**AdS^{2+1}**
		0	$\langle\ ,\ \rangle_t$	$r'_{\mathrm{E}} = \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	**M^{2+1}**
F	$(-1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\mathrm{F}} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	**AdS^{2+1}**
		0	$\langle\ ,\ \rangle_t$	$r'_{\mathrm{F}} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	**M^{2+1}**
G	$(-1,1,1)$	$-\eta^2$	$\frac{(1+\rho^2)}{2\rho^2}\langle\cdot,\cdot\rangle_t + \frac{(1-\rho^2)}{2\eta\rho^2}\langle\cdot,\cdot\rangle_s$	$r'_{\mathrm{G}} = \frac{(1+\rho^2)}{4}(J_1 \wedge P_0 - J_0 \wedge P_1) + \frac{\rho}{2} J_2 \wedge P_2 + \frac{(1-\rho^2)}{4\eta}(\eta^2 J_0 \wedge J_1 + P_0 \wedge P_1)$	**AdS^{2+1}**
		0	None	None	**M^{2+1}**

- Again, the **κ-deformation** $J_0 \wedge P_1 - J_1 \wedge P_0$ appears when combined with a twist.
- Case G is associated to a **superposition of both bilinear forms**.
- However, the bilinear form $\langle\ ,\ \rangle_s$ **cannot be obtained** since $-1 < \rho < 1$.

Many of the quantum groups and noncommutative space-times associated to these r-matrices are not known and are worth to be explored.

THANKS FOR YOUR ATTENTION

Causality and momentum conservation from relative locality

Stefano Bianco

with G. Amelino-Camelia, F. Brighenti and R. J. Buonocore

La Sapienza, Università di Roma

LOOPS 13

Introduction

Context:

- theories with curved momentum space but still relativistic ("DSR-relativistic").

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Issue:

- theories with curved momentum space do not in general preserve causality and global momentum conservation.
- in relativistic theories with curved momentum spaces and relative locality are causality and global momentum conservation preserved?

Concepts:

- framework to study a regime in which $\hbar$ and G are negligible, but $\frac{\hbar}{G}$ is not.
- relativistic framework with the same number and kind of symmetries of special relativity, but deformed, being the parameter $\frac{\hbar}{G}$ associated with the departure from special relativity.

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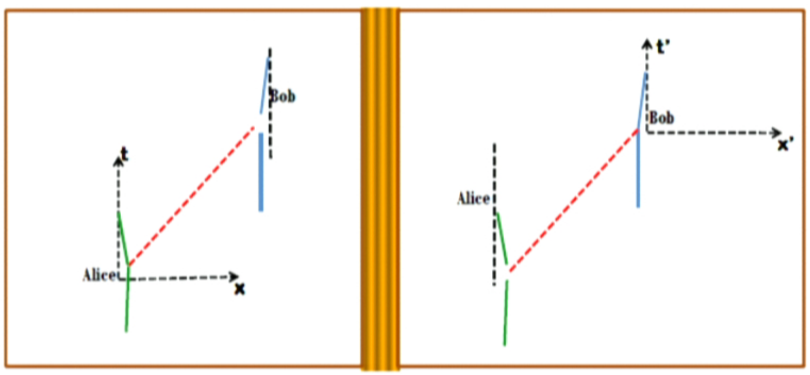
- theory characterized by (possibly deformed) on-shellness and (possibly deformed) law of composition of momenta.
- relative locality:



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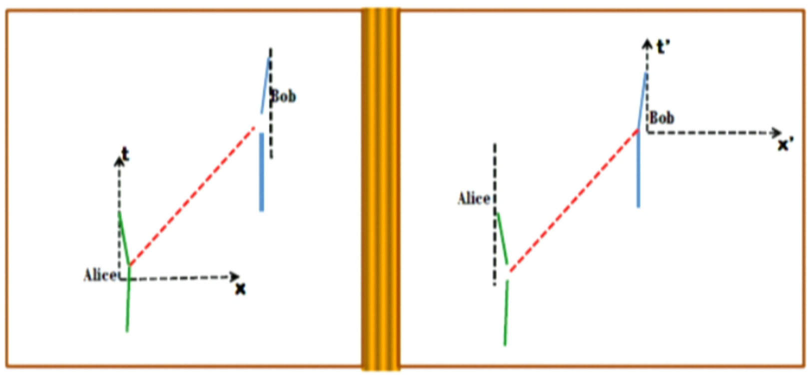


Important: theories on curved momentum space may or may not have relative locality.

k-momentum space example

It is characterized by the de Sitter metric and the k-connection. The on-shellness for a particle of mass m turns out to be

$$p_0^2 - p_1^2 + \ell p_0 p_1^2 = m^2 \tag{1}$$

and the composition of two momenta is

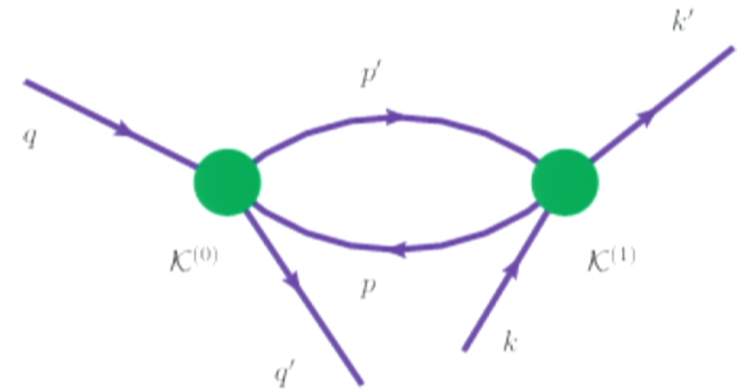
$$(p \oplus q)_0 = p_0 + q_0 \ , \tag{2}$$

$$(p \oplus q)_1 = p_1 + q_1 + \ell p_0 q_1 \ , \tag{3}$$

where ℓ is of the order of the inverse of the Planck mass.

Causal loop

A causal loop process has been studied (L.-Q. Chen, PRD 2013) in a theory with k-momentum space:



$$\begin{aligned} \mathcal{S} \;=\; & \int_{-\infty}^{s_0} ds\,(y^\mu \dot{q}_\mu + \mathcal{N}_q C_q) + \int_{s_0}^{+\infty} ds\,(y'^\mu \dot{q}'_\mu + \mathcal{N}_{q'} C_{q'}) \\ & + \int_{-\infty}^{s_1} ds \left(z^\mu \dot{k}_\mu + \mathcal{N}_k C_k \right) + \int_{s_1}^{+\infty} ds \left(z'^\mu \dot{k}'_\mu + \mathcal{N}_{k'} C_{k'} \right) \\ & + \int_{s_0}^{s_1} ds\,(x'^\mu \dot{p}'_\mu + \mathcal{N}_{p'} C_{p'}) + \int_{s_1}^{s_0} ds\,(x^\mu \dot{p}_\mu + \mathcal{N}_p C_p) + \\ & - \xi^\mu_{(0)} \mathcal{K}^{(0)}_\mu - \xi^\mu_{(1)} \mathcal{K}^{(1)}_\mu , \end{aligned} \qquad (4)$$

with $\mathcal{K}^{(0)}_\mu = [(q \oplus p) \ominus (p' \oplus q')]_\mu$ and $\mathcal{K}^{(1)}_\mu = [(p' \oplus k) \ominus (k' \oplus p)]_\mu$.

Equations of motion

Relevant for the analysis are the following equations of motion

$$\dot{p}_\mu = 0, \qquad \dot{p}'_\mu = 0 \ , \tag{5a}$$

$$\mathcal{C}_p = 0, \qquad \mathcal{C}_{p'} = 0 \ , \tag{5b}$$

$$\dot{x}^\mu(s) = \mathcal{N}_p \frac{\partial \mathcal{C}_p}{\partial p_\mu} \ , \qquad \dot{x}'^\mu(s) = \mathcal{N}_{p'} \frac{\partial \mathcal{C}_{p'}}{\partial p'_\mu} \ , \tag{5c}$$

$$\mathcal{K}^{(0)}_\mu = 0, \qquad \mathcal{K}^{(1)}_\mu = 0 \ , \tag{5d}$$

and the boundary terms

$$x^\mu(s_0) = \xi^\nu_{(0)} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p_\mu} \ , \qquad x^\mu(s_1) = -\xi^\nu_{(1)} \frac{\partial \mathcal{K}^{(1)}_\nu}{\partial p_\mu} \ , \tag{6a}$$

$$x'^\mu(s_0) = -\xi^\nu_{(0)} \frac{\partial \mathcal{K}^{(0)}_\nu}{\partial p'_\mu} \ , \qquad x'^\mu(s_1) = \xi^\nu_{(1)} \frac{\partial \mathcal{K}^{(1)}_\nu}{\partial p'_\mu} \ . \tag{6b}$$

The special relativistic limit

In the special relativistic limit, we have

$$C = p_0^2 - p_i p_i - m^2 \ , \tag{7}$$

$$\mathcal{K}_\mu^{(0)} = q_\mu + p_\mu - p'_\mu - q'_\mu \ , \tag{8}$$

$$\mathcal{K}_\mu^{(1)} = p'_\mu + k_\mu - k'_\mu - p_\mu \ . \tag{9}$$

Combining the equations of motion and the boundary terms, one finds the following condition

$$\Delta\tau \ u^\mu + \Delta\tau' u'^\mu = 0 \ , \tag{10}$$

which has no solution with $\Delta\tau$, $\Delta\tau' > 0$.

Analysis with k-momentum space

In k-momentum space, from the equations of motion and the boundary terms, ones finds the following relation

$$\left[\frac{\partial \mathcal{K}_\nu^{(1)}}{\partial p_\rho}\left(\frac{\partial \mathcal{K}_\nu^{(1)}}{\partial p'_\mu}\right)^{-1} - \frac{\partial \mathcal{K}_\nu^{(0)}}{\partial p_\rho}\left(\frac{\partial \mathcal{K}_\nu^{(0)}}{\partial p'_\mu}\right)^{-1}\right] x'^\mu(s_0) =$$

$$-\frac{\partial \mathcal{K}_\nu^{(1)}}{\partial p_\rho}\left(\frac{\partial \mathcal{K}_\nu^{(1)}}{\partial p'_\mu}\right)^{-1}\Delta\tau' u'^\mu + \Delta\tau u^\rho. \tag{11}$$

At first order in ℓ, it becomes

$$\ell\left[\delta_1^\rho\left(k'_0 - q_0\right) - \delta_0^\rho\left(q'_1 - k_1\right)\right] x'^1(s_0) =$$

$$\Delta\tau u^\rho + \Delta\tau'\left[u'^\rho + u'^1\ell\left(\delta_0^\rho k_1 - \delta_1^\rho k'_0\right)\right] , \tag{12}$$

which has solutions with $\Delta\tau$, $\Delta\tau' > 0$.

Enforcing relative locality

In order to enforce relative locality and to have a relativistic setup, we need (possibly deformed) spacetime-translational invariance:

$$x_B^\mu(s) = x_A^\mu(s) - b^\nu\, T^\mu_{x\nu} \tag{13}$$
$$x_B'^\mu(s) = x_A'^\mu(s) - b^\nu\, T^\mu_{x'\nu} \tag{14}$$
$$y_B^\mu(s) = y_A^\mu(s) - b^\nu\, T^\mu_{y\nu} \tag{15}$$
$$y_B'^\mu(s) = y_A'^\mu(s) - b^\nu\, T^\mu_{y'\nu} \tag{16}$$
$$z_B^\mu(s) = z_A^\mu(s) - b^\nu\, T^\mu_{z\nu} \tag{17}$$
$$z_B'^\mu(s) = z_A'^\mu(s) - b^\nu\, T^\mu_{z'\nu}\ , \tag{18}$$

where we make no assumptions on $T^\mu_{x\nu}$, $T^\mu_{x'\nu}$, $T^\mu_{y\nu}$, $T^\mu_{y'\nu}$, $T^\mu_{z\nu}$ and $T^\mu_{z'\nu}$.

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which has solutions with $\Delta\tau$, $\Delta\tau' > 0$.

Causality from relative locality

Imposing that the equations of motion and the boundary conditions are covariant under the (possibly deformed) spacetime translations, we find that the momenta should satisfy the condition

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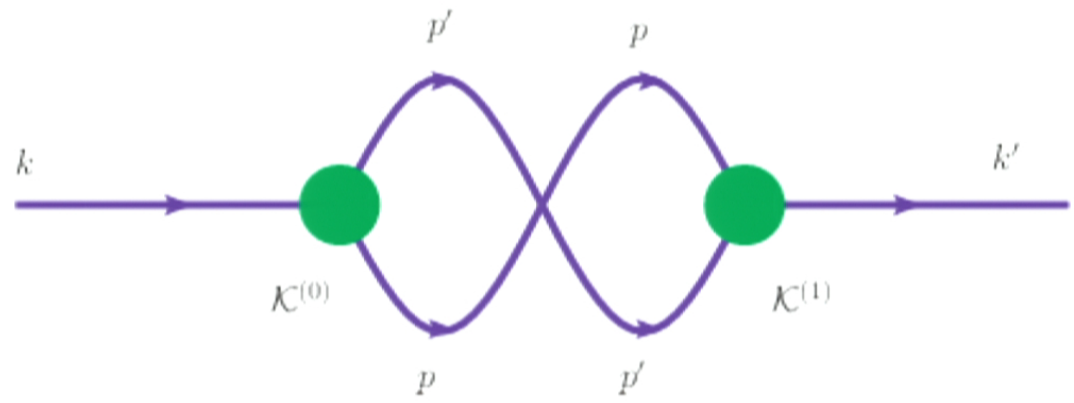
so that we have

$$0 = \Delta\tau u^\rho + \Delta\tau' \left[u'^\rho + u'^1 \ell \left(\delta_0^\rho k_1 - \delta_1^\rho k'_0\right)\right] \ , \qquad (20)$$

which has no solution with $\Delta\tau$, $\Delta\tau' > 0$.

Twisted loop: violation of global momentum conservation?

The following twisted loop has been studied in k-momentum space (A. Banburski, arXiv:1305.7289v1):



$$\begin{aligned} \mathcal{S} \;=\; & \int_{-\infty}^{s_0} ds \left(z^{\mu} \dot{k}_{\mu} + \mathcal{N}_k \mathcal{C}_k \right) + \int_{s_1}^{+\infty} ds \left(z'^{\mu} \dot{k}'_{\mu} + \mathcal{N}_{k'} \mathcal{C}_{k'} \right) + \\ + & \int_{s_0}^{s_1} ds \left(x'^{\mu} \dot{p}'_{\mu} + \mathcal{N}_{p'} \mathcal{C}_{p'} \right) + \int_{s_0}^{s_1} ds \left(x^{\mu} \dot{p}_{\mu} + \mathcal{N}_p \mathcal{C}_p \right) + \\ - & \xi^{\mu}_{(0)} \mathcal{K}^{(0)}_{\mu} - \xi^{\mu}_{(1)} \mathcal{K}^{(1)}_{\mu}, \end{aligned} \tag{21}$$

with $\mathcal{K}^{(0)}_{\mu} = (k \oplus (\ominus (p \oplus p')))_{\mu}$ and $\mathcal{K}^{(1)}_{\mu} = ((p' \oplus p) \oplus (\ominus k'))_{\mu}$.

Twisted loop in k-momentum space

Let us write explicitely the conservation laws $\mathcal{K}_\mu^{(0)} = 0$ and $\mathcal{K}_\mu^{(1)} = 0$:

$$k_\mu = p_\mu + p'_\mu + \delta^1_\mu \ell p'_1 (k_0 - p'_0) \; , \tag{22}$$

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There are solutions such that $k_\mu \neq k'_\mu$.

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Momentum conservation from relative locality

When imposing the (possibly deformed) spacetime translations to be a symmetry, we find the following condition on momenta

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which written explicitely is

$$\ell\left[\delta_1^\mu k_0 - \delta_0^\mu\left(p_1 + p'_1\right)\right] = 0\ . \qquad (26)$$

This implies $k_\mu = 0 + O(\ell)$ and $p_\mu = -p'_\mu + O(\ell)$.

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$$k_\mu = p_\mu + p'_\mu + \delta^{1}_{\mu}\ell p'_1(k_0 - p'_0) \ , \qquad (27)$$

$$k'_\mu = p_\mu + p'_\mu - \delta^{1}_{\mu}\ell p'_0 p_1 \ , \qquad (28)$$

we find $k_\mu = k'_\mu$.

Final remarks

- breaking symmetry for spacetime translations allows for violation of causality and of global conservation of momenta which are not present in special relativity.

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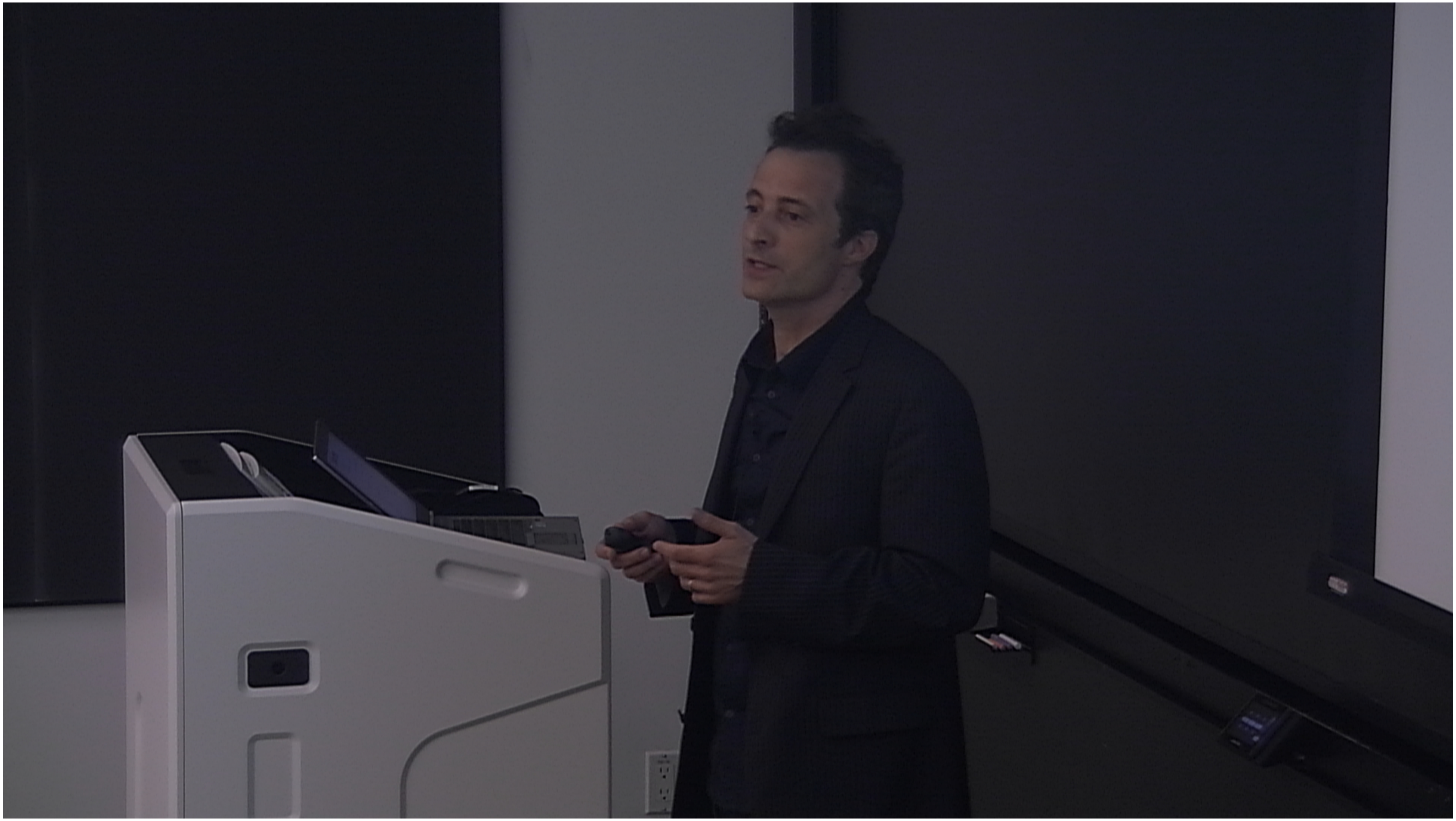
- in relative locality, where translation symmetry is present (even if deformed with respect to special relativity), causality and conservation of momenta are maintained.



A perspective on recent developments in Quantum Gravity Phenomenology

PI 25.7.2013

Giovanni Amelino-Camelia
University of Rome "La Sapienza"

* focus on just a few research lines with significant recent developments

* some comments from objective perspective some from subjective perspective

1. **quantum-gravity cosmology**
2. **quantum-spacetime astrophysics**
3. **relative locality**
4. **macroscopic bodies**

broader picture in GAC, Living Reviews in Relativity 16 (2013) 5

Quantum Gravity Cosmology

strong growth of interest

case for opportunities becoming more solid (see, e.g., Agullo, Barrau)

important to get more quantitative and get better intuition for how lucky we need to be

corrections to standard scenario (with inflation+CMBR+…) are not the only way to go,
and here I can mention as an example the recent work
arXiv:13053153, **Phys.Rev.D87,123532 [GAC+Arzano+Gubitosi+Magueijo]**

$$E^2 = p^2(1 + (\lambda p)^{2\gamma}) \longrightarrow d_S = 1 + \frac{D}{1+\gamma}$$

??

for γ=2 (i.e. d_S=2 in the UV) one gets
unavoidably scale-invariant spectrum
of vacuum fluctuations
(without inflation!!!)

dynamical dimensional reduction also in talks by Eichorn, Loll, Carlip…

Multimessenger Quantum Spacetime Astrophysics

talk by Guetta: annus mirabilis for quantum-spacetime astrophysics!!!

May of this year: IceCube results are announced which amount to the start of astrophysics with high-energy neutrinos!!!!!!!!!!!!

April of this year: Fermi observes GRB130427A setting new records on the high-energy brightness of bursts we can observe!!!!!
(and reassuring us that Fermi is not dead)

Here I have three points:

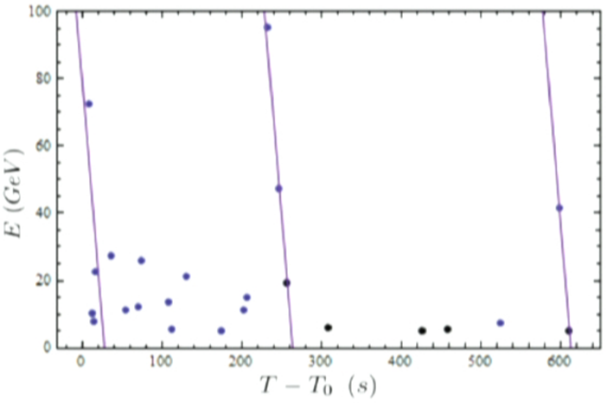
* let us make it so that for once QG community carries weight in the fact that an experimental project becomes reality: GAMMA400 (arXiv:1201.2490)

- An effort is needed for us to understand whether the more refined techniques of data analysis (Granot, Bolmont), which allow more robust handling of the lack of knowledge of the sources, are widely applicable to quantum-spacetime contexts. Techniques inspired by propagation in a medium…

* GRB130427A does have some remarkable features for spectral lags (most likely astrophysical….but they are striking…)

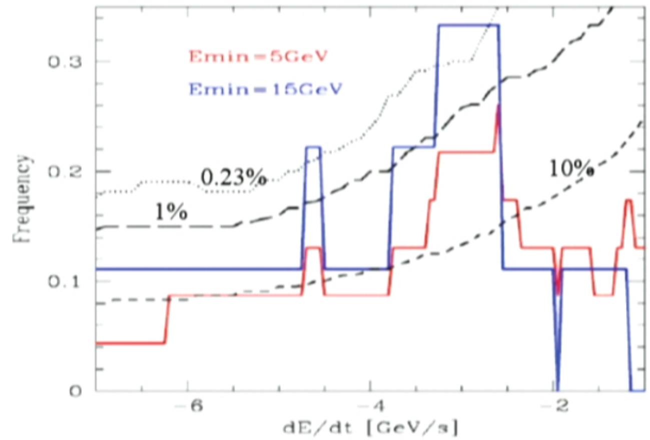
Multimessenger Quantum Spacetime Astrophysics (continues)

GRB130427A, seen above 5 GeV:



GAC + Fabrizio Fiore + Dafne Guetta + Simonetta Puccetti

arXiv.org > astro-ph > arXiv:1305.2626



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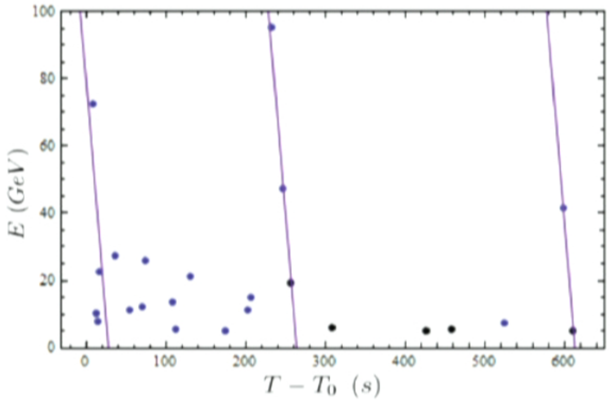

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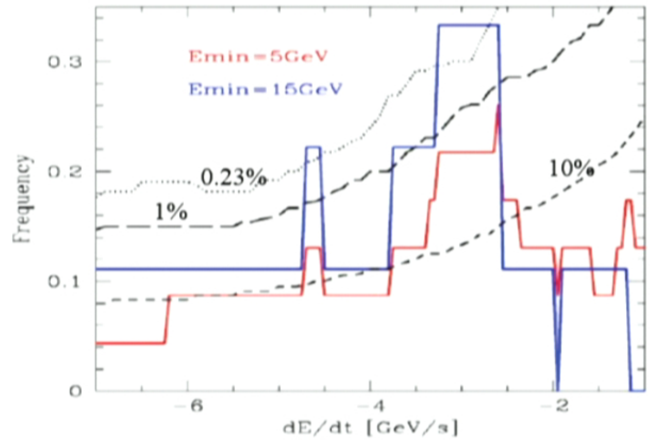

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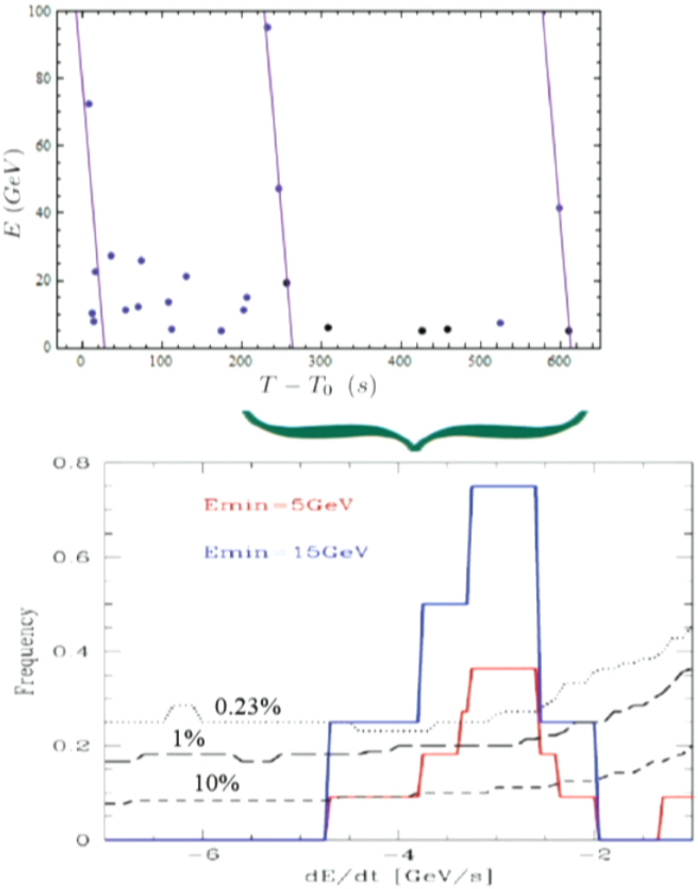

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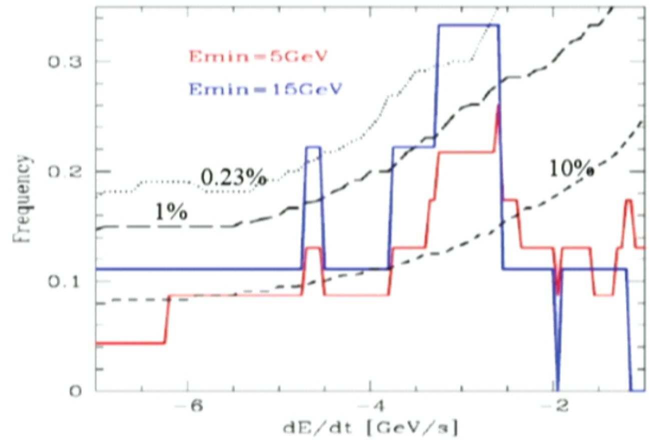

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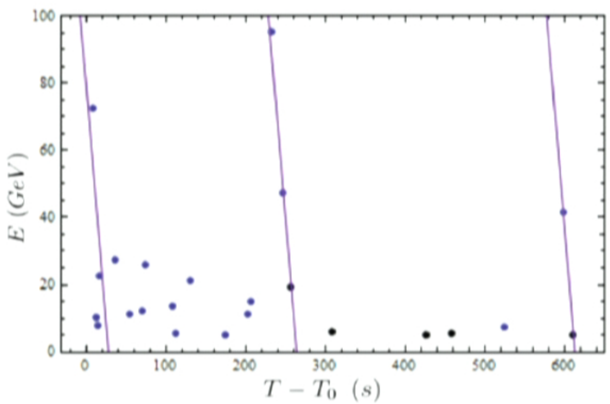

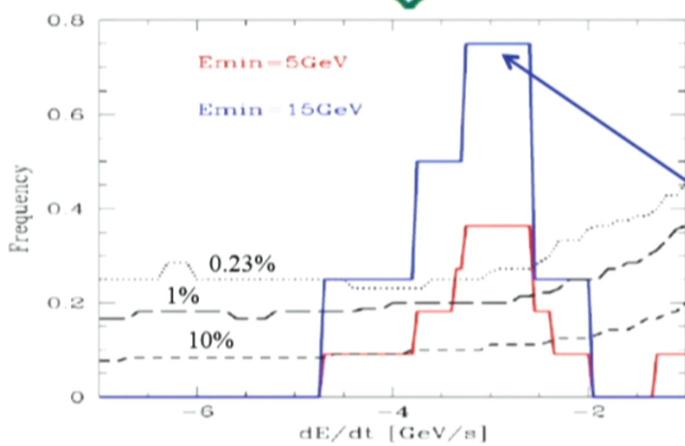

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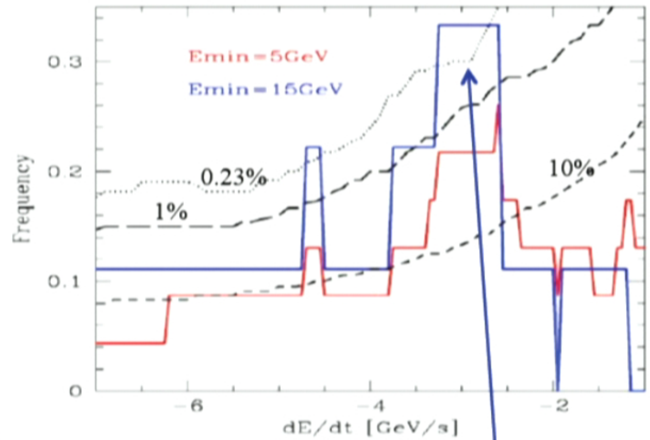

and this is the slope of the IceCube neutrino story in GAC+Guetta+Piran (from March 2013!) but of course it may be just a cruel numerical coincidence

Multimessenger Quantum Spacetime Astrophysics (continues)

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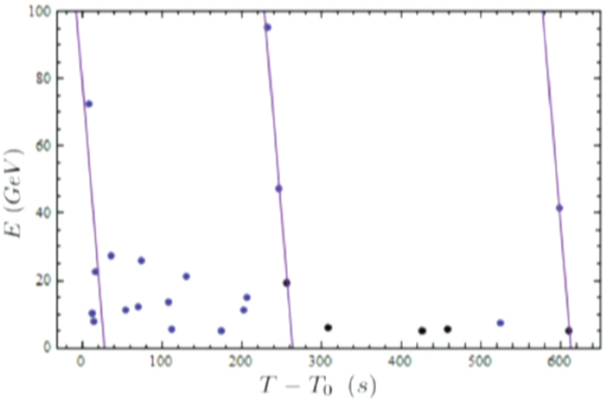

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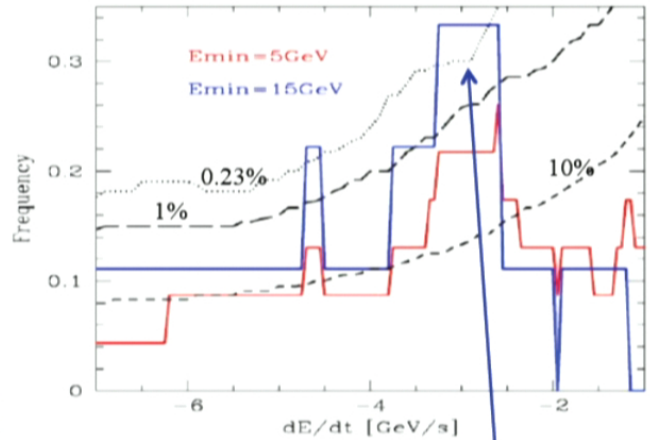
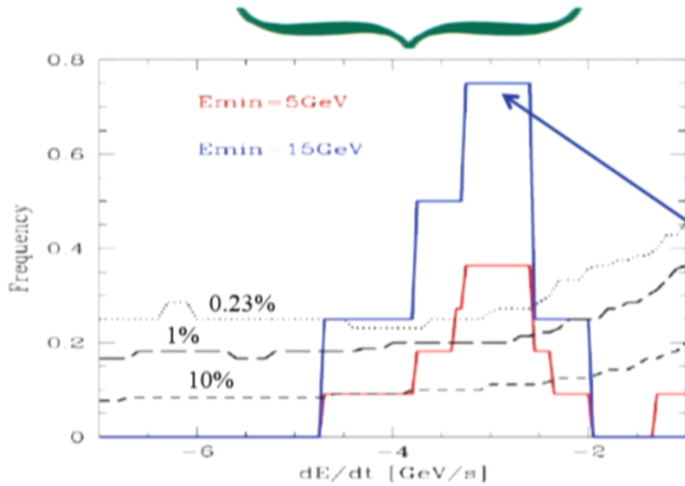
arXiv.org > astro-ph > arXiv:1305.2626





and this is the slope of the IceCube neutrino story in GAC+Guetta+Piran (from March 2013!) but of course it may be just a cruel numerical coincidence

Relative locality

Need to deal with it in order to render the quantum-spacetime astrophysics
we are discussing applicable to the deformed-relativistic-symmetries hypothesis (DSR)

Curvature of momentum space may be one of the "tentative facts" emerging
from quantum-gravity research
It surely is there in 3d gravity (Osei)
and various arguments suggest it for the 4D theory

like it or not,
from curvature of momentum space the relativity of spacetime locality follows (Loret)

Relative locality in string theory?!!! (Minic)
Progress in field-theoretical formulation!!! (Rempel)

Beautiful results on the connection between
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Relative locality (continued)

brief aside on the transition from Galilean to Special Relativity:

	Galilean Relativity	Special Relativity	DSR
boosts	Galilean	c-deformed Galilean	λ,c-deformed Galilean
simultaneity	absolute	relative	relative
locality	absolute	absolute	relative
on-shellness	$E = \frac{p^2}{2m}$ (+m)	$E = \sqrt{p^2 + m^2}$	$E = C_\lambda(p,m)$
composition of momenta	$p_\mu^{(1)} + p_\mu^{(2)}$	$p_\mu^{(1)} + p_\mu^{(2)}$	$p_\mu^{(1)} \oplus_\lambda p_\mu^{(2)}$
composition of velocities	$\vec{w} = \vec{v} + \vec{u}$	$\vec{w} = \vec{v} \oplus_c \vec{u}$	$\vec{w} = \vec{v} \oplus_{c,\lambda} \vec{u}$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \frac{1}{1 + \frac{v_1 u_1 + v_2 u_2 + v_3 u_3}{c^2}} \left\{ \left[1 + \frac{1}{c^2} \frac{\gamma_{\mathbf{v}}}{1 + \gamma_{\mathbf{v}}} (v_1 u_1 + v_2 u_2 + v_3 u_3) \right] \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \frac{1}{\gamma_{\mathbf{v}}} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \right\}$$

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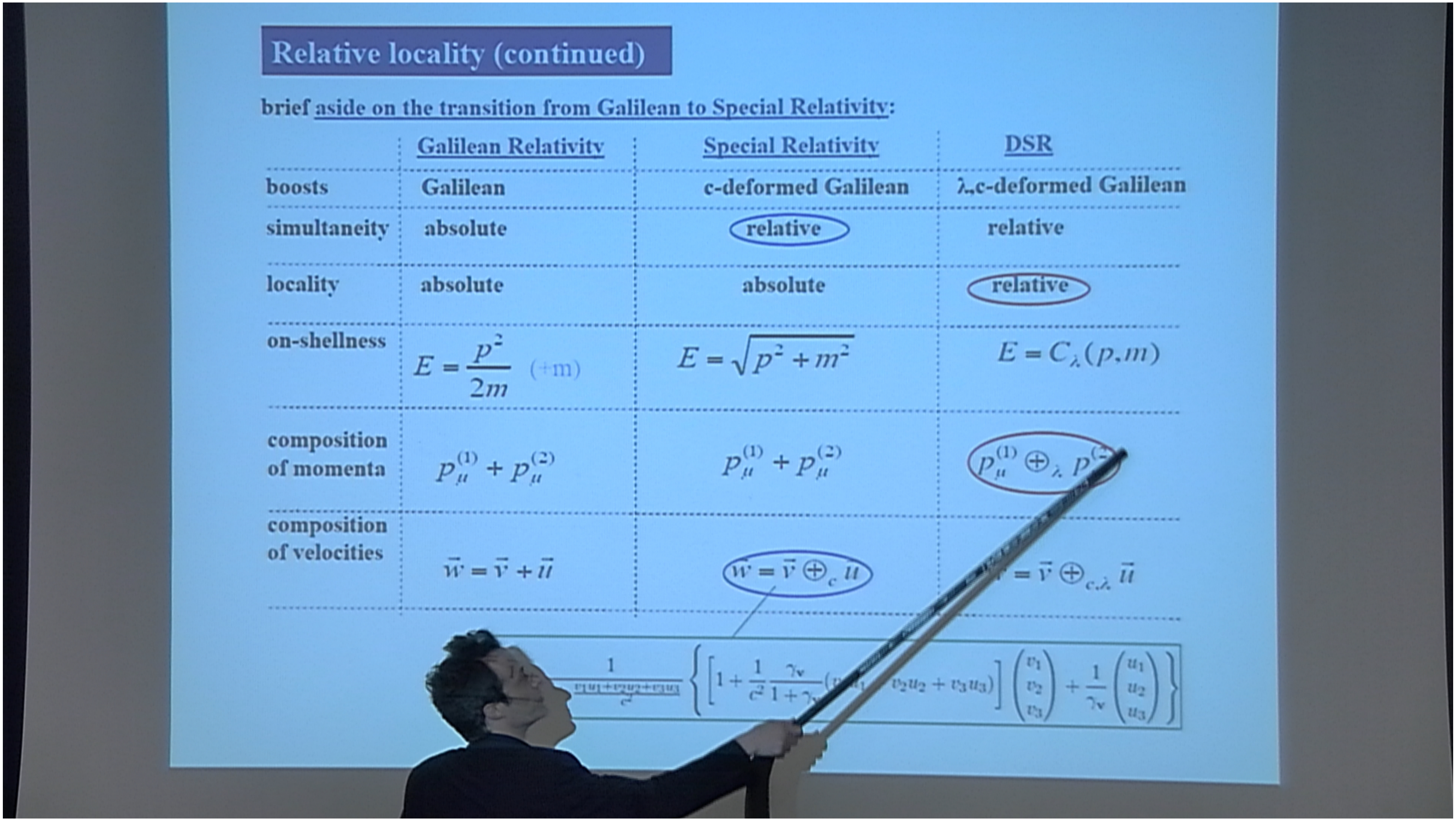
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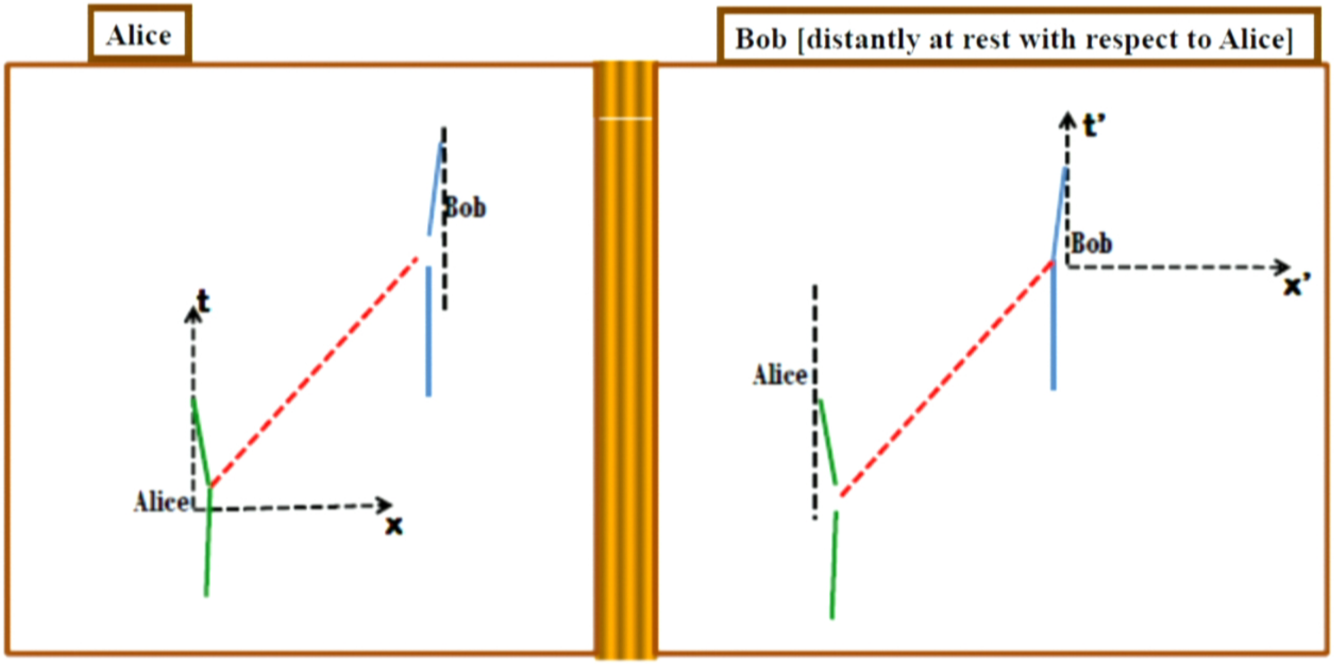
Relative locality (continued)

my preferred take-home message from relative spacetime locality:



Quantum-Gravity Phenomenology with macroscopic bodies

no mention of macroscopic bodies at this meeting

Quantum mechanics could have been discovered (if only the development of experimental techniques had had a different history) not at atomic and subatomic scales, but through tests of, say, the Chandrasekhar limit and superconductivity

Macroscopic bodies in a quantum spacetime can be
a blessing (effects amplified only in some special cases where an experimental signal is found?)
but could also be disastrous (effects generically amplified by number of constituent particles)

Experiments proposed by
Pikovski et al, Nature Physics2012
Bekenstein, PRD2012
Marin et al, Nature Physics2013
assume strong effects for the center of mass of a macroscopic body

Evidence that macroscopic bodies might not be pathological and nonetheless provide Valuable opportunities for phenomenology is found from the dual curved-momentum-space picture when combining results in

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there is no "soccerball problem" → GAC+Freidel+KowalskiGlikman+Smolin, PRD2011; GAC+Freidel+KowalskiGlikman+Smolin, PRD2013

GAC, arXiv:1304.7271
Further confirmation that there is no "soccerball problem": certain observables of certain macroscopic bodies in certain contexts may well be an important opportunity for QG phenomenology, but the center-of-mass observables of a macroscopic body are rather insensitivitive to quantum-spacetime effects

for many years naive (but legitimate) arguments had suggested that center-of-mass degrees of freedom would get strongly amplified effects of spacetime quantization; this was the "soccerball problem" but it is easy to see that those arguments must be wrong

Let me show this by starting with

center-of-mass coordinates the observables X, Y, Z, with

$$X = \frac{1}{N}\sum_{n=1}^{N} x_n \ , \ \ Y = \frac{1}{N}\sum_{n=1}^{N} y_n \ , \ \ Z = \frac{1}{N}\sum_{n=1}^{N} z_n \quad (1)$$

(where of course x_n, y_n, z_n are the coordinates of the n-th composing particle), and I take as center-of-mass momentum the observables P_x, P_y, P_z, with

$$P_x = \sum_{n=1}^{N} p_{x,n} \ , \ \ P_y = \sum_{n=1}^{N} p_{y,n} \ , \ \ P_z = \sum_{n=1}^{N} p_{z,n} \quad (2)$$

(where of course $p_{x,n}, p_{y,n}, p_{z,n}$ are the momentum components of the n-th composing particle).

GAC, arXiv:1304.7271

phase space is already noncommutative in ordinary quantum mechanics

and macroscopic bodies get as much of Heisenberg's uncertainty principle as their constituents:

Heisenberg's

$$[x, p_x] = i\hbar$$

where I focused for simplicity on the x-direction.

Evidently the Heisenberg commutator also applies to a body's center of mass described by (1)-(2):

$$[X, P_x] = \left[\frac{1}{N} \sum_{n=1}^{N} x_n \, , \sum_{m=1}^{N} p_{x,m} \right] \qquad (3)$$

$$= \frac{1}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} \delta_{n,m} i\hbar = \frac{1}{N} \sum_{n=1}^{N} i\hbar = i\hbar$$

then you have to take into account decoherence and the story changes but without decoherence this is the situation of course

GAC, arXiv:1304.7271

And this is not specific to "Lie-algebra noncommutative spacetimes":

for "Moyal noncommutativity"

$$[x, y] = i\ell_M^2$$

one finds for the center-of-mass coordinates of a macroscopic body

$$[X, Y] = \left[\frac{1}{N}\sum_{n=1}^{N} x_n \, , \frac{1}{N}\sum_{m=1}^{N} y_m\right] \tag{6}$$

$$= \frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N} \delta_{n,m} i\ell_M^2 = \frac{1}{N^2}\sum_{n=1}^{N} i\ell_M^2 = i\left(\frac{\ell_M}{\sqrt{N}}\right)^2$$

So again noncommutativity for center-of-mass coordinates gets suppressed by 1/N

thanks Flavio!!!

thanks Flavio!!!

my sign of appreciation: pictures of Jose Altafini, Flavio's favourite player