Title: Discrete Approaches - 2

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Abstract:

Dynamics and (broken) symmetries of discrete gravity models

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Perimeter Institute

Review talk, discrete approaches session, Loops '13 @ Perimeter
July 25th, 2013

Plan of the talk

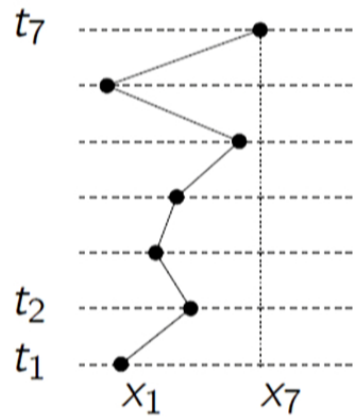
1. Discretizing continuum theories
2. Broken symmetries
3. Canonical dynamics of discrete systems
4. Canonical Regge Calculus
5. Quantization

Discretizing continuum theories

- Broadly:
  1. discretize continuum eoms/constraints in gravity $\Rightarrow$ get 2nd class constraints [Piran, Williams '86; Friedman, Jack '86; Loll '98] which are not preserved by evolution (e.g. numerical relativity)
  2. discretize continuum action $\Rightarrow$ obtain eoms from discrete action

- 2nd option also used in regularizing the path integral in QM

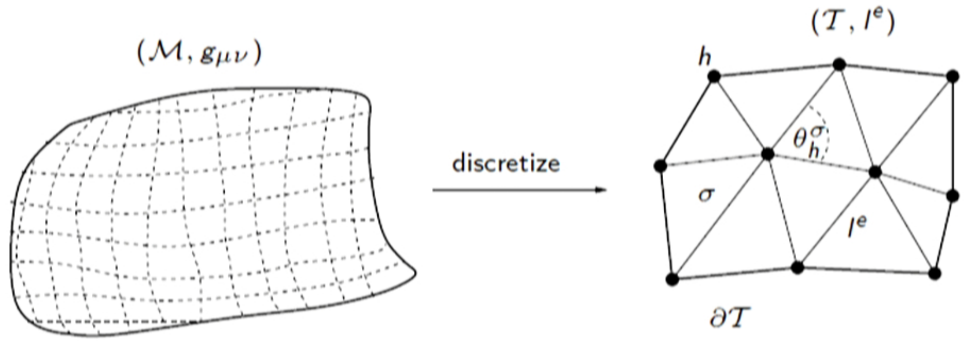
$$\int \mathcal{D}x\, e^{iS} = \lim_{N\to\infty} \int \prod_{k=1}^{N} dx_k\, e^{i\sum_k S_k(x_k, x_{k-1})}$$



- we shall follow 2nd option

- Regge Calculus: replace smooth D–dim. spacetime $(\mathcal{M}, g_{\mu\nu})$ by piecewise-linear flat metric living on triangulation $\mathcal{T}$, comprised of D–simplices σ

h: 'hinge' ($(D-2)$–subsimplex)
θ_h^σ: interior dihedral angle at h in σ
V_h: volume of h
$\epsilon_h := 2\pi - \sum_{\sigma\supset h}\theta_h^\sigma$: deficit angle
$\psi_h := \pi - \sum_{\sigma\supset h}\theta_h^\sigma$: exterior angle



- configuration variables: edge lengths $\{l^e\}_{e\in\mathcal{T}}$, encode complete geometry
- (Euclidean) action $S_{EH} = -\int_{\mathcal{M}}\sqrt{g}Rd^4x - \int_{\partial\mathcal{M}}\sqrt{q}Kd^3x \underset{\text{discretize}}{\longrightarrow} S_R$

$$S_R(\{l^e\}) = -\sum_{h\subset\mathcal{T}\setminus\partial\mathcal{T}} V_h\epsilon_h - \sum_{h\subset\partial\mathcal{T}} V_h\psi_h \qquad \Rightarrow \qquad S_R \text{ additive}$$

Price to pay: breaking symmetries [Marsden, West '01; Dittrich, Bahr '09]

- example: (broken) reparametrization invariance in discrete mechanics
- enlarge system, take t as variable, evolution w.r.t. parameter s

$$S = \int dt\, L\left(x(t), \dot{x}(t)\right) \longrightarrow S_e = \int ds\, L\left(x(s), \frac{x'(s)}{t'(s)}\right) t'(s)$$

- dynamics equivalent (eom for x solved $\Rightarrow$ eom for t solved)
- system invariant under reparametrizations of s
- discretize $s_{in} < \ldots < s_k < \ldots < s_{fin}$, $x_k = x(s_k)$, $t_k = t(s_k)$

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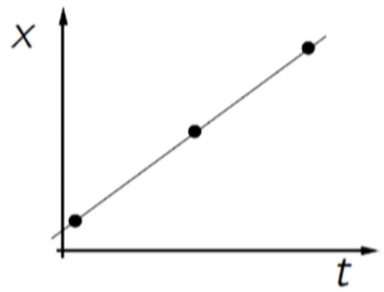

Figure: $V = 0$, sym. preserv.

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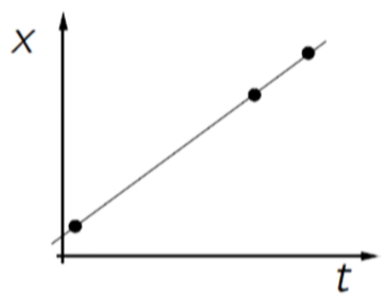

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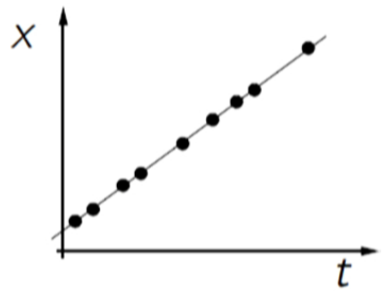

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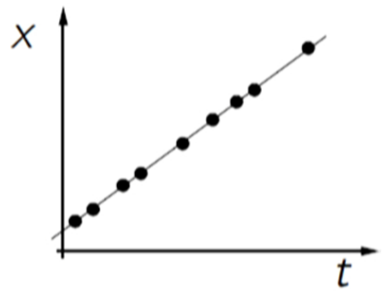

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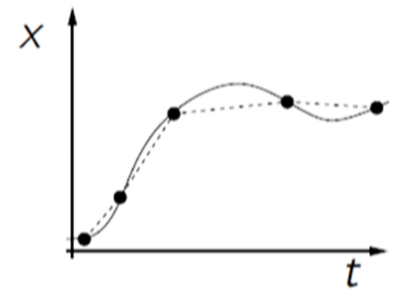

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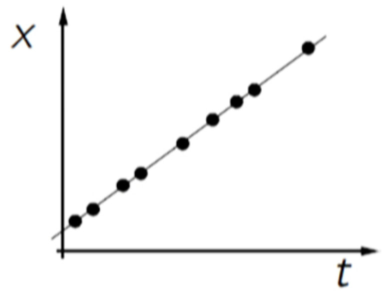

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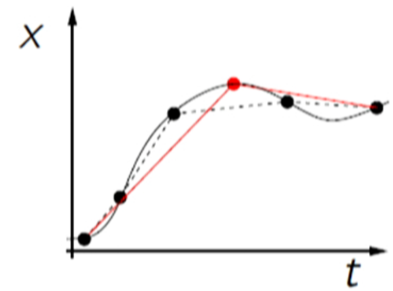

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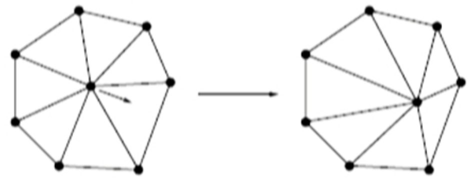
Discretization and diffeomorphism symmetry

- analogous situation in discrete gravity $\Rightarrow$ vertex displacement symmetry in flat sector of Regge Calculus



- symmetry broken in presence of curvature [Rocek, Williams '81; Dittrich '08; Bahr, Dittrich '09]

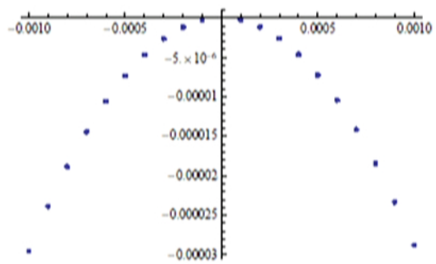


Figure: Bahr, Dittrich, CQG 26 225011 (2009)

- gauge modes of the continuum become propagating in the discrete
- coarse graining/perfect actions [Bahr, Dittrich '09; Bahr, Dittrich, Steinhaus '11]
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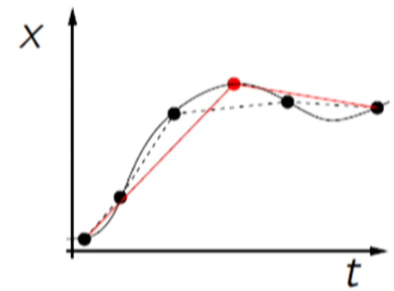

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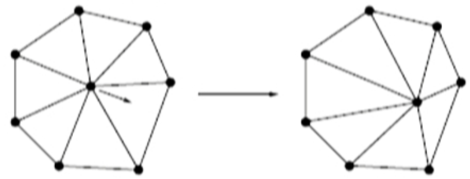

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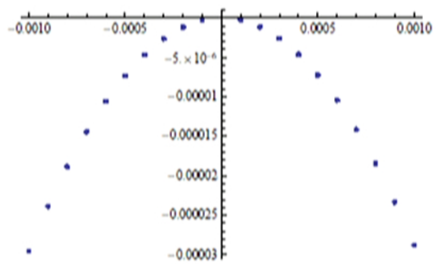

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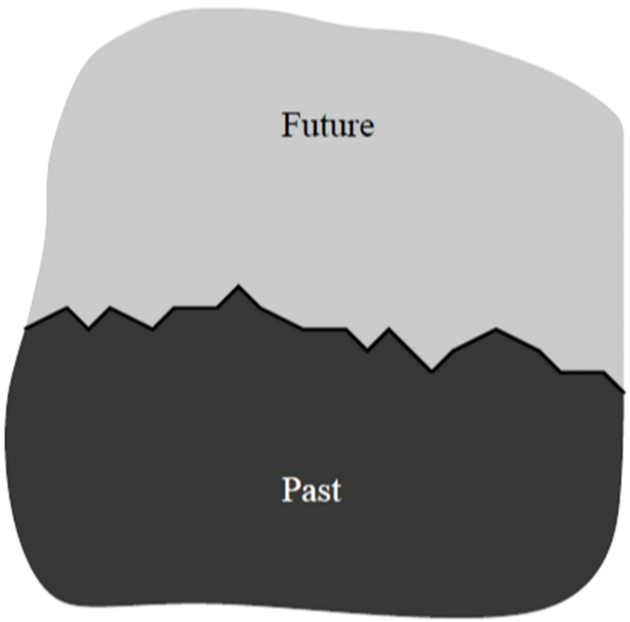
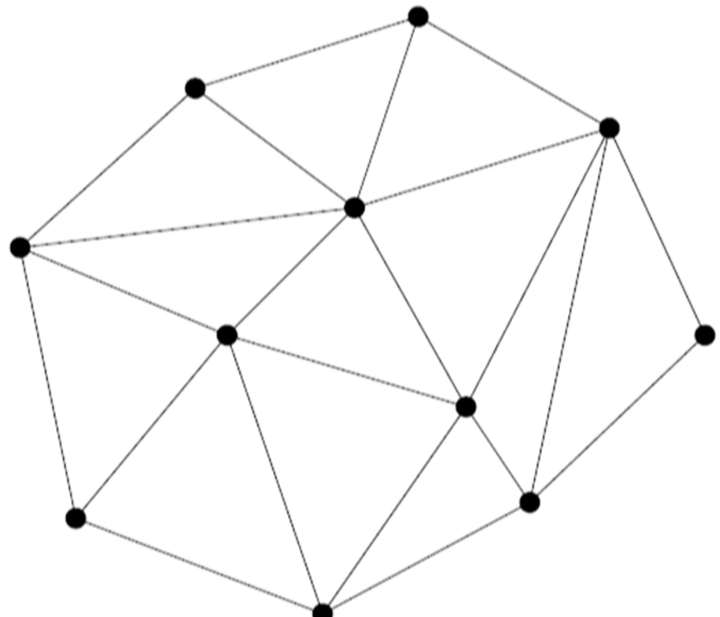
Discrete time evolution in simplicial gravity

- dynamics generated by evolution moves, not constraints/Hamiltonian
- glue pieces of triangulation to triangulated hypersurface Σ_k at each step $k \in \mathbb{Z} \Rightarrow$ add action contributions

step k



Σ_k



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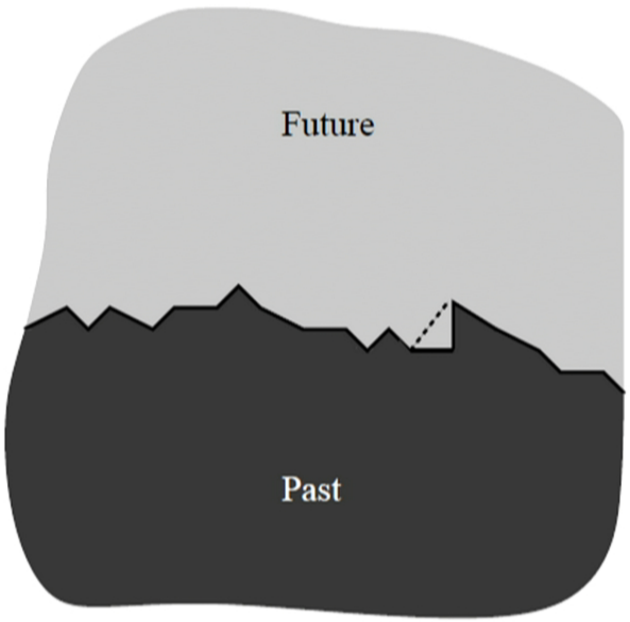
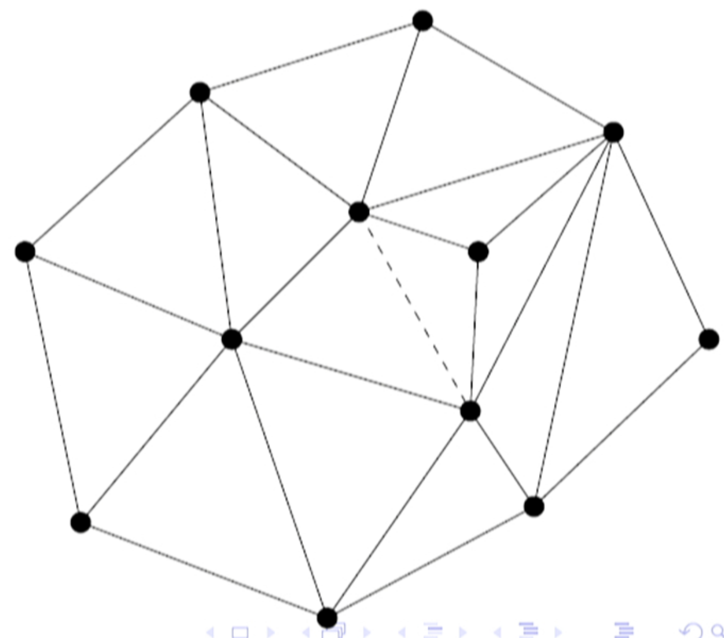
step $k+3$



Σ_{k+3}



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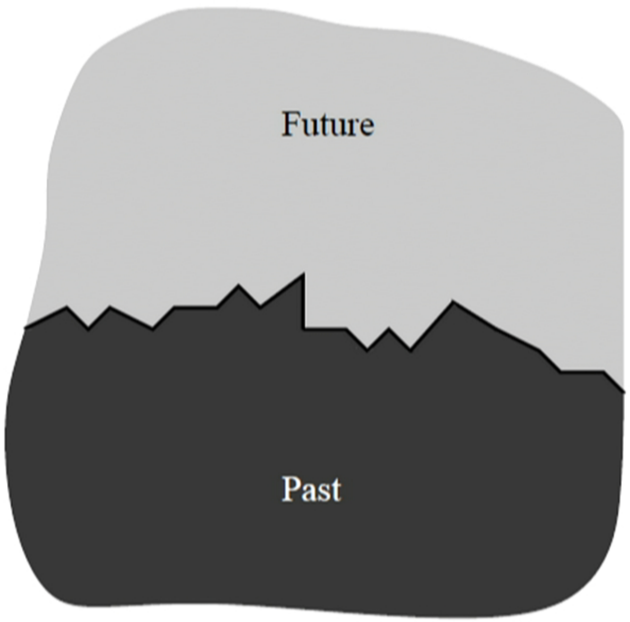
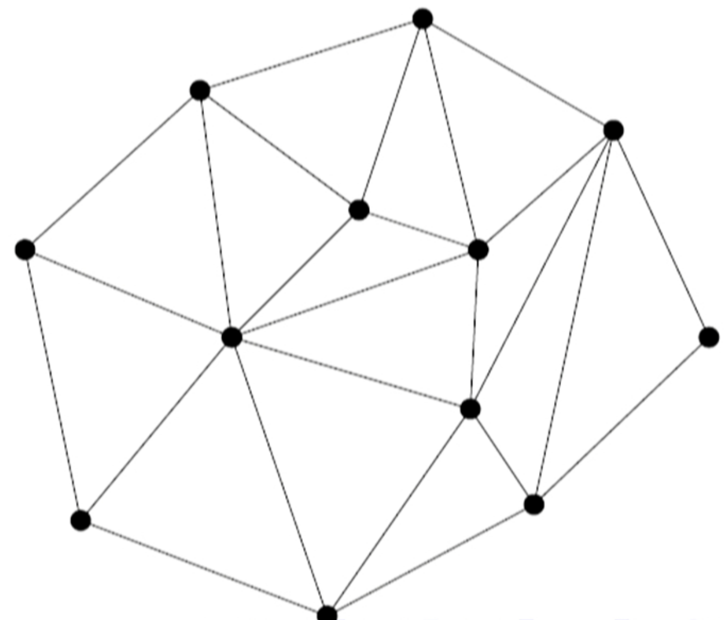
step $k+4$

Σ_{k+4}





Discrete time evolution in simplicial gravity

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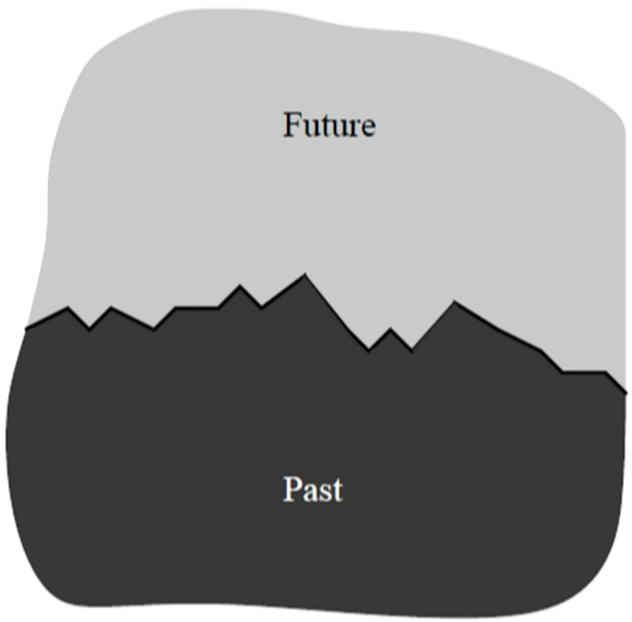
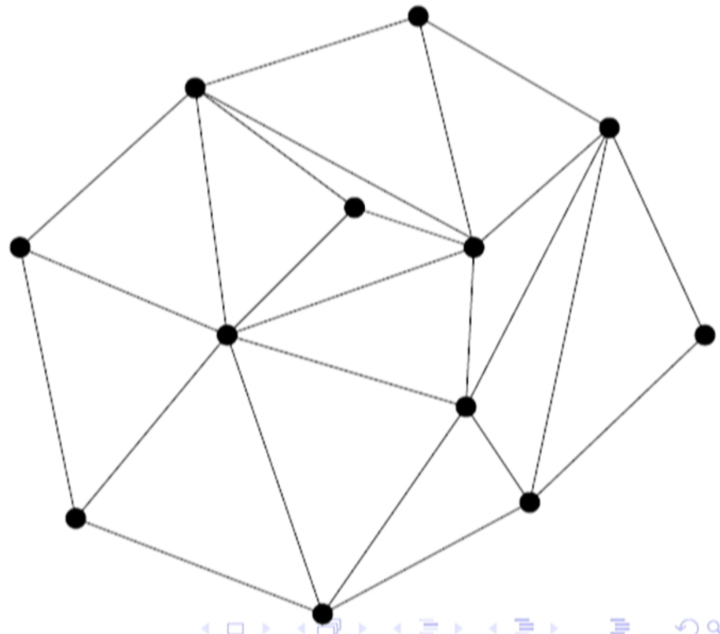
step $k+5$



Σ_{k+5}



Discrete time evolution in simplicial gravity

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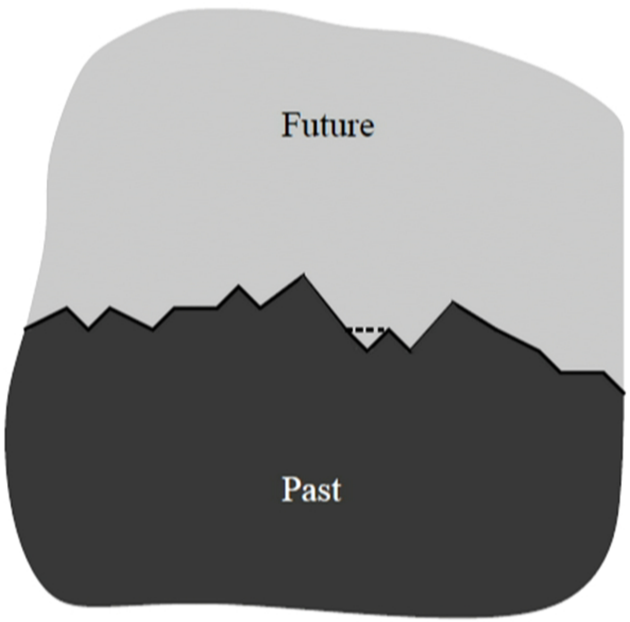
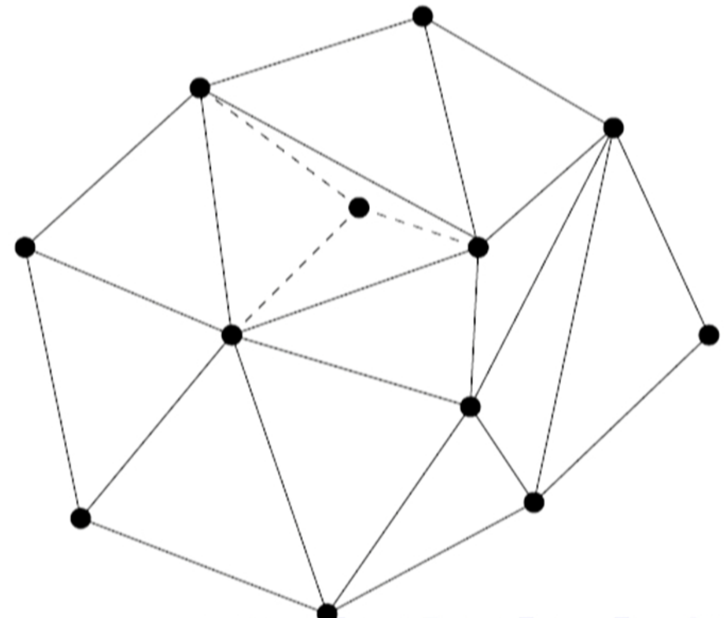
step $k+6$



Σ_{k+6}



Canonical momenta [Marsden, West '01; Gambini, Pullin '03; Dittrich, PH '11,'13]

- discrete action $S = \sum_{k=1}^{N} S_k(x_{k-1}, x_k) \Rightarrow S_k$ as generating fct.

$$ {}^{-}p^{k-1} := -\frac{\partial S_k(x_{k-1}, x_k)}{\partial x_{k-1}} \quad , \quad {}^{+}p^{k} := \frac{\partial S_k(x_{k-1}, x_k)}{\partial x_k} $$

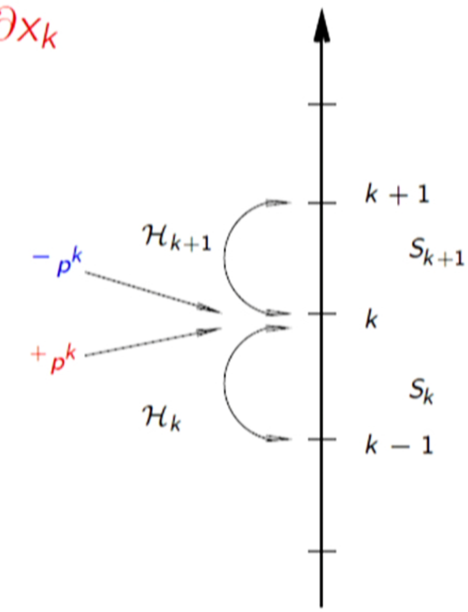
${}^{-}p$: pre–momenta, ${}^{+}p$: post–momenta

- defines time evolution map

$$ \mathcal{H}_k : (x_{k-1}, {}^{-}p^{k-1}) \mapsto (x_k, {}^{+}p^{k}) $$

- similarly, use $S_{k+1}(x_k, x_{k+1})$ as gen. fct.

$$ {}^{-}p^{k} = -\frac{\partial S_{k+1}}{\partial x_k} $$



- eom $\frac{\partial S_k}{\partial x_k} + \frac{\partial S_{k+1}}{\partial x_k} = 0 \Leftrightarrow {}^{+}p^k = {}^{-}p^k$ *momentum matching*

$\Rightarrow$ canon. and covar. formulation equivalent

Constraints [Dittrich, PH '11, '13]

- in cont. $p = \frac{\partial L(q,\dot{q})}{\partial \dot{q}} \Rightarrow$ impl. fct. thm.: if $\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) = 0$ get primary constraints $C_m(q,p) = 0$

- in discrete, $\mathcal{H}_k$ for evolution $(k-1) \to k$ defined by

$$ {}^{-}p^{k-1} := -\frac{\partial S_k(x_{k-1}, x_k)}{\partial x_{k-1}} \quad , \quad {}^{+}p^{k} := \frac{\partial S_k(x_{k-1}, x_k)}{\partial x_k} $$

$\Rightarrow$ obtain <u>two</u> types of constraints if $\det\left(\frac{\partial^2 S_k}{\partial x^i_{k-1} \partial x^j_k}\right) = 0$

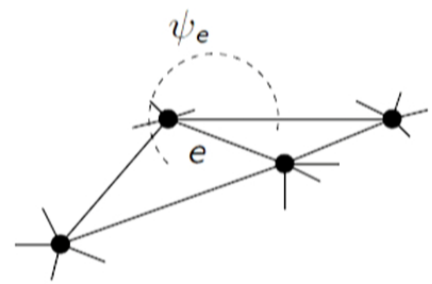
  - ${}^{+}C^k(x_k, {}^{+}p^k) = 0 \quad \Rightarrow$ post–constraints
  - ${}^{-}C^{k-1}(x_{k-1}, {}^{-}p^{k-1}) = 0 \quad \Rightarrow$ pre–constraints

- time evol. map $\mathcal{H}_k$ no longer unique:

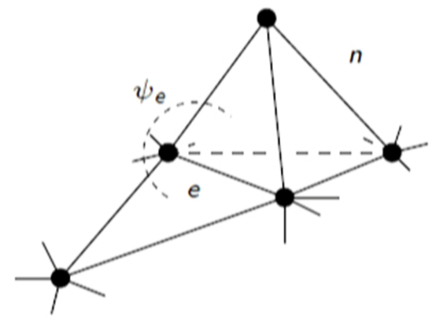
e.g., ${}^{-}C^{k-1}(x_{k-1}, {}^{-}p^{k-1}) = 0 \Rightarrow x_k = x_k(x_{k-1}, {}^{-}p^{k-1}, \lambda^m_k)$,

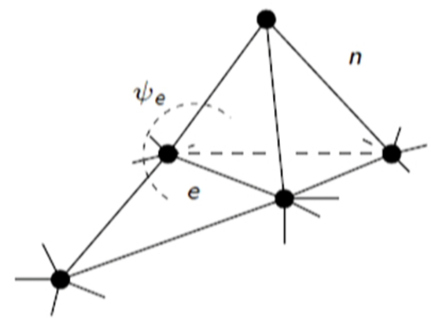
λ_k: *a priori* free parameter

Evolving phase spaces: e.g. 1–3 Pachner move [Dittrich, PH '11; '13]



- 3 new edges, but no eoms for $k \longrightarrow k+1$
  $\Rightarrow$ their lengths l^n_{k+1} are *a priori* free λ_{k+1}





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- extend phase space at step k, add (l^n_k, p^k_n)
- use $S_\tau(l^n_{k+1}, \ldots)$ as type 1 gen. fct. (trivial dep. on l^n_k)

$$p^k_n = -\frac{\partial S_\tau}{\partial l^n_k} = 0 \quad , \quad p^{k+1}_n = \frac{\partial S_\tau}{\partial l^n_{k+1}}$$

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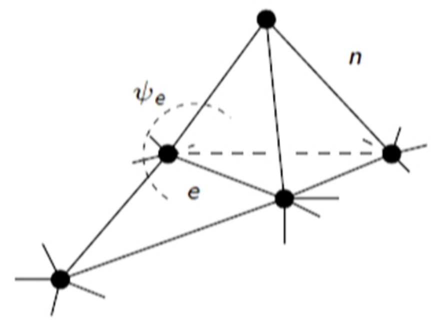

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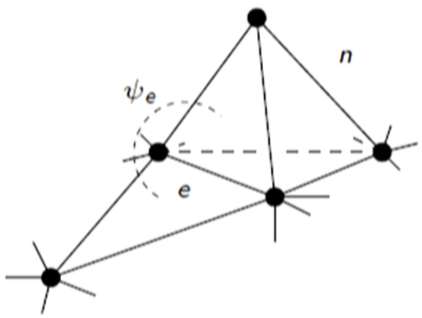

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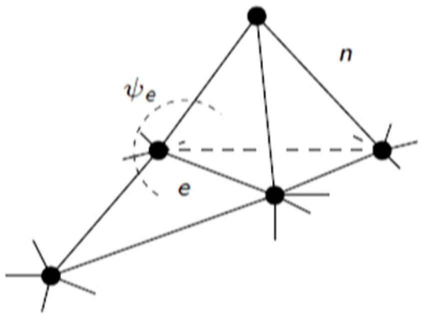

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- 3 pre–constraints at k
- ψ^{k+1}_n only depends on lengths from Σ_{k+1} $\Rightarrow$ obtain 3 post–constraints
- all Pachner moves in 3D/4D analogously $\Rightarrow$ 'pre-symplectic transformations'

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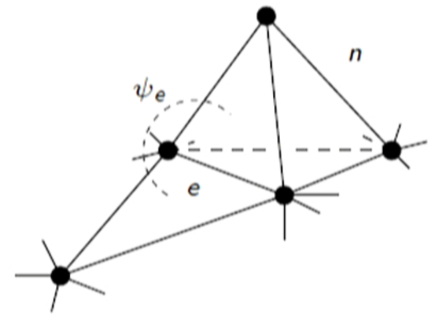

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- use $S_\tau(l^n_{k+1}, \ldots)$ as type 1 gen. fct. (trivial dep. on l^n_k)

$$p^k_n = 0 \qquad , \qquad p^{k+1}_n = \psi^{k+1}_n(l^e_{k+1}, l^n_{k+1})$$

- 3 pre–constraints at k
- ψ^{k+1}_n only depends on lengths from Σ_{k+1} $\Rightarrow$ obtain 3 post–constraints
- all Pachner moves in 3D/4D analogously $\Rightarrow$ 'pre-symplectic transformations'

- evolution $(k-1) \to k \to (k+1)$: generally, ${}^{+}C^{k} \neq {}^{-}C^{k}$
- momentum matching: impose both ${}^{+}C^{k}$ and ${}^{-}C^{k}$ at k
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$$\{{}^{-}C^{k}_{i}, {}^{-}C^{k}_{j}\} \approx 0 \approx \{{}^{+}C^{k}_{i}, {}^{+}C^{k}_{j}\}$$

- generally mixture 2nd class

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- however, if $C^{k} = {}^{-}C^{k} = {}^{+}C^{k}$, then
    1. first class
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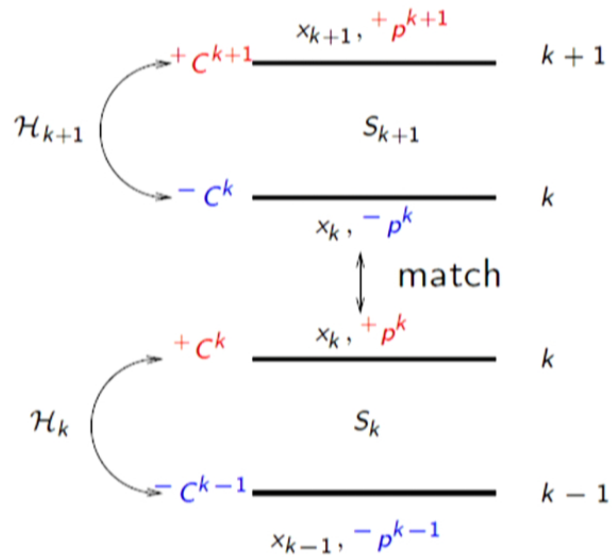

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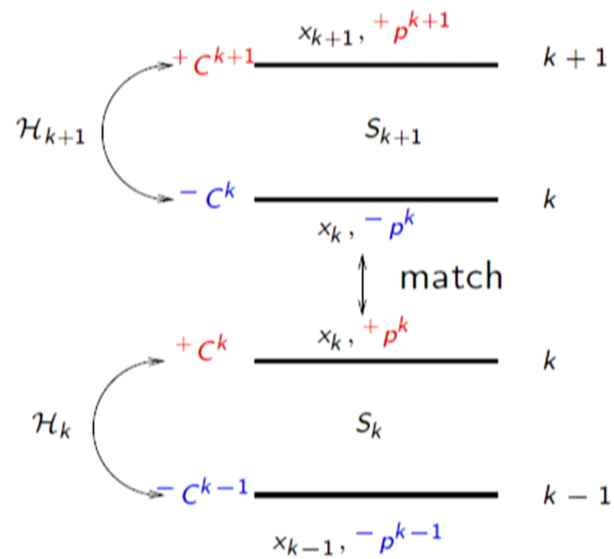

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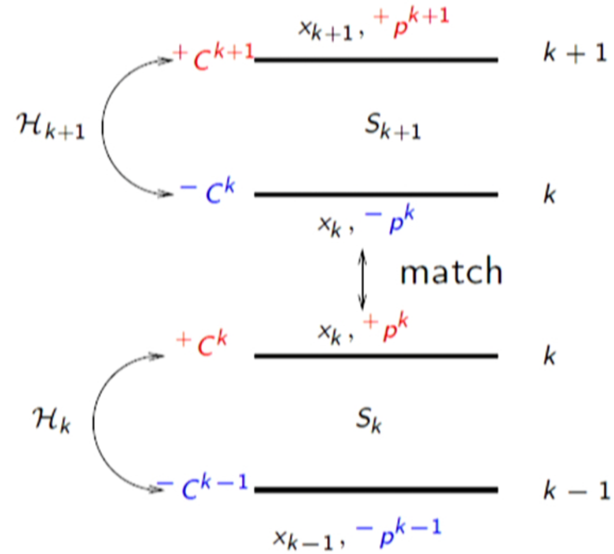

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Propagating degrees of freedom [Dittrich, PH '13]

- need two time steps for propagation, $\mathcal{H}_{k_f} : \mathcal{C}^-_{k_i} \to \mathcal{C}^+_{k_f}$
- data propagating $k_i \longrightarrow k_f$ commutes with pre–constraints at k_i and post–constraints at k_f
- in evolution $k_i \longrightarrow k_f$ number of constraints at k_i depends on k_f (and vice versa)

$\Rightarrow$ number of propagating degrees of freedom, in general, $N_{k_i \to k_f} \neq N_{k'_i \to k'_f}$

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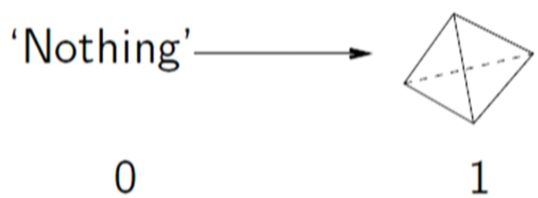

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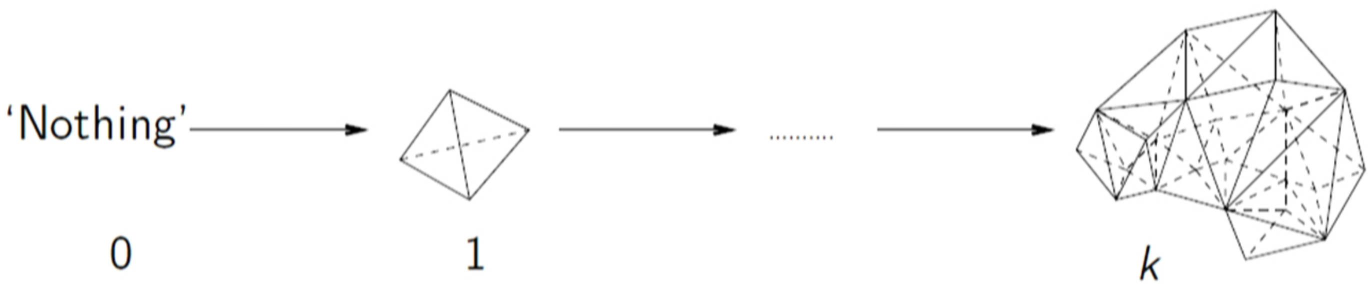


Application: canonical Regge Calculus [Dittrich, PH '09; '11]

- using formalism can implement general time evolution moves in canonical language on evolving phase spaces
- Regge Calculus as discrete dynamics of triangulated hypersurfaces

3D

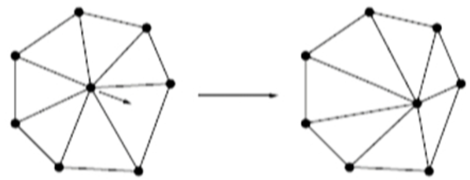
- solutions flat, preserve symmetry
- each vertex equipped with three constraints $C^k = {}^+C^k = {}^-C^k$
- preserved by evolution
- generate vertex displacement symmetry
- 'hyperbolic'

4D

- solutions with curvature possible
- vertices generally not equipped with constraints
- symmetries broken
- generically no hypersurface deformation algebra
- 'non–hyperbolic'

Perturbative 4D Regge Calculus [Dittrich, PH '09; PH to appear]

- expand $l^e = {}^{(0)}l^e + \varepsilon \delta l^e + O(\varepsilon^2)$ around flat solution
- inherits vertex displacement gauge symmetry from flat background



- 4 constraints per vertex $C^k_{vI} = {}^+C^k_{vI} = {}^-C^k_{vI}$, $I = 1, \dots 4$: preserved by dynamics, 1st class $\{C^k_{vI}, C^k_{v'J}\} \approx 0$ and generate symmetry
- 'gravitons': linearized deficit angles $\delta\epsilon_t$ (complete set) and $\{\delta\epsilon_t, C^k_{vI}\} \approx 0 \Rightarrow$ formalism describes their dynamics
- symmetries broken to first non–linear order: background gauge modes become propagating

Quantization for configuration space $\mathcal{Q} \simeq \mathbb{R}^n$ [PH to appear]

- Impose constraints in quantum theory via group averaging

$$^{\pm}\psi_k^{\rm phys} := \prod_I \delta({}^{\pm}\hat{C}_I^k)\psi_k^{\rm kin} = \prod_I \int ds_I \, e^{is^I \, {}^{\pm}\hat{C}_I^k} \psi_k^{\rm kin}$$

- physical inner product
$\langle {}^{\pm}\psi_k^{\rm phys} \big| {}^{\pm}\phi_k^{\rm phys} \rangle_{\rm phys} = \langle \psi_k^{\rm kin} \big| \prod_I \delta({}^{\pm}\hat{C}_I^k)\phi_k^{\rm kin} \rangle_{\rm kin}$
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$$K_{0\to1}(x_0, x_1) = M_{0\to1} \, e^{iS_1(x_0,x_1)} \qquad M_{0\to1} : \text{measure}$$

- construct (improper) projectors from $H_0^{\rm kin}$ to $H_1^{\rm phys+}$ and $H_1^{\rm kin}$ to $H_0^{\rm phys-}$

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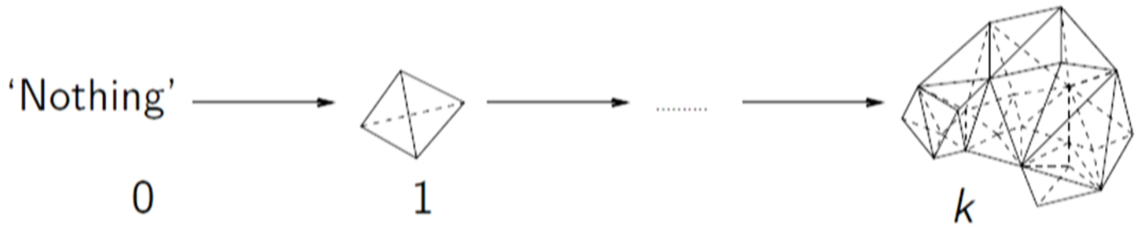
Evolving Hilbert spaces and cylindrical consistency [PH to appear]

- regularized (e.g. Faddeev-Popov) composition yields path integral

$$K^{\mathrm{reg}}_{0\to N} = \int \prod_{j=0}^{N-1} K^{\mathrm{reg}}_{j\to j+1} \prod_{l=1}^{N-1} dx_l$$

- if number of variables varies, extend configuration spaces
  $\Rightarrow$ auxiliary dimension subject to $\hat{p}^k_{aux}\psi^{\mathrm{phys}}_k = 0$
  $\Rightarrow$ ψ^{phys}_k are cylindrical functions on extended configuration spaces, inner product invariant $\Rightarrow$ naturally handles time varying discretization
- toy model for 'no boundary proposal'



for evolution 'Nothing' $\to k$ always get unique physical state $+\psi^{\mathrm{phys}}_k$

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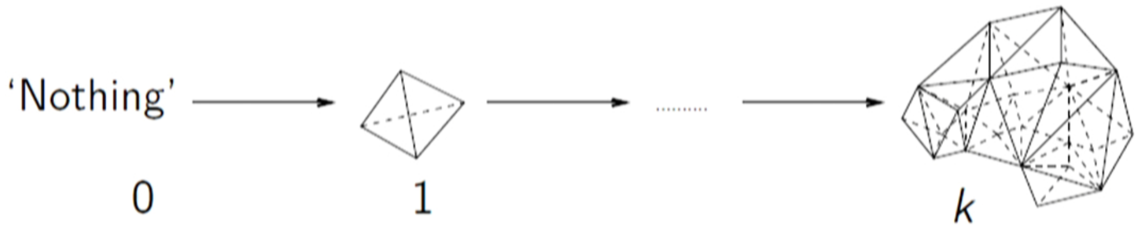

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Summary

- symmetries generically broken in the discrete $\Rightarrow$ consequences for dynamics
- general constraint analysis for variational discrete systems available
  $\Rightarrow$ naturally handles time varying discretizations
  $\Rightarrow$ constraints and propagating dofs evolution move dependent
- can construct general canonical formulation of Regge Calculus
- formalism can be consistently quantized

Transition Amplitudes in Causal Dynamical Triangulations

Jonah M. Miller
Department of Physics, University of Colorado at Boulder

Joshua H. Cooperman
Department of Physics, University of California, Davis

Loops 13
25 July, 2013

What is causal dynamical triangulations?

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What is causal dynamical triangulations?

Toolbox

- lattice regularization
- finite-size scaling
- renormalization

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Lorentzian

$\mathcal{A} = \int \mathcal{D}\mathbf{g}\, e^{iS[\mathbf{g}]}$

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Lorentzian $\mathcal{A} = \int \mathcal{D}\mathbf{g}\, e^{iS[\mathbf{g}]}$ $\xrightarrow{\text{Wick rotation}}$ Euclidean $\mathcal{Z} = \int \mathcal{D}\mathbf{g}\, e^{-S^{(E)}[\mathbf{g}]}$

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$\downarrow$ causal triangulation

$\mathcal{A}_{CDT} = \sum_{\mathcal{T}_c} \mu(\mathcal{T}_c) e^{iS_R[\mathcal{T}_c]}$

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$$\begin{tikzcd}
\begin{array}{l}\text{Lorentzian}\\ \mathcal{A} = \int \mathcal{D}\mathbf{g}\, e^{iS[\mathbf{g}]}\end{array} \arrow[r, dashed, "?"] \arrow[d, dotted, "\begin{array}{c}\text{causal}\\ \text{triangulation}\end{array}"] & \begin{array}{l}\text{Euclidean}\\ \mathcal{Z} = \int \mathcal{D}\mathbf{g}\, e^{-S^{(E)}[\mathbf{g}]}\end{array} \\
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\end{tikzcd}$$

What is a causal triangulation?

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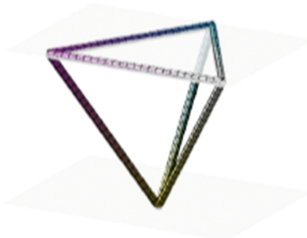
What is a causal triangulation?

$(1,3)$ 3-simplex



$t=1$

$t=0$

$(2,2)$ 3-simplex

$t=1$

$t=0$

$(3,1)$ 3-simplex

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$$l_{SL}^2 = \mathrm{a}^2$$
$$l_{TL}^2 = -\alpha \mathrm{a}^2$$

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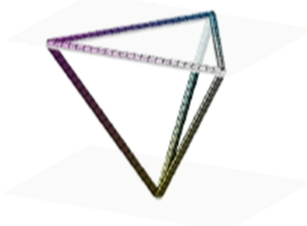

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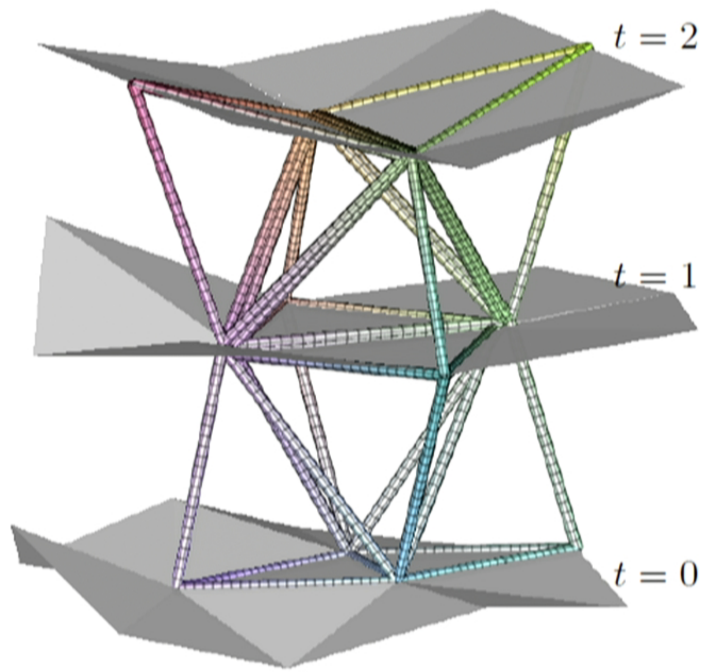
Segment of a causal triangulation



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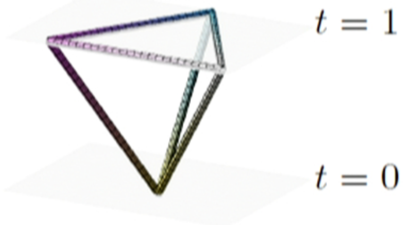

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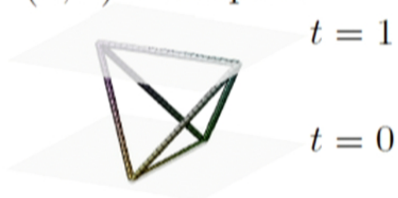

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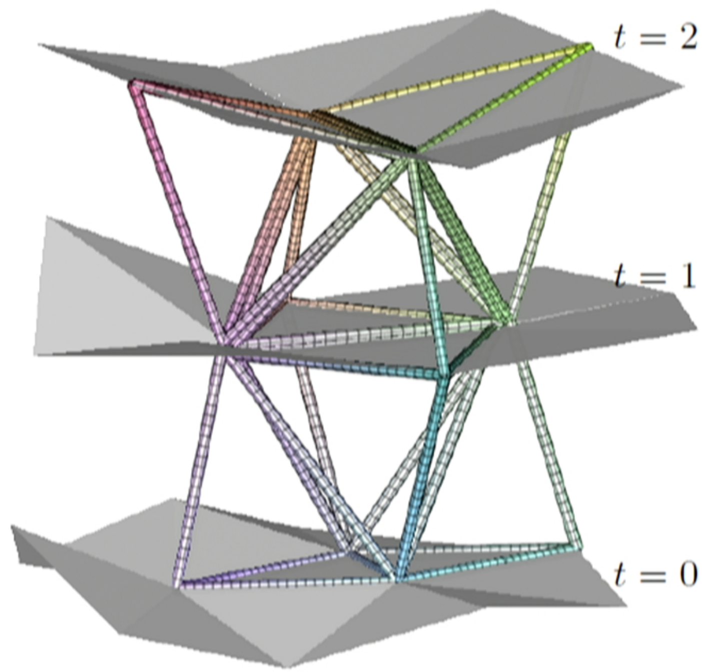
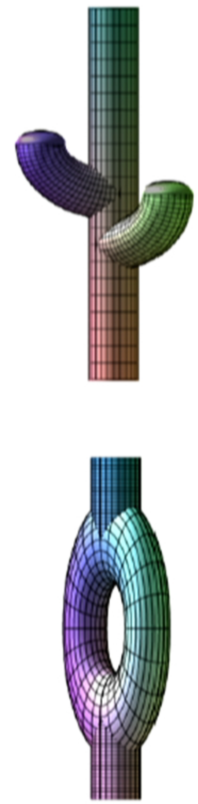


Forbidden spacetimes



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Numerical Simulation

- $\alpha \to -\alpha$

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Markov chain Monte Carlo

The ground state of causal dynamical triangulations

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Quantization of Einstein gravity for spacetime topology $\mathcal{S}^2 \times \mathcal{S}^1$

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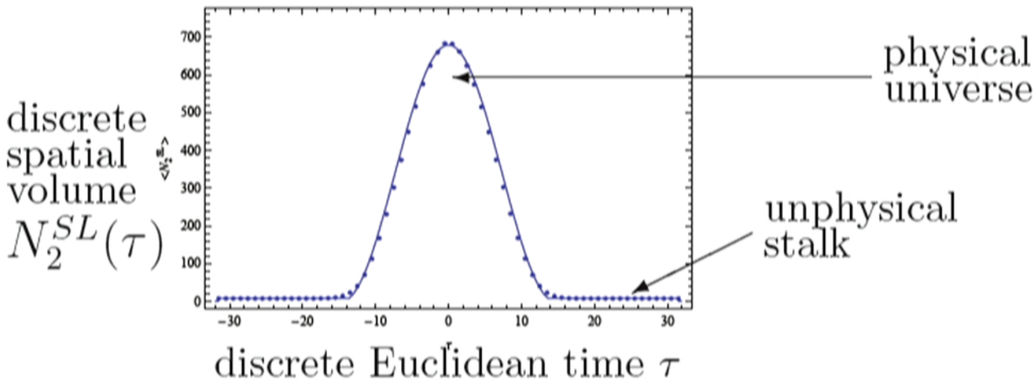

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- Gravitational effective action
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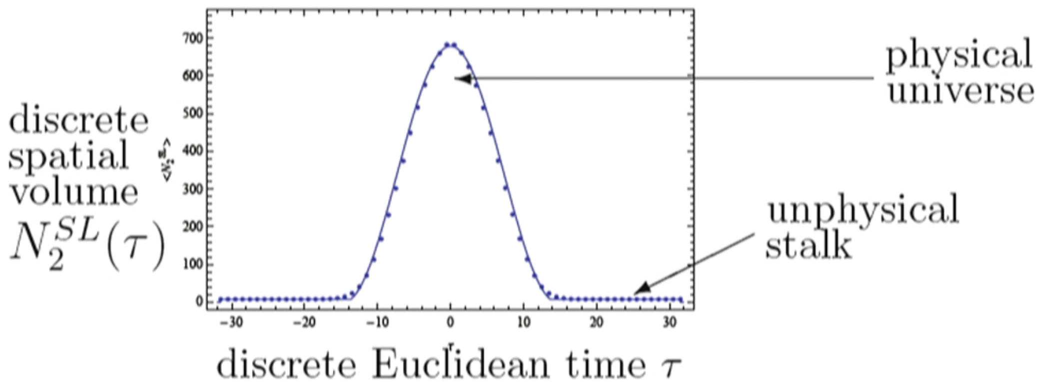

- $$\langle N_2^{SL}(\tau) \rangle = \frac{2}{\pi} \frac{\langle N_3^{(1,3)} \rangle}{\tilde{s}_0 \langle N_3^{(1,3)} \rangle^{1/3}} \cos^2 \left(\frac{\tau}{\tilde{s}_0 \langle N_3^{(1,3)} \rangle^{1/3}} \right)$$

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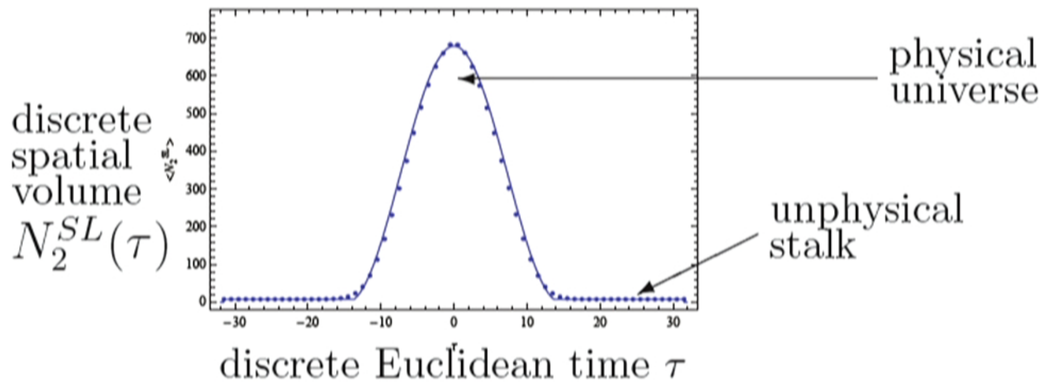
- Gravitational effective action
$$S_{\text{eff}}^{(E)}[N_2^{SL}(\tau)] = c_1 \sum_{\tau=1}^{T} \left\{ \frac{1}{N_2^{SL}(\tau)} \left[\frac{\Delta N_2^{SL}(\tau)}{\Delta \tau} \right]^2 - \lambda N_2^{SL}(\tau) \right\}$$



- $\langle N_2^{SL}(\tau) \rangle = \frac{2}{\pi} \frac{\langle N_3^{(1,3)} \rangle}{\tilde{s}_0 \langle N_3^{(1,3)} \rangle^{1/3}} \cos^2 \left(\frac{\tau}{\tilde{s}_0 \langle N_3^{(1,3)} \rangle^{1/3}} \right)$

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Transition amplitudes

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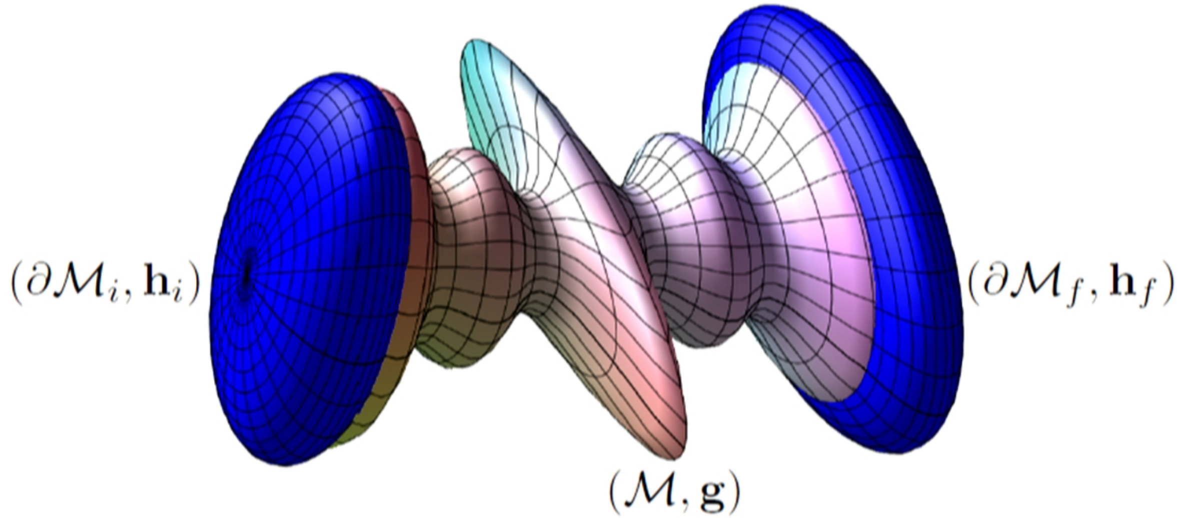
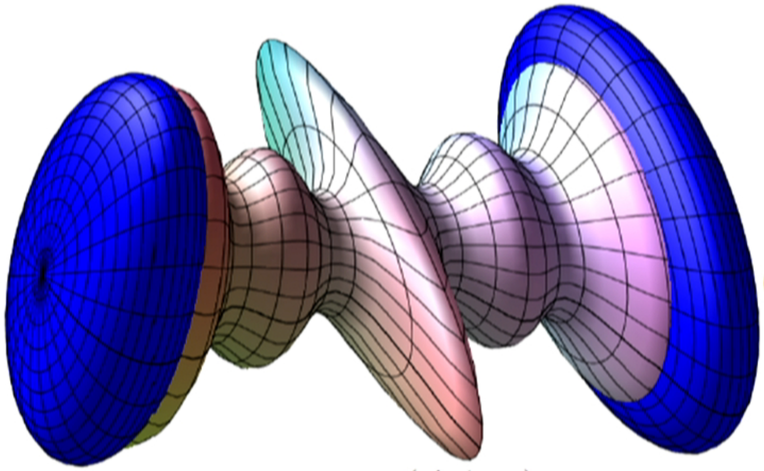
Transition amplitudes

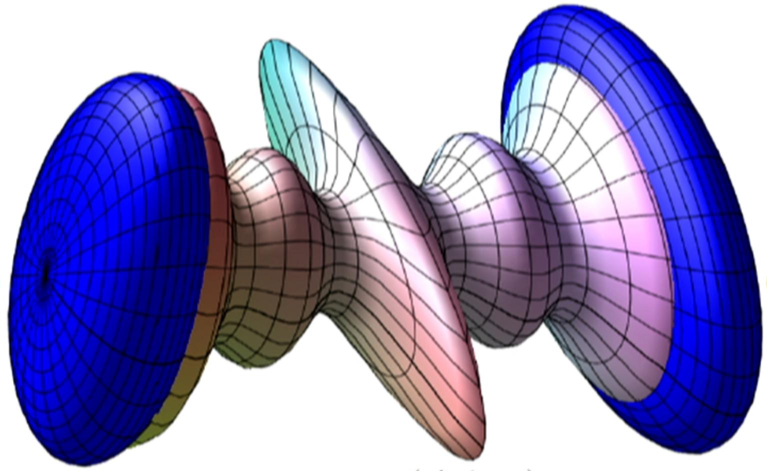


Transition amplitudes



$$S[\mathbf{g}] = \frac{1}{16\pi G}\left[2\int_{\partial\mathcal{M}_i} \mathrm{d}^2y\sqrt{h_i}K_i + \int_{\mathcal{M}} \mathrm{d}^3x\sqrt{-g}(R-2\Lambda) + 2\int_{\partial\mathcal{M}_f} \mathrm{d}^2y\sqrt{h_f}K_f\right]$$

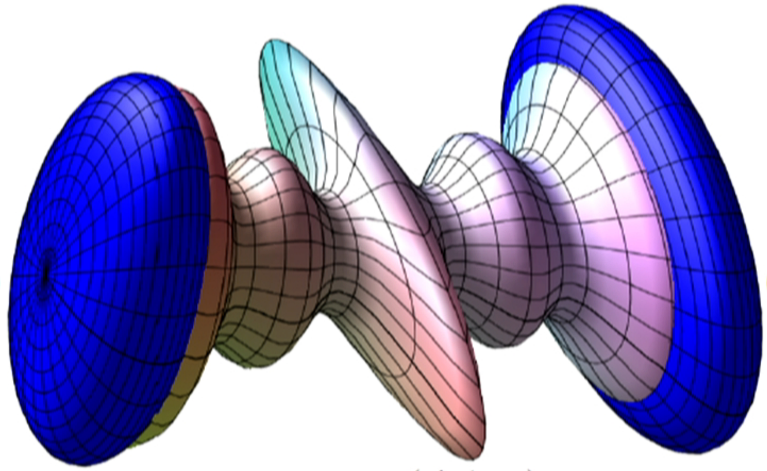
Transition amplitudes



$$S[\mathbf{g}] = \frac{1}{16\pi G}\left[2\int_{\partial\mathcal{M}_i} \mathrm{d}^2y\sqrt{h_i}K_i + \int_{\mathcal{M}} \mathrm{d}^3x\sqrt{-g}(R-2\Lambda) + 2\int_{\partial\mathcal{M}_f} \mathrm{d}^2y\sqrt{h_f}K_f\right]$$

Compute $\mathcal{A}[\mathbf{h}_i, \mathbf{h}_f] = \int_{\mathbf{g}|_{\partial\mathcal{M}_i}=\mathbf{h}_i}^{\mathbf{g}|_{\partial\mathcal{M}_f}=\mathbf{h}_f} \mathcal{D}\mathbf{g}\, e^{iS[\mathbf{g}]}$ given fixed $\mathbf{h}_i$ and $\mathbf{h}_f$

Transition amplitudes



$$S[\mathbf{g}] = \frac{1}{16\pi G}\left[2\int_{\partial\mathcal{M}_i} \mathrm{d}^2y\sqrt{h_i}K_i + \int_{\mathcal{M}} \mathrm{d}^3x\sqrt{-g}(R-2\Lambda) + 2\int_{\partial\mathcal{M}_f} \mathrm{d}^2y\sqrt{h_f}K_f\right]$$

Compute $\mathcal{A}[\mathbf{h}_i, \mathbf{h}_f] = \int_{\mathbf{g}|_{\partial\mathcal{M}_i}=\mathbf{h}_i}^{\mathbf{g}|_{\partial\mathcal{M}_f}=\mathbf{h}_f} \mathcal{D}\mathbf{g}\, e^{iS[\mathbf{g}]}$ given fixed $\mathbf{h}_i$ and $\mathbf{h}_f$

Numerically simulate $Z_{CDT}[\partial\mathcal{T}_{c_i}, \partial\mathcal{T}_{c_f}]$ given fixed $\partial\mathcal{T}_{c_i}$ and $\partial\mathcal{T}_{c_f}$

Semiclassical expectations for transition amplitudes

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Semiclassical expectations for transition amplitudes

No-boundary proposal of Hartle and Hawking

$$\mathcal{A}[\mathbf{h}_f] = \int^{\mathbf{g}|_{\partial\mathcal{M}_f}=\mathbf{h}_f} \mathcal{D}\mathbf{g}\, e^{iS[\mathbf{g}]} \quad \longrightarrow \quad \mathcal{A}[\mathbf{h}_f] = \int_{\mathbf{g}|_{\partial\mathcal{M}_i}=\emptyset}^{\mathbf{g}|_{\partial\mathcal{M}_f}=\mathbf{h}_f} \mathcal{D}\mathbf{g}\, e^{-S^{(E)}[\mathbf{g}]}$$

Semiclassical expectations for transition amplitudes

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- Minisuperspace truncation

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + a^2(\tau)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

Semiclassical expectations for transition amplitudes

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No-boundary proposal of Hartle and Hawking

$$\mathcal{A}[\mathbf{h}_f] = \int^{\mathbf{g}|_{\partial\mathcal{M}_f}=\mathbf{h}_f} \mathcal{D}\mathbf{g}\, e^{iS[\mathbf{g}]} \quad \longrightarrow \quad \mathcal{A}[\mathbf{h}_f] = \int_{\mathbf{g}|_{\partial\mathcal{M}_i}=\emptyset}^{\mathbf{g}|_{\partial\mathcal{M}_f}=\mathbf{h}_f} \mathcal{D}\mathbf{g}\, e^{-S^{(E)}[\mathbf{g}]}$$

- Minisuperspace truncation

$$ds^2 = d\tau^2 + a^2(\tau)(d\theta^2 + \sin^2\theta d\phi^2)$$

- Saddle point approximation

$$\mathcal{A}[a(t)] = \mathcal{N}\, e^{-S^{(E)}[a_{cl}(\tau)]}$$

Semiclassical expectations for transition amplitudes

No-boundary proposal of Hartle and Hawking

$$\mathcal{A}[\mathbf{h}_f] = \int^{\mathbf{g}|\partial\mathcal{M}_f=\mathbf{h}_f} \mathcal{D}\mathbf{g}\, e^{iS[\mathbf{g}]} \quad \longrightarrow \quad \mathcal{A}[\mathbf{h}_f] = \int_{\mathbf{g}|\partial\mathcal{M}_i=\emptyset}^{\mathbf{g}|\partial\mathcal{M}_f=\mathbf{h}_f} \mathcal{D}\mathbf{g}\, e^{-S^{(E)}[\mathbf{g}]}$$

Extrema $a_{\mathrm{cl}}(\tau)$ of the action $S^{(E)}[a(\tau)]$

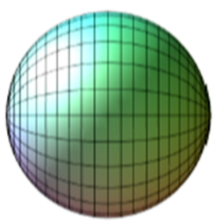
Semiclassical expectations for transition amplitudes

No-boundary proposal of Hartle and Hawking

$$\mathcal{A}[\mathbf{h}_f] = \int^{\mathbf{g}|_{\partial\mathcal{M}_f}=\mathbf{h}_f} \mathcal{D}\mathbf{g}\, e^{iS[\mathbf{g}]} \quad \longrightarrow \quad \mathcal{A}[\mathbf{h}_f] = \int_{\mathbf{g}|_{\partial\mathcal{M}_i}=\emptyset}^{\mathbf{g}|_{\partial\mathcal{M}_f}=\mathbf{h}_f} \mathcal{D}\mathbf{g}\, e^{-S^{(E)}[\mathbf{g}]}$$

Extrema $a_{\mathrm{cl}}(\tau)$ of the action $S^{(E)}[a(\tau)]$

Case 1: $a_i = 0$, $a_f = 0$



Semiclassical expectations for transition amplitudes

No-boundary proposal of Hartle and Hawking

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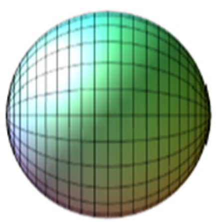
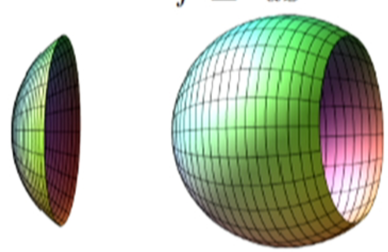


Case 2: $a_i = 0,\ a_f > 0$ with $0 < a_f \leq l_{dS}$



Semiclassical expectations for transition amplitudes

No-boundary proposal of Hartle and Hawking

$$\mathcal{A}[\mathbf{h}_f] = \int^{\mathbf{g}|\partial\mathcal{M}_f = \mathbf{h}_f} \mathcal{D}\mathbf{g}\, e^{iS[\mathbf{g}]} \quad \longrightarrow \quad \mathcal{A}[\mathbf{h}_f] = \int_{\mathbf{g}|\partial\mathcal{M}_i = \emptyset}^{\mathbf{g}|\partial\mathcal{M}_f = \mathbf{h}_f} \mathcal{D}\mathbf{g}\, e^{-S^{(E)}[\mathbf{g}]}$$

Extrema $a_{\mathrm{cl}}(\tau)$ of the action $S^{(E)}[a(\tau)]$

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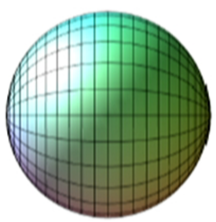
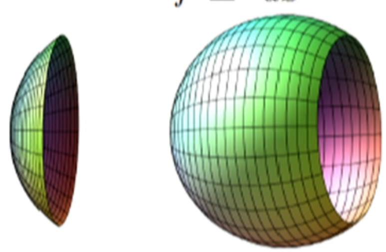
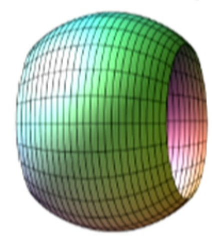


Case 2: $a_i = 0$, $a_f > 0$ with $0 < a_f \leq l_{dS}$



Case 3: $a_i > 0$, $a_f > 0$ with $a_i \leq l_{dS}$, $a_f \leq l_{dS}$



Case 1: Minimal initial to minimal final boundary

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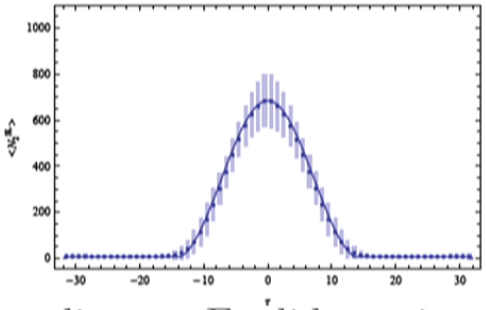
2-sphere spatial topology, periodic in time

$T = 64$

discrete spatial volume $N_2^{SL}(\tau)$



discrete Euclidean time τ

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2-sphere spatial topology, periodic in time

$T = 64$

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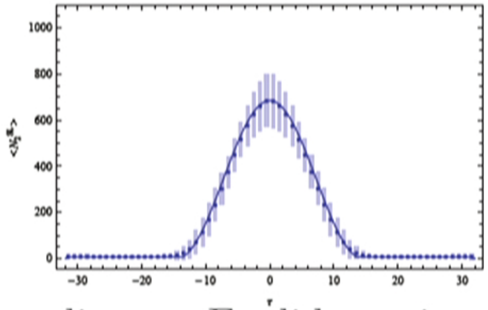
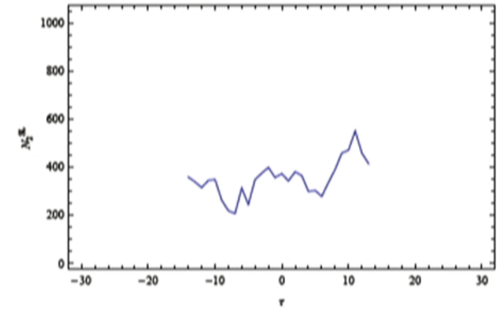


discrete Euclidean time τ

$T = 28$



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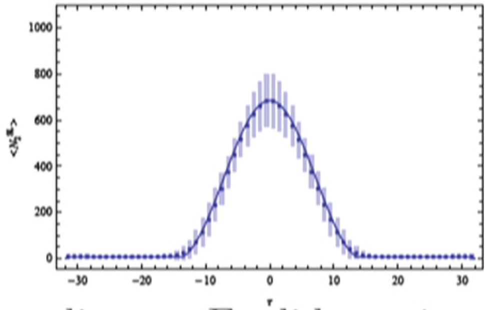

discrete Euclidean time τ

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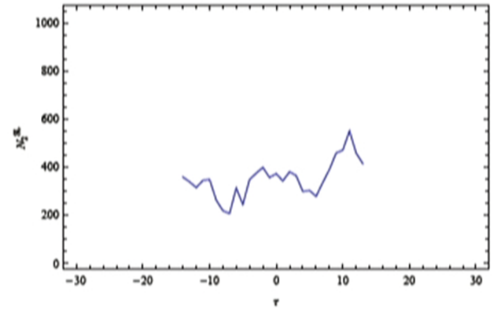
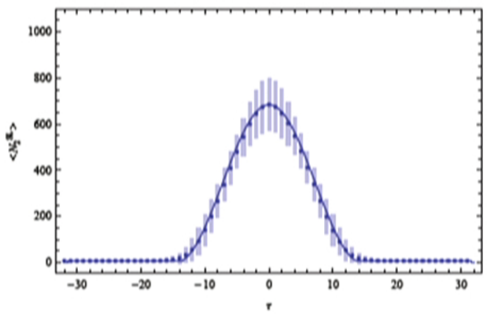


2-sphere spatial topology, finite interval in time



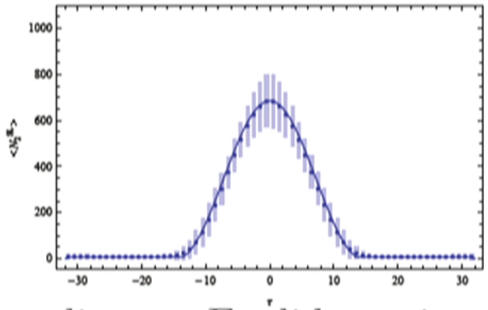
$T = 64$

Case 1: Minimal initial to minimal final boundary

2-sphere spatial topology, periodic in time

$T = 64$

discrete spatial volume $N_2^{SL}(\tau)$



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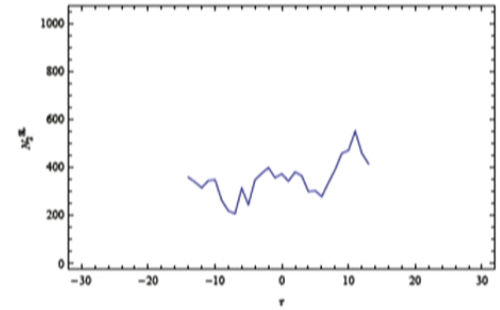
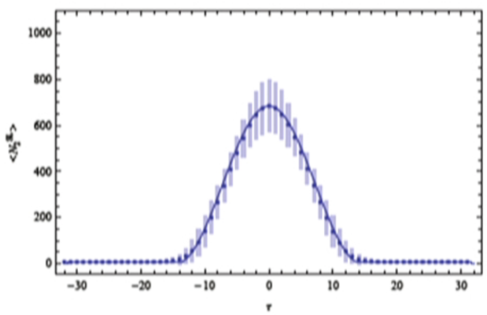
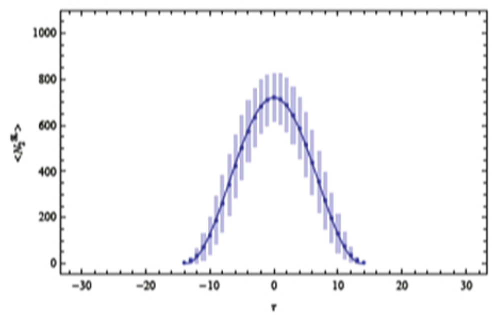


2-sphere spatial topology, finite interval in time



$T = 64$



$T = 28$

Case 1: Minimal initial to minimal final boundary

2-sphere spatial topology, periodic in time

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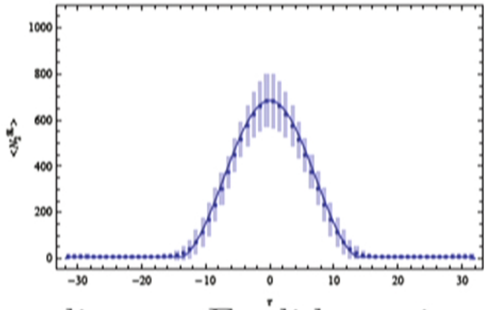
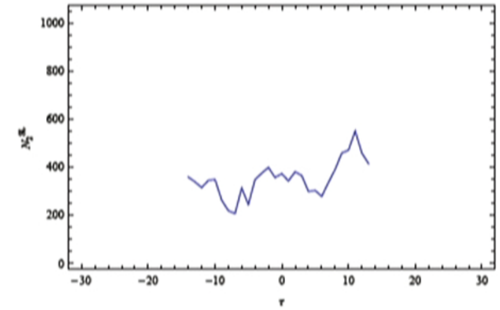
discrete spatial volume $N_2^{SL}(\tau)$

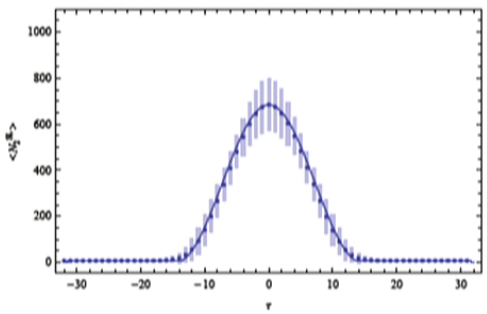


discrete Euclidean time τ

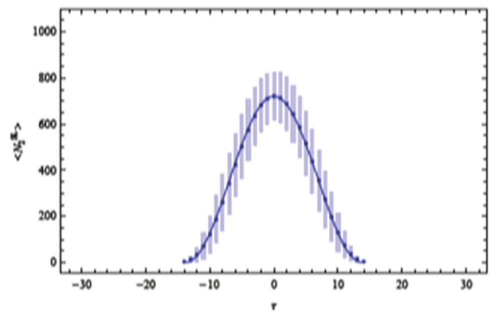
$T = 28$



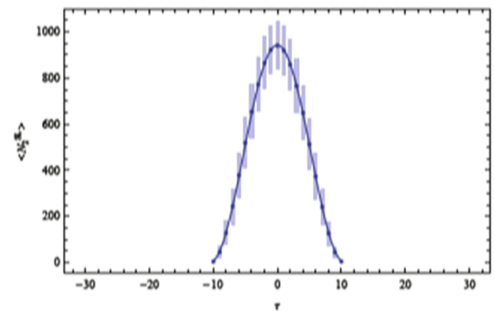
2-sphere spatial topology, finite interval in time



$T = 64$



$T = 28$



$T = 20$

Case 2: Minimal initial to nonminimal final boundary

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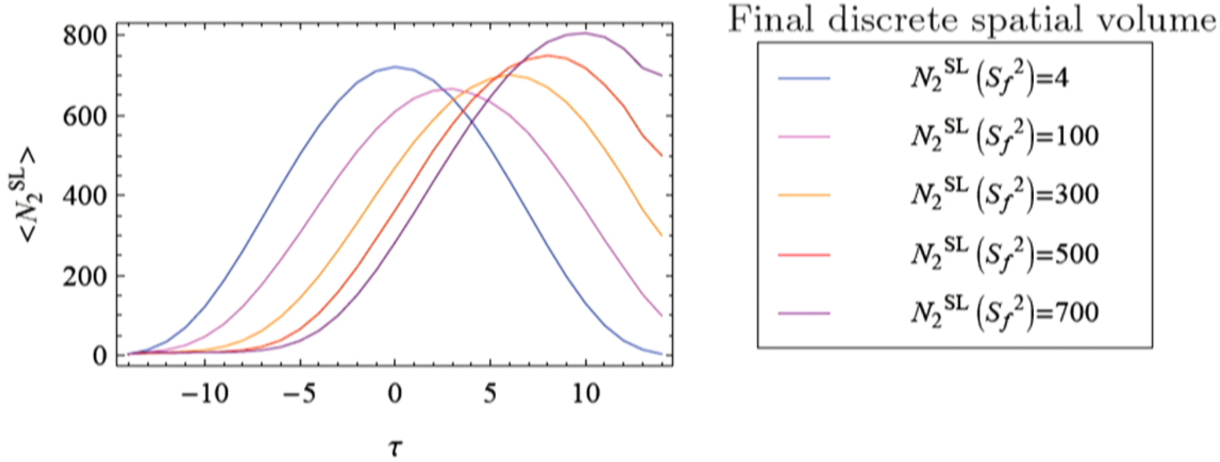
2-sphere spatial topology, finite interval in time

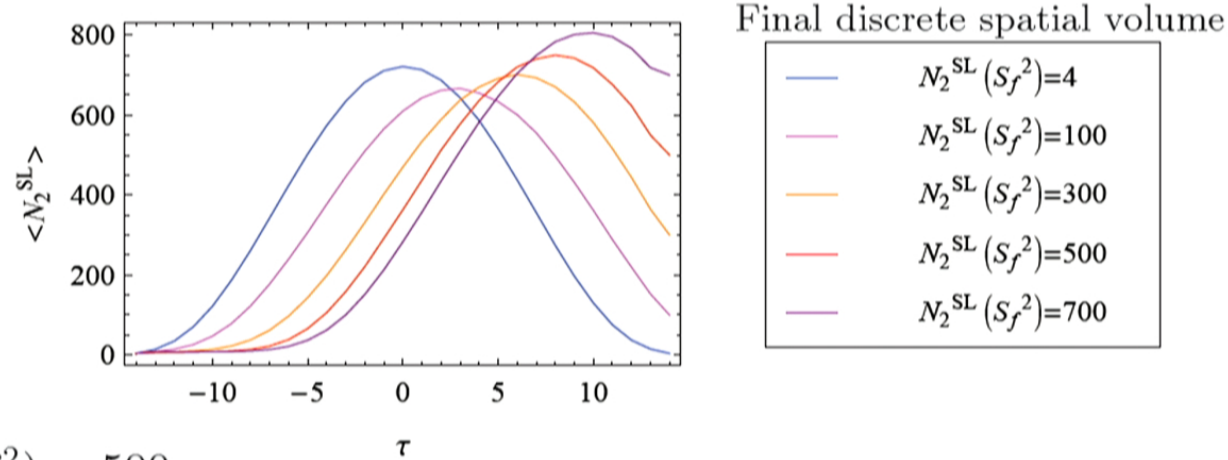
$T = 28$



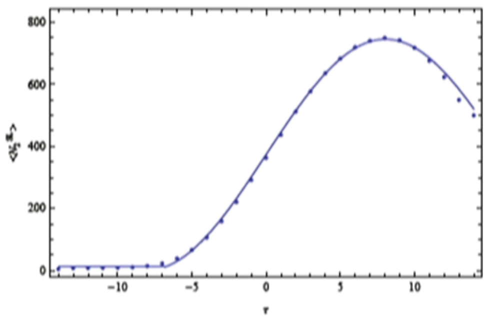
Case 2: Minimal initial to nonminimal final boundary

2-sphere spatial topology, finite interval in time

$T = 28$



For $N_2^{SL}(\mathcal{S}_f^2) = 500$

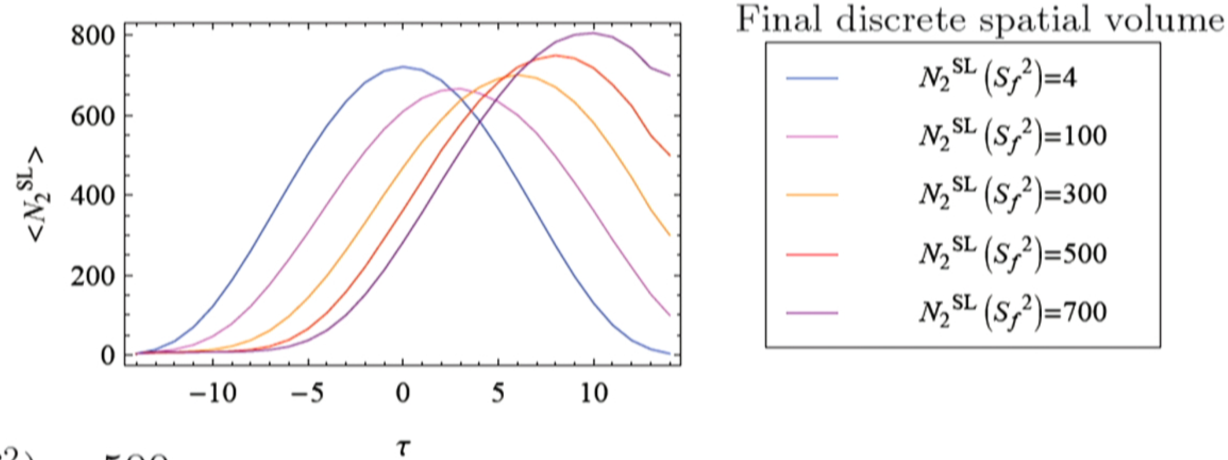


$T = 28$

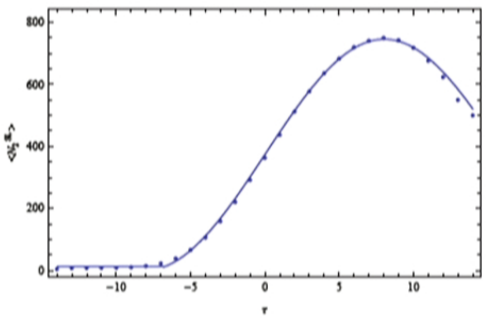
Case 2: Minimal initial to nonminimal final boundary

2-sphere spatial topology, finite interval in time

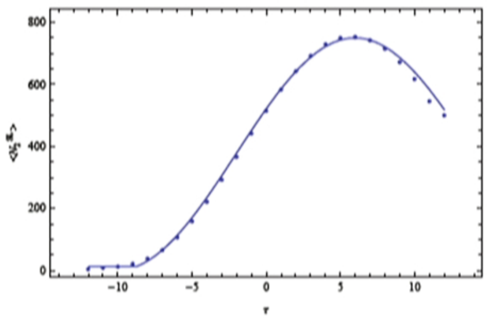
$T = 28$



For $N_2^{SL}(\mathcal{S}_f^2) = 500$



$T = 28$

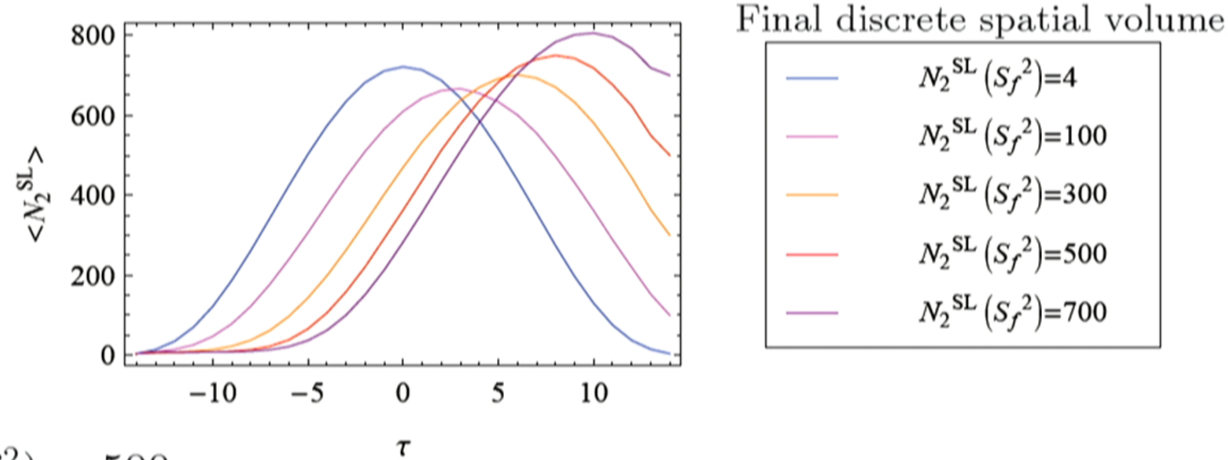


$T = 24$

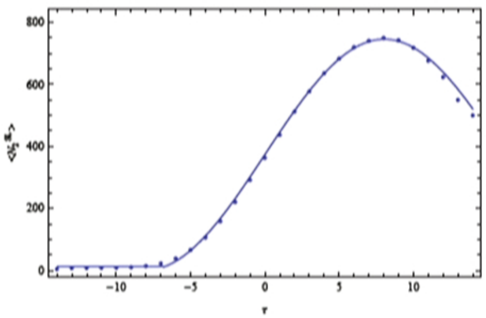
Case 2: Minimal initial to nonminimal final boundary

2-sphere spatial topology, finite interval in time

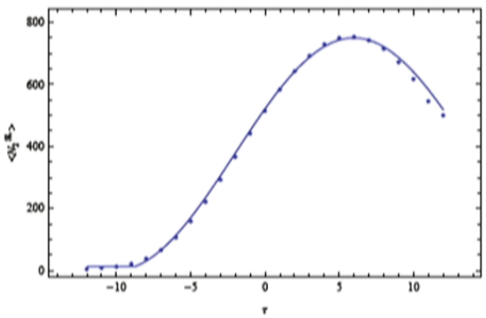
$T = 28$



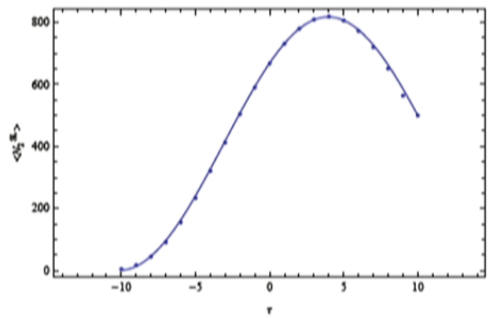
For $N_2^{SL}(\mathcal{S}_f^2) = 500$



$T = 28$



$T = 24$

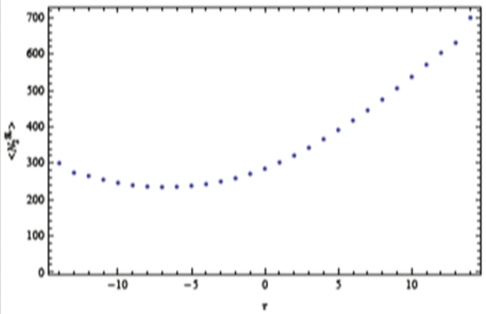


$T = 20$

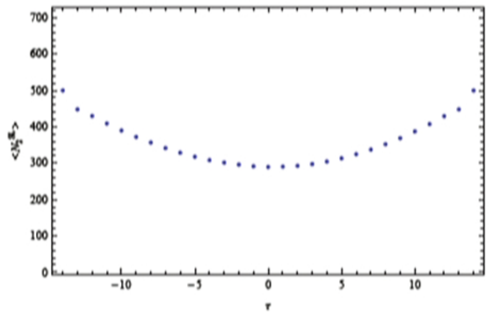
Case 3: Nonminimal initial and final boundaries

2-sphere spatial topology, finite interval in time

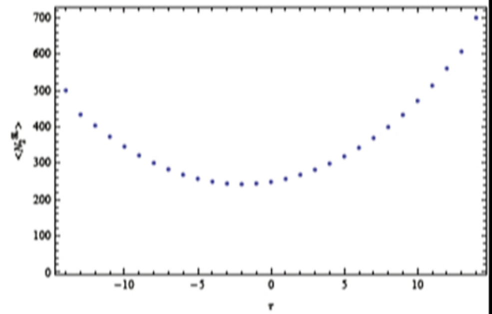
$$N_2^{SL}(\mathcal{S}_i^2) = 300 \qquad N_2^{SL}(\mathcal{S}_f^2) = 700$$



$$N_2^{SL}(\mathcal{S}_i^2) = 500 \qquad N_2^{SL}(\mathcal{S}_f^2) = 500$$



$$N_2^{SL}(\mathcal{S}_i^2) = 500 \qquad N_2^{SL}(\mathcal{S}_f^2) = 700$$



$$T = 28$$

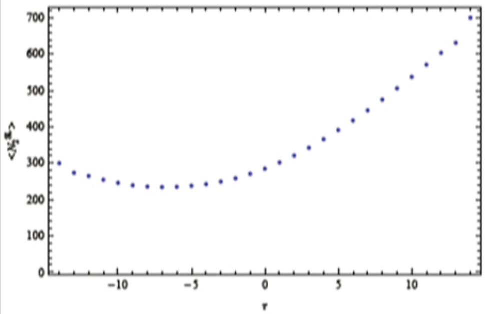
Case 3: Nonminimal initial and final boundaries

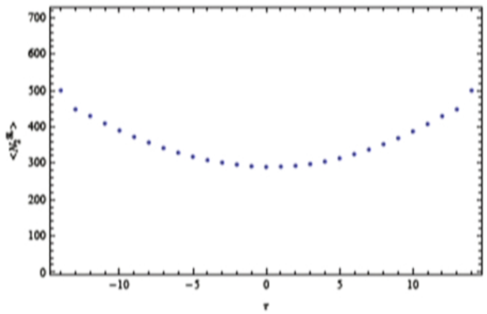
2-sphere spatial topology, finite interval in time

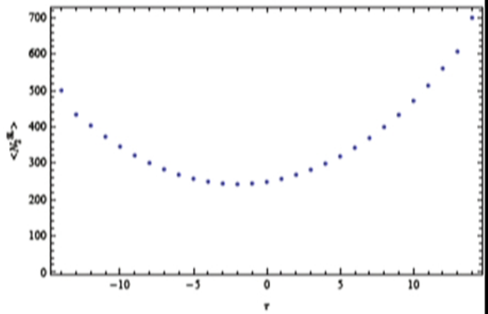
$N_2^{SL}(\mathcal{S}_i^2) = 300$
$N_2^{SL}(\mathcal{S}_f^2) = 700$

$N_2^{SL}(\mathcal{S}_i^2) = 500$
$N_2^{SL}(\mathcal{S}_f^2) = 500$

$N_2^{SL}(\mathcal{S}_i^2) = 500$
$N_2^{SL}(\mathcal{S}_f^2) = 700$







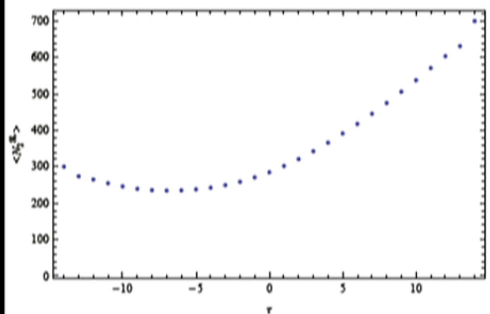
$T = 28$

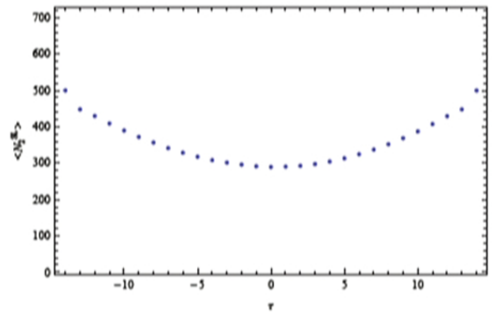
How should we interpret these transition amplitudes?

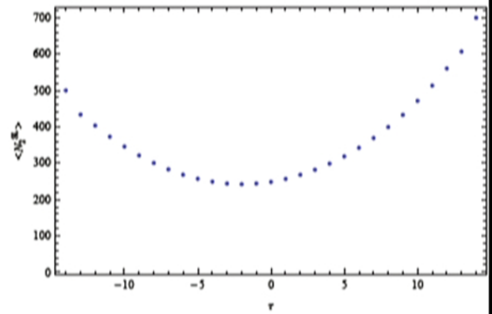
Case 3: Nonminimal initial and final boundaries

2-sphere spatial topology, finite interval in time

$N_2^{SL}(\mathcal{S}_i^2) = 300$	$N_2^{SL}(\mathcal{S}_i^2) = 500$	$N_2^{SL}(\mathcal{S}_i^2) = 500$
$N_2^{SL}(\mathcal{S}_f^2) = 700$	$N_2^{SL}(\mathcal{S}_f^2) = 500$	$N_2^{SL}(\mathcal{S}_f^2) = 700$





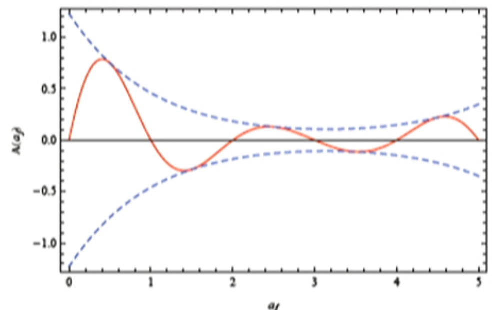


$T = 28$

How should we interpret these transition amplitudes?

Case 2

No-boundary wave function $\mathcal{A}[a_f]$ for $a_f > l_{dS}$



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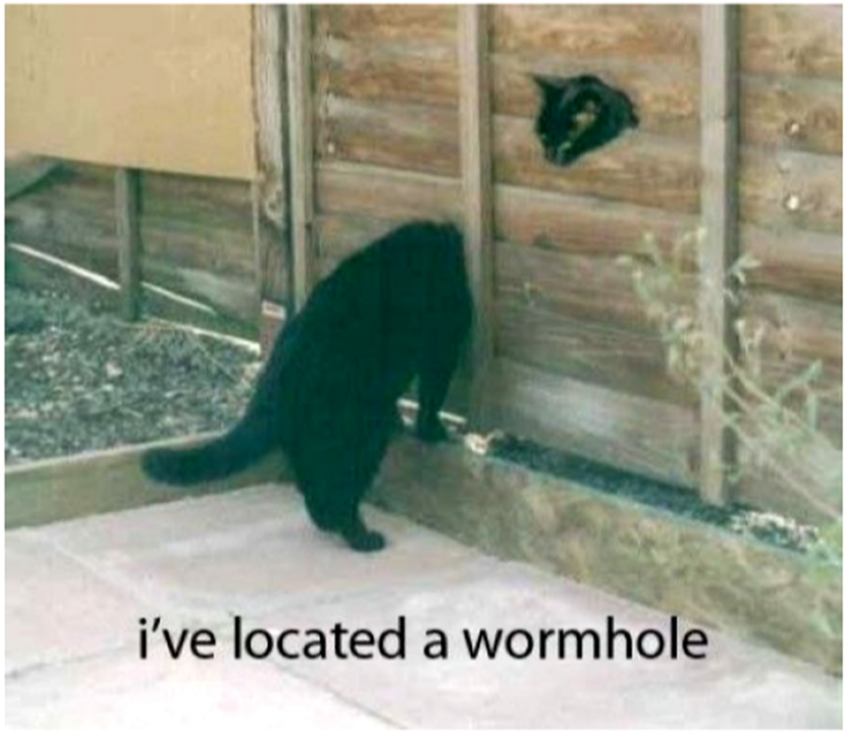
- Can we observe effects beyond the minisuperspace truncation by imposing nonspherically symmetric boundary geometries?
- Is there gauge redundancy in the number T of time slices of a causal triangulation?
  (c.f. this morning's talk)

Thanks to...

- Steve Carlip for tremendous insight and guidance
- Rajesh Kommu for initially developing the Davis group's code
- David Kamensky for developing the algorithm to include fixed boundaries in the Monte Carlo code
- The other members of the Carlip group for many helpful discussions
- You!



(http://zombierobots.net/wormhole-cat)

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Dark Energy from Discrete Spacetime

Aaron Trout

Loops 2013

July 24, 2013

Observational Evidence for Dark Energy

Multiple independent sets of empirical evidence say dark energy (DE) is $\approx 70\%$ of the matter-energy in our universe.

- Cosmic microwave background (Hinshaw et al. 2012)
- Apparent luminosity of supernovae (Kowalski et al. 2008)
- X-ray emissions from galaxy clusters (Allen et al. 2008)
- Large scale distribution of galaxies (Tegmark et al. 2004)

Data are consistent with DE as a *cosmological constant* or equivalently a *uniform vacuum energy density* of

$$\Lambda \approx 10^{-122}$$

in Planck units. The data are *also* consistent with more exotic models, like those where Λ varies with time.

Why is Λ So Tiny?

A theoretical explanation for the magnitude of Λ is difficult.

- Naive quantum field theory (QFT) says $\Lambda \approx 1$.
- Can construct natural theories (e.g. SUSY) where $\Lambda = 0$.
- Very hard to find natural way to get $\Lambda \approx 10^{-122}$.

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Various explanations of DE, for example:

- *Holographic Dark Energy* (HDE): Accelerating expansion driven by entropy on cosmic horizon. (Cohen, et al. 1999)
- *Quintessence*: Accelerating expansion driven by exotic matter field(s). (Caldwell, et al. 1998)
- *Quantum Non-Locality*: Λ is a non-local quantum residue of spacetime discreteness. (Sorkin 1988)
- *Anthropic Principle*: Only universes with $\Lambda \ll 1$ support life. (Weinberg 1987)

Dark Energy from Discrete Spacetime

We present a new model for the origin of DE. The basic story:

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- Spacetime is fundamentally a kind of discrete geometry.
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An intrinsic negative curvature for empty space has the same effect as a positive vacuum energy density.

This story is supported by the basic structure of the Einstein-Hilbert action.

The Einstein-Hilbert Action

$$\mathcal{A}_{EH}(g_{\mu\nu}) = \int_M \left[\frac{1}{16\pi}(R - 2\Lambda) + \mathcal{L}_m\right]\sqrt{-g}\, d^n x.$$

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- **Only the scalar-curvature term R has a physically distinguished zero value.** In QFT on a fixed background

$$\mathcal{L}_m \to \mathcal{L}_m + \text{const}$$

doesn't change the dynamics and we can simply set $\Lambda = 0$. Thus, it is reasonable to argue that a non-zero Λ comes from quantum perturbations on R.

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doesn't change the dynamics and we can simply set $\Lambda = 0$. Thus, it is reasonable to argue that a non-zero Λ comes from quantum perturbations on R.
- We expect an entropic perturbation on the value of a *global* observable (like total R) to be independent of local dynamics. **The cosmological constant term Λ is the only term in $\mathcal{A}_{EH}$ independent of the metric.**

Basics of Our Model

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We compute this effect using a novel variant of the *dynamical triangulations* (DT) theory of quantum gravity and obtain

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- Spacetime states in our model will be triangulations of a fixed compact n-manifold M, just like in DT.
- We use the standard DT action with $\Lambda = 0$.
- However, this theory is *not the same* as DT since we will restrict the set of triangulations which contribute to the partition function. (Like in CDT, but here we include states based on their *action value*.)

Mean DT Action

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We will be concerned with the average DT action (per volume) for triangulations of a fixed region with volume $V = N_n V_n(\ell)$.

$$\overline{\mathcal{A}} := \frac{\mathcal{A}_{DT}}{V} = c_n \ell^{-2} \left(\frac{1}{\mu(T)} - \frac{1}{\mu_n^*} \right)$$

where

- c_n is a constant depending only on the dimension n,
- ℓ is length of all the edges in T,
- $\mu(T) = \frac{1}{N_{n-2}(T)} \sum_{\tau^{n-2} \in T} \deg(\tau^{n-2})$ is the **mean hinge degree**, and
- $\mu_n^* = \frac{2\pi}{\cos^{-1}(1/n)}$ is the **“flat” hinge degree**.

We suppress the ℓ and n dependence writing simply $\overline{\mathcal{A}}(\mu)$ and interpret this quantity as the **mean scalar-curvature** over this region. Note that for any $\mu \neq \mu^*$ this quantity diverges like ℓ^{-2}.

Main Mathematical Result

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Theorem

Let M be a closed 3-manifold and N_3 a fixed number of tetrahedra. Then, there are mean actions

$$\overline{\mathcal{A}}_{min} = \overline{\mathcal{A}}\left(4.5 \cdot \frac{N_3}{N_3 - \frac{1}{2}\gamma^*(M)}\right)$$

and

$$\overline{\mathcal{A}}_{max} = \overline{\mathcal{A}}\left(6 \cdot \frac{N_3}{N_3 + \frac{1}{2}\left(3 + \sqrt{9 + 8N_3}\right)}\right)$$

so that for every integer N_1 with

$$\overline{\mathcal{A}} = \overline{\mathcal{A}}(\mu) = \overline{\mathcal{A}}\left(6N_3/N_1\right) \in \left(\overline{\mathcal{A}}_{min}, \overline{\mathcal{A}}_{max}\right)$$

we know $\overline{\mathcal{A}} = \overline{\mathcal{A}}(T)$ for some triangulation T of M containing N_3 tetrahedra and N_1 edges.

More on Main Result

The $\overline{\mathcal{A}}$ given in the theorem are regularly spaced over $(\overline{\mathcal{A}}_{min}, \overline{\mathcal{A}}_{max})$ with separation

$$\delta\overline{\mathcal{A}} = \frac{1}{8}\left(\frac{\ell}{V}\right).$$

This is the smallest possible separation given fixed N_3 so these are all the possible mean-actions $\overline{\mathcal{A}}$ on this interval.

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When $N_3 \gg 1$ we get

$$\overline{\mathcal{A}}_{min} \approx \overline{\mathcal{A}}(6) \approx -0.19\ell^{-2}$$

and

$$\overline{\mathcal{A}}_{max} \approx \overline{\mathcal{A}}(4.5) \approx 0.17\ell^{-2}.$$

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Constructing the N-Action Model

Since GR vacua at $\Lambda = 0$ have total scalar-curvature zero, we aim to build a model in which $\langle \overline{\mathcal{A}} \rangle = 0$.

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For each ℓ and corresponding N_3, let $\overline{\mathcal{A}}_0$ be the closest attainable mean action to zero. Our model uses triangulations with this mean-action, as well as those having one of the N mean-action values $\overline{\mathcal{A}}_k$ on either side of $\overline{\mathcal{A}}_0$. Let $\mathcal{A}_k$ and μ_k be the corresponding actions and mean edge-degrees.

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By our main result, if $N = N(\ell)$ grows slower than ℓ^{-2} then for small enough ℓ all $\overline{\mathcal{A}}_k$ lie in $(\overline{\mathcal{A}}_{min}, \overline{\mathcal{A}}_{max})$ and our partition function is

$$Z = \sum_{k=-N}^{N} e^{S_k + i(\mathcal{A}_0 + k \cdot \delta\mathcal{A})}$$

where $S_k = \ln(\#\text{ of } T \text{ with } \mathcal{A}_{DT}(T) = \mathcal{A}_k)$ is the entropy at action $\mathcal{A}_k$ and $\delta\mathcal{A} = V\delta\overline{\mathcal{A}} = \frac{1}{8}\ell$ the separation between actions.

Expected Action

The expected action for this model is then

$$\langle \mathcal{A} \rangle = \frac{1}{Z} \sum_{k=-N}^{N} (\mathcal{A}_0 + k \cdot \delta\mathcal{A}) e^{S_k + i(\mathcal{A}_0 + k \cdot \delta\mathcal{A})}.$$

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It is currently impossible to write $\langle \mathcal{A} \rangle$ as an exact closed-form expression.

However, under the **affine entropy approximation**

$$S_k = S_0 + k \cdot \eta$$

where $\eta = \eta(N_3)$ does not depend on k, we get $\langle \mathcal{A} \rangle$ equal to

$$\mathcal{A}_0 - \frac{\delta\mathcal{A}}{e^{\eta + i\delta\mathcal{A}} - 1} + \frac{\delta\mathcal{A}}{e^{(2N+1)(\eta + i\delta\mathcal{A})} - 1} + N\delta\mathcal{A} \coth\left[(2N+1)(\eta + i\delta\mathcal{A})\right].$$

Choosing an Appropriate N

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However, we know enough to guess what such a theory would say. We will assume that $S_k \approx S_0 + k \cdot \eta$ with $\eta(N_3) \not\to 0$ as $N_3 \to \infty$.

- We must have $N \to \infty$ as $\ell \to 0$. Otherwise, all the actions $\mathcal{A}_k$ would go to zero, and we wouldn't have a *quantum* theory.
- The product $N\delta\mathcal{A}$ must converge as $\ell \to 0$. Otherwise, our formula for the expected action $\langle \mathcal{A} \rangle$ would diverge as $\ell \to 0$.

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Since $\delta\mathcal{A} \propto \ell$ and N is dimensionless, we are led to choose $N = \dfrac{V^{1/3}}{\ell}$.

The Cosmological Constant

Using this N along with the affine entropy assumption with $\eta < 0$ gives

$$\lim_{\ell \to 0} \langle \overline{\mathcal{A}} \rangle = \frac{1}{V} \lim_{\ell \to 0} \langle \mathcal{A} \rangle = -\frac{1}{8} V^{-\frac{2}{3}}$$

which, by the Einstein-Hilbert action, implies an effective Λ of

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Can we use this result to estimate Λ in our universe? Considerations of *causality* indicate we should use something like the *Hubble volume* $H(t)^{-3}$ for V, giving

$$\Lambda(t) \approx \frac{1}{16} H(t)^2.$$

In the current era we get $\Lambda \approx 10^{-123}$ in agreement with observation.

Discussion

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Our model shares two key features with HDE approaches:

- Our Λ scales like the area of the cosmic horizon.
- We coordinated the cut-offs ℓ and N so that the entropic perturbation on $\langle \overline{\mathcal{A}} \rangle$ stays bounded as $\ell \to 0$. HDE models typically contain UV and IR field cut-offs which are removed in a way that saturates entropy in the Bekenstein bound.

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Finally, for a Planck-scale universe ($V \approx 1$) we predict $\Lambda \approx 1$ and hence very rapid expansion.

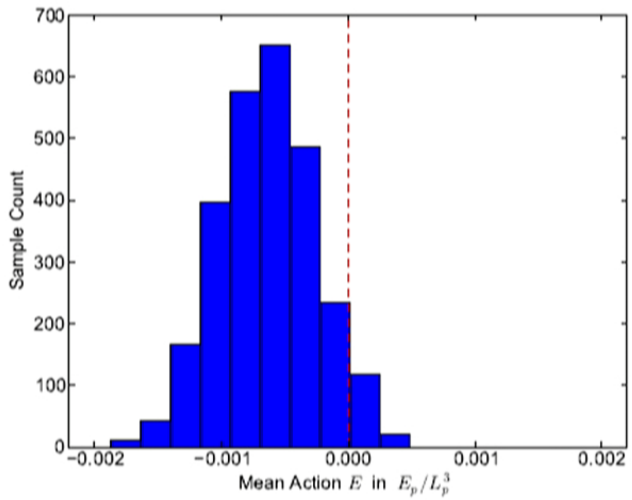
This raises the tantalizing possibility that big-bang inflation and dark-energy are manifestations of a common effect. This possibility is already under active investigation in the HDE context (Easson et al. 2012).

Numerical Evidence for the Entropies S_k

We used the Metropolis algorithm with quadratic objective function

$$U(T) = \alpha \left(\overline{\mathcal{A}}(T) - \overline{\mathcal{A}}^{\dagger}\right)^2 + \beta \left(N_3(T) - N_3^{\dagger}\right)^2$$

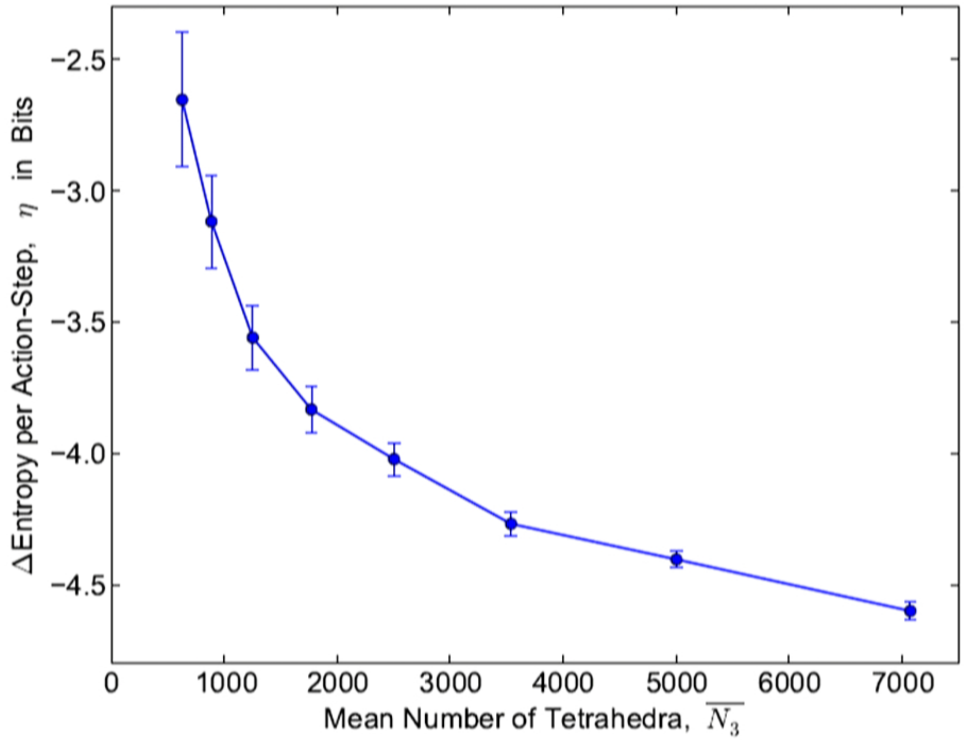
to sample triangulations of the 3-sphere near a target mean-action $\overline{\mathcal{A}}^{\dagger} = 0$ and target number of tetrahedra $N_3^{\dagger}$. Below is a histogram of samples for $N_3^{\dagger} = 1701$, $\alpha = 3.5 \times 10^6$ and $\beta = 1.0 \times 10^{-2}$.



- A Gaussian distribution means S_k depends linearly on k.
- The displacement of the sample mean away from $\overline{\mathcal{A}}^{\dagger} = 0$ implies the slope η is negative.

Evidence for the Entropies S_k

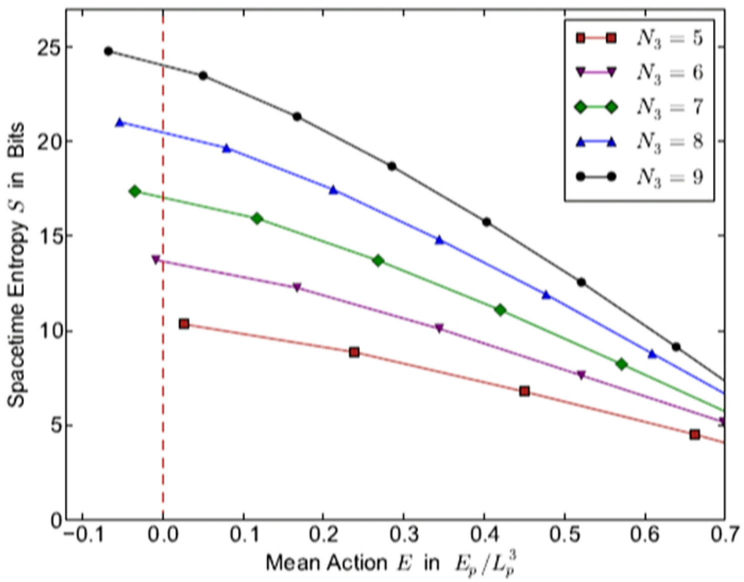
What happens to η as $N_3 \to \infty$?



Evidence for the Entropies S_k

Finally, we can look at censuses of 3-manifold triangulations for small N_3 as a sanity check. Data come from a complete census of the $\approx$ 47 million triangulations of S^3 with at most 9 tetrahedra (Burton 2011).

Technical note: the definition of "triangulation" used here is slightly more general. Allows gluing together of the faces within a single tetrahedron.



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Thank you!

A CDT Hamiltonian from Hořava-Lifshitz gravity

(arXiv:1302.6359)

Jan Ambjørn[1,2], Lisa Glaser[1], Yuki Sato[3], Yoshiyuki Watabiki[4]

[1] Niels Bohr Institute, Copenhagen
[2] Radbaud University Nijmegen
[3] Nagoya University
[4] Tokyo Institute of Technology

July 25, 2013

Outline

Why should they be connected?

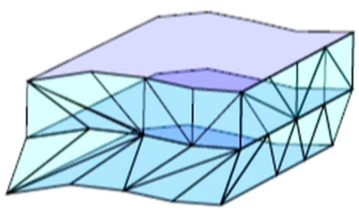
What did we do?

What does this imply?

What will I talk about?

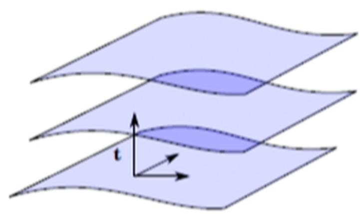
Causal Dynamical Triangulations (CDT)

- path integral
- non-perturbative
- not fundamentally discrete
- euclideanized



Hořava Lifshitz gravity (HL)

- powercounting renormalizable
- preferred time foliation
- continuum theory
- anisotropic scaling



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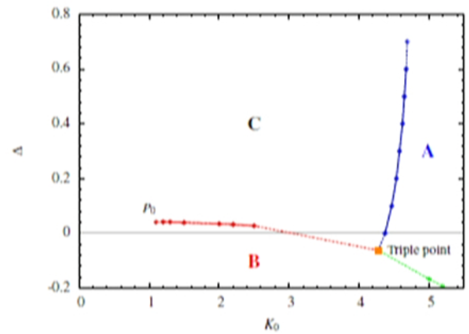
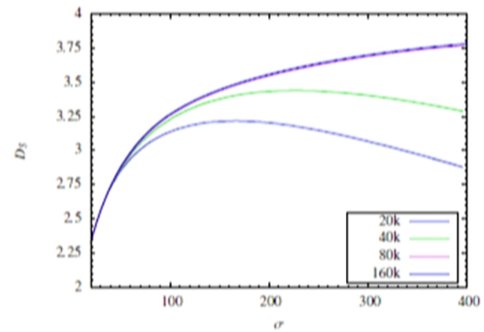
What makes us believe they might be the same?

- phase structure
- spectral dimension
- simulations
- symmetry group

(arXiv:1002.3298)



(arXiv:1203.3591)



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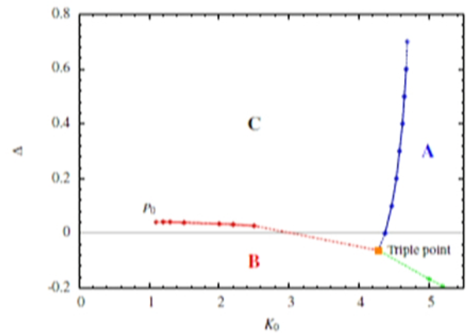

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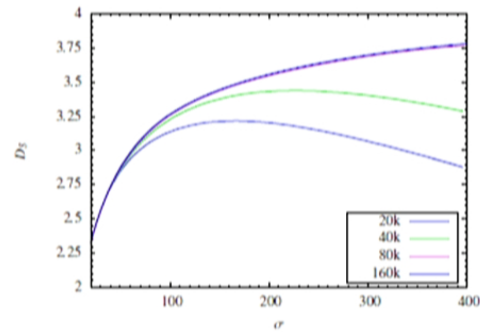

2D disclaimer

The rest of this talk we will be concerned with a 2d universe!

Click on Sign to add text and place signature on a PDF File.

A hamiltonian in CDT

Loop loop correlator

$$G(L_1, L_2, T) = \sum$$

$$= \sum_{\text{geom}} e^{\mu N + \lambda_1 L_1 + \lambda_2 L_2}$$

We can solve this to find:

Hamiltonian

$$\hat{H} = -\frac{\partial}{\partial L} L \frac{\partial}{\partial L} + \Lambda L$$

The HL Lagrangian

$$g_{\mu\nu} = \begin{pmatrix} -N(t)^2 + \gamma^2(x,t)N^{(1)}(x,t)^2 & N^{(1)}(x,t) \\ N^{(1)}(x,t) & \gamma^2(x,t) \end{pmatrix}$$ metric in ADM form

Projectable Hořava Lifshitz

$N(t)$ independent of position!

The HL Lagrangian

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And the action is

$$I = \int dt\ dx\ N\gamma\left((1-\lambda)K^2 - 2\Lambda\right)$$

with $K = \frac{1}{N}\left(\frac{1}{\gamma}\partial_0\gamma - \frac{1}{\gamma^2}\partial_1 N_1 + \frac{N_1}{\gamma^3}\partial_1\gamma\right)$ the external curvature

$$\pi^\gamma = \frac{\partial\mathcal{L}}{\partial\partial_0\gamma} = 2(1-\lambda)K$$

Hamiltonian formalism

The Hamiltonian

$$H = \int \mathrm{d}x \left[N(t)\mathcal{H} + N^{(1)}(x,t)\mathcal{H}_1 \right]$$

$$\mathcal{H} = \gamma \frac{(\pi^\gamma)^2}{4(1-\lambda)} + 2\Lambda$$ Hamiltonian constraint

$$\mathcal{H}_1 = \frac{-\partial_x \pi^\gamma}{\gamma}$$ Momentum constraint

$$\mathcal{H}_1 = 0 \quad \rightarrow \quad \pi^\gamma(t)$$

We can introduce $L(t) = \int \mathrm{d}x \gamma(x,t)$

$$H = N(t) \left(L(t) \frac{\pi^\gamma(t)^2}{4(1-\lambda)} + 2\Lambda L(t) \right)$$

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Quantization

We rescale the hamiltonian

$$H = N(t)\left(L(t)\pi^\gamma(t)^2 + \tilde{\Lambda}L(t)\right)$$

We can gauge fix $N(t) = 1$ and then require

cannonical commutation relations

$$\{L(t), \pi^\gamma(t)\} = 1 \qquad \rightarrow \qquad [\hat{L}, \hat{\pi^\gamma}] = i$$

$$H = \hat{L}\,\hat{\pi^\gamma}^2 + \tilde{\Lambda}\hat{L}$$

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Ordering ambiguity?

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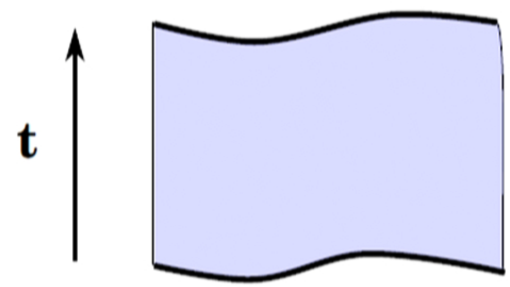
Ordering Ambiguity

Position basis ($\hat{\pi^\gamma} = -i\frac{\partial}{\partial L}$)

$$H = -\frac{\partial}{\partial L} L \frac{\partial}{\partial L} + \Lambda L \quad \rightarrow \quad \text{open boundary + no marked point}$$

$$H = -L\frac{\partial^2}{\partial L^2} + \Lambda L \quad \rightarrow \quad \text{closed boundary + marked point}$$

$$H = -\frac{\partial^2}{\partial L^2} L + \Lambda L \quad \rightarrow \quad \text{closed boundary + no marked point}$$



Ordering Ambiguity

Position basis ($\hat{\pi^\gamma} = -i\frac{\partial}{\partial L}$)

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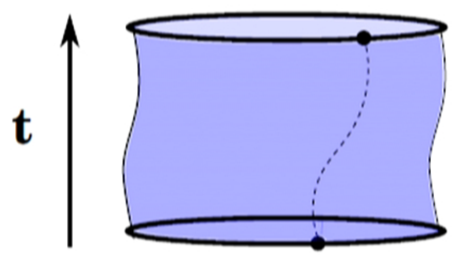

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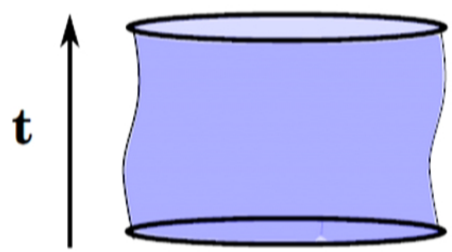

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We have open boundary conditions and no marked point

CDT and HL in 2d are described by the same Hamiltonian!

So is HL the continuum theory for CDT?

- the Hamiltonian agrees with the minisuperspace formulation of GR
- our results show that in 2d HL is the continuum theory

What about 4d?

- HL is a QFT following Wilsonian ideas
  - → all higher order terms that symmetry allows have to be included
- The CDT action is generally covariant
  - → entropic terms do lead to spatial higher derivatves as in HL

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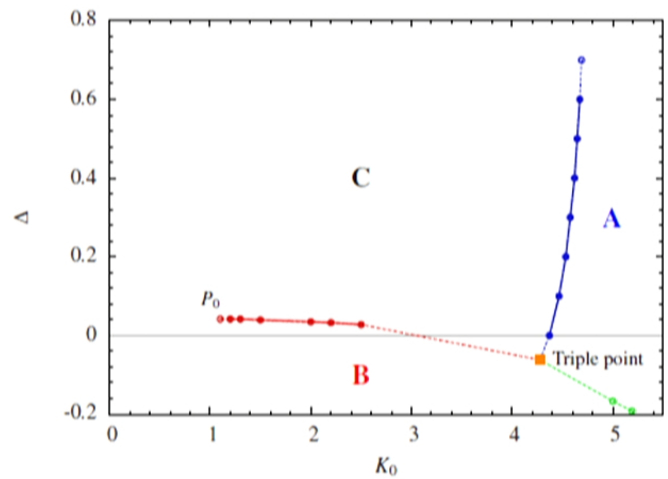
CDT is HL gravity

(isotropic point might still be GR!)

Summary

- CDT and HL have the same symmetries
- in 2d they have the same Hamiltonian
- HL is the continuum theory for part of the CDT phase space



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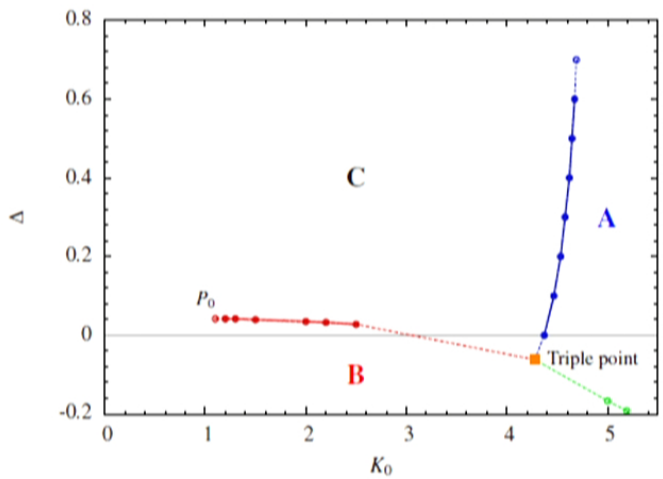

Thank you for your attention.