Title: Canonical Quantum Gravity - 3

Date: Jul 25, 2013 02:30 PM

URL: http://pirsa.org/13070074

Abstract:

A Double Dose of Abelian LQG

Casey Tomlin
with Adam Henderson, Alok Laddha, and Madhavan Varadarajan

Quantum General Covariance

- Every theory with diff symmetry has constraints that generate the "hypersurface deformation" algebra

$$\{D[\vec{N}], D[\vec{M}]\} = D[\mathcal{L}_{\vec{N}}\vec{M}]$$
$$\{D[\vec{N}], H[N]\} = H[\mathcal{L}_{\vec{N}}N]$$
$$\{H[N], H[M]\} = D[q^{ab}(M\partial_b N - N\partial_b M)]$$

encoding 4D spacetime covariance in 3+1 form [HKT]

Quantum General Covariance

- Every theory with diff symmetry has constraints that generate the "hypersurface deformation" algebra

$$\{D[\vec{N}], D[\vec{M}]\} = D[\mathcal{L}_{\vec{N}}\vec{M}]$$
$$\{D[\vec{N}], H[N]\} = H[\mathcal{L}_{\vec{N}}N]$$
$$\{H[N], H[M]\} = D[q^{ab}(M\partial_b N - N\partial_b M)]$$

encoding 4D spacetime covariance in 3+1 form [HKT]

Quantum General Covariance

- Every theory with diff symmetry has constraints that generate the "hypersurface deformation" algebra

$$\{D[\vec{N}], D[\vec{M}]\} = D[\mathcal{L}_{\vec{N}}\vec{M}]$$
$$\{D[\vec{N}], H[N]\} = H[\mathcal{L}_{\vec{N}}N]$$
$$\{H[N], H[M]\} = D[q^{ab}(M\partial_b N - N\partial_b M)]$$

  encoding 4D spacetime covariance in 3+1 form [HKT]

- **Philosophy:** A representation of the HD algebra via quantum operators is *required* in *any* (canonical) theory of quantized geometry, and defines the notion of *quantum spacetime covariance.* E.g.,

$$[\hat{H}[N], \hat{H}[M]] = \mathrm{i}\hbar\hat{D}[\hat{\vec{\omega}}] \qquad (\star)$$

Quantum General Covariance

- Every theory with diff symmetry has constraints that generate the "hypersurface deformation" algebra

$$\{D[\vec{N}], D[\vec{M}]\} = D[\mathcal{L}_{\vec{N}}\vec{M}]$$
$$\{D[\vec{N}], H[N]\} = H[\mathcal{L}_{\vec{N}}N]$$
$$\{H[N], H[M]\} = D[q^{ab}(M\partial_b N - N\partial_b M)]$$

encoding 4D spacetime covariance in 3+1 form [HKT]

- **Philosophy:** A representation of the HD algebra via quantum operators is *required* in *any* (canonical) theory of quantized geometry, and defines the notion of *quantum spacetime covariance.* E.g.,

$$[\hat{H}[N], \hat{H}[M]] = \mathrm{i}\hbar\hat{D}[\hat{\vec{\omega}}] \qquad (\star)$$

- State-of-the-art LQG does not sufficiently capture this relation:

Quantum General Covariance

- Every theory with diff symmetry has constraints that generate the "hypersurface deformation" algebra

$$\{D[\vec{N}], D[\vec{M}]\} = D[\mathcal{L}_{\vec{N}}\vec{M}]$$
$$\{D[\vec{N}], H[N]\} = H[\mathcal{L}_{\vec{N}}N]$$
$$\{H[N], H[M]\} = D[q^{ab}(M\partial_b N - N\partial_b M)]$$

  encoding 4D spacetime covariance in 3+1 form [HKT]

- **Philosophy:** A representation of the HD algebra via quantum operators is *required* in *any* (canonical) theory of quantized geometry, and defines the notion of *quantum spacetime covariance.* E.g.,

$$[\hat{H}[N], \hat{H}[M]] = \mathrm{i}\hbar\hat{D}[\hat{\vec{\omega}}] \qquad (\star)$$

- State-of-the-art LQG does not sufficiently capture this relation:
  - Algebra computed partially "on-shell" [Nikolai et al.]

Quantum General Covariance

- Every theory with diff symmetry has constraints that generate the "hypersurface deformation" algebra

$$\{D[\vec{N}], D[\vec{M}]\} = D[\mathcal{L}_{\vec{N}}\vec{M}]$$
$$\{D[\vec{N}], H[N]\} = H[\mathcal{L}_{\vec{N}}N]$$
$$\{H[N], H[M]\} = D[q^{ab}(M\partial_b N - N\partial_b M)]$$

  encoding 4D spacetime covariance in 3+1 form [HKT]

- **Philosophy:** A representation of the HD algebra via quantum operators is *required* in *any* (canonical) theory of quantized geometry, and defines the notion of *quantum spacetime covariance.* E.g.,

$$[\hat{H}[N], \hat{H}[M]] = \mathrm{i}\hbar\hat{D}[\hat{\vec{\omega}}] \qquad (\star)$$

- State-of-the-art LQG does not sufficiently capture this relation:
  - Algebra computed partially "on-shell" [Nikolai et al.]
  - Density weight responsible for trivial RHS [Lewandowski, Pullin et al.]
  - Ultralocality responsible for trivial LHS

Smolin's Weak-Coupling Limit[1]

Euclidean, self-dual, first order action:

$$S[e,\omega] = \frac{1}{G_{\mathrm{N}}}\int |e| e_I^{\mu} e_J^{\nu} R_{\mu\nu}{}^{IJ}[\omega], \qquad \omega_\mu{}^{IJ} = \tfrac{1}{2}\epsilon^{IJ}{}_{KL}\omega_\mu{}^{KL}$$

Define $A = G_{\mathrm{N}}^{-1}\omega$, take $G_{\mathrm{N}} \to 0$, 3+1 split, get

$$S[A,E] = \int \mathrm{d}t \left(\int_\Sigma \mathrm{d}^3x\; E_i^a \dot{A}_a^i - G[\Lambda] - D[\vec{N}] - H[\underset{\sim}{N}] \right)$$

where

$$G[\Lambda] = \int \Lambda^i \partial_a E_i^a \quad \text{—U(1)}^3 \text{ Gauss}$$

$$D[\vec{N}] = \int E_i^a \mathcal{L}_{\vec{N}} A_a^i \quad \text{—diffeo}$$

$$H[\underset{\sim}{N}] = \tfrac{1}{2}\int \underset{\sim}{N}\epsilon^{ijk} E_i^a E_j^b F_{ab}^k[A]$$ —Euclidean Hamiltonian with **Abelian** curvature $F_{ab}^i := 2\partial_{[a}A_{b]}^i$

Subalgebra of D and H again generates the HD algebra
Goal: Quantize H such that $[H,H] = D$ off-shell

---

[1] CQG **9** 883 1992

Lessons from the Husain-Kuchař Model[2]

- Construct an operator $\hat{D}[\vec{N}]$ such that

$$[\hat{D}[\vec{N}], \hat{D}[\vec{M}]] = \mathrm{i}\hbar\hat{D}[\mathcal{L}_{\vec{N}}\vec{M}] \qquad (\star)$$

[2] Laddha, Varadarajan CQG 29 2011

Lessons from the Husain-Kuchař Model[2]

- Construct an operator $\hat{D}[\vec{N}]$ such that

$$[\hat{D}[\vec{N}], \hat{D}[\vec{M}]] = \mathrm{i}\hbar\hat{D}[\mathcal{L}_{\vec{N}}\vec{M}] \qquad (\star)$$

- Strategy: Use classical geometric hints. Quantize regularized $\hat{D}_\delta[\vec{N}]$ on $\mathcal{H}_{\mathrm{kin}}$ such that

$$\hat{D}_\delta[\vec{N}] \sim \frac{\hat{U}(\phi^\delta_{\vec{N}}) - \mathbf{1}}{\delta}$$

[2] Laddha, Varadarajan CQG 29 2011

Lessons from the Husain-Kuchař Model[2]

- Construct an operator $\hat{D}[\vec{N}]$ such that

$$[\hat{D}[\vec{N}], \hat{D}[\vec{M}]] = \mathrm{i}\hbar\hat{D}[\mathcal{L}_{\vec{N}}\vec{M}] \qquad (\star)$$

- Strategy: Use classical geometric hints. Quantize regularized $\hat{D}_\delta[\vec{N}]$ on $\mathcal{H}_{\mathrm{kin}}$ such that

$$\hat{D}_\delta[\vec{N}] \sim \frac{\hat{U}(\phi^\delta_{\vec{N}}) - \mathbf{1}}{\delta}$$

[2] Laddha, Varadarajan CQG 29 2011

Lessons from the Husain-Kuchař Model[2]

- Construct an operator $\hat{D}[\vec{N}]$ such that

$$[\hat{D}[\vec{N}], \hat{D}[\vec{M}]] = \mathrm{i}\hbar \hat{D}[\mathcal{L}_{\vec{N}} \vec{M}] \qquad (\star)$$

- Strategy: Use classical geometric hints. Quantize regularized $\hat{D}_\delta[\vec{N}]$ on $\mathcal{H}_{\mathrm{kin}}$ such that

$$\hat{D}_\delta[\vec{N}] \sim \frac{\hat{U}(\phi^\delta_{\vec{N}}) - \mathbf{1}}{\delta}$$

- What is required of regularization and quantization choices?
  - Curvature operator aquires non-trivial state-dependence and non-perturbative modifications:

$$\hat{F}^i_\delta = \frac{\mathrm{tr}(h_\Box \tau^i)}{\delta^2} + \frac{3\mathrm{i}}{2\ell_{\mathrm{P}}^2} \frac{\mathrm{tr}(h_\Box - \mathbf{1})}{\delta^2} \hat{E}_i(S_\delta)$$

[2] Laddha, Varadarajan CQG 29 2011

Lessons from the Husain-Kuchař Model[2]

- Construct an operator $\hat{D}[\vec{N}]$ such that

$$[\hat{D}[\vec{N}], \hat{D}[\vec{M}]] = \mathrm{i}\hbar \hat{D}[\mathcal{L}_{\vec{N}}\vec{M}] \qquad (\star)$$

- Strategy: Use classical geometric hints. Quantize regularized $\hat{D}_\delta[\vec{N}]$ on $\mathcal{H}_{\mathrm{kin}}$ such that

$$\hat{D}_\delta[\vec{N}] \sim \frac{\hat{U}(\phi^\delta_{\vec{N}}) - \mathbf{1}}{\delta}$$

- What is required of regularization and quantization choices?
  - Curvature operator aquires non-trivial state-dependence and non-perturbative modifications:

$$\hat{F}^i_\delta = \frac{\mathrm{tr}(h_\square \tau^i)}{\delta^2} + \frac{3\mathrm{i}}{2\ell_{\mathrm{P}}^2}\frac{\mathrm{tr}(h_\square - \mathbf{1})}{\delta^2}\hat{E}_i(S_\delta)$$

[2] Laddha, Varadarajan CQG 29 2011

Lessons from the Husain-Kuchař Model[2]

- Construct an operator $\hat{D}[\vec{N}]$ such that

$$[\hat{D}[\vec{N}], \hat{D}[\vec{M}]] = \mathrm{i}\hbar \hat{D}[\mathcal{L}_{\vec{N}} \vec{M}] \qquad (\star)$$

- Strategy: Use classical geometric hints. Quantize regularized $\hat{D}_\delta[\vec{N}]$ on $\mathcal{H}_{\mathrm{kin}}$ such that

$$\hat{D}_\delta[\vec{N}] \sim \frac{\hat{U}(\phi^\delta_{\vec{N}}) - \mathbf{1}}{\delta}$$

- What is required of regularization and quantization choices?
  - Curvature operator aquires non-trivial state-dependence and non-perturbative modifications:

$$\hat{F}^i_\delta = \frac{\mathrm{tr}(h_\Box \tau^i)}{\delta^2} + \frac{3\mathrm{i}}{2\ell_{\mathrm{P}}^2} \frac{\mathrm{tr}(h_\Box - \mathbf{1})}{\delta^2} \hat{E}_i(S_\delta)$$

[2] Laddha, Varadarajan CQG 29 2011

Lessons from the Husain-Kuchař Model[2]

- Construct an operator $\hat{D}[\vec{N}]$ such that

$$[\hat{D}[\vec{N}], \hat{D}[\vec{M}]] = \mathrm{i}\hbar \hat{D}[\mathcal{L}_{\vec{N}} \vec{M}] \qquad (\star)$$

- Strategy: Use classical geometric hints. Quantize regularized $\hat{D}_\delta[\vec{N}]$ on $\mathcal{H}_{\mathrm{kin}}$ such that

$$\hat{D}_\delta[\vec{N}] \sim \frac{\hat{U}(\phi^\delta_{\vec{N}}) - \mathbf{1}}{\delta}$$

- What is required of regularization and quantization choices?
  - Curvature operator aquires non-trivial state-dependence and non-perturbative modifications:

$$\hat{F}^i_\delta = \frac{\mathrm{tr}(h_\Box \tau^i)}{\delta^2} + \frac{3\mathrm{i}}{2\ell_{\mathrm{P}}^2} \frac{\mathrm{tr}(h_\Box - \mathbf{1})}{\delta^2} \hat{E}_i(S_\delta)$$

[2] Laddha, Varadarajan CQG 29 2011

Lessons from the Husain-Kuchař Model[2]

- Construct an operator $\hat{D}[\vec{N}]$ such that

$$[\hat{D}[\vec{N}], \hat{D}[\vec{M}]] = \mathrm{i}\hbar \hat{D}[\mathcal{L}_{\vec{N}}\vec{M}] \qquad (\star)$$

- Strategy: Use classical geometric hints. Quantize regularized $\hat{D}_\delta[\vec{N}]$ on $\mathcal{H}_{\mathrm{kin}}$ such that

$$\hat{D}_\delta[\vec{N}] \sim \frac{\hat{U}(\phi^\delta_{\vec{N}}) - \mathbf{1}}{\delta}$$

- What is required of regularization and quantization choices?
  - Curvature operator aquires non-trivial state-dependence and non-perturbative modifications:

$$\hat{F}^i_\delta = \frac{\mathrm{tr}(h_\square \tau^i)}{\delta^2} + \frac{3\mathrm{i}}{2\ell_{\mathrm{P}}^2} \frac{\mathrm{tr}(h_\square - \mathbf{1})}{\delta^2} \hat{E}_i(S_\delta)$$

  - Sum over triangulation → product: $\sum_I \delta x_I = \prod_I (1 + \delta x_I) - 1 + O(\delta^2)$

[2] Laddha, Varadarajan CQG 29 2011

Lessons from the Husain-Kuchař Model[2]

- Construct an operator $\hat{D}[\vec{N}]$ such that

$$[\hat{D}[\vec{N}], \hat{D}[\vec{M}]] = \mathrm{i}\hbar \hat{D}[\mathcal{L}_{\vec{N}} \vec{M}] \qquad (\star)$$

- Strategy: Use classical geometric hints. Quantize regularized $\hat{D}_\delta[\vec{N}]$ on $\mathcal{H}_{\mathrm{kin}}$ such that

$$\hat{D}_\delta[\vec{N}] \sim \frac{\hat{U}(\phi^\delta_{\vec{N}}) - \mathbf{1}}{\delta}$$

- What is required of regularization and quantization choices?
  - Curvature operator aquires non-trivial state-dependence and non-perturbative modifications:

$$\hat{F}^i_\delta = \frac{\mathrm{tr}(h_\square \tau^i)}{\delta^2} + \frac{3\mathrm{i}}{2\ell_{\mathrm{P}}^2} \frac{\mathrm{tr}(h_\square - \mathbf{1})}{\delta^2} \hat{E}_i(S_\delta)$$

  - Sum over triangulation $\rightarrow$ product: $\sum_I \delta x_I = \prod_I (1 + \delta x_I) - 1 + O(\delta^2)$
- What space supports the $\delta \rightarrow 0$ limit?

[2] Laddha, Varadarajan CQG 29 2011

Lessons from the Husain-Kuchař Model[2]

- Construct an operator $\hat{D}[\vec{N}]$ such that

$$[\hat{D}[\vec{N}], \hat{D}[\vec{M}]] = \mathrm{i}\hbar \hat{D}[\mathcal{L}_{\vec{N}}\vec{M}] \qquad (\star)$$

- Strategy: Use classical geometric hints. Quantize regularized $\hat{D}_\delta[\vec{N}]$ on $\mathcal{H}_{\rm kin}$ such that

$$\hat{D}_\delta[\vec{N}] \sim \frac{\hat{U}(\phi^\delta_{\vec{N}}) - \mathbf{1}}{\delta}$$

- What is required of regularization and quantization choices?
  - Curvature operator aquires non-trivial state-dependence and non-perturbative modifications:

$$\hat{F}^i_\delta = \frac{\mathrm{tr}(h_\Box \tau^i)}{\delta^2} + \frac{3\mathrm{i}}{2\ell_{\rm P}^2}\frac{\mathrm{tr}(h_\Box - \mathbf{1})}{\delta^2}\hat{E}_i(S_\delta)$$

  - Sum over triangulation $\to$ product: $\sum_I \delta x_I = \prod_I (1 + \delta x_I) - 1 + O(\delta^2)$
- What space supports the $\delta \to 0$ limit?
  - Not $\mathcal{H}_{\rm kin}$, but a well-chosen set of distributions (Lewandowski-Marolf habitat):

$$\hat{D}[\vec{N}]\Psi^f \sim \Psi^{\mathcal{L}_{\vec{N}} f}$$

- Apply these ideas to the $\mathrm{U}(1)^3$ Hamiltonian constraint

[2] Laddha, Varadarajan CQG 29 2011

3+1 Finite-Triangulation Hamiltonian

- Let $N_i^a := q^{-1/3} N E_i^a$. Classically this "electric shift" is a (density weight zero) vector field. The action of H on A can be written

$$\{A_a^i, H[N]\} = -\epsilon^{ijk} \mathcal{L}_{\vec{N}_j} A_a^k + \text{gauge}$$

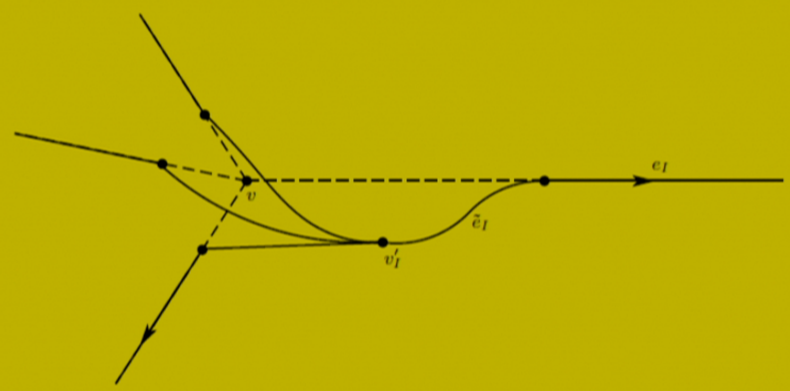
3+1 Finite-Triangulation Hamiltonian

- Let $N_i^a := q^{-1/3} N E_i^a$. Classically this "electric shift" is a (density weight zero) vector field. The action of H on A can be written

$$\{A_a^i, H[N]\} = -\epsilon^{ijk} \mathcal{L}_{\vec{N}_j} A_a^k + \mathrm{gauge}$$

- Since $\hat{N}_i^a(v)|c\rangle$ naturally splits into a sum over edges $\hat{N}_i^a(v)|c\rangle \sim \sum_e \dot{e}^a |c\rangle$, this translates into an action at finite triangulation:

$$\hat{H}_{\delta,v}[N]|c\rangle \sim N(v)\lambda_q^{-1/3} \sum_{e\cap v} \sum_i q_e^i \frac{|c_{e,i,\delta}\rangle - |c\rangle}{\delta}$$



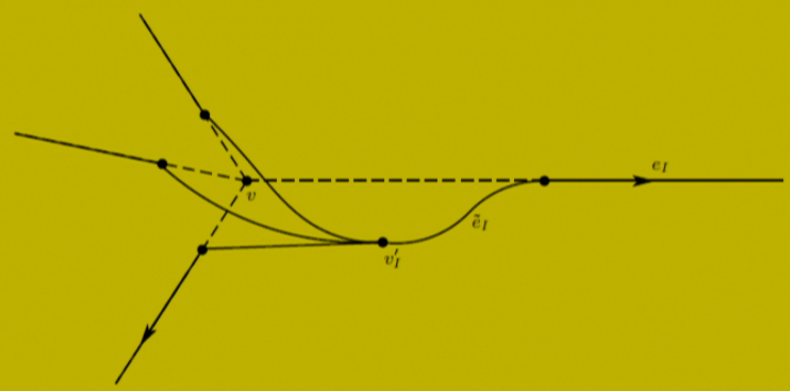
3+1 Finite-Triangulation Hamiltonian

- Let $N_i^a := q^{-1/3} N E_i^a$. Classically this "electric shift" is a (density weight zero) vector field. The action of H on A can be written

$$\{A_a^i, H[N]\} = -\epsilon^{ijk} \mathcal{L}_{\vec{N}_j} A_a^k + \text{gauge}$$

- Since $\hat{N}_i^a(v)|c\rangle$ naturally splits into a sum over edges $\hat{N}_i^a(v)|c\rangle \sim \sum_e \dot{e}^a |c\rangle$, this translates into an action at finite triangulation:

$$\hat{H}_{\delta,v}[N]|c\rangle \sim N(v)\lambda_q^{-1/3} \sum_{e\cap v} \sum_i q_e^i \frac{|c_{e,i,\delta}\rangle - |c\rangle}{\delta}$$



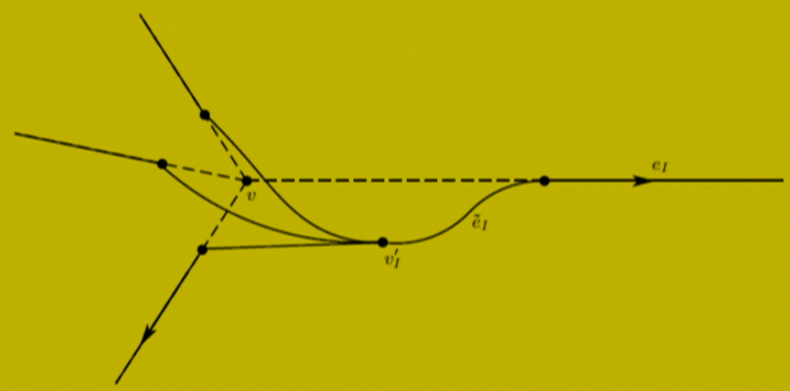
3+1 Finite-Triangulation Hamiltonian

- Let $N_i^a := q^{-1/3} N E_i^a$. Classically this "electric shift" is a (density weight zero) vector field. The action of H on A can be written

$$\{A_a^i, H[N]\} = -\epsilon^{ijk} \mathcal{L}_{\vec{N}_j} A_a^k + \mathrm{gauge}$$

- Since $\hat{N}_i^a(v)|c\rangle$ naturally splits into a sum over edges $\hat{N}_i^a(v)|c\rangle \sim \sum_e \dot{e}^a |c\rangle$, this translates into an action at finite triangulation:

$$\hat{H}_{\delta,v}[N]|c\rangle \sim N(v)\lambda_q^{-1/3} \sum_{e\cap v} \sum_i q_e^i \frac{|c_{e,i,\delta}\rangle - |c\rangle}{\delta}$$



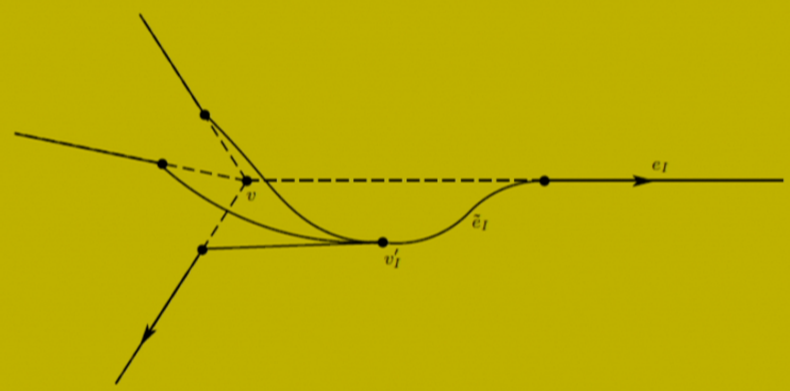
3+1 Finite-Triangulation Hamiltonian

- Let $N_i^a := q^{-1/3} N E_i^a$. Classically this "electric shift" is a (density weight zero) vector field. The action of H on A can be written

$$\{A_a^i, H[N]\} = -\epsilon^{ijk} \mathcal{L}_{\vec{N}_j} A_a^k + \text{gauge}$$

- Since $\hat{N}_i^a(v)|c\rangle$ naturally splits into a sum over edges $\hat{N}_i^a(v)|c\rangle \sim \sum_e \dot{e}^a |c\rangle$, this translates into an action at finite triangulation:

$$\hat{H}_{\delta,v}[N]|c\rangle \sim N(v)\lambda_q^{-1/3} \sum_{e\cap v} \sum_i q_e^i \frac{|c_{e,i,\delta}\rangle - |c\rangle}{\delta}$$



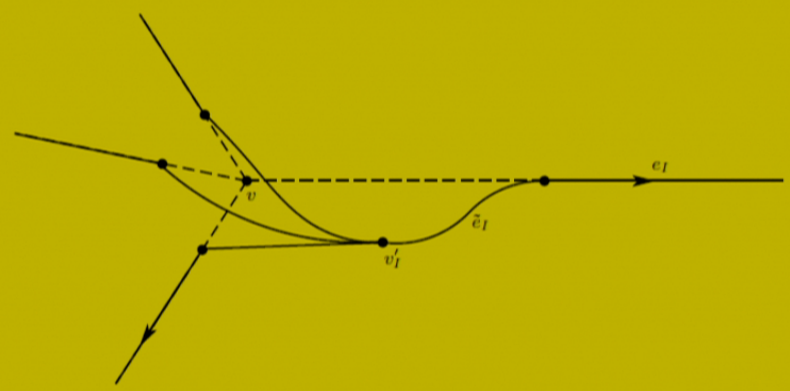
3+1 Finite-Triangulation Hamiltonian

- Let $N_i^a := q^{-1/3} N E_i^a$. Classically this "electric shift" is a (density weight zero) vector field. The action of H on A can be written

$$\{A_a^i, H[N]\} = -\epsilon^{ijk} \mathcal{L}_{\vec{N}_j} A_a^k + \text{gauge}$$

- Since $\hat{N}_i^a(v)|c\rangle$ naturally splits into a sum over edges $\hat{N}_i^a(v)|c\rangle \sim \sum_e \dot{e}^a |c\rangle$, this translates into an action at finite triangulation:

$$\hat{H}_{\delta,v}[N]|c\rangle \sim N(v)\lambda_q^{-1/3} \sum_{e\cap v} \sum_i q_e^i \frac{|c_{e,i,\delta}\rangle - |c\rangle}{\delta}$$



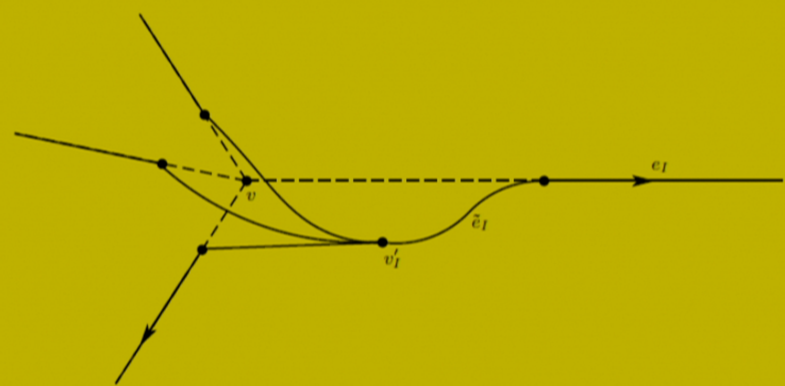
3+1 Finite-Triangulation Hamiltonian

- Let $N^a_i := q^{-1/3} N E^a_i$. Classically this "electric shift" is a (density weight zero) vector field. The action of H on A can be written

$$\{A^i_a, H[N]\} = -\epsilon^{ijk} \mathcal{L}_{\vec{N}_j} A^k_a + \mathrm{gauge}$$

- Since $\hat{N}^a_i(v)|c\rangle$ naturally splits into a sum over edges $\hat{N}^a_i(v)|c\rangle \sim \sum_e \dot{e}^a |c\rangle$, this translates into an action at finite triangulation:

$$\hat{H}_{\delta,v}[N]|c\rangle \sim N(v)\lambda_q^{-1/3} \sum_{e\cap v} \sum_i q^i_e \frac{|c_{e,i,\delta}\rangle - |c\rangle}{\delta}$$



- Non-trivial vertex is moved $\Rightarrow$ 2[nd] $\hat{H}$ in commutator acts at displaced vertex, removing ultralocality problem

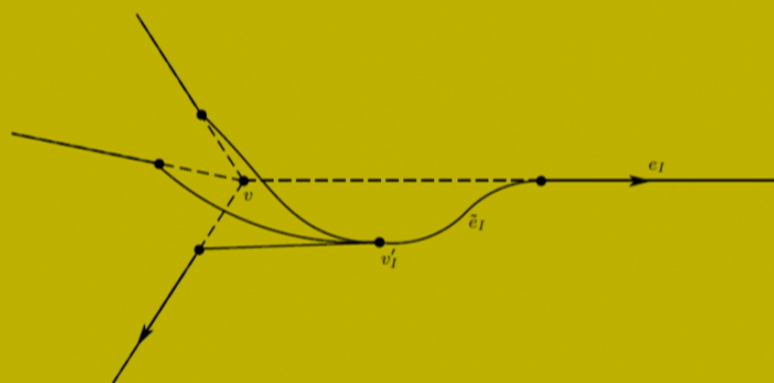
3+1 Finite-Triangulation Hamiltonian

- Let $N_i^a := q^{-1/3} N E_i^a$. Classically this "electric shift" is a (density weight zero) vector field. The action of H on A can be written

$$\{A_a^i, H[N]\} = -\epsilon^{ijk} \mathcal{L}_{\vec{N}_j} A_a^k + \mathrm{gauge}$$

- Since $\hat{N}_i^a(v)|c\rangle$ naturally splits into a sum over edges $\hat{N}_i^a(v)|c\rangle \sim \sum_e \dot{e}^a |c\rangle$, this translates into an action at finite triangulation:

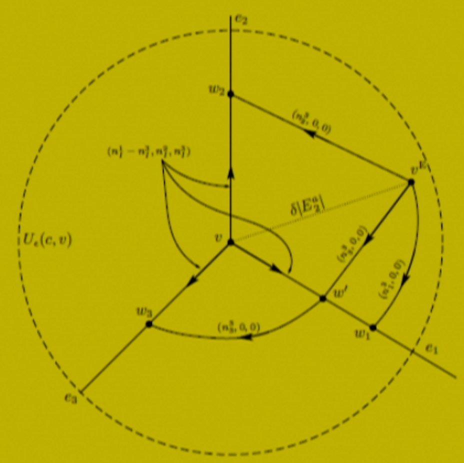
$$\hat{H}_{\delta,v}[N]|c\rangle \sim N(v)\lambda_q^{-1/3} \sum_{e \cap v} \sum_i q_e^i \frac{|c_{e,i,\delta}\rangle - |c\rangle}{\delta}$$



- Non-trivial vertex is moved $\Rightarrow$ 2nd $\hat{H}$ in commutator acts at displaced vertex, removing ultralocality problem

2+1 Finite-Triangulation Hamiltonian

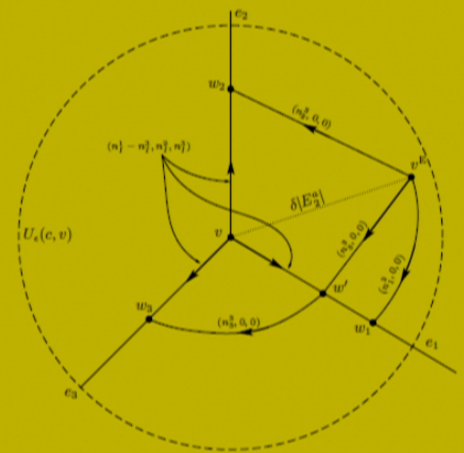
- One electric shift per vertex
- Switch to product form over edges at a vertex



- Deformations feature a new vertex, but it is only charged in a single factor of U(1)—possible ultra-locality problem

2+1 Finite-Triangulation Hamiltonian

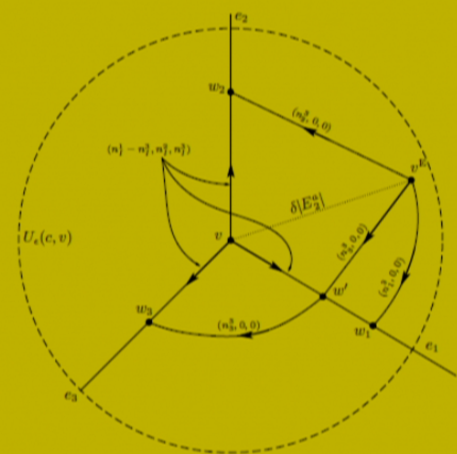
- One electric shift per vertex
- Switch to product form over edges at a vertex



- Deformations feature a new vertex, but it is only charged in a single factor of U(1)—possible ultra-locality problem

2+1 Finite-Triangulation Hamiltonian

- One electric shift per vertex
- Switch to product form over edges at a vertex



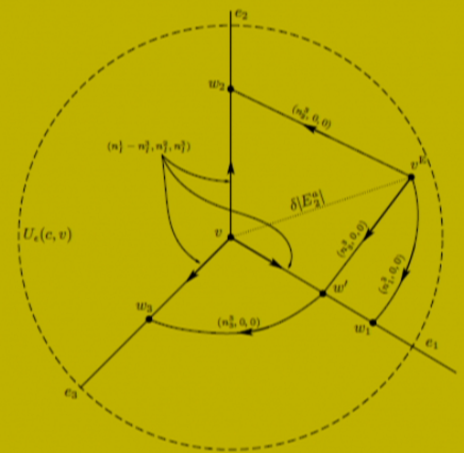
- Deformations feature a new vertex, but it is only charged in a single factor of U(1)—possible ultra-locality problem
  - ⇒ Exploit availability of non-perturbative corrections to ensure that this vertex moves under the action of a second Hamiltonian: Classically

$$\{E_i^a, H^{(i)}[N]\} \sim \epsilon^{ijk} E_j^a E_k^b \partial_b N + \cdots$$

Nontrivial action of 2nd $\hat{H}$ on quantum shift components which gave the first deformation

2+1 Finite-Triangulation Hamiltonian

- One electric shift per vertex
- Switch to product form over edges at a vertex



- Deformations feature a new vertex, but it is only charged in a single factor of U(1)—possible ultra-locality problem
  - ⇒ Exploit availability of non-perturbative corrections to ensure that this vertex moves under the action of a second Hamiltonian: Classically

$$\{E_i^a, H^{(i)}[N]\} \sim \epsilon^{ijk} E_j^a E_k^b \partial_b N + \cdots$$

Nontrivial action of 2nd $\hat{H}$ on quantum shift components which gave the first deformation

Remarks

- Off-shell closure in a precise sense

$$\lim_{\delta,\delta'\to 0}(\Psi|[\hat{H}[N],\hat{H}[M]]_{\delta,\delta'}|c\rangle = \lim_{\delta,\delta'\to 0}(\Psi|\hat{D}[\vec{\omega}]_{\delta,\delta'}|c\rangle$$

  and there is more than one way to do it.
- SU(2) looks promising
- Speculation: $\mathrm{U}(1)^3$ E-representation has linear-in-momenta constraints. Could investigate in flux rep of Dittrich et al.

Remarks

- Off-shell closure in a precise sense

$$\lim_{\delta,\delta'\to 0}(\Psi|[\hat{H}[N],\hat{H}[M]]_{\delta,\delta'}|c\rangle = \lim_{\delta,\delta'\to 0}(\Psi|\hat{D}[\vec{\omega}]_{\delta,\delta'}|c\rangle$$

  and there is more than one way to do it.
- SU(2) looks promising
- Speculation: $U(1)^3$ E-representation has linear-in-momenta constraints. Could investigate in flux rep of Dittrich et al.

Remarks

- Off-shell closure in a precise sense

$$\lim_{\delta,\delta'\to 0}(\Psi|[\hat{H}[N],\hat{H}[M]]_{\delta,\delta'}|c\rangle = \lim_{\delta,\delta'\to 0}(\Psi|\hat{D}[\vec{\omega}]_{\delta,\delta'}|c\rangle$$

  and there is more than one way to do it.
- SU(2) looks promising
- Speculation: $\mathrm{U}(1)^3$ E-representation has linear-in-momenta constraints. Could investigate in flux rep of Dittrich et al.

Remarks

- Off-shell closure in a precise sense

$$\lim_{\delta,\delta'\to 0}(\Psi|[\hat{H}[N],\hat{H}[M]]_{\delta,\delta'}|c\rangle = \lim_{\delta,\delta'\to 0}(\Psi|\hat{D}[\vec{\omega}]_{\delta,\delta'}|c\rangle$$

  and there is more than one way to do it.
- SU(2) looks promising
- Speculation: $U(1)^3$ E-representation has linear-in-momenta constraints. Could investigate in flux rep of Dittrich et al.

Remarks

- Off-shell closure in a precise sense

$$\lim_{\delta,\delta'\to 0}(\Psi|[\hat{H}[N],\hat{H}[M]]_{\delta,\delta'}|c\rangle = \lim_{\delta,\delta'\to 0}(\Psi|\hat{D}[\vec{\omega}]_{\delta,\delta'}|c\rangle$$

  and there is more than one way to do it.
- SU(2) looks promising
- Speculation: $U(1)^3$ E-representation has linear-in-momenta constraints. Could investigate in flux rep of Dittrich et al.

Why use Effective Equations?

- Correlation functions are calculated with an absolutely generalized initial state, as required for cosmology.
- Can avoid several technical difficulties like the exact structure of inner products on the Hilbert space, or the non-unique nature of self-adjoint extensions.
- Systematic way to realize higher derivative corrections in the equations of motion for a canonically quantized system.
- New perspective on known features of QFT, like renormalization, which may prove to be useful while quantizing with a dynamical background.
- Being canonical, applicable to certain models of LQG and LQC.

Why use Effective Equations?

- Correlation functions are calculated with an absolutely generalized initial state, as required for cosmology.
- Can avoid several technical difficulties like the exact structure of inner products on the Hilbert space, or the non-unique nature of self-adjoint extensions.
- Systematic way to realize higher derivative corrections in the equations of motion for a canonically quantized system.
- New perspective on known features of QFT, like renormalization, which may prove to be useful while quantizing with a dynamical background.
- Being canonical, applicable to certain models of LQG and LQC.

The New Variables

[M. Bojowald and A. Skirzewski, 2006]

- Define expectation values, with respect to some state, as:

$$\tilde{G}^{a,n} := \langle (\hat{p} - \langle \hat{p} \rangle)^a (\hat{q} - \langle \hat{q} \rangle)^{n-a} \rangle_{\mathrm{Weyl}} \tag{2.1}$$

- Begin with a Hamiltonian operator: $\widehat{H} = \widehat{H}(\hat{q}, \hat{p})$
  Take its expectation value with respect to the same state to define an 'effective' Quantum Hamiltonian

$$H_Q := \langle \widehat{H} \rangle = \left\langle \widehat{H}(\langle \hat{q} \rangle + (\hat{q} - \langle \hat{q} \rangle), \langle \hat{p} \rangle + (\hat{p} - \langle \hat{q} \rangle)) \right\rangle$$
$$= \sum_{n=0}^{\infty} \sum_{a=0}^{n} \frac{1}{n!} \binom{n}{a} \frac{\partial^n H(q, p)}{\partial p^a \partial q^{n-a}} \tilde{G}^{a,n} \tag{2.2}$$

- A point in this infinite dimensional space is completely specified by $(\langle \hat{q} \rangle, \langle \hat{p} \rangle, \tilde{G}^{a,n})$

The Equations of Motion

Let $q := \langle \hat{q} \rangle$ and $p := \langle \hat{p} \rangle$.
The Hamilton's equations of motion gives us

$$\dot{q} = \{q, H_Q\} \tag{2.5}$$

$$\dot{p} = \{p, H_Q\} \tag{2.6}$$

$$\dot{\tilde{G}}^{a,n} = \{\tilde{G}^{a,n}, H_Q\} \tag{2.7}$$

Instead of solving the Schrödinger's partial differential equation, we have to solve this infinite set of coupled ordinary differential equations.

- The validity of the solutions to these equations of motion are subject to certain 'Uncertainty Relations', imposed on the moments.

The Effective Quantum Hamiltonian

The Hamiltonian for an oscillator with a perturbation term is

$$\widehat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2 + \widehat{U}(\hat{q})$$

The corresponding 'effective' Quantum Hamiltonian is

$$H_Q = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2 + U(q) + \frac{\hbar\omega}{2}(G^{0,2} + G^{2,2}) + \sum_n \frac{1}{n!}(\hbar/m\omega)^{n/2}U^{(n)}(q)G^{0,n} \qquad (2.8)$$

where $G^{a,n} = \hbar^{-n/2}(m\omega)^{n/2-a}\tilde{G}^{a,n}$ are now dimensionless quantities.

We need to make two approximations:

- Moments need to be solved perturbatively in $\left(\frac{\hbar}{L}\right)^{1/2}$. Here L is some angular momentum scale provided by the perturbing potential.
- Need to make an adiabatic approximation for the moments where we assume they are slowly varying with time but the evolution of q and p are free. Derivatives with respect to time in equations of motion are rescaled as $\frac{d}{dt} \to \lambda \frac{d}{dt}$. In the end, we shall set $\lambda = 1$

Thus, we can expand the moments as

$$G^{a,n} = \sum_e \sum_i G^{a,n}_{e,i} \left(\frac{\hbar}{L}\right)^{e/2} \lambda^i \tag{2.10}$$

At a given order in $\sqrt{\hbar/L}$, denoted by the index e, the adiabatic approximation gives

$$0 = \{G^{a,n}_{e,0}, H_0\} \tag{2.11}$$

to leading order, and

$$\dot{G}^{a,n}_{e,i} = \{G^{a,n}_{e,i+1}, H_0\} \tag{2.12}$$

for higher orders.

We need to make two approximations:

- Moments need to be solved perturbatively in $\left(\frac{\hbar}{L}\right)^{1/2}$. Here L is some angular momentum scale provided by the perturbing potential.
- Need to make an adiabatic approximation for the moments where we assume they are slowly varying with time but the evolution of q and p are free. Derivatives with respect to time in equations of motion are rescaled as $\frac{d}{dt} \to \lambda \frac{d}{dt}$. In the end, we shall set $\lambda = 1$

Thus, we can expand the moments as

$$G^{a,n} = \sum_e \sum_i G^{a,n}_{e,i} \left(\frac{\hbar}{L}\right)^{e/2} \lambda^i \tag{2.10}$$

At a given order in $\sqrt{\frac{\hbar}{L}}$, denoted by the index e, the adiabatic approximation gives

$$0 = \left\{ G^{a,n}_{e,0}, H_Q \right\} \tag{2.11}$$

to leading order, and

$$\dot{G}^{a,n}_{e,i} = \left\{ G^{a,n}_{e,i+1}, H_Q \right\} \tag{2.12}$$

for higher orders.

Equation of motion for q up to $\hbar^{3/2}$ and fourth adiabatic order

We may now rewrite the equation of motion as:

$$\ddot{q} = -\omega^2 q - U'(q)/m - \frac{\hbar}{2m^2\omega} U'''(q)\left[f(q,\dot{q}) + f_1(q,\dot{q})\ddot{q} + f_2(q)\ddot{q}^2 + f_3(q,\dot{q})\dddot{q} + f_4(q)\ddddot{q}\right] + \mathcal{O}(\hbar^2) \quad (2.16)$$

where

$$\begin{aligned} f(q,\dot{q}) = {} & \frac{1}{2}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-1/2} + \frac{U''''(q)\dot{q}^2}{16m\omega^4}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-5/2} - \frac{5(U'''(q))^2\dot{q}^2}{64m^2\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} \\ & - \frac{U''''''(q)\dot{q}^4}{64m\omega^6}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-7/2} + \frac{21(U''''(q))^2\dot{q}^4}{256m^2\omega^8}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-9/2} \\ & + \frac{7U'''''(q)U'''(q)\dot{q}^4}{64m^2\omega^8}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-9/2} - \frac{231U''''(q)(U'''(q))^2\dot{q}^4}{512m^3\omega^{10}}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-11/2} \\ & + \frac{1155(U'''(q))^4\dot{q}^4}{4096m^4\omega^{12}}\left(1+\frac{U''(q)}{m\omega^2}\right)^{-13/2} \end{aligned} \quad (2.17)$$

CW Potential for a $(0+1)$-dimensional system

[S. Coleman and E. Weinberg, 1973]

For a given Lagrangian $L(q,\dot{q},t) = \frac{1}{2}m\dot{q}^2 - V(q)$, with a vev defined by $\langle 0|\, q\, |0\rangle := q_0$, the Effective Coleman-Weinberg potential is given by

$$V_{\text{eff}}(q) = V(q_0) + \frac{\hbar}{2\sqrt{m}} \int \frac{\mathrm{d}k}{2\pi} \log\left(\frac{k^2 + V''(q_0)}{k^2}\right) + O(\hbar^2) \qquad (2.22)$$

This integral is obviously convergent and it gives:

$$V_{\text{eff}}(q) = V(q_0) + \frac{\hbar}{2\sqrt{m}}\sqrt{V''(q_0)} + O(\hbar^2) \qquad (2.23)$$

The Setup

- Use the 'in-in' formalism to get equal-time correlation functions
- The 'phi-fourth' Hamiltonian
  $\widehat{H} = \int \mathrm{d}^3x \left[\frac{\hat{\pi}^2(x)}{2} + \frac{m^2}{2}\hat{\phi}^2(x) + \frac{1}{2}\left(\nabla\hat{\phi}(x)\right)^2 + \lambda\hat{\phi}^4(x)\right]$

Define

$$G^{a,b}(x_1,\dots,x_a;y_1,\dots,y_b,t) := \Big\langle (\hat{\pi}(x_1,t) - \langle\hat{\pi}(x_1,t)\rangle)\dots(\hat{\pi}(x_a,t) - \langle\hat{\pi}(x_a,t)\rangle) \times (\hat{\phi}(y_1,t) - \langle\hat{\phi}(y_1,t)\rangle)\dots(\hat{\phi}(y_b,t) - \langle\hat{\phi}(y_b,t)\rangle)\Big\rangle_{\mathrm{Weyl}} \quad (3.1)$$

$$\nabla_{x_i}\nabla_{y_j}\left[G^{a,b}(x_1,\dots,x_a;y_1,\dots,y_b,t)\right] := \Big\langle (\hat{\pi}(x_1,t) - \langle\hat{\pi}(x_1,t)\rangle)\dots\nabla_{x_i}(\hat{\pi}(x_i,t) - \langle\hat{\pi}(x_i,t)\rangle)\dots \times (\hat{\phi}(y_1,t) - \langle\hat{\phi}(y_1,t)\rangle)\dots\nabla_{y_j}(\hat{\phi}(y_j,t) - \langle\hat{\phi}(y_j,t)\rangle)\dots\Big\rangle_{\mathrm{Weyl}} \quad (3.2)$$

With $\langle\hat{\pi}(x)\rangle := \pi(x)$ and $\langle\hat{\phi}(x)\rangle := \phi(x)$,

$$H_Q = \frac{1}{2}\int \mathrm{d}^3x\Big[\pi^2(x) + G^{2,0}(x,x) + m^2\left(\phi^2(x) + G^{0,2}(x,x)\right)$$
$$+\nabla_x^2\left(G^{0,2}(x,x)\right) + (\nabla\phi(x))^2 + 2\lambda\{\phi^4(x)$$
$$+6\phi^2(x)G^{0,2}(x,x) + 4\phi(x)G^{0,3}(x,x,x) + G^{0,4}(x,x,x,x)\}\Big] \quad (3.3)$$

The (equal time) Poisson Algebra is defined as:

$$\{\phi(x),\pi(y)\} := \frac{1}{i\hbar}\left\langle\left[\hat{\phi}(x),\hat{\pi}(x)\right]\right\rangle = \delta^3(x-y) \quad (3.4)$$

The equations of motion are derived as:

$$\frac{d}{dt}[\mathcal{O}] := \{H_Q,\mathcal{O}\} \quad (3.5)$$

EOM (Higher Order Moments)

The general scheme for equations of higher order moments

$$
\begin{aligned}
\dot{G}^{0,n}(y_1,\ldots,y_n,t) &\sim G^{1,n-1}(y_1,\ldots,y_n,t)\\
\dot{G}^{1,n-1}(y_1,\ldots,y_n,t) &\sim G^{2,n-2}(y_1,\ldots,y_n,t) + G^{0,n}(y_1,\ldots,y_n,t)\\
&\quad + \lambda G^{0,n+2}(y_1,\ldots,y_n,t)\\
&\vdots\\
\dot{G}^{n-1,1}(y_1,\ldots,y_n,t) &\sim G^{n-2,2}(y_1,\ldots,y_n,t) + G^{n,0}(y_1,\ldots,y_n,t)\\
&\quad + \lambda G^{n-2,4}(y_1,\ldots,y_n,t)\\
\dot{G}^{n,0}(y_1,\ldots,y_n,t) &\sim G^{n-1,1}(y_1,\ldots,y_n,t) + \lambda G^{n-1,3}(y_1,\ldots,y_n,t) \qquad (3.8)
\end{aligned}
$$

With $\langle\hat{\pi}(x)\rangle := \pi(x)$ and $\langle\hat{\phi}(x)\rangle := \phi(x)$,

$$H_Q = \frac{1}{2}\int \mathrm{d}^3x \Big[\pi^2(x) + G^{2,0}(x,x) + m^2\left(\phi^2(x) + G^{0,2}(x,x)\right) + \nabla_x^2\left(G^{0,2}(x,x)\right) + \left(\nabla\phi(x)\right)^2 + 2\lambda\{\phi^4(x) + 6\phi^2(x)G^{0,2}(x,x) + 4\phi(x)G^{0,3}(x,x,x) + G^{0,4}(x,x,x,x)\}\Big] \tag{3.3}$$

The (equal time) Poisson Algebra is defined as:

$$\{\phi(x),\pi(y)\} := \frac{1}{i\hbar}\left\langle \left[\hat{\phi}(x),\hat{\pi}(x)\right]\right\rangle = \delta^3(x-y) \tag{3.4}$$

The equations of motion are derived as:

$$\frac{d}{dt}[\mathcal{O}] := \{H_Q,\mathcal{O}\} \tag{3.5}$$

The Setup

- Use the 'in-in' formalism to get equal-time correlation functions
- The 'phi-fourth' Hamiltonian

  $\hat{H} = \int d^3x \left[\frac{\hat{\pi}^2(x)}{2} + \frac{m^2}{2}\hat{\phi}^2(x) + \frac{1}{2}\left(\nabla\hat{\phi}(x)\right)^2 + \lambda\hat{\phi}^4(x) \right]$

Define

$$G^{a,b}(x_1,\ldots,x_a;y_1,\ldots,y_b,t) := \Big\langle \left(\hat{\pi}(x_1,t) - \langle\hat{\pi}(x_1,t)\rangle\right)\ldots\left(\hat{\pi}(x_a,t) - \langle\hat{\pi}(x_a,t)\rangle\right) \times \left(\hat{\phi}(y_1,t) - \langle\hat{\phi}(y_1,t)\rangle\right)\ldots\left(\hat{\phi}(y_b,t) - \langle\hat{\phi}(y_b,t)\rangle\right)\Big\rangle_{\text{Weyl}} \tag{3.1}$$

$$\nabla_{x_i}\nabla_{y_j}\left[G^{a,b}(x_1,\ldots,x_a;y_1,\ldots,y_b,t)\right] := \Big\langle \left(\hat{\pi}(x_1,t) - \langle\hat{\pi}(x_1,t)\rangle\right)\ldots\nabla_{x_i}\left(\hat{\pi}(x_i,t) - \langle\hat{\pi}(x_i,t)\rangle\right)\ldots \times \left(\hat{\phi}(y_1,t) - \langle\hat{\phi}(y_1,t)\rangle\right)\ldots\nabla_{y_j}\left(\hat{\phi}(y_j,t) - \langle\hat{\phi}(y_j,t)\rangle\right)\ldots\Big\rangle_{\text{Weyl}} \tag{3.2}$$

Solving these equations

- Expand the moments in powers of the coupling constant, $G^{a,b} = \sum_e \bar{\lambda} G_e^{a,b}$
- Solve for the moments in lower orders in $\bar{\lambda}$, starting with the free field solutions.
- Plug the (solved) lower order $\bar{\lambda}$ moments, in the equations containing higher order in $\bar{\lambda}$.
- In this way, perturbatively solve for the moments, which shall give us the required correlation functions.

Solving these equations

- Expand the moments in powers of the coupling constant, $G^{a,b} = \sum_e \bar{\lambda} G_e^{a,b}$
- Solve for the moments in lower orders in $\bar{\lambda}$, starting with the free field solutions.
- Plug the (solved) lower order $\bar{\lambda}$ moments, in the equations containing higher order in $\bar{\lambda}$.
- In this way, perturbatively solve for the moments, which shall give us the required correlation functions.

Cancellation of the tadpole term

For ϕ^4 theory,

$$\ddot{\phi}(y,t) = -(m^2 - \nabla_y^2)\phi(y,t) + 4\lambda\phi^3(y,t) + 12\lambda\phi(y,t)G^{0,2}(y,y,t) + 4\lambda G^{0,3}(y,y,y,t) \tag{3.9}$$

In this case, $\phi(y,t) = 0$ is easily a solution up to any order since all odd moments (including $G^{0,3}(y_1,y_2,y_3,t)$) are zero up to any order.
For ϕ^3 theory,

$$\ddot{\phi}(y,t) = -(m^2 - \nabla_y^2)\phi(y,t) + 3\lambda\phi^2(y,t) + 3\lambda G^{0,2}(y,y,t) \tag{3.10}$$

In order for $\phi(y,t) = 0$ to be a solution of this equation, we require an additional term (proportional to ϕ) in the Hamiltonian (or equivalently, Lagrangian) which will cancel off the $G^{0,2}(y,y,t)$ up to whichever order we want.

Cancellation of the tadpole term

For ϕ^4 theory,

$$\begin{aligned}\ddot{\phi}(y,t) &= -(m^2 - \nabla_y^2)\phi(y,t) \\ &+ 4\lambda\phi^3(y,t) + 12\lambda\phi(y,t)G^{0,2}(y,y,t) \\ &+ 4\lambda G^{0,3}(y,y,y,t) \end{aligned} \quad (3.9)$$

In this case, $\phi(y,t) = 0$ is easily a solution up to any order since all odd moments (including $G^{0,3}(y_1, y_2, y_3, t)$) are zero up to any order.
For ϕ^3 theory,

$$\begin{aligned}\ddot{\phi}(y,t) &= -(m^2 - \nabla_y^2)\phi(y,t) \\ &+ 3\lambda\phi^2(y,t) + 3\lambda G^{0,2}(y,y,t) \end{aligned} \quad (3.10)$$

In order for $\phi(y,t) = 0$ to be a solution of this equation, we require an additional term (proportional to ϕ) in the Hamiltonian (or equivalently, Lagrangian) which will cancel off the $G^{0,2}(y,y,t)$ up to whichever order we want.

For a particular initial value of the moments, given by

$$G^{0,2}(y,z,0) = \frac{1}{2\pi^3}\int \frac{\mathrm{d}^3k}{2\sqrt{k^2+m^2}} e^{i\vec{k}.(\vec{y}-\vec{z})} \quad \text{and}$$
$$\dot{G}^{0,2}(y,z,0) = 0 \tag{3.13}$$

we reproduce the usual result from QFT, that is,

$$G^{0,2}(y,z,t) = \int \frac{\mathrm{d}^3k}{2(2\pi^3)\sqrt{k^2+m^2}} e^{i\vec{k}.(\vec{y}-\vec{z})} \tag{3.14}$$

The unique factorization of $\omega = \omega_y - \omega_z$ is why the two results (rightly) match up.

- The propagator has been calculated to agree up to one loop order with QFT.

Important lessons and looking ahead

So, why Effective Equations?

- Using these canonical techniques for effective action, we recover the usual QFT results and also extend them, for instance, by including more general states.
- There is well defined systematic way to derive the higher derivative corrections while avoiding some technical difficulties.

Where are these useful?

- Currently being applied to certain models of isotropic, homogeneous cosmology and also to a de Sitter background.
- Current work is underway to include (perturbative) quantum corrections in the Scalar and Diffeomorphism constraints of spherical LQG, and see what effects they have on the hyperspace deformation algebra. In the high curvature regime, these might be of the same order as that of other non-perturbative corrections (like holonomy corrections), and hence they should be included for a full analysis.

Born-Oppenheimer decomposition for non-commuting slow variables

Alexander Stottmeister

(work with T. Thiemann (forthcoming))

Institute for Theoretical Physics III
FAU Erlangen-Nuremberg

July 25, 2013

Perimeter Institute for Theoretical Physics

Waterloo, Canada

EMERGING FIELDS INITIATIVE

ERLANGEN CENTRE FOR ASTROPARTICLE PHYSICS

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
NATURWISSENSCHAFTLICHE FAKULTÄT

Outline

The Born-Oppenheimer decomposition in canonical LQG

The Born-Oppenheimer decomposition has a long tradition in quantum gravity (cf. [Kiefer, 2004]). It consist in a splitting of gravity-matter system $\mathfrak{H}$ in slow $\mathfrak{H}_S$ and fast $\mathfrak{H}_F$ sectors.

$$\mathfrak{H} = \mathfrak{H}_S \otimes \mathfrak{H}_F \quad (1)$$

Approximation schemes in this setting are governed by one or more adiabatic scales, e.g. $\frac{m_\Phi}{m_P}$.

Its direct application to loop quantum gravity is prevented by the noncommutativity of the fluxes, if one intends to work in a representation admitting a parametrisation by classical metrics (q_{ab}) in the fast sector, e.g. quantum fields on classical backgrounds (cf. [Giesel, Tambornino, Thiemann, 2009]).

It is important to note, that the Born-Oppenheimer decomposition is not a semiclassical approximation scheme per se (cf. [Teufel, 2003]), but we will combine it with coherent state methods (cf. [Faure & Zhilinskii, 2001], in the context of spin systems) enabling us to consider the semiclassical $(\hbar \to 0)$ and the adiabatic limit simultaneously. Furthermore, these methods allow us to surpass the difficulties imposed by the noncommutativity to some extent.

Remarks

1. It is interesting to note that the passage from a theory of quantum gravity to quantum field theory on curved spacetimes can be interpreted as a measurement problem in disguise.

   Recently, the possible need for a simultaneous consideration of the adiabatic and semiclassical limit w.r.t. the measurement problem has be pointed out (cf. [Landsman & Reuvers, 2013]).
2. The application of the framework to deparametrised models seems to be especially interesting (cf. [Giesel & Thiemann, 2012] for an overview).
3. From a mathematical point of view intriguing connections with complex geometry, microlocal analysis and pseudo-differential calculus arise (cf. [Teufel, 2003, Hörmander, 1983, Hörmander, 1985]).

The (generalised) Born-Oppenheimer decomposition

(cf. [Chruściński & Jamiolkowski, 2004, Kiefer, 2004])

Let us shortly the discuss the setup of the traditional Born-Oppenheimer scheme:
Consider a quantum mechanical system described by a triple

$$H\ ,\ \mathfrak{H},\ \mathfrak{A} = \{Q, P, q, p\}\,. \tag{2}$$

Assume a splitting

$$\mathfrak{H} = \mathfrak{H}_S \otimes \mathfrak{H}_F,\ \mathfrak{A} = \mathfrak{A}_S \otimes \mathfrak{A}_F \tag{3}$$

and a decomposition

$$H = H_S(P, Q) \otimes \mathbb{1}_F + H_{S\otimes F}(Q, p, q). \tag{4}$$

If the subset $\{Q\} \subset \mathfrak{A}_S$ is a commutative subalgebra ($\cong C(\sigma(Q))$), one considers the restriction of $H_{S\otimes F}$ to its joint spectral subspaces $\mathfrak{H}_Q \cong \mathfrak{H}_F$

$$H_{S\otimes F|\mathfrak{H}_Q} : \mathfrak{H}_Q \longrightarrow \mathfrak{H}_Q. \tag{5}$$

The eigenvalue problem of H is parametrised by the eigenvalue problems of the restrictions $H_{S\otimes F|\mathfrak{H}_Q}$.

$$(\lambda_\Psi - E_{n^F}(Q))\,\Psi_{n^F}(Q) = \left(H_S(P + A^F(Q), Q) \circ \Psi\right)_{n^F}(Q) \quad (BOE) \tag{6}$$

The (generalised) Born-Oppenheimer decomposition

(cf. [Chruściński & Jamiolkowski, 2004, Kiefer, 2004])

Let us shortly the discuss the setup of the traditional Born-Oppenheimer scheme:

Consider a quantum mechanical system described by a triple

$$H\ ,\ \mathfrak{H},\ \mathfrak{A} = \{Q, P, q, p\}\,. \tag{2}$$

Assume a splitting

$$\mathfrak{H} = \mathfrak{H}_S \otimes \mathfrak{H}_F,\ \mathfrak{A} = \mathfrak{A}_S \otimes \mathfrak{A}_F \tag{3}$$

and a decomposition

$$H = H_S(P, Q) \otimes \mathbb{1}_F + H_{S\otimes F}(Q, p, q). \tag{4}$$

If the subset $\{Q\} \subset \mathfrak{A}_S$ is a commutative subalgebra ($\cong C(\sigma(Q))$), one considers the restriction of $H_{S\otimes F}$ to its joint spectral subspaces $\mathfrak{H}_Q \cong \mathfrak{H}_F$

$$H_{S\otimes F|\mathfrak{H}_Q} : \mathfrak{H}_Q \longrightarrow \mathfrak{H}_Q. \tag{5}$$

The eigenvalue problem of H is parametrised by the eigenvalue problems of the restrictions $H_{S\otimes F|\mathfrak{H}_Q}$.

$$(\lambda_\Psi - E_{n^F}(Q))\,\Psi_{n^F}(Q) = \left(H_S(P + A^F(Q), Q) \circ \Psi\right)_{n^F}(Q) \quad (BOE) \tag{6}$$

The (generalised) Born-Oppenheimer decomposition

(cf. [Chruściński & Jamiolkowski, 2004, Kiefer, 2004])

Let us shortly the discuss the setup of the traditional Born-Oppenheimer scheme:
Consider a quantum mechanical system described by a triple

$$H\ ,\ \mathfrak{H},\ \mathfrak{A} = \{Q, P, q, p\}\,. \tag{2}$$

Assume a splitting

$$\mathfrak{H} = \mathfrak{H}_S \otimes \mathfrak{H}_F,\ \mathfrak{A} = \mathfrak{A}_S \otimes \mathfrak{A}_F \tag{3}$$

and a decomposition

$$H = H_S(P, Q) \otimes \mathbb{1}_F + H_{S\otimes F}(Q, p, q). \tag{4}$$

If the subset $\{Q\} \subset \mathfrak{A}_S$ is a commutative subalgebra ($\cong C(\sigma(Q))$), one considers the restriction of $H_{S\otimes F}$ to its joint spectral subspaces $\mathfrak{H}_Q \cong \mathfrak{H}_F$

$$H_{S\otimes F|\mathfrak{H}_Q} : \mathfrak{H}_Q \longrightarrow \mathfrak{H}_Q. \tag{5}$$

The eigenvalue problem of H is parametrised by the eigenvalue problems of the restrictions $H_{S\otimes F|\mathfrak{H}_Q}$.

$$(\lambda_\Psi - E_{n^F}(Q))\,\Psi_{n^F}(Q) = \left(H_S(P + A^F(Q), Q) \circ \Psi\right)_{n^F}(Q) \quad (BOE) \tag{6}$$

Remarks

1. The object $A^F = A^F(Q)$ is a (generalised) *Berry-Simon connection*.
2. The Born-Oppenheimer equation (6) is exact, but there are several approximation schemes, e.g. the Born-Oppenheimer approximation ($A^F = 0$) or the adiabatic or no-mixing approximation (A^F preserves the eigenspaces of (5)).
3. Effective Hamiltonians for the slow variables arise, if we restrict the slow dynamics to spectral subspaces of $H_{S\otimes F|\mathfrak{H}_Q}$, i.e.

   $$H_{E^F}(P,Q)_{nm} := < n^F|H|m^F > . \tag{7}$$

   In the adiabatic approximation, we are led to the quantum geometric forces in $\sigma(Q)$.
4. Additional complications arise due to spectral instabilities, e.g. eigenvalue crossings or bifurcations.
5. The quality of the approximations is controlled by spectral gap conditions (cf. [Teufel, 2003]).

Remarks

1. The object $A^F = A^F(Q)$ is a (generalised) *Berry-Simon connection*.
2. The Born-Oppenheimer equation (6) is exact, but there are several approximation schemes, e.g. the Born-Oppenheimer approximation ($A^F = 0$) or the adiabatic or no-mixing approximation (A^F preserves the eigenspaces of (5)).
3. Effective Hamiltonians for the slow variables arise, if we restrict the slow dynamics to spectral subspaces of $H_{S\otimes F|\mathfrak{H}_Q}$, i.e.

$$H_{E^F}(P,Q)_{nm} := < n^F|H|m^F > . \tag{7}$$

   In the adiabatic approximation, we are led to the quantum geometric forces in $\sigma(Q)$.
4. Additional complications arise due to spectral instabilities, e.g. eigenvalue crossings or bifurcations.
5. The quality of the approximations is controlled by spectral gap conditions (cf. [Teufel, 2003]).

The coherent state form of the Born-Oppenheimer decomposition I

Again, we consider a quantum mechanical system given in terms of a triple

$$H,\ \mathfrak{H},\ \mathfrak{A} = \{A, A^*, q, p\}, \tag{8}$$

a splitting

$$\mathfrak{H} = \mathfrak{H}_S \otimes \mathfrak{H}_F,\ \mathfrak{A} = \mathfrak{A}_S \otimes \mathfrak{A}_F \tag{9}$$

and a decomposition

$$H = H_S(A, A^*) \otimes \mathbb{1}_F + H_{S\otimes F}(A, A^*, q, p). \tag{10}$$

Furthermore, we require the existence of a complete set of coherent states in the slow sector, but there are NO commutativity assumptions

$$A|z\rangle = z|z\rangle,\ \mathbb{1}_S = \int_{\Gamma^S_{\mathbb{C}}} d\mu(z,\bar{z})|z\rangle\langle z|,\ \langle z|O_S|z\rangle = 0 \Leftrightarrow O_S = 0. \tag{11}$$

This allows us to obtain a diagonal form for $H_{S\otimes F}$ in $\mathfrak{H}_S$

$$H_{S\otimes F} = \int_{\Gamma^S_{\mathbb{C}}} d\mu(z,\bar{z}) P_{H_{S\otimes F}}(z,\bar{z})|z\rangle\langle z|. \tag{12}$$

The coherent state form of the Born-Oppenheimer decomposition I

Again, we consider a quantum mechanical system given in terms of a triple

$$H,\ \mathfrak{H},\ \mathfrak{A} = \{A, A^*, q, p\}, \tag{8}$$

a splitting

$$\mathfrak{H} = \mathfrak{H}_S \otimes \mathfrak{H}_F,\ \mathfrak{A} = \mathfrak{A}_S \otimes \mathfrak{A}_F \tag{9}$$

and a decomposition

$$H = H_S(A, A^*) \otimes \mathbb{1}_F + H_{S\otimes F}(A, A^*, q, p). \tag{10}$$

Furthermore, we require the existence of a complete set of coherent states in the slow sector, but there are NO commutativity assumptions

$$A|z>= z|z>,\ \mathbb{1}_S = \int_{\Gamma^S_{\mathbb{C}}} d\mu(z,\bar{z})|z><z|,\ <z|O_S|z>= 0 \Leftrightarrow O_S = 0. \tag{11}$$

This allows us to obtain a diagonal form for $H_{S\otimes F}$ in $\mathfrak{H}_S$

$$H_{S\otimes F} = \int_{\Gamma^S_{\mathbb{C}}} d\mu(z,\bar{z}) P_{H_{S\otimes F}}(z,\bar{z})|z><z|. \tag{12}$$

The coherent state form of the Born-Oppenheimer decomposition I

Again, we consider a quantum mechanical system given in terms of a triple

$$H,\ \mathfrak{H},\ \mathfrak{A} = \{A, A^*, q, p\}, \tag{8}$$

a splitting

$$\mathfrak{H} = \mathfrak{H}_S \otimes \mathfrak{H}_F,\ \mathfrak{A} = \mathfrak{A}_S \otimes \mathfrak{A}_F \tag{9}$$

and a decomposition

$$H = H_S(A, A^*) \otimes \mathbb{1}_F + H_{S\otimes F}(A, A^*, q, p). \tag{10}$$

Furthermore, we require the existence of a complete set of coherent states in the slow sector, but there are NO commutativity assumptions

$$A|z>= z|z>,\ \mathbb{1}_S = \int_{\Gamma^S_{\mathbb{C}}} d\mu(z,\bar{z})|z><z|,\ <z|O_S|z>= 0 \Leftrightarrow O_S = 0. \tag{11}$$

This allows us to obtain a diagonal form for $H_{S\otimes F}$ in $\mathfrak{H}_S$

$$H_{S\otimes F} = \int_{\Gamma^S_{\mathbb{C}}} d\mu(z,\bar{z}) P_{H_{S\otimes F}}(z,\bar{z})|z><z|. \tag{12}$$

The coherent state form of the Born-Oppenheimer decomposition I

Again, we consider a quantum mechanical system given in terms of a triple

$$H,\ \mathfrak{H},\ \mathfrak{A} = \{A, A^*, q, p\}, \tag{8}$$

a splitting

$$\mathfrak{H} = \mathfrak{H}_S \otimes \mathfrak{H}_F,\ \mathfrak{A} = \mathfrak{A}_S \otimes \mathfrak{A}_F \tag{9}$$

and a decomposition

$$H = H_S(A, A^*) \otimes \mathbb{1}_F + H_{S\otimes F}(A, A^*, q, p). \tag{10}$$

Furthermore, we require the existence of a complete set of coherent states in the slow sector, but there are NO commutativity assumptions

$$A|z>= z|z>,\ \mathbb{1}_S = \int_{\Gamma^S_{\mathbb{C}}} d\mu(z,\bar{z})|z><z|,\ <z|O_S|z>= 0 \Leftrightarrow O_S = 0. \tag{11}$$

This allows us to obtain a diagonal form for $H_{S\otimes F}$ in $\mathfrak{H}_S$

$$H_{S\otimes F} = \int_{\Gamma^S_{\mathbb{C}}} d\mu(z,\bar{z}) P_{H_{S\otimes F}}(z,\bar{z})|z><z|. \tag{12}$$

The coherent state form of the Born-Oppenheimer decomposition II

In the following, we will use the spectral decomposition of the upper symbol

$$P_{H_{S\otimes F}}(z,\bar{z}) : \mathfrak{H}_F(z,\bar{z}) \longrightarrow \mathfrak{H}_F(z,\bar{z}) \tag{13}$$

as input for the eigenvalue problem of H.

$$(K \circ U((\lambda_\Psi - E_{n^F}))\Psi)_{n^F}(z,\bar{z}) = \left(H_S(z,\bar{z},\partial^{A^F},\bar{\partial}^{A^F})) \circ \Psi\right)_{n^F}(z,\bar{z}) \quad (CSBOE) \tag{14}$$

The resolution of unity of the coherent states system identifies a fibre bundle structure

$$\mathfrak{H}_S \otimes \mathfrak{H}_F \cong \mathcal{H}L^2\left(\Gamma^S_{\mathbb{C}}, d\mu, \mathfrak{H}_F\right) \tag{15}$$

and typically leads to additional flatness constraints on the solutions

$$\partial^{A^F}(\rho\Psi)_{n^F}(z,\bar{z}) = 0. \tag{16}$$

Remarks

1. The upper symbol is generically not unique. It can be determined from the lower symbol $< z|H_{S\otimes F}|z >$ by duality (non trivial), and exists for a (strongly) dense set of operators (cf. [Simon, 1980, Klauder & Skagerstam, 1985]).
2. The solutions to the eigenvalue equation of H can be considered as holomorphic, horizontal sections of a $\mathfrak{H}_F$-bundle on the complexified (slow) phase space $\Gamma^S_{\mathbb{C}}$.
3. The main difference of this approach arises through the integral operator $K \circ U$, which is composed of the coherent states kernel K and the bundle transition operator U.
4. Similar to the traditional approach, we obtain a (generalised) *Berry-Simon connection* $A^F = A^F(z, \bar{z})$.
5. Completeness can be conveniently achieved in complexifier framework exploiting holomorphicity (cf. [Thiemann, 2006]). The associated covariant differential can be interpreted as covariant Dolbeault operator.
6. Approximation schemes similar to those of the traditional Born-Oppenheimer approach are conceivable in the semiclassical limit, but require a detailed asymptotic expansion of $K \circ U$ and the upper symbol $P_{H_{S\otimes F}} = P_{H_{S\otimes F}}(z, \bar{z})$ (heat kernel analysis, microlocal analysis).
   Essentially the same is required for effective Hamiltonians, spectral stability and gap conditions.

Remarks

1. The upper symbol is generically not unique. It can be determined from the lower symbol $<z|H_{S\otimes F}|z>$ by duality (non trivial), and exists for a (strongly) dense set of operators (cf. [Simon, 1980, Klauder & Skagerstam, 1985]).
2. The solutions to the eigenvalue equation of H can be considered as holomorphic, horizontal sections of a $\mathfrak{H}_F$-bundle on the complexified (slow) phase space $\Gamma^S_{\mathbb{C}}$.
3. The main difference of this approach arises through the integral operator $K \circ U$, which is composed of the coherent states kernel K and the bundle transition operator U.
4. Similar to the traditional approach, we obtain a (generalised) *Berry-Simon connection* $A^F = A^F(z, \bar{z})$.
5. Completeness can be conveniently achieved in complexifier framework exploiting holomorphicity (cf. [Thiemann, 2006]). The associated covariant differential can be interpreted as covariant Dolbeault operator.
6. Approximation schemes similar to those of the traditional Born-Oppenheimer approach are conceivable in the semiclassical limit, but require a detailed asymptotic expansion of $K \circ U$ and the upper symbol $P_{H_{S\otimes F}} = P_{H_{S\otimes F}}(z, \bar{z})$ (heat kernel analysis, microlocal analysis).
   Essentially the same is required for effective Hamiltonians, spectral stability and gap conditions.

Remarks

1. The upper symbol is generically not unique. It can be determined from the lower symbol $<z|H_{S\otimes F}|z>$ by duality (non trivial), and exists for a (strongly) dense set of operators (cf. [Simon, 1980, Klauder & Skagerstam, 1985]).
2. The solutions to the eigenvalue equation of H can be considered as holomorphic, horizontal sections of a $\mathfrak{H}_F$-bundle on the complexified (slow) phase space $\Gamma^S_{\mathbb{C}}$.
3. The main difference of this approach arises through the integral operator $K \circ U$, which is composed of the coherent states kernel K and the bundle transition operator U.
4. Similar to the traditional approach, we obtain a (generalised) *Berry-Simon connection* $A^F = A^F(z, \bar{z})$.
5. Completeness can be conveniently achieved in complexifier framework exploiting holomorphicity (cf. [Thiemann, 2006]). The associated covariant differential can be interpreted as covariant Dolbeault operator.
6. Approximation schemes similar to those of the traditional Born-Oppenheimer approach are conceivable in the semiclassical limit, but require a detailed asymptotic expansion of $K \circ U$ and the upper symbol $P_{H_{S\otimes F}} = P_{H_{S\otimes F}}(z, \bar{z})$ (heat kernel analysis, microlocal analysis).
   Essentially the same is required for effective Hamiltonians, spectral stability and gap conditions.

$T^*U(1)$ coherent states I

(cf. [Hall, 1994, Varadarajan, 2000, Thiemann & Winkler, 2001a, Hall & Mitchell, 2002], [Kowalski, Rembielinski, Papaloucas, 1996])

If we consider theories with phase space $T^*U(1) \cong U(1)_{\mathbb{C}} \cong \mathbb{C}^*$ for the slow sector, the (slow) Hilbert space will be given by

$$\mathfrak{H}_S = L^2(\mathbb{R}_{\text{Bohr}}, d\mu_H) \cong \bigoplus_{\theta \in S^1} \mathfrak{H}_\theta. \quad (17)$$

Two sets of elementary operators are important for the construction

$$\{U, J \mid J^* = J, U^* = U^{-1}, [J, U] = U\}, \ \{X, X^* \mid X = e^{-\frac{1}{2}J^2} U e^{\frac{1}{2}J^2} = U e^{-J-\frac{1}{2}}\}. \quad (18)$$

The coherent state system is constructed by heat kernel methods and adapted to the decomposition of $\mathfrak{H}_S$:

$$X|\xi>_{j_0} = \xi|\xi>_{j_0}, \ j_0 = \frac{\theta}{2\pi}, \ |\xi>_{j_0} = c_0^{j_0}(\xi) \sum_{j \in \mathbb{Z}} e^{-\frac{1}{2}j^2} (e^{j_0}\xi)^{-j} |j>, \quad (19)$$

$$\mathbb{1}_{\mathfrak{H}_\theta} = \int_{\mathbb{C}^*} \frac{d\xi \wedge d\bar{\xi}}{4\pi^{\frac{3}{2}} i |\xi|^2} e^{-(\ln|\xi| + j_0)^2} |\xi>_{j_0} \, {}_{j_0}\!<\xi|, \ \xi = e^{-l+i\varphi}.$$

$T^*SU(2)$ coherent states I

(cf. [Hall, 1994, Thiemann, 2001, Thiemann & Winkler, 2001a], [Bianchi, Magliaro, Perini, 2010])

Theories admitting a phase space $T^*SU(2) \cong SU(2)_{\mathbb{C}} \cong SL_2(\mathbb{C})$ for the slow sector, will be formulated on the (slow) Hilbert space

$$\mathfrak{H}_S = L^2(SU(2), d\mu_H). \tag{23}$$

Similar to the $U(1)$-case, we need to sets of elementary operators

$$\{h, \vec{p} \mid [h,h] = 0, [\vec{p}, h] = -\frac{1}{2}\vec{\sigma}h, \vec{p}\times\vec{p} = i\vec{p}\},\ \{A, A^* \mid A = e^{\frac{1}{2}\vec{p}^2} h e^{-\frac{1}{2}\vec{p}^2} = e^{-\frac{1}{2}(\vec{p}\cdot\vec{\sigma}-\frac{3}{4})}h\} \tag{24}$$

Again, the coherent state system is constructed by heat kernel methods:

$$A|g>= g|g>,\ |g>= \sum_{j\in\frac{1}{2}\mathbb{N}_0} (2j+1)\, e^{-\frac{1}{2}j(j+1)} \sum_{m,n=-j}^{j} \pi_j(g)_{mn}|jmn> \tag{25}$$

$$\mathbb{1}_{H_S} = \int_{SL_2(\mathbb{C})} d\mu(g,\bar{g})\, |g><g|,\ g = e^{-\vec{l}\cdot\vec{\sigma}}h,\ h\in SU(2).$$

Applications in quantum gravity

1. In a first attempt, the method has been applied to WdW theory, as it makes computations simpler.

   A specific model that was considered is FLRW-cosmology coupled to a scalar field leading to the reduced action

$$S_{\text{red}} = \int_{\mathbb{R}} \Bigg[p_a \dot{a} + \int_{\Sigma} d^3x \; p_\Phi \dot{\Phi} - N \Bigg\{ \Bigg[-\frac{\kappa'}{4a} p_a^2 + \frac{1}{\kappa'} \left\{ \frac{\Lambda}{3} a^3 - a \right\} \Bigg] - \Bigg[\int_{\Sigma} d^3x \left(\frac{\lambda'}{2\sqrt{a^3 \tilde{q}}} p_\Phi^2 + \frac{a^3}{\lambda'} \sqrt{\tilde{q}} \left\{ \frac{1}{2a^2} \tilde{q}^{ab} (\tilde{D}_a \Phi)(\tilde{D}_b \Phi) + V(\Phi) \right\} \right) \Bigg] \Bigg\} \Bigg], \tag{29}$$

which leads to the following CSBOE

$$\lambda_\Psi \Psi_{\{n_k\}}(z, \bar{z}) = H_G(\hat{z}, \hat{z}^*) \Psi_{\{n_k\}}(z, \bar{z}) + \int_{\mathbb{C}} \frac{d^2 z}{\pi} E_{\{n_k\}}(z, \bar{z}) < z|z > \Psi_{\{n_k\}}(z, \bar{z}), \tag{30}$$

$$E_{\{n_k\}}(z, \bar{z}) = \frac{a(z, \bar{z})^3}{2\lambda'} e^{-\frac{9}{2}\hbar} \sum\!\!\!\!\!\!\!\int d\mu(k)(k^2 + m^2 a(z, \bar{z})^2 e^{-\frac{16}{2}\hbar}) n_k.$$

Applications to LQG, LQC & WdW

2. The formulas for canonical LQG on a single graph are under consideration (truncated dynamics).

   Earlier work in this respect makes use of the lower symbol (cf. [Sahlmann & Thiemann, 2006a, Sahlmann & Thiemann, 2006b]).

   An extension to infinite graphs along the lines of the infinite tensor product construction is conceivable, but probably needs a revision to make the semiclassical limit feasible ([Thiemann & Winkler, 2001b], [Sahlmann, Thiemann, Winkler, 2001]).
3. The $U(1)$-case applies to LQC as well as linearized gravity ($U(1)^3$) or Maxwell theory in the parametrized field theory framework. (cf. [Ashtekar & Singh, 2011, Varadarajan, 2000])
4. It would be interesting to consider the gauge invariant sector, but it is not strictly necessary, since the coherent state mainly serve as a technical tool in the formulation of the method [Bahr & Thiemann, 2009].

   Although, it could be advantageous to go to the gauge invariant sector in the approximations schemes.
5. The volume operator in relation to coherent states will be crucial in this approach [Flori, 2009, Flori & Thiemann, 2008], as it enters in all vacuum & matter coupling Hamiltonians in an essential way.

Outlook & Final Remarks

- The asymptotic analysis of $K \circ U$ and the upper symbol $P_{H_{S\otimes F}} = P_{H_{S\otimes F}}(z, \bar{z})$ is technically conceivable, but more involved than in the original BOE.
- The proposed method applies to general Hamiltonian systems that admit a fast-slow decomposition.

  Especially, in LQG it applies to totally constrained as well as deparametrised models.
- The (nonlinear) Fock space structures of the $T^*U(1)$ & $T^*SU(2)$ theories differ from those discussed in the (mathematical) literature (cf. [Hall, 2001]). Nevertheless, these constructions generalise to the case of Lie groups of compact type K, as well.
- Similarly, the inversion formulas developed for $T^*U(1)$ & $T^*SU(2)$ generalise to arbitrary T^*K, but the determination of the upper symbol of the ground state projection $P_{|0>}$ is non trivial.

Outlook & Final Remarks

- The asymptotic analysis of $K \circ U$ and the upper symbol $P_{H_{S\otimes F}} = P_{H_{S\otimes F}}(z,\bar{z})$ is technically conceivable, but more involved than in the original BOE.
- The proposed method applies to general Hamiltonian systems that admit a fast-slow decomposition.

  Especially, in LQG it applies to totally constrained as well as deparametrised models.
- The (nonlinear) Fock space structures of the $T^*U(1)$ & $T^*SU(2)$ theories differ from those discussed in the (mathematical) literature (cf. [Hall, 2001]). Nevertheless, these constructions generalise to the case of Lie groups of compact type K, as well.
- Similarly, the inversion formulas developed for $T^*U(1)$ & $T^*SU(2)$ generalise to arbitrary T^*K, but the determination of the upper symbol of the ground state projection $P_{|0>}$ is non trivial.

The SL(2,R) totally constrained model within the Uniform Discretizations quantization approach

Javier Olmedo

Instituto de Física, Facultad de Ciencias (UDELAR)
in collaboration with R. Gambini

LOOPS13
July 2013

Classical system

1) The system consists in two pairs (u_i, p_i) and (v_i, π_i), with $i = 1, 2$, and three constraints

$$H_1 = \frac{1}{2}(p_1^2 + p_2^2 - v_1^2 - v_2^2), \quad H_2 = \frac{1}{2}(\pi_1^2 + \pi_2^2 - u_1^2 - u_2^2),$$
$$D = u_1 p_1 + u_2 p_2 - v_1 \pi_1 - v_2 \pi_2,$$

$$\{H_1, H_2\} = D, \quad \{H_1, D\} = -2H_1, \quad \{H_2, D\} = 2H_2.$$

2) The constants of motion –$so(2,2)$ Lie algebra–

$$O_{12} = u_1 p_2 - p_1 u_2, \quad O_{23} = u_2 v_1 - p_2 \pi_1, \quad O_{14} = u_1 v_2 - p_1 \pi_2,$$
$$O_{13} = u_1 v_1 - p_1 \pi_1, \quad O_{24} = u_2 v_2 - p_2 \pi_2, \quad O_{34} = \pi_1 v_2 - v_1 \pi_2.$$

Classical system: observables

3) A more convenient choice $-so(2,1)\times so(2,1)$ algebra–

$$Q_1=\frac{1}{2}(O_{23}+O_{14}),\quad Q_2=\frac{1}{2}(-O_{13}+O_{24}),\quad Q_3=\frac{1}{2}(O_{12}-O_{34}),$$
$$P_1=\frac{1}{2}(O_{23}-O_{14}),\quad P_2=\frac{1}{2}(-O_{13}-O_{24}),\quad P_3=\frac{1}{2}(O_{12}+O_{34}).$$
$$\{Q_i,Q_j\}=\epsilon_{ij}{}^kQ_k,\quad \{P_i,P_j\}=\epsilon_{ij}{}^kP_k,\quad \{Q_i,P_j\}=0.$$

4) Identities between observables and constraints

$$Q_1^2+Q_2^2-Q_3^2=P_1^2+P_2^2-P_3^2=\tfrac{1}{4}(D^2+4H_1H_2)=:\mathcal{C},$$
$$4Q_3P_3=(u_1^2+u_2^2)H_1-(u_1p_1+u_2p_2+v_1\pi_1+v_2\pi_2)D-(v_1^2+v_2^2)H_2.$$

5) Solution space: four cones joined in the origin

a) $P_i=0$ and $Q_3\in\mathbb{R}$, with $Q_1^2+Q_2^2=Q_3^2$,
b) $Q_i=0$ and $P_3\in\mathbb{R}$, with $P_1^2+P_2^2=P_3^2$,
c) $Q_i=0$ and $P_i=0$.

Kinematical Hilbert space

1) Kinematical Hilbert space $\mathcal{H}_{\text{kin}} = \mathcal{L}^2(\mathbb{R}^4)$ (and $\hbar = 1$).

2) Operator representation

$$\hat{p}_i\psi(u,v) = -i\partial_{u_i}\psi(u,v), \quad \hat{\pi}_i\psi(u,v) = -i\partial_{v_i}\psi(u,v),$$
$$\hat{u}_i\psi(u,v) = u_i\psi(u,v), \quad \hat{v}_i\psi(u,v) = v_i\psi(u,v).$$

3) Quantum constraints

$$\hat{H}_1 = -\frac{1}{2}(\partial^2_{u_1} + \partial^2_{u_2} + v_1^2 + v_2^2), \quad \hat{H}_2 = -\frac{1}{2}(\partial^2_{v_1} + \partial^2_{v_2} + u_1^2 + u_2^2),$$
$$\hat{D} = -i(u_1\partial_{u_1} + u_2\partial_{u_2} - v_1\partial_{v_1} - v_2\partial_{v_2}).$$

(factor ordering of $\hat{D}$ yields anomaly free constraint algebra)

$$[\hat{H}_1, \hat{H}_2] = i\hat{D}, \quad [\hat{H}_1, \hat{D}] = -2i\hat{H}_1, \quad [\hat{H}_2, \hat{D}] = 2i\hat{H}_2.$$

Uniform discretizations: quantum description

1) Simultaneous diagonalization of $\hat{H}$, $\hat{H}_-$, $\hat{Q}_3$ and $\hat{P}_3$ on $\mathcal{H}_{\text{kin}}$:

a) $k = \sigma_p(\hat{H}_-) \in \mathbb{Z}$, b) $2q_3 = \sigma_p(\hat{Q}_3) \in \mathbb{Z}$, c) $2p_3 = \sigma_p(\hat{P}_3) \in \mathbb{Z}$.

d) The continuous spectrum is

$$\sigma_c(\hat{H}) = \lambda_{\text{cont}} = \tfrac{1}{2} + \tfrac{1}{2}x^2 + k^2 > 0, \quad x \in \big[0, \infty\big).$$

Otherwise, if $k > 0$ and $|q_3 + p_3| - |q_3 - p_3| \geq 2$ or $k < 0$ for $|q_3 + p_3| - |q_3 - p_3| \leq 2$ the discrete counterpart:

$$\sigma_p(\hat{H}) = \lambda_{\text{discr}} = 2t(1-t) + k^2,$$

with $\quad t = 1, 2, \ldots, \frac{1}{2}\min(|k|, ||q_3 + p_3| - |q_3 - p_3||)$ for even k,

and $\quad t = \frac{3}{2}, \frac{5}{2} \ldots, \frac{1}{2}\min(|k|, ||q_3 + p_3| - |q_3 - p_3||)$ for odd k.

Uniform discretizations: modified observable algebra description

1) Let us define

$$\hat{t} = \frac{1}{2}\hat{I} + \sqrt{\frac{1}{4}\hat{I} - \hat{C}_{\text{disc}}}, \quad \hat{\varepsilon}_q := \hat{I} - \delta_{|\hat{Q}_3|,\hat{t}}, \quad \hat{\varepsilon}_p := \hat{I} - \delta_{|\hat{P}_3|,\hat{t}} \,.$$

A more convenient family of observables is $\tilde{Q}_\pm := \hat{\varepsilon}_q \hat{Q}_\pm \hat{\varepsilon}_q$, and $\tilde{P}_\pm := \hat{\varepsilon}_p \hat{P}_\pm \hat{\varepsilon}_p$,

$$\tilde{Q}_\pm |q_3, p_3\rangle_{t,k} = \pm(1 - \delta_{|q_3|,t})(1 - \delta_{|q_3 \pm 1|,t}) \frac{-i}{\sqrt{2}} [q_3 \pm t] \, |q_3 \pm 1, p_3\rangle_{t,k},$$

$$\tilde{P}_\pm |q_3, p_3\rangle_{t,k} = \pm(1 - \delta_{p_3|,t})(1 - \delta_{|p_3 \pm 1|,t}) \frac{-i}{\sqrt{2}} [p_3 \pm t] \, |q_3, p_3 \pm 1\rangle_{t,k},$$

2) The subspaces $\{|q_3 = \pm t, p_3\rangle_{t,k}\}$ and $\{|q_3, p_3 = \pm t\rangle_{t,k}\}$, with $q_3, p_3 \in (-\infty, -t] \cup [t, \infty)$ respectively, remain invariant.

Master constraint programme

1) $2\hat{\mathbf{M}} = \hat{H}_+^2 + \hat{H}_-^2 + \hat{D}^2$ has a minimum nonvanishing eigenvalue in σ_c ($k = 0$ and $\lambda_{\text{cont}} = 1/2$) and the operators $\hat{Q}_3$ and $\hat{P}_3$ are unbounded on that subspace.

2) The prescription in this case

$$\hat{\mathbf{M}}'' = \hat{\mathbf{M}} - \frac{1}{2}\hat{I} + \frac{1}{2}(\hat{Q}_3\hat{P}_3)^2$$

with the (on shell) observables $\hat{Q}'_i := |\text{sgn}(\hat{Q}_3)|\hat{Q}_i|\text{sgn}(\hat{Q}_3)|$ and $\hat{P}'_i := |\text{sgn}(\hat{P}_3)|\hat{P}_i|\text{sgn}(\hat{P}_3)|$.

3) Physical Hilbert space

$$\{|\lambda_{\text{cont}} = 1/2, k = 0, q_3, p_3\rangle\} \quad \text{with} \quad q_3 = 0 \quad \text{or} \quad p_3 = 0$$

Conclusions and outlook

1) Within the uniform discretizations approach (as well as in the MC) we provide a prescription for the quantization of an $SL(2,\mathbb{R})$ model by:
   a) Considering the whole infrared spectrum of the discrete Hamiltonian (discrete dynamics).
   b) Together with a suitable choice of observable algebra.
2) The physical Hilbert space is a subspace of the kinematical one.
3) Study of the discrete (quantum) dynamics (comparison between evolving constants and conditional probabilities).

Conclusions and outlook

1) Within the uniform discretizations approach (as well as in the MC) we provide a prescription for the quantization of an $SL(2,\mathbb{R})$ model by:
   a) Considering the whole infrared spectrum of the discrete Hamiltonian (discrete dynamics).
   b) Together with a suitable choice of observable algebra.
2) The physical Hilbert space is a subspace of the kinematical one.
3) Study of the discrete (quantum) dynamics (comparison between evolving constants and conditional probabilities).

Constraint Lie algebra and true local Hamiltonian for the CGHS model

Saeed Rastgoo

CCM, UNAM, Mexico

July 25, 2013

Why the CGHS model?

The Callan-Giddings-Harvey-Strominger model is 2D dilatonic (Φ) model which (coupled to matter field f) reads

$$S_{\text{CGHS}} = \int d^2x\sqrt{-|g|}\left(\frac{1}{8}\Phi^2 R + \frac{1}{2}g^{ab}\partial_a\Phi\partial_b\Phi + \frac{1}{2}\Phi^2\lambda^2\right) - \int d^2x\sqrt{-|g|}g^{ab}\partial_a f\partial_b f$$

- It contain black hole solutions, FRW cosmological models, Hawking radiation etc.,

What do we address?

We propose a classical formulation of the CGHS model where we have:

- **CGHS in Ashtekar-like variables**: very similar canonical transformation from a generic 2D action $\Longrightarrow$ CGHS/3+1 in Ashtekar-like variables.

What do we address?

We propose a classical formulation of the CGHS model where we have:

- **CGHS in Ashtekar-like variables**: very similar canonical transformation from a generic 2D action $\Longrightarrow$ CGHS/3+1 in Ashtekar-like variables.
- **Possibility of traditional Dirac quantization**: the constraint algebra is a Lie algebra even in presence of matter,

A generic 2D dilatonic Lagrangian

- An "almost" generic diffeomorphism invariant action yielding 2nd order differential equations for the metric g and a scalar (dilaton) field Φ in 2D

$$L_{\rm g} = \sqrt{-|g|}\left\{Y(\Phi)R + \frac{1}{2}g^{ab}\partial_a\Phi\partial_b\Phi + V(\Phi)\right\},$$

$$L_{\rm m} = -\sqrt{-|g|}W(\Phi)g^{ab}\partial_a f\partial_b f.$$

A generic 2D dilatonic Lagrangian

- An "almost" generic diffeomorphism invariant action yielding 2nd order differential equations for the metric g and a scalar (dilaton) field Φ in 2D

$$L_{\mathrm{g}} = \sqrt{-|g|}\left\{Y(\Phi)R + \frac{1}{2}g^{ab}\partial_a\Phi\partial_b\Phi + V(\Phi)\right\}.$$
$$L_{\mathrm{m}} = -\sqrt{-|g|}W(\Phi)g^{ab}\partial_a f\partial_b f.$$

  - $Y(\Phi)$, $V(\Phi)$ and $W(\Phi)$ model specific functions of the dilaton field.
  - Contains CGHS (Φ=dilaton field), 3+1 spherically symmetric ($ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \Phi^2(d\theta^2 + \sin^2(\theta)d\phi^2)$), etc.

The CGHS Hamiltonian in new variables

Write the theory in tetrads

The CGHS Hamiltonian in new variables

Write the theory in tetrads

⇓

Make a Legendre transformation to a generic Hamiltonian

⇓

Use specific $Y(\Phi)$, $V(\Phi)$ and $W(\Phi)$

⇓

Make a canonical transformation to new variables for CGHS:

$$P_1 = 2\cosh(\eta)E^{\varphi}, \qquad P_2 = 2\sinh(\eta)E^{\varphi}, \qquad P_\omega = E^x,$$

The CGHS Hamiltonian in new variables

Write the theory in tetrads

$\Downarrow$

Make a Legendre transformation to a generic Hamiltonian

$\Downarrow$

Use specific $Y(\Phi)$, $V(\Phi)$ and $W(\Phi)$

$\Downarrow$

Make a canonical transformation to new variables for CGHS:

$$P_1 = 2\cosh(\eta)E^{\varphi}, \qquad P_2 = 2\sinh(\eta)E^{\varphi}, \qquad P_\omega = E^x,$$

$\Downarrow$

A second class Hamiltonian $H(K_x, E^x, K_\varphi, E^\varphi, \Phi, P_\Phi, f, P_f)$
with two second class constraints α and μ: $\{\mu, \alpha\} \not\approx 0$.

The CGHS Hamiltonian in new variables

Write the theory in tetrads

Similar transformations for the 3+1 case

This is much like what we do to get the 3+1 case from generic action:

$$P_1 = 2\frac{\cosh(\eta)}{E^{x\frac{1}{4}}}E^{\varphi}, \qquad P_2 = 2\frac{\sinh(\eta)}{E^{x\frac{1}{4}}}E^{\varphi}, \qquad P_\omega = \frac{E^x}{2}$$

but we get a first class system.

Make a canonical transformation to new variables for CGHS:

$$P_1 = 2\cosh(\eta)E^{\varphi}, \qquad P_2 = 2\sinh(\eta)E^{\varphi}, \qquad P_\omega = E^x,$$

$$\Downarrow$$

A second class Hamiltonian $H(K_x, E^x, K_\varphi, E^\varphi, \Phi, P_\Phi, f, P_f)$
with two second class constraints α and μ: $\{\mu, \alpha\} \not\approx 0$.

Second class procedure

Similar transformations for the 3+1 case

Finally we get a total Hamiltonian density in new variables

$$H = N\mathcal{H} + N^1\mathcal{D}$$

with $\mathcal{H}$ and $\mathcal{D}$ being the Hamiltonian and diffeomorphism constraints and N and N^1 lapse and shift.

Second class procedure

Similar transformations for the 3+1 case

Finally we get a total Hamiltonian density in new variables

$$H = N\mathcal{H} + N^1\mathcal{D}$$

with $\mathcal{H}$ and $\mathcal{D}$ being the Hamiltonian and diffeomorphism constraints and N and N^1 lapse and shift.

Transforming the Hamiltonian constraint

Rescale lapse and shift

$$\overline{N}^1 = N^1 + \frac{NK_\varphi}{E^{x\prime}}, \qquad \overline{N} = N\frac{E^\varphi E^x}{E^{x\prime}}$$

$\Downarrow$

The total Hamiltonian density will become

$$H_T = \overline{N}\Bigg[\overbrace{\partial_x\left(\frac{E^{x\prime 2}}{2E^{\varphi 2}E^x} - 2E^x\lambda^2 - \frac{K_\varphi^2}{2E^x}\right) - \frac{f'P_fK_\varphi}{E^xE^\varphi} + \frac{E^{x\prime}\left(f'^2 + P_f^2\right)}{2E^{\varphi 2}E^x}}^{\mathcal{H}}\Bigg] + \overline{N}^1\Bigg[\underbrace{-U_xE^{x\prime} + f'P_f + E^\varphi K_\varphi'}_{\mathcal{D}}\Bigg]$$

Lie algebra of constraints

Now $\mathcal{H}$ constraint has strongly Abelian algebra with itself in both vacuum and coupled-to-matter cases:

$$\{\mathcal{H}(x), \mathcal{H}(y)\}_D = 0,$$

$$\left\{ \mathcal{H}(x)\Big|_{f=0, P_f=0}, \mathcal{H}(y)\Big|_{f=0, P_f=0} \right\}_D = 0$$

chance of implementing back-reaction?

Lie algebra of constraints

Now $\mathcal{H}$ constraint has strongly Abelian algebra with itself in both vacuum and coupled-to-matter cases:

$$\{\mathcal{H}(x),\mathcal{H}(y)\}_D = 0,$$

$$\left\{\mathcal{H}(x)\Big|_{f=0,P_f=0},\mathcal{H}(y)\Big|_{f=0,P_f=0}\right\}_D = 0$$

chance of implementing back-reaction?

$\Downarrow$

Constraints algebra is now a Lie algebra,

$$\{\mathcal{D},\mathcal{D}\} = \mathcal{D}$$
$$\{\mathcal{D},\mathcal{H}\} = \mathcal{H}$$
$$\{\mathcal{H},\mathcal{H}\} = 0$$

and the same method works for vacuum 3+1 model.

Possibility of resolution of singularity?

CGHS Hamiltonian in this formulation is very similar to the $3+1$ model.

Possibility of resolution of singularity?

CGHS Hamiltonian in this formulation is very similar to the 3+1 model.

$\Downarrow$

Gambini-Pullin method of eliminating the singularity for 3+1 can be used here? (work under development).

The metric becomes an evolving constant operator $\Longrightarrow$ self-adjointness of this operator leads to removal of singularity.

Deparametrizing and gauge fixing

- Deparametrizing the constraints by transformation to new canonical coordinates (Brown, Kuchar, Thiemann, Giesel, Gambini, Pullin, ...).
- **Step 1:** fix the first gauge $\zeta_1 = E^x - h(x) \approx 0$ to get

$$\dot{\zeta}_1 = 0 \Longrightarrow \overline{N}^1 = 0$$

so that

$$H_{PF} = \overline{N}\left[\partial_x\left(\frac{1}{2}\frac{h'^2}{hE^{\varphi 2}} - 2h\lambda^2 - \frac{1}{2}\frac{K_\varphi^2}{h}\right) - \frac{f' P_f K_\varphi}{hE^\varphi} + \frac{1}{2}\frac{P_f^2 h'}{hE^{\varphi 2}} + \frac{1}{2}\frac{h' f'^2}{hE^{\varphi 2}}\right]$$

Deparametrizing and gauge fixing

- Deparametrizing the constraints by transformation to new canonical coordinates (Brown, Kuchar, Thiemann, Giesel, Gambini, Pullin, ...).
- **Step 1:** fix the first gauge $\zeta_1 = E^x - h(x) \approx 0$ to get

$$\dot{\zeta}_1 = 0 \Longrightarrow \overline{N}^1 = 0$$

so that

$$H_{PF} = \overline{N}\left[\partial_x\left(\frac{1}{2}\frac{h'^2}{hE^{\varphi 2}} - 2h\lambda^2 - \frac{1}{2}\frac{K_\varphi^2}{h}\right) - \frac{f'P_f K_\varphi}{hE^\varphi} + \frac{1}{2}\frac{P_f^2 h'}{hE^{\varphi 2}} + \frac{1}{2}\frac{h'f'^2}{hE^{\varphi 2}}\right]$$

Deparametrizing and gauge fixing

- **Step 3:** find the conjugate momentum

$$P_X = -\frac{hh'}{\Omega(K_\varphi, X)\left(K_\varphi + \Omega(K_\varphi, X)\right)}$$

- **Step 4:** From $\mathcal{H} \approx 0$ find $K_\varphi = K_\varphi(X, f, P_f)$. Substituting this in above gives the new total Hamiltonian

$$H_{\text{tot}} = \bar{N}\left(P_X + \frac{hh'}{\Omega(X, f, P_f)\left(K_\varphi(X, f, P_f) + \Omega(X, f, P_f)\right)}\right)$$

The true dynamics

- The true local Hamiltonian gives the correct evolution: since f and P_f commute with P_X, we get

$$\dot{f} = \left\{ f, \int dx H_{\text{tot}} \right\}_D = \left\{ f, \int dx H_{\text{true}} \right\}_D ,$$
$$\dot{P}_f = \left\{ P_f, \int dx H_{\text{tot}} \right\}_D = \left\{ P_f, \int dx H_{\text{true}} \right\}_D ,$$

- It is a local Hamiltonian density: $\bar{N}$ not an integral of canonical variables, the Hamiltonian not an integral of an integral (non-local), eqs. of motion local.

Summary

- The CGHS can be written in a similar Ashtekar-like variables as in the 3+1, from a generic 2D dilatonic Lagrangian. In this formulation:

Summary

- The CGHS can be written in a similar Ashtekar-like variables as in the 3+1, from a generic 2D dilatonic Lagrangian. In this formulation:
- The constraint algebra can be cast into a Lie algebra: possibility of completing the Dirac quantization / implementing back-reaction?
- Similarity to the 3+1 Hamiltonian: possibility to eliminate the singularity a la Gambini and Pullin?