Title: Spinor and Twistor Networks in Loop Gravity

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Abstract: I will review the reformulation of the loop gravity phase space in terms of spinor networks and twistor networks, and present how these techniques can be used to write spinfoam amplitudes as discretized path integrals and to study the dynamics that they define (recursion, Hamiltonian constraints as differential equations).

Spinor and Twistor Networks for Loop Quantum Gravity

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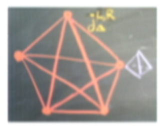
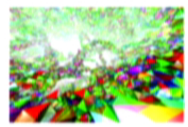
July 2013 in Waterloo @ LOOPS '13

Understanding the Geometry of Loop Quantum Gravity

Part of the programme:

Understand the geometrical interpretation of the quantum states of LQG and of their transition amplitude defined by spinfoam models and parameterize systematically the deformations (diffeos?) of their geometry in order to describe the dyn of the theory


- LQG spin network states define a quantum geometry. . . but can it be effectively interpreted as a classical discrete geometry?
- Spinfoam models are discrete path integrals defining transition amplitudes and projector on physical states, constructed algebraically as state-sums for BF theory plus constraints at the quantum level. . . but a discrete space-time geometry?

The Rise of the Spinor Formalism for Loop Gravity & Spinfoams

A Basic Technical Idea:

Introduce spinor variables as Darboux coordinates for the holonomy-flux phase space of loop gravity on a fixed graph

Many developments:

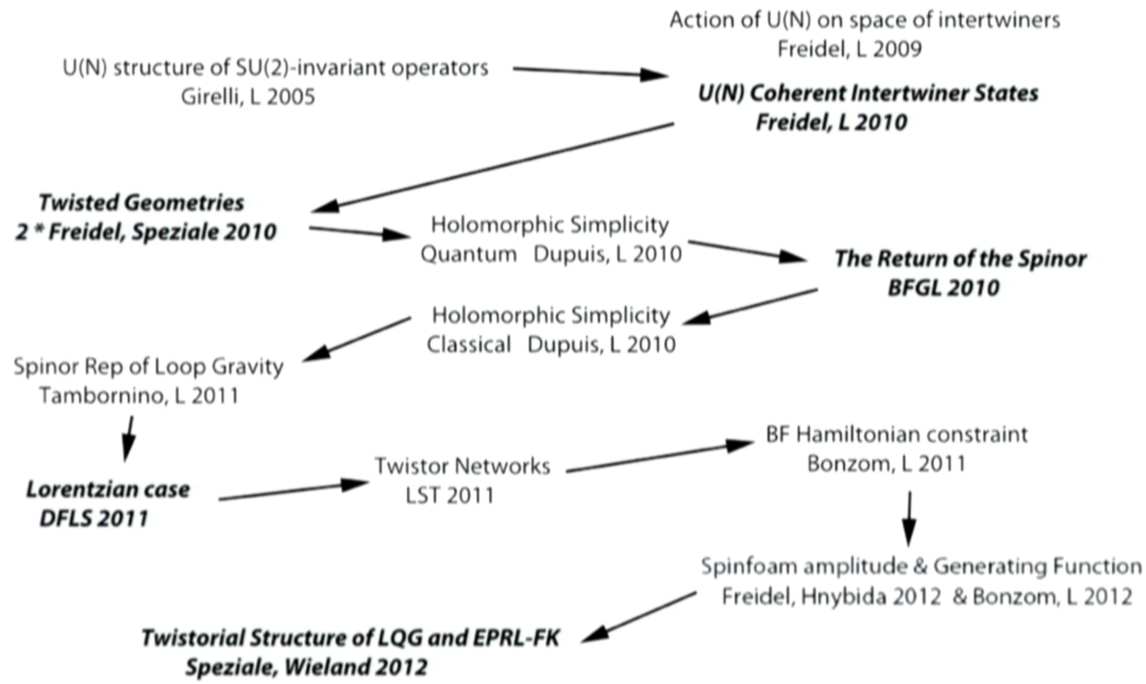
- From spinor networks to twisted geometry
- $\mathrm{U}(N)$ framework and coherent intertwiners, generating functions and asymptotics for 3nj symbols & spinfoam ampl
- Twistor networks for $\mathrm{SL}(2,\mathbb{C})$ and simple twistor networks
- Spinfoam amplitude as path integral in spinor/twistor space
- Extension to quantum groups for loop gravity with $\Lambda \neq 0$

Work by many people

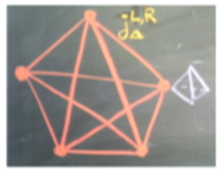
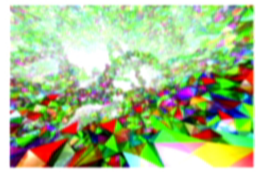
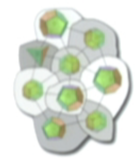
L. Freidel, S. Speziale, V. Bonzom, W. Wieland, M. Dupuis, F. Girelli, J. Tambornino, ...

The Rise of the Spinor Formalism for Loop Gravity & Spinfoams

Historically, we worked completely backwards. . .



Spinor and Twistor Networks for Loop Gravity

- Overview of spinor network formalism

Then more details on:

1. Spinor phase space and Quantization
2. U(N) framework for polyhedra and intertwiners
3. Spinor path integral and Hamiltonian constraints for Spinfoams
4. Twistor networks and Simplicity constraints

Spinor Networks for Loop Gravity

A fresh look on the phase space of loop gravity

Always interesting to study new variables encoding the (quantum) geometry...

- **Canonical Variables** $\{z, \bar{z}\} = -i\delta$ on a fixed graph
  - Simpler Poisson bracket and kinematical structure
  - Straightforward quantization and efficient framework to investigate quantization ambiguities
  - Straightforward construction of coherent spin network states from harmonic oscillators

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Spinor Networks for Loop Gravity

A formalism easy to handle and allowing a detailed analysis of the degrees of freedom...

- **A new perspective on the geometry degrees of freedom**

- A single set of variables: holonomies g and fluxes X both as composite variables in the spinor variables
- Clear geometrical interpretation in terms of framed polyhedra and twisted geometries
- Easy to switch focus between edge d.o.f. and vertex d.o.f. $\hookrightarrow$ two complementary points of view: holonomy formalism for the parallel transport along edges vs. U(N) formalism for polyhedra and intertwiners around vertices
- A framework allowing a simpler exploration and analysis of algebra of constraints and Hamiltonian dynamics

Spinor Networks for Loop Gravity

Beyond the canonical framework, we apply these techniques to spinfoams. . .

- **A mathematical tool for Spinfoam amplitudes**

- Natural boundary data for spinfoam amplitudes computing transition amplitudes between coherent spin network states
- Express spinfoam amplitudes as path integral in spinor space
- Powerful mathematical tool to evaluate amplitudes and understand NC structure
  $\hookrightarrow$ to study $\{3nj\}$ symbols, generating functions and asymptotics
- Hamiltonian constraints implemented as differential equations in the spinors and satisfied by spinfoam amplitudes (3d, 4d BF, FRW cosmo, . . .)

Upgraded to Twistor Networks for Loop Gravity *à la* Spinfoam

Twistors (or Dirac spinors) for representing Lorentz group $\mathrm{SL}(2,\mathbb{C})\dots$

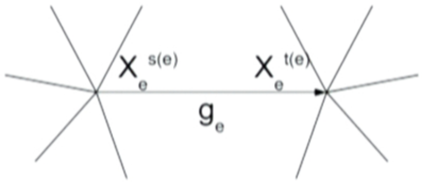
- **Developing a 4d covariant point of view:**
- Clear structure of algebra of constraints, allowing to solve explicitly the simplicity constraints (reducing BF theory to GR)
  $\hookrightarrow$ Holomorphic simplicity constraints & Simple twistor networks
- Equivalence of $\mathrm{SL}(2,\mathbb{C})$ structures and $\mathrm{SU}(2)$ phase space
  $\hookrightarrow$ Non-trivial embedding of $T^*\mathrm{SU}(2)$ in $T^*\mathrm{SL}(2,\mathbb{C})$ with $\mathrm{SU}(2)$ holonomy carrying data about extrinsic curvature
- Explore 4d geometry d.o.f. (time normal, 4d dihedral angles, ...)
- EPRL-FK amplitudes as path integral over twistor space
  $\hookrightarrow$ a good framework to investigate coarse-graining and effective spinfoam actions

Very nice work by Simone and Wolfgang [Speziale, Wieland 2012]

The Holonomy-Flux Poisson algebra on a fixed graph

Let's go back to the standard holonomy-flux variables of loop gravity... On a fixed oriented graph:

- Holonomy along edge e
  $g_e \in \mathrm{SU}(2)$
- Flux for edge e around vertex v
  $X_e^v \in \mathfrak{su}(2) \sim \mathbb{R}^3$



$T^*\mathrm{SU}(2)$ Poisson bracket :

$$\{g_e, g_{e'}\} = 0$$
$$\{X_e^a, X_{e'}^b\} = i\delta_{ee'}\epsilon^{abc}X_e^c$$
$$\{\vec{X}_e, g_e\} = \vec{\sigma} g_e$$

With constraints:

- Parallel transport of vectors by 3d rotations along edges:
  $X_e^{t(e)} = -g_e \triangleright X_e^{s(e)}$
- Closure constraint around vertices: $\sum_{e \ni v} X_e^v = 0$
  generating local $\mathrm{SU}(2)$-inv

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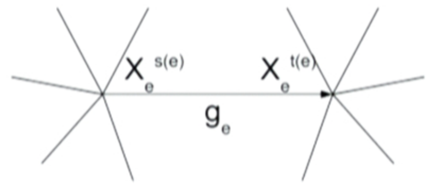

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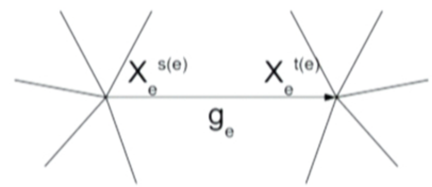

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The Holonomy-Flux Poisson algebra on a fixed graph

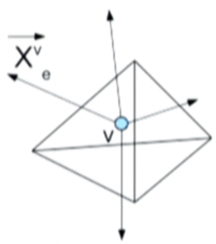
Where does it come from? What does it mean?...

Discretization of connection-triad fields A and E

$$g_e = \mathcal{P}\exp(i\int_e A)$$
$$X^v_{\mathcal{S}} = \int_{p\in\mathcal{S}} g_{v\to p}\, E\, g^{-1}_{v\to p}$$

Geometrical Interpretation as Twisted Geometries

- Closure constraint $\Rightarrow$ convex polyhedra dual to each vertex



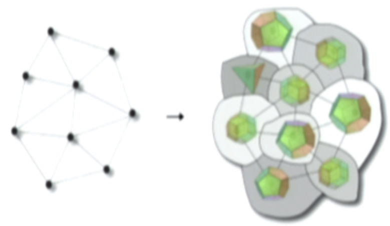
Each edge dual to a face with $\vec{X}_e$ as the normal vector and norm $|\vec{X}_e|$ giving the area of the face

- Polyhedra glued along the edges by area matching

$$\hookrightarrow \exists g_e \in \mathrm{SO}(3),\ \vec{X}^t_e = -g_e \triangleright \vec{X}^s_e \Leftrightarrow |\vec{X}^t_e| = |\vec{X}^s_e|$$

A word on Twisted Geometries

Twisted geometries generalize Regge geometries



On an edge, the vectors $X_e^{s,t}$ do not entirely determine the holonomy g_e:

$$g_e = n_{t(e)}\, e^{i\,\xi_e \frac{\sigma_3}{2}}\, \epsilon\, n_{s(e)}^{-1}$$

Angle $\xi \sim \gamma K$ gives extrinsic curvature

Chunks of flat space (living in tangent space) glued together with curvature and torsion

$\rightsquigarrow$ Can impose gluing constraints to restrict to Regge geometries

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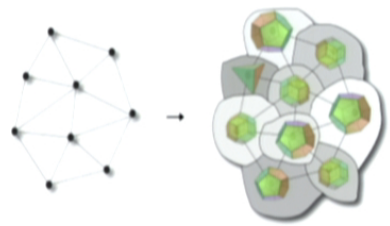

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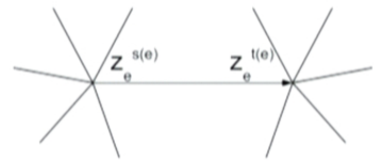
Chunks of flat space (living in tangent space) glued together with curvature and torsion
$\leadsto$ Can impose gluing constraints to restrict to Regge geometries

The Spinor Network phase space

Let us re-visit this phase space replacing vectors $\vec{X}_e^v \in \mathbb{R}^3$ by spinors $|z_e^v\rangle \in \mathbb{C}^2$ ...

We describe twisted geometries as Spinor Networks



- Can reconstruct both g's and X's from spinors :
  - ▶ $\vec{X} = \langle z|\vec{\sigma}|z\rangle$ with Pauli matrices, area as norm $|\vec{X}| = \langle z|z\rangle$
  - ▶ Unique $\mathrm{SU}(2)$ element g_e mapping z_e^s to z_e^t since we are in the fundamental representation of $\mathrm{SU}(2)$

$$g_e = \frac{|z_e^t]\langle z_e^s| - |z_e^t\rangle[z_e^s|}{\sqrt{\langle z_e^s|z_e^s\rangle\langle z_e^t|z_e^t\rangle}}, \quad |z] = \epsilon|\bar{z}\rangle = \begin{pmatrix} -\bar{z}^1 \\ \bar{z}^0 \end{pmatrix}$$

- Darboux coordinates with canonical Poisson bracket $\{z_a, \bar{z}_b\} = -i\delta_{ab}$

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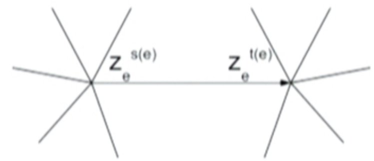

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The Spinor Network phase space

We keep area-matching constraints and closure constraints :

- around vertices: $\sum_{e\ni v} |z_e^v\rangle\langle z_e^v| \propto \mathbb{I}$
  $\Rightarrow$ Generates $\mathrm{SU}(2)$ transformations on spinors
- along edges: $\langle z_e^s|z_e^s\rangle = \langle z_e^t|z_e^t\rangle$
  $\Rightarrow$ Generates inverse $\mathrm{U}(1)$ phase transformations on spinors

New feature: $\mathrm{U}(1)$ phase of spinors . . . New d.o.f.'s?

$\hookrightarrow$ Vectors X invariant under $\mathrm{U}(1)$ transformations

$\hookrightarrow$ Holonomies g inv under joint $\mathrm{U}(1)$ on both spinors

$\hookrightarrow$ Relative phase gives angle ξ, conjugate to area !

Quantizing: from Spinor Networks to Spin Networks

We have canonical Poisson brackets, so quantization is direct ...

1. Raise spinor components to creation/annihilation operators :

$$z^0 \to a^0, \quad z^1 \to a^1, \quad \bar{z}^0 \to a^{0\dagger}, \quad \bar{z}^1 \to a^{1\dagger}$$

2. Gives Schwinger's representation of $\mathfrak{su}(2)$ in terms of two HOs, with spin j being total energy
3. Quantizing matching and closure constraints, we recover the Hilbert space of spin network states as (holomorphic) wave-functions in the spinors
4. We project HOs' coherent states and define coherent spin network peaked on classical spinor networks :

$$\psi_{\{z_e^v\}}(g_e) = \int [dh_v]\, e^{\sum_e [z_e^{s(e)}| h_{s(e)}^{-1}\, g_e\, h_{t(e)} | z_e^{t(e)}\rangle}$$

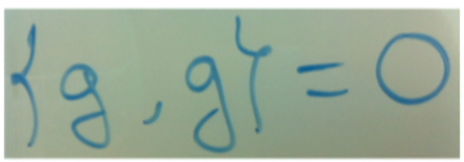
Semi-classical Twisted Geometries for LQG

A remark on quantization ambiguities

Actually there is no quantization ambiguity : use normal ordering with annihilation a's on R and creation $a^\dagger$ on L.
$\Rightarrow$ Quantum commutators reproducing exactly classical Poisson algebra for quadratic observables such as vectors X's and with known corrections for polynomials

But what happens for non-polynomial observables? ...

I.e.g. holonomy on an edge $g = \frac{|z\rangle\langle\tilde{z}| - |z][\tilde{z}|}{\sqrt{\langle z|z\rangle\langle\tilde{z}|\tilde{z}\rangle}}$??



We know the action of $\hat{g}$ on spin basis of $L^2(\mathrm{SU}(2))$: obvious quantization of numerator and norm factor quantized simply as $(2j+1)^{-1}$ with $+1$ shift and split in $\sqrt{\ }$ on left and right.

But could try Thiemann's trick? Generating them from bracket $\{\sqrt{\langle z|z\rangle}, \cdot\}$. But then anomaly $\{g, g\} \neq 0$! Bad bad bad...

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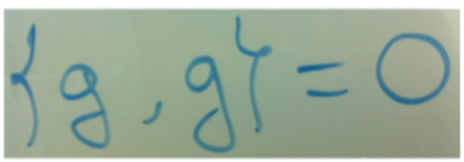

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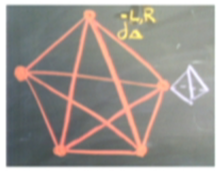

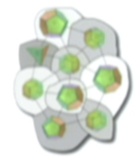

1. Spinor phase space and Quantization
2. U(N) **framework for polyhedra and intertwiners**
3. Spinor path integral and Hamiltonian constraints for Spinfoams
4. Twistor networks and Simplicity constraints

$\mathrm{U}(N)$ Formalism for Polyhedra and Intertwiners

We focus on a single vertex of a spin(or) network

Why? Intertwiners are the basic building blocks of spin networks and represent chunks of 3d volume. Essential to understand their structure and geometrical interpretation.

What? $\mathrm{U}(N)$ transformations deforming and exploring the whole space of polyhedra and intertwiners (with N faces) at fixed boundary area.

More? Basic $\mathrm{SU}(2)$-invariant operators generating all deformations, fixing or not the area.

So. . . ? Count intertwiners (for BHs), Define coherent intertwiners, Decompose holonomies in elementary operators, Average over polyhedra and trace on intertwiners by integral over $\mathrm{U}(N)$, Link with Itzykson-Zuber formula and matrix models.

Framed Polyhedra: Spinor Phase Space for Intertwiners

A simple classical setting:

1. Consider N spinors $z_i \in \mathbb{C}^2$ with canonical bracket $\{z^A, \bar{z}^B\} = -i\,\delta^{AB}$

2. Map them on 3-vectors: $|z\rangle \in \mathbb{C}^2 \longmapsto \vec{V} \equiv \langle z|\vec{\sigma}|z\rangle \in \mathbb{R}^3$.
3. Define closure constraints $\vec{\mathcal{C}} \equiv \sum_i \langle z_i|\vec{\sigma}|z_i\rangle = \sum_i \vec{V}_i$.

- Defines a unique dual (convex) polyhedron with the $\vec{V}_i$ being the normal vectors are the N faces, with area $V_i = \langle z_i|z_i\rangle$
- $\vec{\mathcal{C}}$ are 1st class and generates global $\mathrm{SU}(2)$ transf onspinors

$\mathcal{P}_N = \mathbb{C}^{2N}//\mathrm{SU}(2)$ as space of *framed* polyhedra up to 3d rotations

$\hookrightarrow$ Dim of $\mathcal{P}_N$ is $(3N-6)+N$. We have extra phases on each face!

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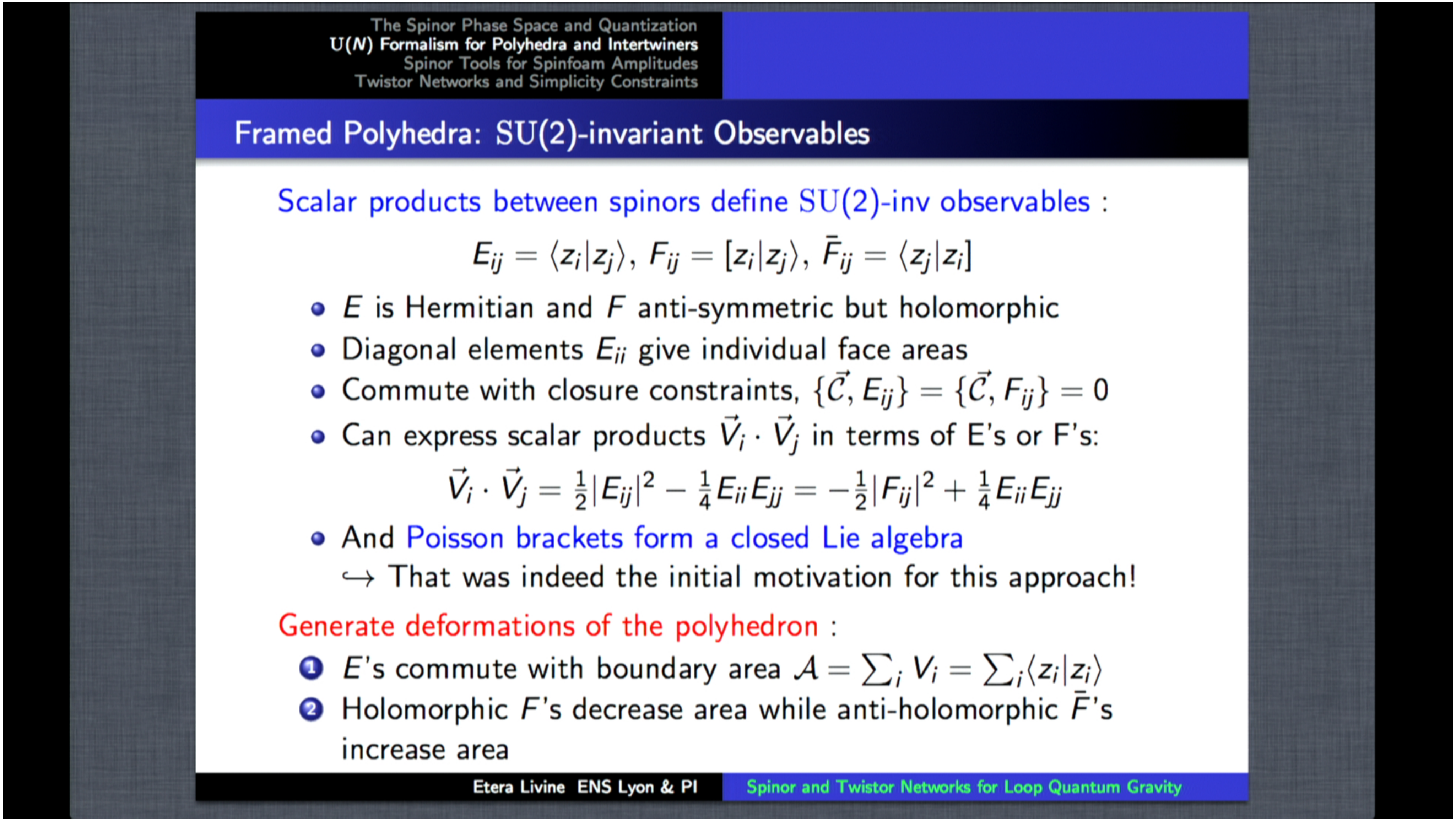
The Spinor Phase Space and Quantization
U(N) Formalism for Polyhedra and Intertwiners
Spinor Tools for Spinfoam Amplitudes
Twistor Networks and Simplicity Constraints
Framed Polyhedra: SU(2)-invariant Observables
Scalar products between spinors define SU(2)-inv observables :
$E_{ij} = \langle z_i | z_j \rangle,\ F_{ij} = [z_i | z_j \rangle,\ \bar{F}_{ij} = \langle z_j | z_i]$
E is Hermitian and F anti-symmetric but holomorphic
Diagonal elements E_{ii} give individual face areas
Commute with closure constraints, $\{\vec{\mathcal{C}}, E_{ij}\} = \{\vec{\mathcal{C}}, F_{ij}\} = 0$
Can express scalar products $\vec{V}_i \cdot \vec{V}_j$ in terms of E's or F's:
$\vec{V}_i \cdot \vec{V}_j = \frac{1}{2}|E_{ij}|^2 - \frac{1}{4}E_{ii}E_{jj} = -\frac{1}{2}|F_{ij}|^2 + \frac{1}{4}E_{ii}E_{jj}$
And Poisson brackets form a closed Lie algebra
↪ That was indeed the initial motivation for this approach!
Generate deformations of the polyhedron :
1 E's commute with boundary area $\mathcal{A} = \sum_i V_i = \sum_i \langle z_i | z_i \rangle$
2 Holomorphic F's decrease area while anti-holomorphic $\bar{F}$'s increase area
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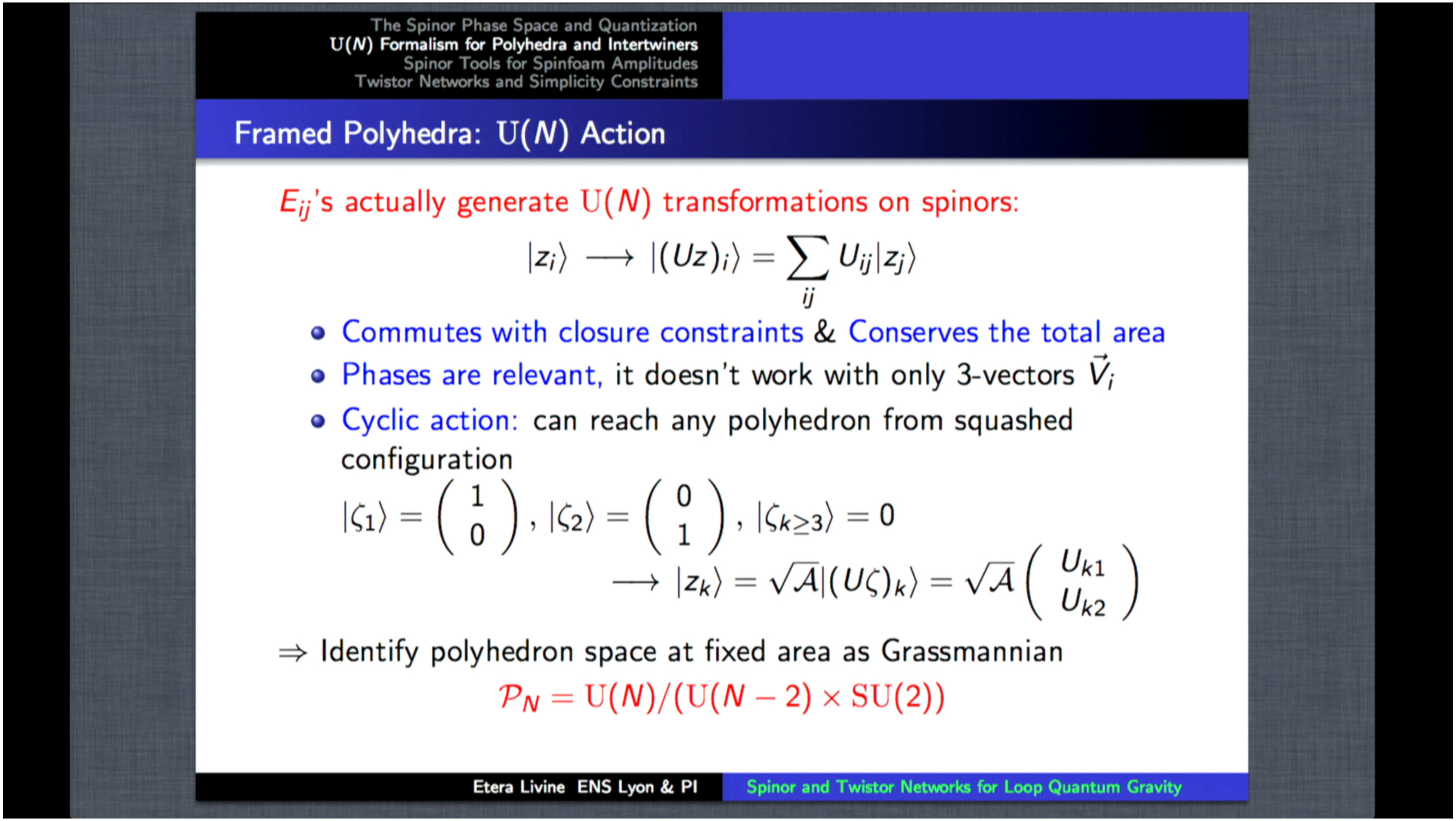
Spinor and Twistor Networks for Loop Quantum Gravity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Framed Polyhedra: U(N) Action
E_{ij}'s actually generate U(N) transformations on spinors:
$|z_i\rangle \longrightarrow |(Uz)_i\rangle = \sum_{ij} U_{ij}|z_j\rangle$
Commutes with closure constraints & Conserves the total area
Phases are relevant, it doesn't work with only 3-vectors $\vec{V}_i$
Cyclic action: can reach any polyhedron from squashed configuration
$|\zeta_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\zeta_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |\zeta_{k\geq 3}\rangle = 0$
$\longrightarrow |z_k\rangle = \sqrt{\mathcal{A}}|(U\zeta)_k\rangle = \sqrt{\mathcal{A}} \begin{pmatrix} U_{k1} \\ U_{k2} \end{pmatrix}$
$\Rightarrow$ Identify polyhedron space at fixed area as Grassmannian
$\mathcal{P}_N = \mathrm{U}(N)/(\mathrm{U}(N-2)\times \mathrm{SU}(2))$
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Framed Polyhedra: U(N) Action

E_{ij}'s actually generate $\mathrm{U}(N)$ transformations on spinors:

$$|z_i\rangle \longrightarrow |(Uz)_i\rangle = \sum_{ij} U_{ij}|z_j\rangle$$

- Commutes with closure constraints & Conserves the total area
- Phases are relevant, it doesn't work with only 3-vectors $\vec{V}_i$
- Cyclic action: can reach any polyhedron from squashed configuration

$$|\zeta_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\zeta_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |\zeta_{k\geq 3}\rangle = 0$$
$$\longrightarrow |z_k\rangle = \sqrt{\mathcal{A}}|(U\zeta)_k\rangle = \sqrt{\mathcal{A}}\begin{pmatrix} U_{k1} \\ U_{k2} \end{pmatrix}$$

$\Rightarrow$ Identify polyhedron space at fixed area as Grassmannian

$$\mathcal{P}_N = \mathrm{U}(N)/(\mathrm{U}(N-2)\times \mathrm{SU}(2))$$

Framed Polyhedra: $\mathrm{U}(N)$ Action

E_{ij}'s actually generate $\mathrm{U}(N)$ transformations on spinors:

$$|z_i\rangle \longrightarrow |(Uz)_i\rangle = \sum_{ij} U_{ij}|z_j\rangle$$

- Commutes with closure constraints & Conserves the total area
- Phases are relevant, it doesn't work with only 3-vectors $\vec{V}_i$
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Framed Polyhedra: Using the U(N) tool and Computing Averages

- e.g. distribution of individual face area:

$$\langle V^n \rangle = \mathcal{A}^n \frac{(n+1)!(N-1)!}{(N+n-1)!}$$

- Average of shape tensor $\Theta_{ab} = \sum_i V_i^a V_i^b - \frac{1}{3}\delta^{ab} V_i V_i$

$$\Rightarrow \langle \Theta_{ab} \rangle = 0, \ \langle \mathrm{Tr}\Theta^2 \rangle \sim \mathcal{A}^4 N^{-3} \underset{N\to\infty}{\longrightarrow} 0$$

$\rightsquigarrow$ Polyhedra peaked around spherical config. Concentration of measure?

- Study other observables, e.g. (squared) volume?
- Link with matrix models through the Itzykson-Zuber formula?
- Dynamics of framed polyhedra by unitary transformations?

Coherent Intertwiners

These are semi-classical intertwiner states peaked on classical framed polyhedra

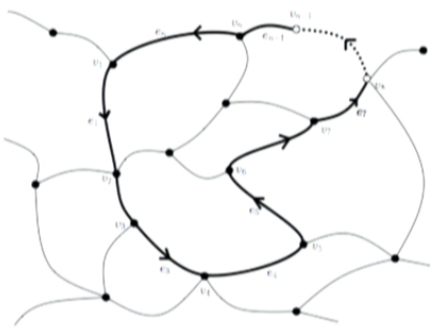
- Allow a simple decomposition of the identity:

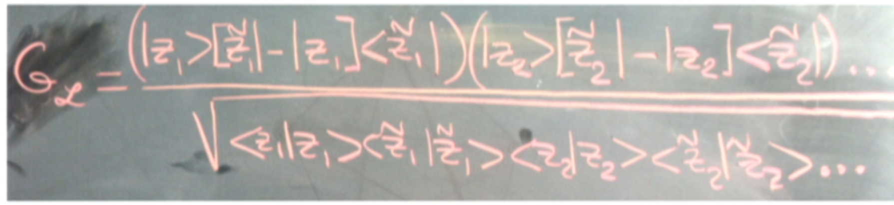
$$\mathbb{I}_N^J = \frac{1}{J!(J+1)!} \int_{\mathbb{C}^{2N}} \prod_i \frac{e^{-\langle z_i | z_i \rangle} d^4 z_i}{\pi^2} |J, \{z_i\}\rangle\langle J, \{z_i\}|$$

- Can glue them together into semi-classical coherent spin network states.
- All $\mathrm{SU}(2)$-invariant operators can be decomposed in terms of $\hat{E}$ and $\hat{F}$ operators and they act on spinor wave-functions and coherent spin networks as differential operators in the spinors z's.

(Generalized) Holonomy Operators

Let us have a look at the LQG holonomy operator around a loop $\mathcal{L}$:





Re-group terms by vertices instead of edges

$$\Rightarrow \quad \mathrm{Tr}\, G_{\mathcal{L}} = \frac{(F^{v_1} F^{v_2} \dots F^{v_n} + FEF \dots + \dots)}{\sqrt{\prod_i \langle z_i^s | z_i^s \rangle \langle z_i^t | z_i^t \rangle}}$$

Up to norm factors, each sequence of E, F's correspond to a definite shift $+\frac{1}{2}$ or $-\frac{1}{2}$ shift in the spins j_i on each link.
$\hookrightarrow$ generalized holonomies allow for finer analysis of deformations & dynamics

Spinor formalism & Coherent Spin Networks
Ready to be used in Spinfoams !

Spinfoams in terms of Spinors: effective Hamiltonian

Compute spinfoam amplitudes for coherent spin network ...

- Gives quantum transition amplitude $\mathcal{A}_{SF}[\{z_{in}\},\{z_{out}\}]$ for classical spinor networks and twisted geometries on the canonical boundary
- From this, we extract the effective dynamics and Hamiltonian for spinor networks as prescribed by a spinfoam model:

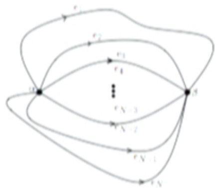
  $\hookrightarrow$ Hamiltonian constraints as diff ops in the z's satisfied by amplitude $\mathcal{A}_{SF}[\{z_{in}\},\{z_{out}\}]$, expressing inv under certain deformations of the boundary (diffeo inv?)

Already applied in a few cases with exact results:

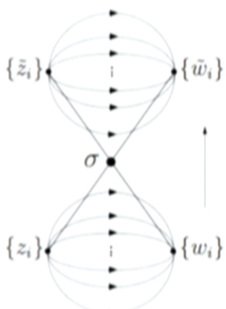
1. Spinfoam cosmology or transition from $\varnothing$ to isotropic states on 2-vertex graph $\Rightarrow$ modified FRW equations (un-improved LQC dynamics)

2. 3d and 4d BF $\Rightarrow$ flatness constraint as eqn $\langle z|z'\rangle = \langle z|G_{\mathcal{L}}|z'\rangle$

Just a Simple Model: The 2-vertex Model for Spinfoam Cosmology



SF amplitude @ LO given by evaluation of coh spinnet at $\mathbb{I}$:
$\mathcal{W}(z_i, w_i) = \psi_{\{z_i,w_i\}}(\mathbb{I}) = \langle \{\epsilon \bar{w}_i\} | \{z_i\} \rangle$
given by Bessel function: $\mathcal{W} = \sum_J \frac{1}{J!(J+1)!} \left(\det \sum_i |z_i\rangle [w_i|\right)^J$



- Hamiltonian constraint as diff operator in the z's:
  $$\widehat{\mathcal{H}} \sim \sum_{j,k} \left(2\hat{E}^{\alpha}_{jk} \hat{E}^{\beta}_{jk} + \hat{F}^{\alpha}_{jk} \hat{F}^{\beta}_{jk} + \hat{F}^{\alpha\dagger}_{jk} \hat{F}^{\beta\dagger}_{jk} \right) - 2(\hat{E} + N - 1)^2$$
  Sum of holonomy ops up to area factors $\rightsquigarrow$ discretization of LQG Hamiltonian constraint on 2-vertex graph
- Isotropic ansatz by requiring inv under $\mathrm{U}(N)$: z_i equal to w_i up to global phase ϕ conjugate to total area λ
- Reduction to single complex variable $z = \sqrt{\lambda} e^{i\phi}$
  with amplitude $\mathcal{W}(z) \propto I_1(z^2)/z^2$
  and effective classical Hamiltonian $\mathcal{H} \propto \lambda^2 \sin^2 \phi$

Obtain flat $\Lambda = 0$ FRW LQC without matter...

Spinfoams in terms of Spinors: effective Hamiltonian

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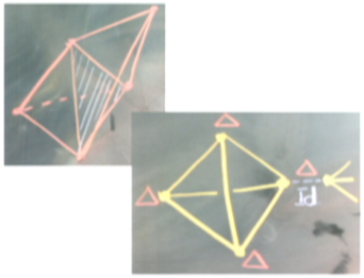
Exact results possible because of spinor variables and simple structure of coherent spin network wave-functions!

Spinfoams in terms of Spinors

And can we express the whole spinfoam amplitudes as a path integral over spinor space with spinors in the bulk?



Look at $\mathrm{SU}(2)$ BF theory as a start ...
In 3d, Tetrahedra glued together by their triangles

1. Representation on edges
2. Intertwiners on triangles glued by identity on intertwiner space.

Works the same in 4d: 4-simplices glued by intertwiner $\mathbb{I}$ on tetrahedra

Instead of using spin basis, use decomposition of identity in terms of coherent intertwiners $\rightsquigarrow$ spinfoam as history of coherent spinnet slices

See it directly at the algebraic level by decomposing $\delta(g)$ on a loop

$$\delta(g) = \sum_j (2j+1)\chi_j(g) \;\rightarrow\; \delta(g) = \int_{\mathbb{C}^2} \frac{(\langle z|z\rangle - 1)\, d^2 z}{\pi^2}\, e^{\langle z|g-\mathbb{I}|z\rangle}$$

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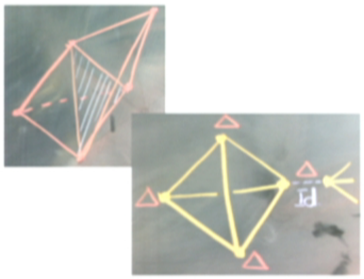

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Spinfoams in terms of Spinors: Exact Evaluations

We obtain as basic vertex amplitudes a coherent version of the $\{3nj\}$ symbols... $\rightsquigarrow$ $\{2\times 3nz\}$ symbols defined as evaluation of coherent spinnet at $\mathbb{I}$

$$\{2\times 3nz\}_\Gamma = \psi_{\{z_e^v\}}(\mathbb{I}) = \int [dh_v]\, e^{\sum_e [z_e^{s(e)}|h_{s(e)}^{-1} h_{t(e)}|z_e^{t(e)}\rangle}$$

Can map $\int$ over $\mathrm{SU}(2)$ to $\int$ over spinors:

$$\{2\times 3nz\}_\Gamma^{(gaussian)} = \widetilde{\psi}_{\{z_e^v\}}(\mathbb{I}) = \int [dw_v]\, e^{\sum_e [z_e^{s(e)}|(|w^{s(e)}\rangle\langle w^{t(e)}| + |w^{s(e)}][w^{t(e)}|)|z_e^{t(e)}\rangle}$$

- Fixed points corresponds to twisted geometry
- Can compute Gaussian integrals exactly Freidel, Hnybida 2012
- Evaluation of Coherent Spinnet are not only semi-classical versions of the $\{3nj\}$ symbols but also Generating Functions for the $\{3nj\}$'s

  $\hookrightarrow$ Here modification from ψ to $\widetilde{\psi}$ similar to

$$e^x = \sum_n \frac{x^n}{n!} \sim \int dy\, e^{-y^2+yx} \rightarrow \frac{1}{\sqrt{1-x}} = \sum_n \frac{(2n)!x^n}{2^{2n}(n!)^2} \sim \int dy\, e^{-y^2+y^2x}$$

Spinfoams in terms of Spinors: a Word on Spinfoam Asymptotics

The generating function (behavior under poles) contain the information about the asymptotics of spinfoam amplitudes.

But why are those asymptotics important?...

For large spins or large spinor norm, Spinfoam amplitudes are related to the exp or cos of Regge action with modifications

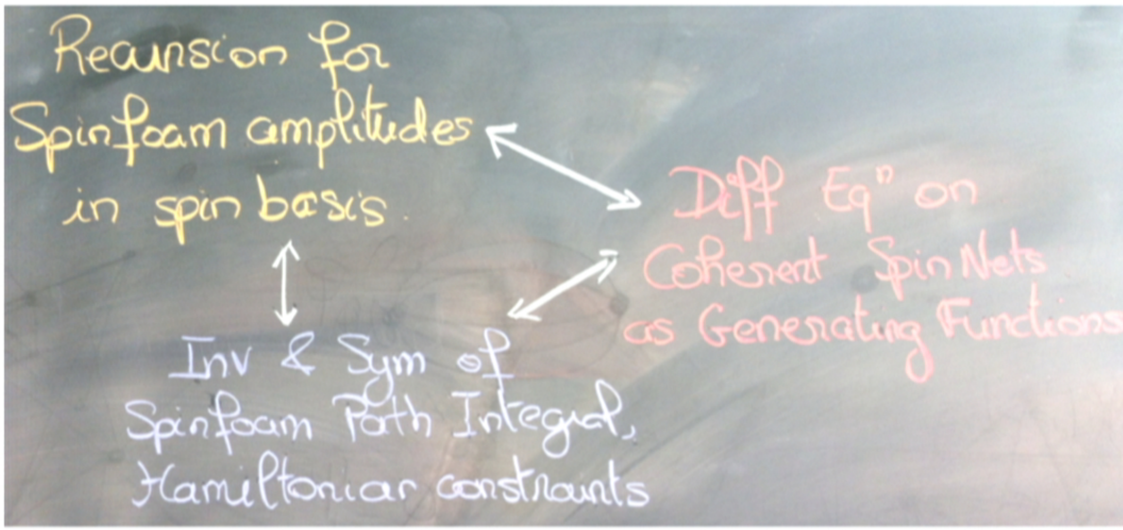
$$\mathcal{A}_{SF} \underset{j,|z|\to\infty}{\sim} e^{iS_{Regge}}$$

$\rightsquigarrow$ This provides a geometrical interpretation of spinfoam as discrete space-time !

At least it works for sure on a single 4-simplex...

Spinfoams in terms of Spinors: Generating Functions and Dynamics

Thus Coherent SpinNets are Generating functions useful to study large spin asymptotics and compute $\{3nj\}$ symbols...
But they also have physical interpretation as semi-classical amplitudes!



Spinfoams in terms of Twistors: Going beyond SU(2)

To go beyond SU(2) and investigate models like EPRL-FK and the role of simplicity constraints at the quantum level, have to move on to bi-sipnors or . . . twistors !

Three (selected) papers with "everything":

1. *Holomorphic Lorentzian Simplicity Constraints* DFLS 11
2. *Twistor Networks and Covariant Twisted Geometries* LST 11
3. *Twistorial Structure of LG transtition amplitudes* Speziale &Wieland 12

Go to Simone's great review talk on Monday if you want to know more!

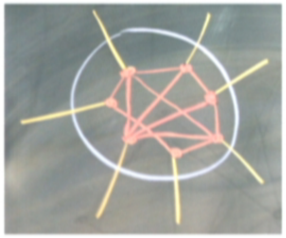
Twistor Networks and Spinfoams

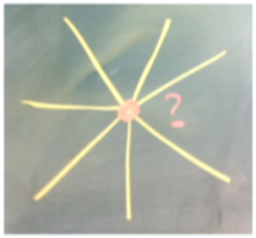
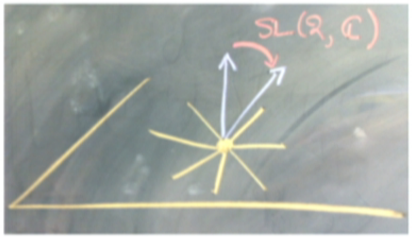
We use twistor networks, instead of spinor networks, as classical phase space for $\mathrm{SL}(2,\mathbb{C})$ holonomy & flux.

- Twistor as left plus right spinors $|t_e^v\rangle, |u_e^v\rangle$ on each edge around a vertex
- Define Twistor networks with complex area matching and closure constraints
- Introduce the simplicity constraints and solve them:
  $|t_e^v\rangle = e^{i\frac{\theta}{2}} \Lambda^v \, |u_e^v\rangle$ with pure boost Λ^v and *Imm angle* $e^{i\theta} = \frac{1+i\gamma}{1-i\gamma}$
  Effectively reduces twistors to spinors...
  ...but with non-trivial embedding
- Can quantize the whole thing...and write the EPRL-FK amplitude as path integrals over twistor space
  $\rightsquigarrow$ Can also investigate alternative quantization schemes of simplicity constraints based on this phase space

An idea that I like. . .

Coarse-graining spin(or) networks and relaxing the closure constraints. . .



. . . or why we need to go from spinors to twistors

Conclusion: "Little" Projects for the Future...

In Progress:

- Large N limit, refinement limit, superposition of N's?
- More on the asymptotics and geometry of $\{2 \times 3nz\}$ symbols
- q-deformation, $\Lambda \neq 0$, hyperbolic discrete geometry Dupuis

Hopefully:

- Exact evaluations of twistor integrals for spinfoam amplitudes?
- Flow and fate of simplicity constraints under coarse-graining?
- Recursion relations, diff equations and symmetries for non-topological spinfoams, application to cosmology?
- Which twisted geometry for which physical context? Coherent spin network states for cosmology? for BHs?
- Relation to matrix models, Look at dynamics of bounded regions of SpinNets, a CFT description of LQG boundary dynamics?
- Link with twistorial methods used in other approaches to QG?