Title: Canonical Quantum Gravity - 2

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Abstract:

Introduction

- The states and operators of LQG can be developed from phase spaces P_Γ associated to graphs Γ.
- These phase spaces represent the kinematics of gravity, so we seek to understand them as spatial geometries.
- Three such representations are twisted geometries[1], flat-cell geometries[2] and singular geometries[2].
- For gravity we would like a geometry which possesses a continuous frame field and connection.
- To this end we introduce spinning geometries.

[1] L. Freidel and S. Speziale (2010)

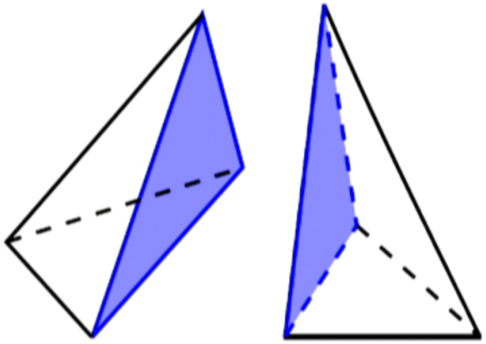
[2] L. Freidel, M. Geiller, and JZ (2012)

Outline

1. Review twisted and flat-cell geometries.
2. Introduce angular momentum variables on edges.
3. Reduce the ambiguity in the flat-cell edge shapes.
4. Show that the resulting edge shapes are compatible with the gluing maps.
5. Discuss the results.

Twisted geometry

- Composed of polyhedra 'glued' together at faces.
- On each face we have: $\boldsymbol{X}_f \in \mathfrak{su}(2), \quad h_f \in \mathrm{SU}(2)$.



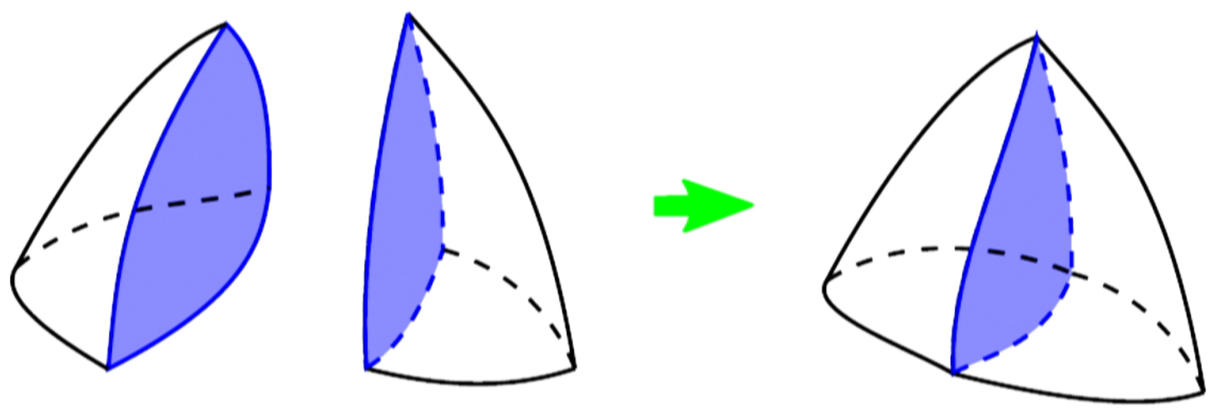
- Fluxes on each cell obey a closure relation: $\sum_f \boldsymbol{X}_f = 0$.
- Cells are glued along faces using: $\boldsymbol{X}_{c'c} = -h_{cc'}^{-1} \boldsymbol{X}_{cc'} h_{cc'}$.
- This geometry admits a torsionless connection[3] but is discontinuous.

[3] H. M. Haggard, C. Rovelli, W. Wieland and F. Vidotto (2013)

Flat-cell geometry

- A collection of three-dimensional cells c, each diffeomorphic to a polyhedron.
- There exist invertible gluing maps for each face f:

$$s_{cc'} : \ \bar{f}_{cc'} \to \bar{f}_{c'c}.$$



Flat-cell geometry

- Each cell possesses a coordinate function:

$$\boldsymbol{z}^c : \bar{c} \to \mathbb{R}^3,$$

which defines a flat metric $(g^c)_{\mu\nu} := \partial_\mu \boldsymbol{z}^c \cdot \partial_\nu \boldsymbol{z}^c$.

- Coordinate functions between cells are related by:

$$\boldsymbol{z}^{c'}(s_{cc'}(x)) = h_{cc'}^{-1}(\boldsymbol{z}^c(x) + \boldsymbol{a}_{cc'})h_{cc'}, \qquad \forall x \in \bar{f}_{cc'}.$$

- This geometry is isomorphic to a twisted geometry[2]. It is continuous but may have torsion.

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Flat, torsionless spaces

- There are two interesting subclasses of the flat-cell geometry:
  - Regge geometry: torsion vanishes everywhere and the induced metric on all of the faces is flat.
  - Spinning geometry: torsion is non-zero on edges (and only on edges), and the cell faces are generally curved.
- The Regge geometry cells are polyhedra, but at this point the spinning geometry cell shapes are ambiguous.
- Let us now reduce the ambiguity in the shape of spinning geometry cells.

Angular Momentum

- One can define fluxes associated to faces:

$$\boldsymbol{X}_{cc'} := \frac{1}{2}\int_{f_{cc'}} [\mathrm{d}\boldsymbol{z}^c, \mathrm{d}\boldsymbol{z}^c] = \frac{1}{2}\int_{\partial f_{cc'}} [\boldsymbol{z}^c, \mathrm{d}\boldsymbol{z}^c].$$

- Fluxes are associated with angular momentum due to their Poisson algebra.
- The Gauss law allows for a new relationship with angular momentum:

$$\boldsymbol{X}_{cc'} = \sum_{\ell \in \partial f_{cc'}} J^c_\ell, \qquad J^c_\ell := \frac{1}{2}\int_\ell \mathrm{d}s \boldsymbol{z}^c \times \dot{\boldsymbol{z}}^c.$$

- Each link momentum J^c_ℓ is the angular momentum for a point particle integrated along the world line ℓ.

Deformation of links

- A flux can be defined in terms of link momenta; the choice of face is irrelevant.
- Any deformation of the links which keeps the link momenta fixed will not change the fluxes.
- Each edge of a Regge geometry is the shortest path between the endpoints.

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$\Rightarrow$ Let us minimize link lengths while keeping momenta fixed.

Equations of motion

- Consider the action $I = \sum_\ell I_\ell$ where:

$$I_\ell = \int_\ell |\dot{\boldsymbol{z}}^c| \mathrm{d}s + \boldsymbol{\omega}_\ell^c \cdot \left(\boldsymbol{J}_\ell^c - \frac{1}{2} \int_\ell (\boldsymbol{z}_\ell^c \times \dot{\boldsymbol{z}}_\ell^c) \mathrm{d}s \right) .$$

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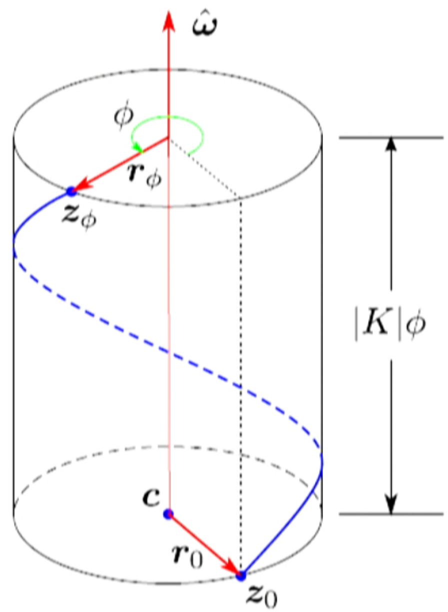
- The solution is a helix:

$$\boldsymbol{z}_\phi = \boldsymbol{c} + K\phi\hat{\boldsymbol{\omega}} + \boldsymbol{r}_\phi.$$



Analysis of a single link

- Let us define a helix basis $\boldsymbol{\sigma}_i \equiv (\hat{\boldsymbol{\omega}}, \hat{\boldsymbol{r}}_0, \hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{r}}_0)$.
- The displacement vector between nodes can be written as:

$$\boldsymbol{D} \equiv \boldsymbol{z}_\Phi - \boldsymbol{z}_0 \equiv 2K\varphi\boldsymbol{\sigma}_0 + 2r\sin\varphi\boldsymbol{\sigma}_\varphi.$$

- One finds that the link momentum contains two parts:

$$\begin{aligned} \boldsymbol{J} &= \boldsymbol{L} + \boldsymbol{S}, \qquad \boldsymbol{L} = \frac{1}{2}\boldsymbol{z}_0 \times \boldsymbol{D}, \\ \boldsymbol{S} &= r^2 f_\varphi \boldsymbol{\sigma}_0 + 2rK\varphi g_\varphi \boldsymbol{\sigma}_\varphi, \end{aligned}$$

where $\varphi \equiv \Phi/2$, $\boldsymbol{\sigma}_\varphi \equiv (-\sin\varphi\boldsymbol{\sigma}_1 + \cos\varphi\boldsymbol{\sigma}_2)$ and:

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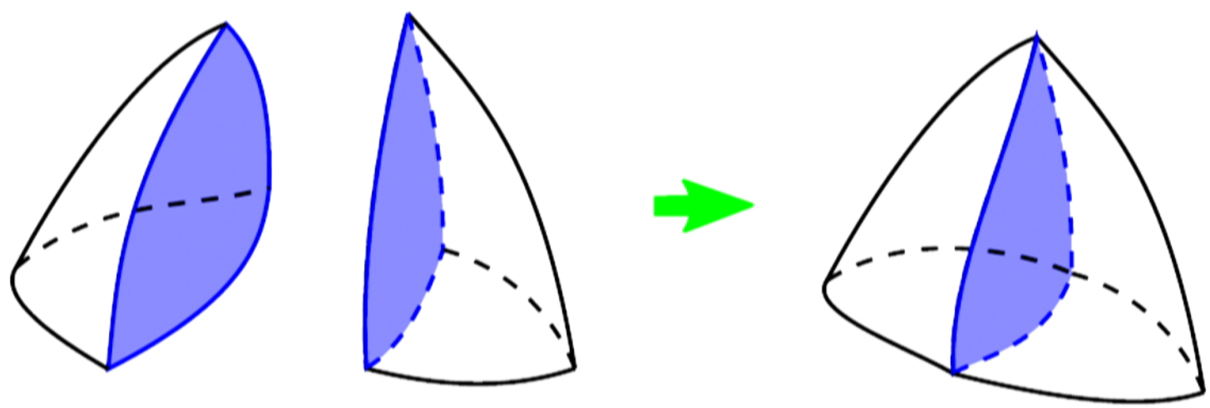

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A helix for any $(\boldsymbol{D}, \boldsymbol{S})$?

- We have a map:

$$(\boldsymbol{z}_0, r, K, \varphi, \boldsymbol{\sigma}_i) \to (\boldsymbol{z}_0, \boldsymbol{D}, \boldsymbol{S}).$$

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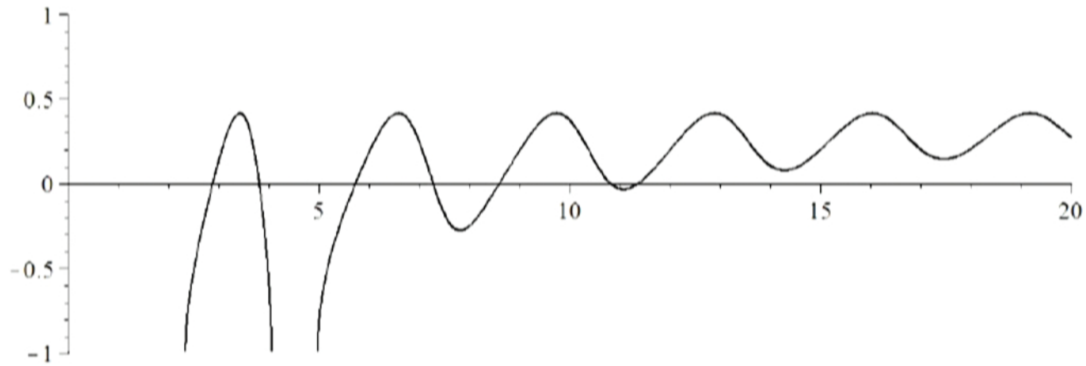
- Is there a helix for any $(\boldsymbol{D}, \boldsymbol{S})$ data, i.e. can we invert this map?
- If we take $(\boldsymbol{D}, \boldsymbol{S})$ as given, we can find $(r, K, \boldsymbol{\sigma}_i)$ in terms of this data and φ.
- The problem boils down to solving the equation:

$$2(r_\varphi^2 K_\varphi \varphi)(f_\varphi + 2g_\phi \sin\varphi) - \boldsymbol{S} \cdot \boldsymbol{D} = 0.$$

Many helices for a given $(\boldsymbol{D}, \boldsymbol{S})$!

- A typical plot for $\boldsymbol{S} \cdot \boldsymbol{D}/|D|^3 = |\boldsymbol{S} \times \boldsymbol{D}|/|D|^3 = 1$.



- We checked numerically for solutions over the range $-1000 \leq \frac{\boldsymbol{S}\cdot\boldsymbol{D}}{|D|^3} \leq 1000, \qquad 0 \leq \frac{|\boldsymbol{S}\times\boldsymbol{D}|}{|D|^3} \leq 1000.$
- There is a helix for any $(\boldsymbol{D}, \boldsymbol{S})$!

Analysis over a cell

- Given $(\boldsymbol{D}_\ell, \boldsymbol{S}_\ell)$, we can find a helix for each edge of a single cell.
- There are many choices of $(\boldsymbol{D}_\ell, \boldsymbol{S}_\ell)$ for a given set of fluxes.
- Each choice leads to different helices in boundary.
- Under what conditions for $(\boldsymbol{D}_\ell, \boldsymbol{S}_\ell)$ can we consistently glue cells together?

Analysis around an edge

- Consider a single link at the intersection of a number of cells.
- Recall the relation between the coordinate functions of neighbouring cells:

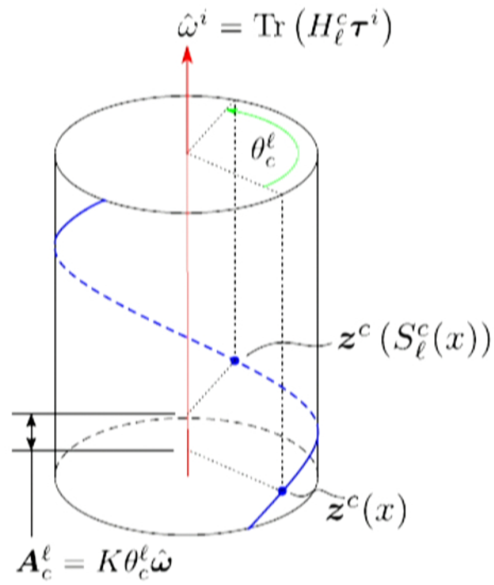
$$\boldsymbol{z}^{c'}(s_{cc'}(x)) = h_{cc'}^{-1}(\boldsymbol{z}^{c}(x) + \boldsymbol{a}_{cc'})h_{cc'}, \qquad \forall x \in \bar{f}_{cc'}.$$

- Repeatedly using this to go completely around the edge:

$$\boldsymbol{z}^{c}(S_c^{\ell}(x)) = H_c^{\ell}\boldsymbol{z}^{c}(x)H_c^{\ell} + \boldsymbol{A}_c^{\ell}, \qquad \forall x \in \ell.$$

A helix again!

- This equation is solved by a helix!
- $\hat{\boldsymbol{\omega}}$ is the axis of rotation defined by H_c^ℓ.
- The translation is $\boldsymbol{A}_c^\ell = K\theta_c^\ell \hat{\boldsymbol{\omega}}$.



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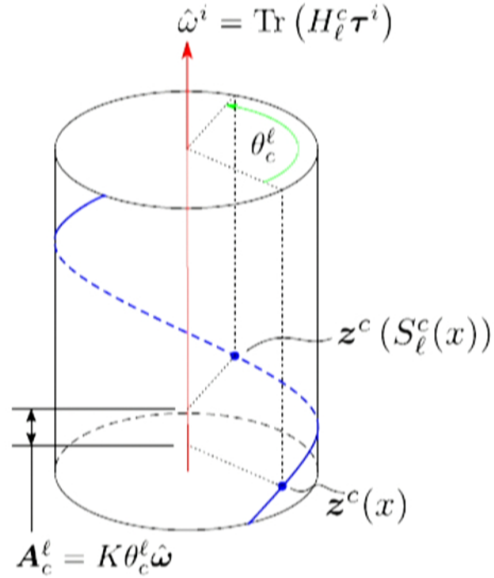

Given this restriction, can a closed network of helices can be constructed for any set of $(\boldsymbol{X}_f, h_f)$?

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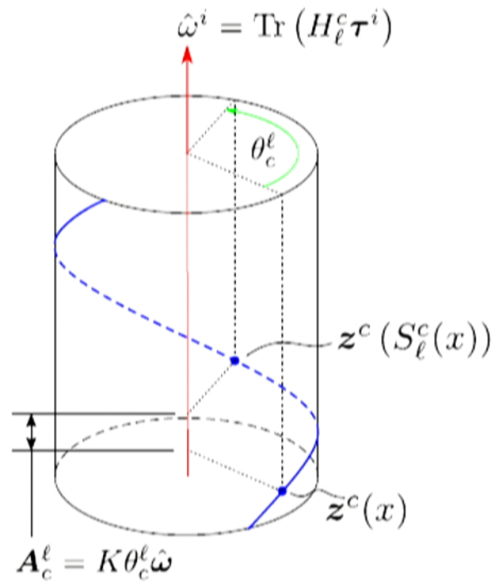
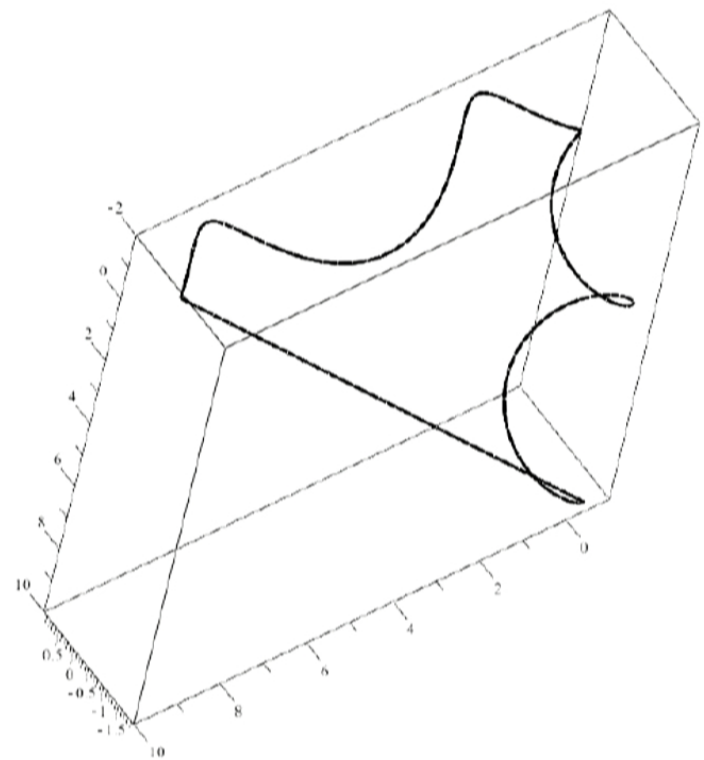


Given this restriction, can a closed network of helices can be constructed for any set of $(\boldsymbol{X}_f, h_f)$?

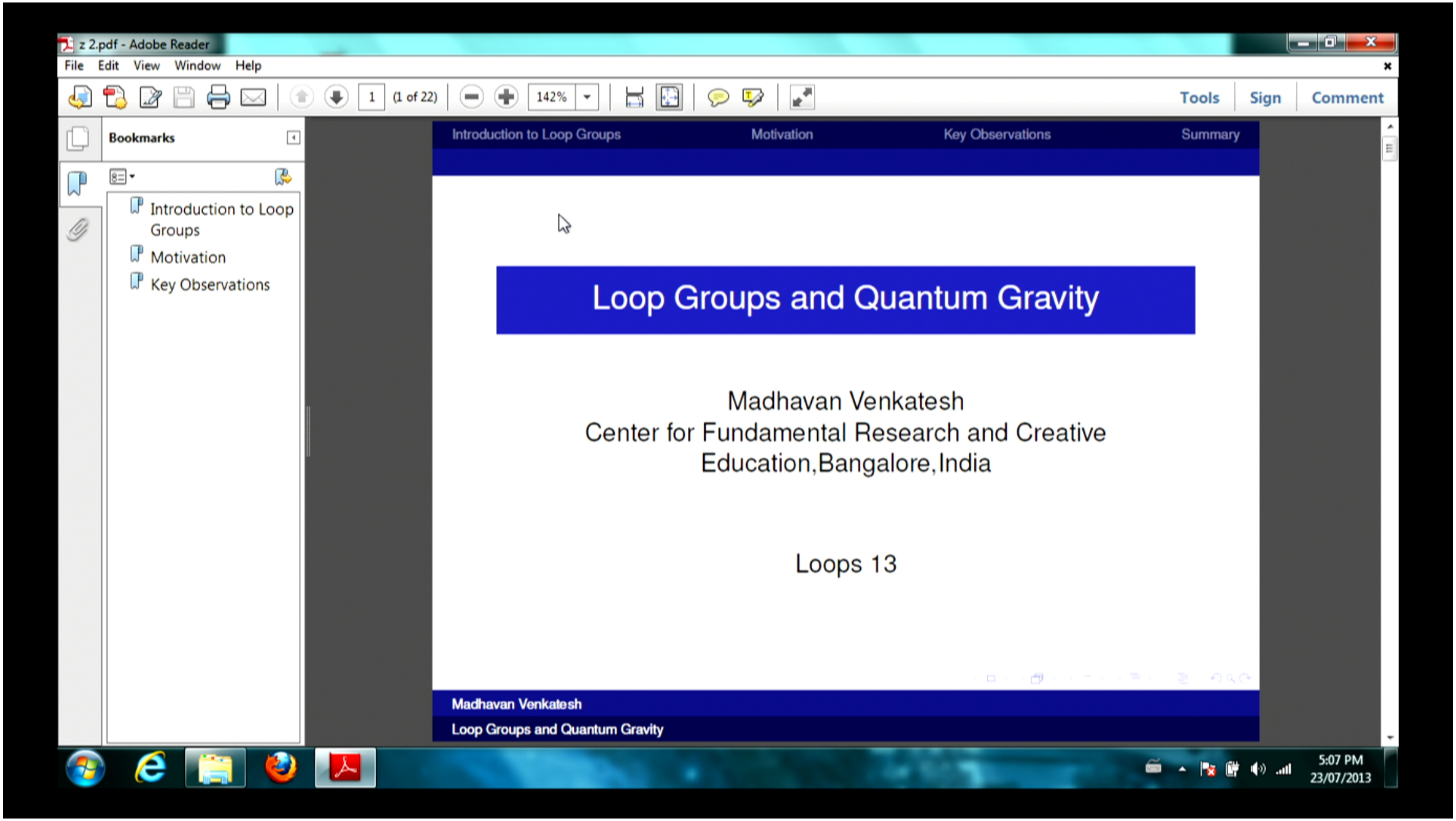
Conclusions

- Spinning geometries are isomorphic to twisted geometries, and represent the loop gravity phase space.
- They are continuous, and have torsion and curvature supported on a closed network of helices.
- The axes of the helices are defined by the holonomy data.
- This is the most general cellular space with vanishing curvature and torsion outside of edges.
- Spinning geometries provide a means to define continuous $(\boldsymbol{A}, \boldsymbol{e})$ fields from holonomy-flux data.
- This opens a new door to dynamics, allowing us to draw from the general relativistic equations of motion.

Thank you.



Jonathan Ziprick Spinning geometries = Twisted geometries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Introduction to Loop Groups
Motivation
Key Observations
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Summary
Loop Groups and Quantum Gravity
Madhavan Venkatesh
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Loops 13
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Loop Groups and Quantum Gravity
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23/07/2013

Outline

1. Introduction to Loop Groups
2. Motivation
3. Key Observations

Loop Groups and Properties

- A Loop Group, LG is the group of maps from the circle S^1 into a topological group G.
- A new equivalence relation, the cobordism, is introduced on a subgroup of this loop group. We denote the Loop Group with the equivalence relation as $L_C G$.
- One can describe a Chas-Sullivan type product on the cobordism.
- The composition $\circ$ and an associated operator Δ make the loop homology into a Batalin Vilkovisky algebra.

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The Loop Products

- Vertical Composition

$$\alpha \oplus \beta = (\alpha \circ \beta) \circ \gamma.$$

- Horizontal Composition

$$\alpha \ominus \beta = (\alpha \circ \gamma) + (\beta \circ \gamma).$$

- 'Total Product'

$$\alpha \circledast \beta = (\alpha \oplus \beta) \circ (\alpha \ominus \beta).$$

Loop Products and Curvature

- γ is the 'holonomy average' of the two loops given by

$$\gamma = \mathrm{P}exp\left\{\frac{1}{2}\left(\oint_\alpha \phi_{ab}dx^a dx^b + \oint_\beta \phi_{ab}dy^a dy^b\right)\right\}.$$

- Connection:

$$\phi_{ab} = \phi_a\phi_b - \phi_b\phi_a + \phi_{[a,b]}.$$

- Proposition:

$$d\phi = \int_{\Omega G} \alpha \circledast \beta.$$

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Concise, sketchy Proof

- This is proved by defining the inner product on the group suitably.
$$\langle \alpha, \beta \rangle = (1 + \Delta)^s (\alpha \circledast \beta) .$$
- We have the symplectic form on the loop space, due to the Kähler structure of ΩG as
$$\omega(\alpha, \beta) = \int_{\Omega G} \langle \alpha, \beta \rangle .$$
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Link with GR

- Kähler structure of the Loop space and Ricci flatness indicate Calabi-Yau.
- By the Campbell-Magaard embedding theorem, one can embed n-dimensional spacetime into an n+1 dimensional Ricci-flat manifold.

Link with QM

- Loop Groups can be thought of as groups of operators on Hilbert Space. (See Pressley-Segal)
- Plücker embedding.

Some definitions (- See Gambini and Pullin)

- The path variational :

$$\delta\alpha = \varrho_0^x \circ \delta u \circ \varrho_{x+\epsilon u}^0$$

- The Loop Derivative:

$$\Delta_{ab}\left(\alpha_0^x\right) = \partial_a\delta_b\left(x\right) - \partial_b\delta_a\left(x\right) + \left[\delta_a\left(x\right), \delta_b\left(x\right)\right]$$

- The Mandelstam Derivative:

$$D_a\alpha\left(\varrho_0^x\right) = \partial_a\alpha\left(x\right) + i\phi_a\left(x\right)\alpha\left(x\right)$$

- The connection functional :

$$\frac{\delta\alpha}{\delta\phi^a\left(x\right)} = \oint_\alpha dx^b\delta\left(y-x\right)\Delta_{ab}\left(\alpha_0^x\right)\phi\left(x\right)$$

Action

- Due to the proof of the Proposition, one can write an action:

$$S(\alpha, \beta) = \int \{(\alpha \oplus \beta) + (\alpha \ominus \beta)\} \sqrt{g} d^3 x.$$

- Following, the action can be varied, with respect to the loops, in order to obtain the equations of motion.

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Dynamics

- By varying the action and making use of the loop techniques, we have the 'momenta':

$$\tilde{\pi} = \left[[\int \left(\frac{\delta \alpha}{\delta \phi} \circ \beta \circ \gamma \right) + \left(\alpha \circ \frac{\delta \beta}{\delta \phi} \circ \gamma \right) \right\} + \left\{ \int \left(\frac{\delta \alpha}{\delta \phi} \circ \gamma \right) + \int \left(\frac{\delta \beta}{\delta \phi} \circ \gamma \right)] \right]$$

$$\tilde{\pi} = [\int \left(\left(\left(\oint_\alpha dx^b \delta\left(y - x\right) \Delta_{ab}\left(\alpha_0^x\right) \phi^a \right) \circ \beta \circ \gamma \right) \right.$$
$$\left. + \left(\alpha \circ \left(\oint_\beta dy^b \delta\left(x - y\right) \Delta_{ab}\left(\beta_0^y\right) \phi^a \right) \circ \gamma \right) \right)]$$
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- Now, we define a quantity called 'velocity' as:

$$\varpi = \int \mathrm{i}_X \left\{ \int (\alpha \oplus \beta) + \int (\alpha \ominus \beta) \right\}.$$

- Following this, we are enabled to define an 'energy' function in terms of the momenta and velocity:

$$\mathcal{Q} = \int_{\Omega G} \tilde{\pi} \circledast \varpi.$$

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Higher Dimensions

- Why?
  - Definition of the Energy function.
  - It has been proved that the 'momenta' and 'velocity behave as cobordant loops in dimension 5 and above.
- So, we can write down curvature in higher dimensions in terms of the 'momenta' and 'velocity' in ordinary dimensions.
  - For example

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Kähler (Calabi-Yau) Structure of the Loop Group

- The Loop Space ΩG has a manifest Kähler Structure. This combined with Ricci flatness leads to Calabi-Yau properties.
- The scalar curvature on it is given by:

$$R = \int \tilde{\pi} \oplus \varpi + \int \tilde{\pi} \ominus \varpi.$$

- And the 'averaged scalar curvature':

$$\hat{R} = \int \tilde{\pi} \circledast \varpi.$$

Calabi-Yau

- The Calabi energy is given by:

$$C = \int_{\Omega G} \left(R - \hat{R}\right)^2 \omega.$$

- This corresponds to the energy operator, that the Loop Group is equipped with, given by:

$$\mathcal{E}\left(\alpha\right) = < \psi_\alpha, i\frac{d}{d\theta}\psi_\alpha > .$$

Quantizability and the Projective Hilbert Space

- The connection is quantization compatible.
- A holomorphic embedding can be constructed from the Loop Space to the Projective Hilbert Space:

$$\pi : \Omega G \to P(\mathrm{H}) .$$

- For a Hilbert Space H with polarization $\mathrm{H} = \mathrm{H}_{+} \oplus \mathrm{H}_{-}$.
- Plücker embedding of the resultant Grassmannian.
- The Plücker co-ordinates define a holomorphic embedding.
- Cobordism invariant knots can be constructed.(See Turaev)
- This is necessary to make void the effect of the group equivalence relation of the loops (ie. cobordism).

Quantizability and the Projective Hilbert Space

- The connection is quantization compatible.
- A holomorphic embedding can be constructed from the Loop Space to the Projective Hilbert Space:

$$\pi : \Omega G \to P(\mathrm{H}) .$$

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Main Messages

- LGQG Loop Groups as a means for Quantum Gravity
- Consistency of quantum with classical Prospect for Quantization: Berezin-Toeplitz
- Basis An Overcomplete basis can be sidestepped.
- Possible Questions
  - Is the classical loop theory really GR?
  - Uniqueness in cobordance between loops.

Thank You for your Attention.
PS: The Fredholm index is cobordism invariant!!!

Deformed Phase Space for Hyperbolic Surfaces

Maïté Dupuis

July, 23rd 2013

Work in progress
in collaboration with
V. Bonzom, F. Girelli, E. Livine.

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Motivations: Why deforming the phase space of Loop Quantum Gravity?

- M.D and F. Girelli: $\mathcal{U}_q(\mathfrak{su}(2))$ spinnetworks = quantization of hyperbolic discrete geometries. [Phys.Rev.D.87.121502(R)]
- Poisson Lie group symmetries = classical analogues of quantum group symmetries.

$$\Downarrow$$

How to deform the phase space of Loop Quantum Gravity?

- Symplectic structure constructed on $\mathrm{SL}(2,\mathbb{C}) \simeq \mathrm{SU}(2) \times \mathrm{SB}(2,\mathbb{C})$ parametrized by $\kappa \in \mathbb{R}$.

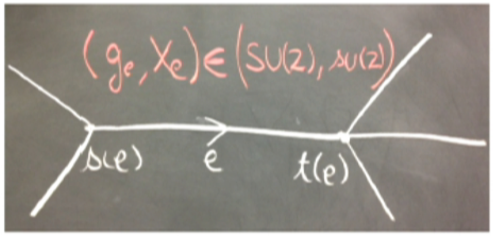
$\Rightarrow$ Symmetries are $\mathrm{SU}(2)$ Poisson-Lie group symmetries,

& after quantization a $\mathcal{U}_q(\mathfrak{su}(2))$ gauge symmetry.

1. Canonical phase space for LQG
2. Deformed phase space
3. Constraints and geometrical insights

Loop Quantum Gravity

- For a given graph Γ with E edges, $\mathcal{H}_\Gamma = L^2(\mathrm{SU}(2)^E, d^E g)$, is the quantization of the classical space $[T^*\mathrm{SU}(2)]^E$.
- For a given edge, e, **phase space**: $T^*\mathrm{SU}(2) \simeq \mathrm{SU}(2) \times \mathfrak{su}(2)$ parametrized by $(g_e, X_e = \vec{X}_e \cdot \vec{\sigma})$.



$$\{g_{IJ}, g_{KL}\} = 0,$$
$$\{X^i, X^j\} = \epsilon^{ij}_k X^k,$$
$$\{X^i, g_{IJ}\} = -\sigma^i g_{IJ}.$$

- **Symmetries**:
$$g_e \longrightarrow h_{s(e)} g_e h_{t(e)}^{-1}, \;\; h_{s(e)}, h_{t(e)} \in \mathrm{SU}(2)$$
- **Constraints**:
  - Gauss constraint, $\vec{C} = \sum_{i=1}^N \vec{X}_i$ implements the $\mathrm{SU}(2)$ invariance at each vertex.
  - Vectorial and Hamiltonian constraints... Or in (2+1)D gravity: flatness constraint.

An alternative Hamiltonian formulation?

We modify

- **the phase space**

$$T^*\mathrm{SU}(2) \longrightarrow \mathrm{SL}(2,\mathbb{C}),$$

- **the nature of the symmetries**

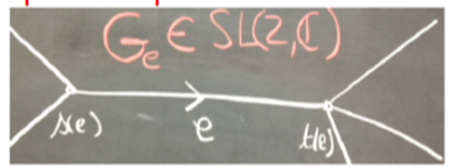
$$\underset{\text{Standard transformations}}{\mathrm{SU}(2)} \longrightarrow \underset{\text{Poisson Lie group symmetries.}}{\mathrm{SU}(2)}$$

- **new Gauss constraint**
- Vectorial and Hamiltonian constraints ?? For (2+1)D gravity= **new flatness constraint**...

→ gravity with a cosmological constant?

The deformed phase space

We focus on one oriented edge, e, of a network.



- **Phase space** $= \mathrm{SL}(2,\mathbb{C}) \simeq \mathrm{SU}(2) \times \mathrm{SB}(2,\mathbb{C})$
- **Iwasawa decomposition**: $G = \ell u$ with $u \in \mathrm{SU}(2)$, $\ell \in \mathrm{SB}(2,\mathbb{C})$,

$$u = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(2), \quad \ell = \begin{pmatrix} \lambda & 0 \\ z & \lambda^{-1} \end{pmatrix} \in \mathrm{SB}(2,\mathbb{C}),\ \lambda \in \mathbb{R}_+^*,\ z \in \mathbb{C}.$$

- Non trivial quadratic **Poisson structure** for $G \in \mathrm{SL}(2,\mathbb{C})$ [Marmo, Simoni, Stern, '93]:

$$\{G_1, G_2\} = -rG_1G_2 - G_1G_2r^\dagger \text{ with } G_1 = G \otimes \mathbb{I},\ G_2 = \mathbb{I} \otimes G,\ r = r(\kappa).$$

- **Deformation** of the Poisson brackets on $T^*(\mathrm{SU}(2))$ for LQG:

$$\kappa \to 0 \text{ in } \ell = e^{i\kappa X^i \tau_i},\ r(\kappa).$$

- Switching orientation of the edge: $G^{-1} = \tilde{\ell}^{-1}\tilde{u}^{-1}$

The Poisson-Lie group symmetries

- **Rotations** by $\mathrm{SU}(2)$ group elements:

$$G = \ell u \longrightarrow v_L G\, v_R^{-1} \Rightarrow \text{ for } v_L \left\{ \begin{array}{lcl} \ell & \rightarrow & \ell^{(v_L)} = v_L \ell v'^{-1} \\ u & \rightarrow & v' u \end{array} \right.$$

- Generator of a left $\mathrm{SU}(2)$ rotation,
$v_L = \mathbb{I} + i\vec{\epsilon}\cdot\vec{\sigma} = \mathbb{I} + i(V - \frac{1}{2}\mathrm{tr}(V)\mathbb{I})$,

$$\kappa^{-1}\mathrm{tr}(V\ell\ell^\dagger) \Rightarrow \left\{ \begin{array}{l} \delta\ell = -\lambda^{-2}\{\kappa^{-1}\mathrm{tr}(V\ell\ell^\dagger), \ell\}, \\ \delta u = -\lambda^{-2}\{\kappa^{-1}\mathrm{tr}(V\ell\ell^\dagger), u\}, \end{array} \right.$$

i.e. $\mathrm{SU}(2)$ rotations generated by the Poisson brackets with the Hermitian matrix $\ell\ell^\dagger$.

- **Translations** by multiplication by triangular matrices

$G \longrightarrow m_L G m_R^{-1}$; translations generated by Poisson brackets with u.

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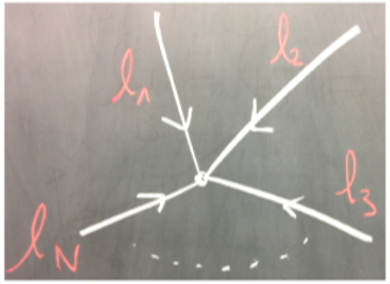
The Gauss constraint



- Gauss constraint (notation: $l = (\ell^\dagger)^{-1}$)

$$\left.\begin{array}{l}\mathcal{G}_+ = \ell_1 \cdots \ell_N = \mathbb{I} \\ \mathcal{G}_- = l_1 \cdots l_N = \mathbb{I}\end{array}\right\} = \begin{array}{c}\text{first-class} \\ \text{constraints}\end{array} \begin{array}{l}\{\mathcal{G}_{+1}, \mathcal{G}_{+2}\} = -[r, \mathcal{G}_{+1}\mathcal{G}_{+2}]|_{\mathcal{G}_+=\mathbb{I}} = 0, \\ \{\mathcal{G}_{-1}, \mathcal{G}_{-2}\} = -[r^\dagger, \mathcal{G}_{-1}\mathcal{G}_{-2}]|_{\mathcal{G}_-=\mathbb{I}} = 0, \\ \{\mathcal{G}_{+1}, \mathcal{G}_{-2}\} = -[r^\dagger, \mathcal{G}_{+1}\mathcal{G}_{-2}]|_{\mathcal{G}_\pm=\mathbb{I}} = 0.\end{array}$$

Geometrical interpretation (3D Euclidean gravity with $\Lambda < 0$)

- Cartan decomposition,

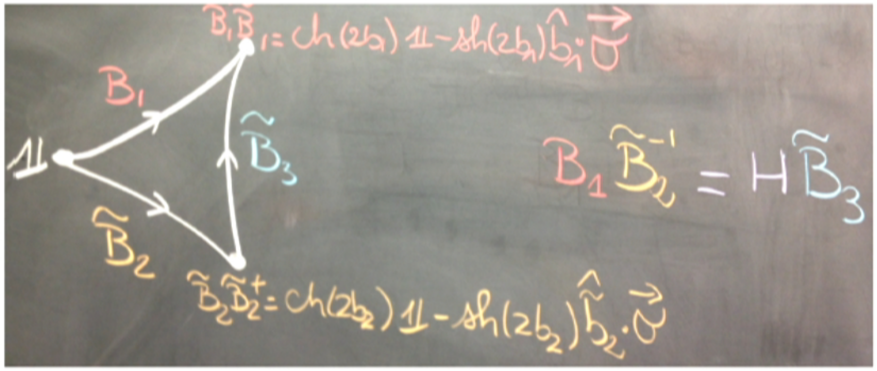
$$G = \ell u = (Bh^{-1})u, \text{ with } \left\{\begin{array}{l} B = \cosh(b)\mathbb{I} - \sinh(b)\hat{b}\cdot\vec{\sigma} \in \mathrm{SL}(2,\mathbb{C}), \\ h \in \mathrm{SU}(2).\end{array}\right.$$

The Gauss constraint and the hyperbolic cosine law

- N=3, Gauss law,

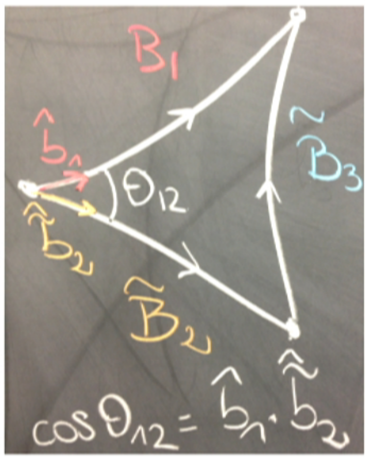
$$\ell_1\ell_2\ell_3 = \mathbb{I} = B_1\underbrace{(h_1^{-1}B_2h_1)}_{\widetilde{B}_2^{-1}}\underbrace{(h_1^{-1}h_2^{-1}B_3h_2h_1)}_{\widetilde{B}_3^{-1}}\underbrace{(h_1^{-1}h_2^{-1}h_3^{-1})}_{H^{-1}}$$

$$\Rightarrow \begin{cases} (1) & B_1\widetilde{B}_2^{-1} = H\widetilde{B}_3 \\ (2) & \widetilde{B}_2H\widetilde{B}_3H^{-1} = B_1H^{-1} \\ (3) & B_1\widetilde{B}_3^{-1} = \widetilde{B}_2H \end{cases}$$



- Hyperbolic triangle, totally specified by three angles.
- 3 different ways ((1), (2), (3)) to write the Gauss law → 3 angles; e.g. using (1):

$$B_1\widetilde{B}_2^{-1} = H\widetilde{B}_3 \Rightarrow \mathrm{tr}(B_1\widetilde{B}_2^{-1}(B_1\widetilde{B}_2^{-1})^\dagger) = \mathrm{tr}(\widetilde{B}_3(\widetilde{B}_3)^\dagger)$$



$$\left[\begin{array}{c} \cosh(2b_1)\cosh(2b_2) - \sinh(2b_1)\sinh(2b_2)\hat{b}_1 \cdot \hat{\tilde{b}}_2 \\ = \\ \cosh(2b_3). \end{array}\right.$$

- Hyperbolic triangle, totally specified by three angles.
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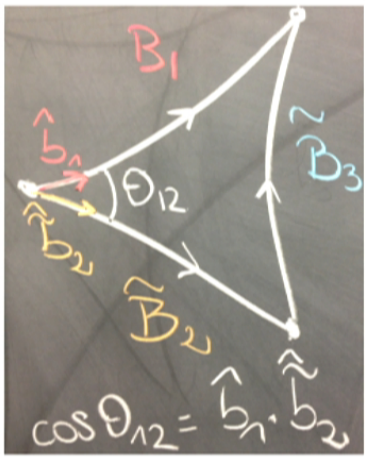

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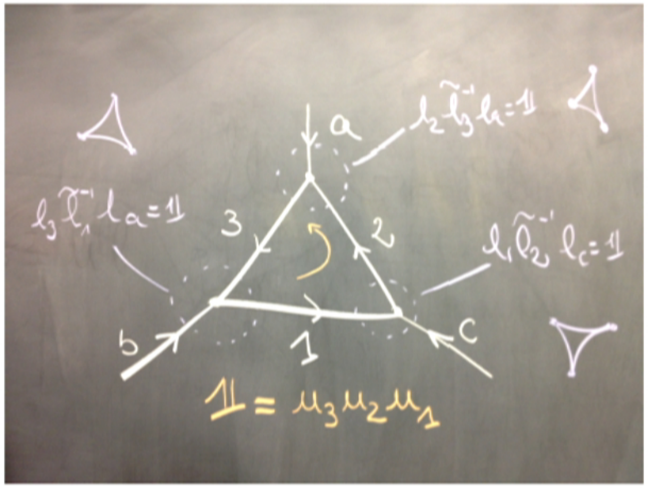
The flatness constraint

- A proposition: $u_N \dots u_1 = \mathbb{I}$: first-class constraint and $\mathrm{SU}(2)$ gauge invariant.
- Gluing of triangles



Conclusion

Some preliminary results,

- New phase space parametrized by κ (related to Λ),
- Propositions for the constraints,
- Some geometrical insights; characterization of some hyperbolic geometries.

To explore further,

- Continuum limit,
- Gauss + flatness constraints: solutions for a given topology?,
- Spinor variables,
- Quantization,
- To compare with the combinatorial quantization formalism,
- ...

TENSOR OPERATORS FOR THE LORENTZ GROUP IN $2+1$ LOOP QUANTUM GRAVITY*

Giuseppe Sellaroli

University of Waterloo

July 23, 2013

*Work in collaboration with Florian Girelli.

Spinor approach to LQG

The so-called spinor approach is a way to treat Loop Quantum Gravity with gauge group $\mathrm{SU}(2)$ using tensor operators, in particular spinor ones, either explicitly or implicitly through the Jordan–Schwinger construction. (cf. Livine plenary talk)

Some advantages of this approach:

- Closed algebra for the generators of observables (Freidel, Girelli, Livine)
- Construction of Hamiltonian constraint in 3D (Bonzom, Freidel)
- Treatment of LQG with cosmological constant, i.e. gauge group $\mathcal{U}_q(\mathrm{SU}(2))$ (Dupuis, Girelli)

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Spinor approach to LQG

The Lie algebra $su(2)$ has generators

$$[J_z, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_z.$$

They can be rewritten in the Schwinger–Jordan representation introducing two uncoupled harmonic oscillators

$$[a, a^\dagger] = [b, b^\dagger] = \mathbb{1},$$

so that

$$J_+ = a^\dagger b, \quad J_- = b^\dagger a, \quad J_z = \frac{1}{2}(a^\dagger a - b^\dagger b)$$

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Spinor approach to LQG

Considering a single intertwiner with N legs, one can introduce a couple of harmonic oscillator (a_i, b_i) for the leg (i). All of the observables can be generated by the operators

$$E_{ij} = a_i^\dagger a_j + b_i^\dagger b_j;$$

the diagonal ones $E_i \equiv E_{ii}$ give the area associated to the leg (i), with the total area given by

$$E = \sum_{i=1}^{N} E_i.$$

The action of E_{ij} on the intertwiner is to take quanta of area from leg (j) to leg (i), without changing the total area. They generate the closed algebra $\mathfrak{u}(N)$.

Is there a spinor approach in the (3D) Lorentzian case?

Representations of $\mathrm{SL}(2,\mathbb{R})$

The gauge group for 3D Lorentzian LQG is the non-compact $\mathrm{SL}(2,\mathbb{R})$. Its Lie algebra is generated by

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_0,$$

with Casimir

$$Q = \mathbf{J}\cdot\mathbf{J} = -J_0^2 + \tfrac{1}{2}(J_- J_+ + J_+ J_-).$$

It acts on its representations as

$$\begin{cases} J_0 \left| j\, \varepsilon\, m \right\rangle = m \left| j\, \varepsilon\, m \right\rangle \\ J_\pm \left| j\, \varepsilon\, m \right\rangle = C_\pm(j,m) \left| j\, \varepsilon\, m \pm 1 \right\rangle \\ Q \left| j\, \varepsilon\, m \right\rangle = -j(j+1) \left| j\, \varepsilon\, m \right\rangle \end{cases}$$

with $C_\pm(j,m) = \sqrt{-(j \mp m)(j \pm m + 1)}$.

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The irreducible representations are classified as

- Discrete positive (negative) series $\mathcal{D}_j^{\pm}$:
  $j = -\frac{1}{2}, 0, \frac{1}{2}, \dots \quad j = \varepsilon \,(\mathrm{mod.}\, 1) \quad m = \pm(j+1), \pm(j+2), \dots$
- Continuous series $\mathcal{C}_j^{\varepsilon}$:
  $j \in \mathbb{C} - \mathbb{Z}/2 \quad \varepsilon = 0, \frac{1}{2} \quad m \in \varepsilon + \mathbb{Z}$
- Finite dimensional series $\mathcal{V}_\gamma$:
  $\gamma = 0, \frac{1}{2}, 1, \dots \quad \gamma = \varepsilon \,(\mathrm{mod.}\, 1) \quad |\mu| \leq \gamma$

Spin networks only carry the ones appearing in the Plancherel decomposition, i.e. $D_j^{\pm}$ with $j \geq 0$ and C_j^{ε} with $j = -\frac{1}{2} + is$, $s > 0$, which are unitary.

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Tensor operators

Tensor operators are a particular type of operators acting between two (possibily different) representations, which transform as covectors in a finite-dimensional representation. A rank γ irreducible tensor operator T^γ transforms as covectors in $\mathcal{V}_\gamma$ (which is non-unitary), and its components satisfy

$$[J_0, T^\gamma_\mu] = \mu\, T^\gamma_\mu, \quad [J_\pm, T^\gamma_\mu] = C_\pm(\gamma,\mu)\, T^\gamma_{\mu\mp1}, \quad |\mu| \leq \gamma.$$

The Lie algebra generators form a rank 1 tensor operator (vector operator)

$$T^1_0 = J_0, \quad T^1_{\pm1} = -\frac{1}{\sqrt{2}} J_\pm.$$

Moreover, observables in LQG are in 1-to-1 correspondence with (hermitian) rank 0 tensor operators (scalar operators).

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Two tensor operators can be combined to get another one using the Clebsch–Gordan coefficients of $\mathcal{V}_{\gamma_1} \otimes \mathcal{V}_{\gamma_2}$, which are the same as the $\mathrm{SU}(2)$ ones. Explicitly, the quantity

$$\sum_{\mu_1,\mu_2} \langle \gamma\, \mu | \gamma_1\, \mu_1\, \gamma_2\, \mu_2 \rangle\, T^{\gamma_1}_{\mu_1} T^{\gamma_2}_{\mu_2}$$

is the μ-th component of a rank γ tensor operator.

In the spinor approach to LQG, we look for two $\frac{1}{2}$ operators (spinor operators) which can be combined to construct the J operators. For $\mathrm{SU}(2)$ this is achieved through the Jordan–Schwinger representation.

For $\mathrm{SL}(2,\mathbb{R})$, one can construct a Jordan–Schwinger representation for both the discrete series (Schwinger 1952), but until now an analogous construction for the continuous series was unknown. We will fill this gap with the aid of the Wigner–Eckart theorem.

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Wigner–Eckart theorem

Tensor operators have particularly simple matrix elements. For the continuous series, one can prove that, as long as $j \notin \mathbb{Z}/2$, the matrix elements of a tensor operator T^γ_μ are given by

$$\langle j'\,\varepsilon'\,m'|T^\gamma_\mu|j\,\varepsilon\,m\rangle = B(j'\,\varepsilon'\,m'|\gamma\,\mu\,j\,\varepsilon\,m)\,\langle j'\,\varepsilon'\|T^\gamma\|j\,\varepsilon\rangle\,,$$

where $\langle j'\,\varepsilon'\|T^\gamma\|j\,\varepsilon\rangle$ is a quantity which does not depend on m, m' and μ.

$B(j'\,\varepsilon'\,m'|\gamma\,\mu\,j\,\varepsilon\,m)$ is the inverse Clebsch–Gordan coefficient of the coupling $\mathcal{V}_\gamma \otimes \mathcal{C}^\varepsilon_j$, which satisfy the selection rules

$$j-\gamma \leq j' \leq j+\gamma, \qquad m' = m+\mu.$$

Remark: half integral operators always take us out of the Plancherel decomposition.

Spinor approach for the continuous series

Using the Wigner–Eckart theorem, we can prove that the generators can be constructed, in the continuous series, as

$$J_+ = A^\top B, \quad J_- = AB^\top, \quad J_0 = \frac{1}{2}(A^\top A + BB^\top),$$

with

$$[A, A^\top] = [B, B^\top] = \mathbb{1}.$$

Observables are generated by the scalar operators

$$E_{ij} = A_i^\top A_j - B_i B_j^\top,$$

which incidentally still form a $\mathfrak{u}(N)$ algebra. The same is true if we also include the discrete series in the picture.

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Outlook

Further points to investigate:

- First-order polynomials in the E_{ij} can be observables in the $\mathrm{SU}(2)$ case, while in the Lorentzian case they must be at least second order. Why?
- Can the Hamiltonian constraint be implemented in 3D Lorentzian LQG with the spinor approach?

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