

Title: Canonical Quantum Gravity - 1

Date: Jul 23, 2013 02:30 PM

URL: <http://pirsa.org/13070053>

Abstract:

The canonical LOOPS'13

Jerzy Lewandowski
Warszawa / Erlangen

The canonical LOOPS'13

Jerzy Lewandowski
Warszawa / Erlangen

The canonical LOOPS'13 – p.1

The canonical LOOPS'13

Jerzy Lewandowski

Warszawa / Erlangen

The canonical LOOPS'13 – p.1

*“It is the mark of the educated man
to look for precision”*

Lee Smolin

The canonical LOOPS'13 – p.2

*“It is the mark of the educated man
to look for precision”*

Lee Smolin

after the lecture on LQG by Jerzy Lewandowski

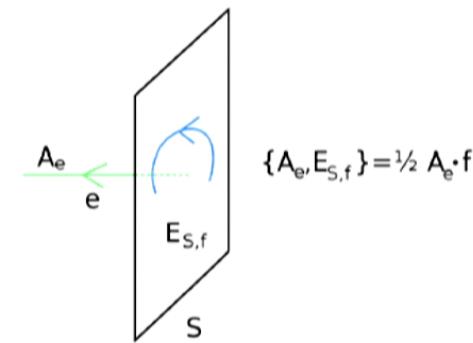
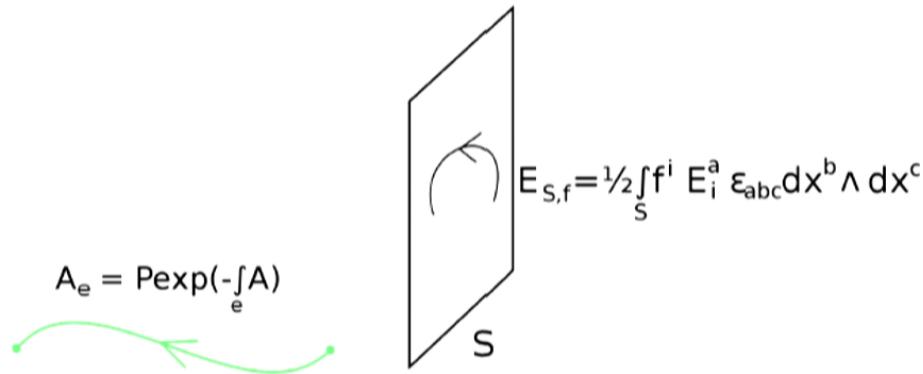
The canonical LOOPS'13 – p.3

Plan of the talk

- ➊ The tools provided by LQG: the Hilbert spaces, quantum representations, infinity free regularizations of quantum operators of geometry, of the constraints, of the matter hamiltonians, the habitat...
- ➋ Pure GR:
 - ➌ a new scheme,
 - ➌ a new $\hat{C}^{\text{gr}}(x)$
- ➌ GR coupled to a real scalar field:
 - ➌ normalizable solutions to the quantum constraints
 - ➌ accomodating all the cases:
$$\pi \geq 0,$$
$$\pi \leq 0,$$
$$\pi^2 \geq \phi_{,a}\phi_{,b}q^{ab}q,$$
$$\pi^2 \leq \phi_{,a}\phi_{,b}q^{ab}q.$$
 - ➌ observables
- ➌ Outlook

The canonical LOOPS'13 – p.4

The holonomy-flux variables Lie algebra



Rovelli, Smolin, Pulin, Husain... the end of 1980s

The canonical LOOPS'13 – p.6

Cylindrical functions, cylindrical consistency

(Ashtekar, L 1992):

Quantum states:

$$\Psi(A) = \psi(A_{e_1}, \dots, A_{e_n}) = \psi'(A'_{e'_1}, \dots, A'_{e'_{n'}})$$

with some graphs $\Gamma = \{e_1, \dots\}$, and respectively $\Gamma' = \{e'_1, \dots\}$ embedded in Σ .

Cylindrically consistent measures define $\int d\mu(A)$

$$\int_{SU(2)^n} \psi d\mu_\Gamma = \int_{SU(2)^{n'}} \psi' d\mu'_{\Gamma'} =: \int \Psi(A) d\mu(A)$$

$$\mathcal{H}_{\text{gr}} = L^2(d\mu)$$

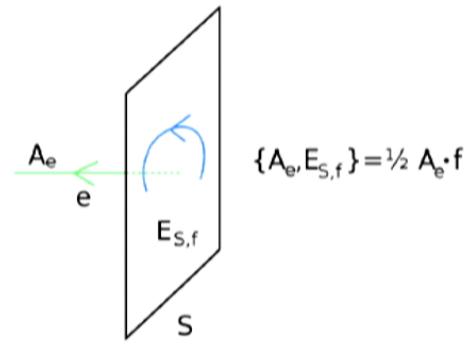
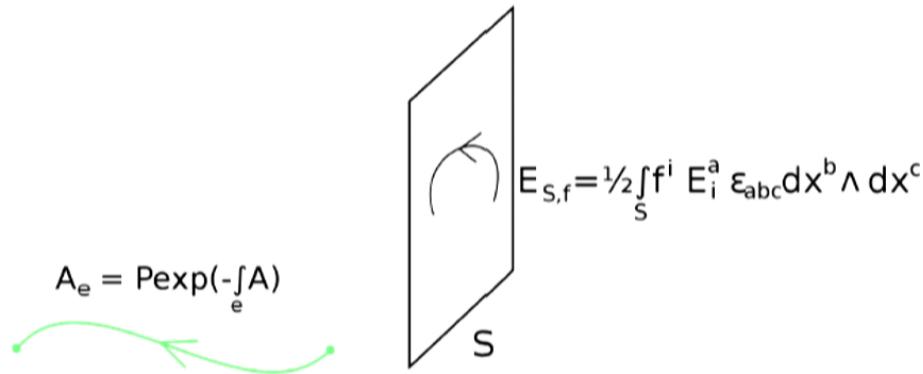
Cylindrically consistent operators define $\mathcal{O}\Psi$:

$$\hat{\mathcal{O}}_\Gamma \psi(A_{e_1}, \dots, A_{e_n}) = \hat{\mathcal{O}}'_{\Gamma'} \psi(A'_{e'_1}, \dots, A'_{e'_{n'}}) =: \hat{\mathcal{O}}\Psi(A)$$

example: $\hat{E}_{S,f} \Psi(A) := \frac{\hbar}{i} \{ \Psi(A), E_{S,f} \}$

The canonical LOOPS'13 – p.7

The holonomy-flux variables Lie algebra



Rovelli, Smolin, Pulin, Husain... the end of 1980s

The canonical LOOPS'13 – p.6

Cylindrical functions, cylindrical consistency

(Ashtekar, L 1992):

Quantum states:

$$\Psi(A) = \psi(A_{e_1}, \dots, A_{e_n}) = \psi'(A'_{e'_1}, \dots, A'_{e'_{n'}})$$

with some graphs $\Gamma = \{e_1, \dots\}$, and respectively $\Gamma' = \{e'_1, \dots\}$ embedded in Σ .

Cylindrically consistent measures define $\int d\mu(A)$

$$\int_{SU(2)^n} \psi d\mu_\Gamma = \int_{SU(2)^{n'}} \psi' d\mu'_{\Gamma'} =: \int \Psi(A) d\mu(A)$$

$$\mathcal{H}_{\text{gr}} = L^2(d\mu)$$

Cylindrically consistent operators define $\mathcal{O}\Psi$:

$$\hat{\mathcal{O}}_\Gamma \psi(A_{e_1}, \dots, A_{e_n}) = \hat{\mathcal{O}}'_{\Gamma'} \psi(A'_{e'_1}, \dots, A'_{e'_{n'}}) =: \hat{\mathcal{O}}\Psi(A)$$

example: $\hat{E}_{S,f} \Psi(A) := \frac{\hbar}{i} \{ \Psi(A), E_{S,f} \}$

The canonical LOOPS'13 – p.7

Cylindrical functions, cylindrical consistency

(Ashtekar, L 1992):

Quantum states:

$$\Psi(A) = \psi(A_{e_1}, \dots, A_{e_n}) = \psi'(A'_{e'_1}, \dots, A'_{e'_{n'}})$$

with some graphs $\Gamma = \{e_1, \dots\}$, and respectively $\Gamma' = \{e'_1, \dots\}$ embedded in Σ .

Cylindrically consistent measures define $\int d\mu(A)$

$$\int_{SU(2)^n} \psi d\mu_\Gamma = \int_{SU(2)^{n'}} \psi' d\mu'_{\Gamma'} =: \int \Psi(A) d\mu(A)$$

$$\mathcal{H}_{\text{gr}} = L^2(d\mu)$$

Cylindrically consistent operators define $\mathcal{O}\Psi$:

$$\hat{\mathcal{O}}_\Gamma \psi(A_{e_1}, \dots, A_{e_n}) = \hat{\mathcal{O}}'_{\Gamma'} \psi(A'_{e'_1}, \dots, A'_{e'_{n'}}) =: \hat{\mathcal{O}}\Psi(A)$$

example: $\hat{E}_{S,f} \Psi(A) := \frac{\hbar}{i} \{ \Psi(A), E_{S,f} \}$

The canonical LOOPS'13 – p.7

The unique state

The quantum *-algebra (abstract), defined by:

$$A_B^A \mapsto \hat{A}_B^A, \quad E_{S,f} \mapsto \hat{E}_{S,f}, \quad \{\cdot, \cdot\} \mapsto \frac{1}{i\hbar}[\cdot, \cdot]$$

and the natural relations satisfied by $E_{S,f}$,
admits a unique $\text{Diff}(M)$ invariant state

$$\omega : \hat{A}_{B_1 e_1}^{A_1}, \dots, \hat{A}_{B_n e_n}^{A_n} \hat{E}_{S_1, f_1}, \dots, \hat{E}_{S_m, f_m} \mapsto \mathbb{C}.$$

(L.Okolow, Sahlmann, Thiemann 2005)

It is defined as follows:

$$\omega(\dots E_{S,f}) = 0, \quad \omega(f(A_{e_1}, \dots, A_{e_n})) := \int f(A) d\mu(A).$$

Open questions:

- ➊ $\{E_{S,f}, E_{S',f'}\} \neq 0$
- ➋ diffeo covariant representations of the QHF algebra which do not admit a diffeo invariant state?

The canonical LOOPS'13 – p.8

The unique state

The quantum *-algebra (abstract), defined by:

$$A_B^A \mapsto \hat{A}_B^A, \quad E_{S,f} \mapsto \hat{E}_{S,f}, \quad \{\cdot, \cdot\} \mapsto \frac{1}{i\hbar}[\cdot, \cdot]$$

and the natural relations satisfied by $E_{S,f}$,
admits a unique $\text{Diff}(M)$ invariant state

$$\omega : \hat{A}_{B_1 e_1}^{A_1}, \dots, \hat{A}_{B_n e_n}^{A_n} \hat{E}_{S_1, f_1}, \dots, \hat{E}_{S_m, f_m} \mapsto \mathbb{C}.$$

(L.Okolow, Sahlmann, Thiemann 2005)

It is defined as follows:

$$\omega(\dots E_{S,f}) = 0, \quad \omega(f(A_{e_1}, \dots, A_{e_n})) := \int f(A) d\mu(A).$$

Open questions:

- ➊ $\{E_{S,f}, E_{S',f'}\} \neq 0$
- ➋ diffeo covariant representations of the QHF algebra which do not admit a diffeo invariant state?

The canonical LOOPS'13 – p.8

The unique state

The quantum *-algebra (abstract), defined by:

$$A_B^A \mapsto \hat{A}_B^A, \quad E_{S,f} \mapsto \hat{E}_{S,f}, \quad \{\cdot, \cdot\} \mapsto \frac{1}{i\hbar}[\cdot, \cdot]$$

and the natural relations satisfied by $E_{S,f}$,
admits a unique $\text{Diff}(M)$ invariant state

$$\omega : \hat{A}_{B_1 e_1}^{A_1}, \dots, \hat{A}_{B_n e_n}^{A_n} \hat{E}_{S_1, f_1}, \dots, \hat{E}_{S_m, f_m} \mapsto \mathbb{C}.$$

(L, Okolow, Sahlmann, Thiemann 2005)

It is defined as follows:

$$\omega(\dots E_{S,f}) = 0, \quad \omega(f(A_{e_1}, \dots, A_{e_n})) := \int f(A) d\mu(A).$$

Open questions:

- ➊ $\{E_{S,f}, E_{S',f'}\} \neq 0$
- ➋ diffeo covariant representations of the QHF algebra which do not admit a diffeo invariant state?

The canonical LOOPS'13 – p.8

The Ashtekar's canonical gravity

- ➊ The canonically conjugate coordinate and momentum variables defined on a 3-manifold Σ :
 - (A_a^i, E_i^a) $a, b = 1, 2, 3, i = 1, 2, 3$ - the gravitational field, the Ashtekar-Barbero variables
 - (ϕ_I, π^I) $I = 1, \dots$ - the matter fields
 - $N(t, x), N^a(t, x)$ the laps and shift functions
- ➋ the constraints
 - $C^{\text{tot}}(x) := C^{\text{gr}} + C^\phi = 0$ scalar constraint
 - $C_a^{\text{tot}}(x) = 0$ vector constraint - $\text{Diff}(\Sigma)$ generators
 - $C_{\text{Gauss}}^{\text{tot}}{}^i(x) = 0$ the Gauss constraint - the frame rotation generators
 - $C_{\text{other}}^{\text{tot}}$ other matter constraints (for example the generators of the Maxwell/Yang-Mills field gauge transformations)
- ➌ Dirac observables: \mathcal{O} such that
$$\{\mathcal{O}, C^{\text{tot}}\} = 0$$
restricted to the solutions to the constraints. **No natural dynamics.**
Extra input needed.

The canonical LOOPS'13 – p.5

The unique state

The quantum *-algebra (abstract), defined by:

$$A_B^A \mapsto \hat{A}_B^A, \quad E_{S,f} \mapsto \hat{E}_{S,f}, \quad \{\cdot, \cdot\} \mapsto \frac{1}{i\hbar}[\cdot, \cdot]$$

and the natural relations satisfied by $E_{S,f}$,
admits a unique $\text{Diff}(M)$ invariant state

$$\omega : \hat{A}_{B_1 e_1}^{A_1}, \dots, \hat{A}_{B_n e_n}^{A_n} \hat{E}_{S_1, f_1}, \dots, \hat{E}_{S_m, f_m} \mapsto \mathbb{C}.$$

(L, Okolow, Sahlmann, Thiemann 2005)

It is defined as follows:

$$\omega(\dots E_{S,f}) = 0, \quad \omega(f(A_{e_1}, \dots, A_{e_n})) := \int f(A) d\mu(A).$$

Open questions:

- ➊ $\{E_{S,f}, E_{S',f'}\} \neq 0$
- ➋ diffeo covariant representations of the QHF algebra which do not admit a diffeo invariant state?

The canonical LOOPS'13 – p.8

Infinity free regularization of quantum operators

(Rovelli, Smolin, Ashtekar, L 1993, Thiemann 1997)

$$\begin{aligned}\mathcal{O}(E, A, F) &\mapsto \mathcal{O}^{(N)}(E_{S_I, f_I}, A_{e_i}, A_{l_j} - (A_{l_j})^{-1}) \\ &\mapsto \mathcal{O}^{(N)}(\hat{E}_{S_I, f_I}, \hat{A}_{e_i}, \hat{A}_{l_j} - (\widehat{A_{l_j}})^{-1}) \\ &\mapsto \lim_{N \rightarrow \infty} \mathcal{O}^{(N)}(\hat{E}_{S_I, f_I}, \hat{A}_{e_i}, \hat{A}_{l_j} - (\widehat{A_{l_j}})^{-1}) \\ &=: \hat{\mathcal{O}}(E, A, F)\end{aligned}\tag{1}$$

The condition: the existence of the limit, or of the dual limit:

$$\langle \text{diff inv } |\mathcal{O}^{(N)}(\hat{E}_{S_I, f_I}, \hat{A}_{e_i}, \hat{A}_{l_j} - (\widehat{A_{l_j}})^{-1})| \cdot \rangle$$

the diffeomorphism covariance of the result.

Additional (Thiemann's) tricks: express $\frac{1}{\sqrt{\cdot}}$ by $\{\cdot, \sqrt{\cdot}\}$

The advantage: no ϵ , no $\frac{1}{\epsilon}$, no infinities, generalises to coupled matter:

$$\mathcal{O}(A, E, \phi_I, \pi^I).$$

The canonical LOOPS'13 – p.9

Infinity free regularization of quantum operators

(Rovelli, Smolin, Ashtekar, L 1993, Thiemann 1997)

$$\begin{aligned}\mathcal{O}(E, A, F) &\mapsto \mathcal{O}^{(N)}(E_{S_I, f_I}, A_{e_i}, A_{l_j} - (A_{l_j})^{-1}) \\ &\mapsto \mathcal{O}^{(N)}(\hat{E}_{S_I, f_I}, \hat{A}_{e_i}, \hat{A}_{l_j} - (\widehat{A_{l_j}})^{-1}) \\ &\mapsto \lim_{N \rightarrow \infty} \mathcal{O}^{(N)}(\hat{E}_{S_I, f_I}, \hat{A}_{e_i}, \hat{A}_{l_j} - (\widehat{A_{l_j}})^{-1}) \\ &=: \hat{\mathcal{O}}(E, A, F)\end{aligned}\tag{1}$$

The condition: the existence of the limit, or of the dual limit:

$$\langle \text{diff inv } |\mathcal{O}^{(N)}(\hat{E}_{S_I, f_I}, \hat{A}_{e_i}, \hat{A}_{l_j} - (\widehat{A_{l_j}})^{-1})| \cdot \rangle$$

the diffeomorphism covariance of the result.

Additional (Thiemann's) tricks: express $\frac{1}{\sqrt{\cdot}}$ by $\{\cdot, \sqrt{\cdot}\}$

The advantage: no ϵ , no $\frac{1}{\epsilon}$, no infinities, generalises to coupled matter:

$$\mathcal{O}(A, E, \phi_I, \pi^I).$$

The canonical LOOPS'13 – p.9

Infinity free regularization of quantum operators

(Rovelli, Smolin, Ashtekar, L 1993, Thiemann 1997)

$$\begin{aligned}\mathcal{O}(E, A, F) &\mapsto \mathcal{O}^{(N)}(E_{S_I, f_I}, A_{e_i}, A_{l_j} - (A_{l_j})^{-1}) \\ &\mapsto \mathcal{O}^{(N)}(\hat{E}_{S_I, f_I}, \hat{A}_{e_i}, \hat{A}_{l_j} - (\widehat{A_{l_j}})^{-1}) \\ &\mapsto \lim_{N \rightarrow \infty} \mathcal{O}^{(N)}(\hat{E}_{S_I, f_I}, \hat{A}_{e_i}, \hat{A}_{l_j} - (\widehat{A_{l_j}})^{-1}) \\ &=: \hat{\mathcal{O}}(E, A, F)\end{aligned}\tag{1}$$

The condition: the existence of the limit, or of the dual limit:

$$\langle \text{diff inv } |\mathcal{O}^{(N)}(\hat{E}_{S_I, f_I}, \hat{A}_{e_i}, \hat{A}_{l_j} - (\widehat{A_{l_j}})^{-1})| \cdot \rangle$$

the diffeomorphism covariance of the result.

Additional (Thiemann's) tricks: express $\frac{1}{\sqrt{\cdot}}$ by $\{\cdot, \sqrt{\cdot}\}$

The advantage: no ϵ , no $\frac{1}{\epsilon}$, no infinities, generalises to coupled matter:

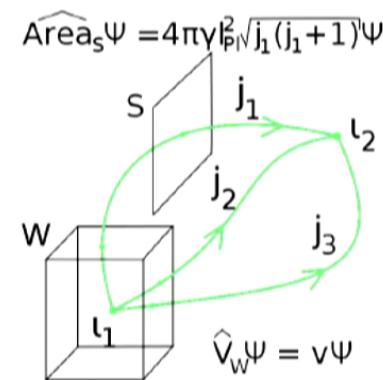
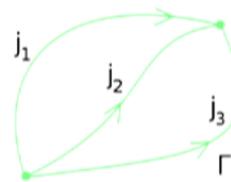
$$\mathcal{O}(A, E, \phi_I, \pi^I).$$

The canonical LOOPS'13 – p.9

Quantum Geometry

A convenient generalised spin-network decomposition:

$$\mathcal{H}_{\text{gr}} = \bigoplus_{\Gamma} \mathcal{H}_{\text{gr},\Gamma} = \bigoplus_{\Gamma,j} \mathcal{H}_{\text{gr},\Gamma,j}$$

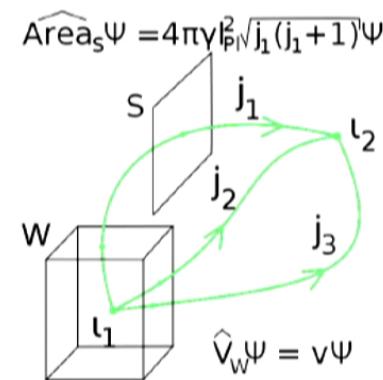
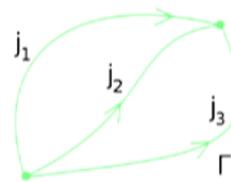


The canonical LOOPS'13 - p.10

Quantum Geometry

A convenient generalised spin-network decomposition:

$$\mathcal{H}_{\text{gr}} = \bigoplus_{\Gamma} \mathcal{H}_{\text{gr},\Gamma} = \bigoplus_{\Gamma,j} \mathcal{H}_{\text{gr},\Gamma,j}$$



The canonical LOOPS'13 - p.10

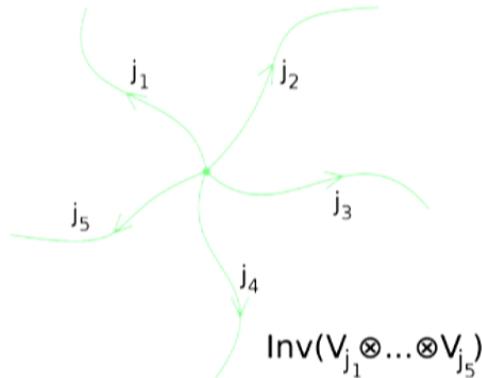
Pure GR: The YM gauge transformations

The Yang-Mills gauge transformations:

$$\hat{A}_e \mapsto g(e_{\text{target}})^{-1} \hat{A}_e g(e_{\text{source}}),$$

$$\hat{E}_{S,f} \mapsto \hat{E}_{S,g^{-1}fg}$$

The invariant elements in \mathcal{H}_{gr} are defined by the invariants of the product of the representations $\rho_{j_1} \otimes \dots \otimes \rho_{j_m}$: the spin networks Penrose 1970s, Rovelli-Smolin 1994, Baez 1994,....

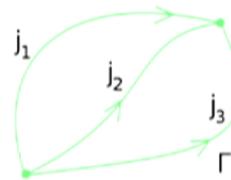


The canonical LOOPS'13 - p.11

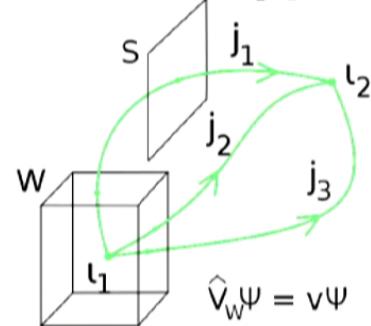
Quantum Geometry

A convenient generalised spin-network decomposition:

$$\mathcal{H}_{\text{gr}} = \bigoplus_{\Gamma} \mathcal{H}_{\text{gr},\Gamma} = \bigoplus_{\Gamma,j} \mathcal{H}_{\text{gr},\Gamma,j}$$



$$\widehat{\text{Area}}_S \Psi = 4\pi r_p^2 \sqrt{j_1(j_1+1)} \Psi$$



The canonical LOOPS'13 - p.10

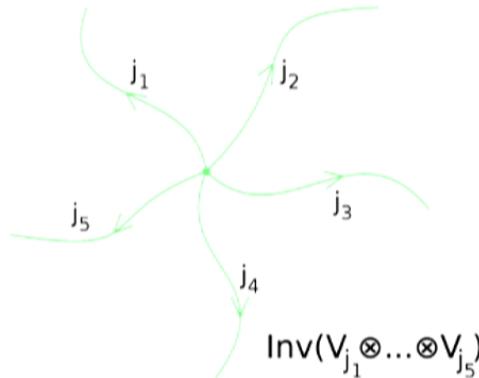
Pure GR: The YM gauge transformations

The Yang-Mills gauge transformations:

$$\hat{A}_e \mapsto g(e_{\text{target}})^{-1} \hat{A}_e g(e_{\text{source}}),$$

$$\hat{E}_{S,f} \mapsto \hat{E}_{S,g^{-1}fg}$$

The invariant elements in \mathcal{H}_{gr} are defined by the invariants of the product of the representations $\rho_{j_1} \otimes \dots \otimes \rho_{j_m}$: the spin networks Penrose 1970s, Rovelli-Smolin 1994, Baez 1994,....



The canonical LOOPS'13 – p.11

Pure GR: The $\text{Diff}(\Sigma)$ gauge transformations

- $\mathcal{H}_{\text{diff}}$ - solutions to the diffeo constraint.

$$u : \text{Diff}(\Sigma) \rightarrow U(\mathcal{H}_{\text{gr}})$$

The diffeomorphism averaging map (Ashtekar, L, Marolf, Mourao, Thiemann 1995)

$$\eta_{\text{diff}} : \mathcal{H}_{\text{gr}} \supset \text{Cyl} \rightarrow \text{Cyl}^*$$

$$\eta(\Psi) := \langle \int_{\text{Diff}} d\mu(\varphi)'' \Psi^* u(\varphi)^*,$$

and a new, averaged scalar product

$$(\eta(\Psi)|\eta(\Psi''))_{\text{diff}} := \langle \eta(\Psi), \Psi' \rangle,$$

and the resulting Hilbert space $\mathcal{H}_{\text{gr,diff}}$.

- Habitat: the span of the states Ψ averaged with an extra weight factor f which depends smoothly on location of the vertices of Ψ in Σ ,

$$\eta(f, \Psi) := \langle \int_{\text{Diff}} d\mu(\varphi)'' f \Psi^* u(\varphi)^* \rangle_{\text{Cyl}^*}$$

The canonical LOOPS'13 – p.12

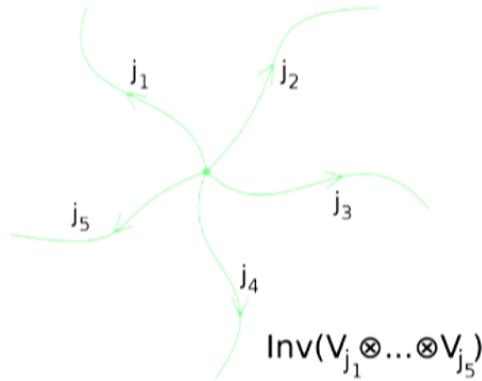
Pure GR: The YM gauge transformations

The Yang-Mills gauge transformations:

$$\hat{A}_e \mapsto g(e_{\text{target}})^{-1} \hat{A}_e g(e_{\text{source}}),$$

$$\hat{E}_{S,f} \mapsto \hat{E}_{S,g^{-1}fg}$$

The invariant elements in \mathcal{H}_{gr} are defined by the invariants of the product of the representations $\rho_{j_1} \otimes \dots \otimes \rho_{j_m}$: the spin networks Penrose 1970s, Rovelli-Smolin 1994, Baez 1994,....



The canonical LOOPS'13 - p.11

Pure GR: The $\text{Diff}(\Sigma)$ gauge transformations

- $\mathcal{H}_{\text{diff}}$ - solutions to the diffeo constraint.

$$u : \text{Diff}(\Sigma) \rightarrow U(\mathcal{H}_{\text{gr}})$$

The diffeomorphism averaging map (Ashtekar, L, Marolf, Mourao, Thiemann 1995)

$$\eta_{\text{diff}} : \mathcal{H}_{\text{gr}} \supset \text{Cyl} \rightarrow \text{Cyl}^*$$

$$\eta(\Psi) := \langle \int_{\text{Diff}} d\mu(\varphi)'' \Psi^* u(\varphi)^*,$$

and a new, averaged scalar product

$$(\eta(\Psi)|\eta(\Psi''))_{\text{diff}} := \langle \eta(\Psi), \Psi' \rangle,$$

and the resulting Hilbert space $\mathcal{H}_{\text{gr,diff}}$.

- Habitat: the span of the states Ψ averaged with an extra weight factor f which depends smoothly on location of the vertices of Ψ in Σ ,

$$\eta(f, \Psi) := \langle \int_{\text{Diff}} d\mu(\varphi)'' f \Psi^* u(\varphi)^* \rangle_{\text{Cyl}^*}$$

The canonical LOOPS'13 – p.12

The tools provided by LQG

- ➊ the kinematical Hilbert space for the geometric degrees of freedom \mathcal{H}_{gr} , the polymer Hilbert space of non-gauge fields, and the loop Hilbert space of the gauge fields
- ➋ the quantum operators of the geometry of Σ and of the matter fields (no divergences)
- ➌ the quantum operators of the Gauss constraint and the Hilbert space $\mathcal{H}_{\text{gr},\text{Gauss}}$ of the invariants (the same with matter)
- ➍ the unitary action of the diffeomorphism groups $\text{Diff}(\Sigma)$ and the Hilbert spaces $\mathcal{H}_{\text{gr,diff}}$ of the invariant distributions (the same with matter).
- ➎ the quantum scalar constraint operators \hat{C}^{gr} defined in $\mathcal{H}_{\text{gr,diff}}$ (no divergences - the same with matter) (*Thiemann*)
- ➏ the quantum scalar constraint operators defined in \mathcal{H}_{gr} within the AQGR (and discretized matter) (*Giesel, Thiemann*)

The canonical LOOPS'13 – p.15

The tools provided by LQG

- ➊ the kinematical Hilbert space for the geometric degrees of freedom \mathcal{H}_{gr} , the polymer Hilbert space of non-gauge fields, and the loop Hilbert space of the gauge fields
- ➋ the quantum operators of the geometry of Σ and of the matter fields (no divergences)
- ➌ the quantum operators of the Gauss constraint and the Hilbert space $\mathcal{H}_{\text{gr},\text{Gauss}}$ of the invariants (the same with matter)
- ➍ the unitary action of the diffeomorphism groups $\text{Diff}(\Sigma)$ and the Hilbert spaces $\mathcal{H}_{\text{gr,diff}}$ of the invariant distributions (the same with matter).
- ➎ the quantum scalar constraint operators \hat{C}^{gr} defined in $\mathcal{H}_{\text{gr,diff}}$ (no divergences - the same with matter) (*Thiemann*)
- ➏ the quantum scalar constraint operators defined in \mathcal{H}_{gr} within the AQGR (and discretized matter) (*Giesel, Thiemann*)

The canonical LOOPS'13 – p.15

$$F = dA + A \wedge A$$

$$\sqrt{-2g} C^{\nu} dx^{\gamma}$$



$$F = dA + A \wedge A$$

$$\rho = \sqrt{-2\sqrt{g} C^{\mu\nu}} dx^\gamma$$

$$\sqrt{-2\sqrt{g} C_{\gamma}}(x)$$

Quantum \hat{C}^{gr} operator

- ➊ Progress in understanding Thiemann's operator

$$\frac{c_k^{ij} \widehat{F_{ab}^k E_i^a E_j^b}}{\sqrt{q}}$$

Zipfel, Alesci 2013

- ➋ A new simpler proposal for the Lorentzian case:

- ➌ use Barbero's 1991 formula :

$$C^{\text{gr}}(x) = -\frac{c_k^{ij} F_{ab}^k E_i^a E_j^b}{\sqrt{q}}(x) - 2\sqrt{q}R^{(3)}(x) \quad (2)$$

- ➍ construct the 3-Ricci scalar operator $\widehat{\sqrt{q}R^{(3)}}$ Alesci, Assanoussi, L 2013 (see Assanoussi's talk).
 - ➎ The resulting $\widehat{C^{\text{gr}}(x)}$ contains terms adding / annihilating a loop, and not changing a graph, similarly to the scalar constraint in the LQC

The canonical LOOPS'13 – p.17

Pure GR: The scalar constraint

- Starting point: the space of solutions $\mathcal{H}_{\text{gr}, \text{Gauss}, \text{diff}}$ to the Gauss and to the vector constraint.
- Pass the Gauss and diffeo invariant operators defined on the kinematical \mathcal{H}_{gr} into $\mathcal{H}_{\text{gr}, \text{Gauss}, \text{diff}}$.
- In $\mathcal{H}_{\text{gr}, \text{Gauss}, \text{diff}}$ consider the scalar constraint operator $\hat{C}_{\text{gr}}(x)$.

Problem: we want to consider a self-adjoint operator, but any $\hat{C}_{\text{gr}}(x)$ will map us out of $\mathcal{H}_{\text{gr}, \text{Gauss}, \text{diff}}$! Possible ways out:

- define an operator $\int d^3 \sqrt{-\sqrt{q} C^{\text{gr}}}$ and look for the zero eigen value subspace (Rovelli-Smolin, 1994 see later)
- define an operator $\int d^3 \frac{C^{\text{gr}}}{\sqrt{q}}$ and look for the zero eigen value subspace (Thiemann, 2003, the master constraint program)
- New: define for each x a Hilbert space $\mathcal{H}_{\text{gr}, \text{Gauss}, \text{diff}_x}$ in which a self-adjoint operator $\hat{C}^{\text{gr}}(x)$ can be well defined, solve therein
$$\hat{C}^{\text{gr}}(x)\Psi_x = 0,$$
use the solutions Ψ_x to construct $\Psi \in \mathcal{H}_{\text{gr}, \text{Gauss}, \text{diff}}$ such that
$$\hat{C}^{\text{gr}}(x)\Psi = 0$$
 (see Domagala, Dziendzikowski, L 2012)

The canonical LOOPS'13 – p.16

Quantum \hat{C}^{gr} operator

- ➊ Progress in understanding Thiemann's operator

$$\frac{c_k^{ij} \widehat{F_{ab}^k E_i^a E_j^b}}{\sqrt{q}}$$

Zipfel, Alesci 2013

- ➋ A new simpler proposal for the Lorentzian case:

- ➌ use Barbero's 1991 formula :

$$C^{\text{gr}}(x) = -\frac{c_k^{ij} F_{ab}^k E_i^a E_j^b}{\sqrt{q}}(x) - 2\sqrt{q}R^{(3)}(x) \quad (2)$$

- ➍ construct the 3-Ricci scalar operator $\widehat{\sqrt{q}R^{(3)}}$ Alesci, Assanoussi, L 2013 (see Assanoussi's talk).
 - ➎ The resulting $\widehat{C^{\text{gr}}(x)}$ contains terms adding / annihilating a loop, and not changing a graph, similarly to the scalar constraint in the LQC

The canonical LOOPS'13 – p.17

Gravity coupled to a potentialless scalar field

A, E - gravitational dof, ϕ, π - a real scalar field

The scalar constraint can be solved:

$$\pi(x) = h(A(x), E(x)), \quad \text{where}$$

- for non-rotating dust

$$h(x) = C^{gr}(x)$$

- for massless scalar field

$$h = \begin{cases} \sqrt{-\sqrt{q}C^{\text{gr}} + \sqrt{q}\sqrt{(C^{\text{gr}})^2 - q^{ab}C_a^{\text{gr}}C_b^{\text{gr}}}}, & \pi \geq 0, \pi^2 \geq q^{ab}\phi_{,a}\phi_{,b}q \\ \sqrt{-\sqrt{q}C^{\text{gr}} - \sqrt{q}\sqrt{(C^{\text{gr}})^2 - q^{ab}C_a^{\text{gr}}C_b^{\text{gr}}}}, & \pi \geq 0, \pi^2 \leq q^{ab}\phi_{,a}\phi_{,b}q \\ -\sqrt{-\sqrt{q}C^{\text{gr}} + \sqrt{q}\sqrt{(C^{\text{gr}})^2 - q^{ab}C_a^{\text{gr}}C_b^{\text{gr}}}}, & \pi \leq 0, \pi^2 \geq q^{ab}\phi_{,a}\phi_{,b}q \\ -\sqrt{-\sqrt{q}C^{\text{gr}} - \sqrt{q}\sqrt{(C^{\text{gr}})^2 - q^{ab}C_a^{\text{gr}}C_b^{\text{gr}}}}, & \pi \leq 0, \pi^2 \leq q^{ab}\phi_{,a}\phi_{,b}q \end{cases} \quad (3)$$

Kuchar-Romano 1995, Markopoulou 1996, Rovelli-Smolin 1997,
Thiemann 2006, Domagała, Giesel, Kamiński, L, Dziendzikowski
2010, 2012, Pawłowski, Husain 2011, Giesel, Thiemann 2013

The canonical LOOPS'13 – p.18

Quantum scalar field compatible with LQG

Scalar field defined on Σ :

$$\{\phi(x), \pi(x)\} = \delta(x, y)$$

The polymer variables compatible with LQG [Ashtekar, L, Sahlmann 2002](#):

$$N_{\lambda,x}(\phi) = \exp(i\lambda\phi(x)), \quad \pi(f) = \int_{\Sigma} \pi f$$
$$\{N_{\lambda,x}, \pi(f)\} = i\lambda f(x)N_{\lambda,x}.$$

The quantum *-algebra:

$$[\hat{N}_{\lambda,x}, \hat{\pi}(f)] = -\lambda f(x)\hat{N}_{\lambda,x}$$

A unique $\text{Diff}(\Sigma)$ invariant state:

$$\omega(\hat{a}) = 0 \text{ unless } \hat{a} = 1$$

$$\mathcal{H}_{\phi} = \text{Span}(|\lambda_1, x_1; \dots, \lambda_n, x_n\rangle : x_i \in \Sigma, \lambda_i \in \mathbb{R}, n \in (N))$$

$$\langle \lambda'_1, x'_1; \dots, \lambda'_{n'}, x'_{n'} | \lambda_1, x_1; \dots, \lambda_n, x_n \rangle = \delta_{nn'} \dots \delta_{x_n x'_{n'}}$$

The canonical LOOPS'13 – p.19

GR+scalar field: quantum solutions

- ➊ The diffeomorphism constraint:

There is in the LQG framework the well defined diff-averaging

$$\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_\phi \ni |\Psi\rangle \mapsto \langle \int_{\text{Diff}} D\varphi'' \langle \Psi | U_\varphi$$
 (4)

for Ψ from suitable domain. It gives:

$$(\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_\phi)_{\text{diff}}$$
 (5)

- ➋ the scalar constraint:

- ➌ In $(\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_\phi)_{\text{diff}}$ we define $e^{i \int d^3x \widehat{\phi(x)} h(x)}$ (in the physical subspace)

- ➍ The general solution to the quantum scalar constraint is:

$$\Psi(\phi) = e^{i \int d^3x \widehat{\phi(x)} h(x)} \psi,$$
 (6)

where

$$\frac{\delta}{\delta \phi(x)} \psi = 0 \Rightarrow \psi \in \mathcal{H}_{\text{gr,diff}}.$$

The canonical LOOPS13 - p.20

GR+scalar field: quantum solutions

- ➊ The diffeomorphism constraint:

There is in the LQG framework the well defined diff-averaging

$$\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_\phi \ni |\Psi\rangle \mapsto \langle \int_{\text{Diff}} D\varphi'' \langle \Psi | U_\varphi$$
 (4)

for Ψ from suitable domain. It gives:

$$(\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_\phi)_{\text{diff}}$$
 (5)

- ➋ the scalar constraint:

- ➌ In $(\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_\phi)_{\text{diff}}$ we define $e^{i \int \widehat{d^3x \phi(x)} h(x)}$ (in the physical subspace)
- ➍ The general solution to the quantum scalar constraint is:

$$\Psi(\phi) = e^{i \int \widehat{d^3x \phi(x)} h(x)} \psi,$$
 (6)

where

$$\frac{\delta}{\delta \phi(x)} \psi = 0 \Rightarrow \psi \in \mathcal{H}_{\text{gr,diff}}.$$

The canonical LOOPS13 - p.20

GR+scalar field: all the sectors

Actually, we have to consider 4 operators:

$$e^{\int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)}$$

and the physical states of the 4 types:

$$e^{i \int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)} \psi \in (\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_{\phi})_{\text{diff}}$$

The scalar product between the solutions is still that of $(\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_{\phi})_{\text{diff}}$.

An operator \hat{L} in $(\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_{\phi})_{\text{diff}}$ defines a Dirac observable:

$$\mathcal{O}(\hat{L}) e^{i \int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)} \psi = e^{i \int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)} \hat{L} \psi \quad (8)$$

Those observables form an algebra

$$\mathcal{O}(\hat{L}) \mathcal{O}(\hat{L}') = \mathcal{O}(\hat{L} \hat{L}') \quad (9)$$

(suppose the operators \hat{L} are bounded).

The canonical LOOPS'13 – p.21

GR+scalar field: $C_a^{\text{gr}} = ?$

The operator we need is

$$e^{\pm i \int \phi \sqrt{-\sqrt{q} C^{\text{gr}}} \pm \widehat{\sqrt{q} \sqrt{(C^{\text{gr}})^2 - q^{ab} C_a^{\text{gr}} C_b^{\text{gr}}}}}$$

What do we do with the terms C_a^{gr} ? Assume the consistency with first solving the constraint classically, gauge fixing and next quantising. In the $+-$ and $--$ case, the “first solve” case is the Rovelli-Smolin model. For the consistency we need

$$e^{\pm i \int \phi \sqrt{-\sqrt{q} C^{\text{gr}}} \pm \widehat{\sqrt{q} \sqrt{(C^{\text{gr}})^2 - q^{ab} C_a^{\text{gr}} C_b^{\text{gr}}}}} \psi = e^{\pm i \int \phi \sqrt{-2\sqrt{q} C^{\text{gr}}}} \psi$$

This is equivalent to the ordering “ \hat{C}_a^{gr} to the right”, because ψ is diff invariant. For the other sectors we need to perform “first solve next quantise” to guess the consistent quantization.

GR+scalar field: all the sectors

Actually, we have to consider 4 operators:

$$e^{\int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)}$$

and the physical states of the 4 types:

$$e^{i \int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)} \psi \in (\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_{\phi})_{\text{diff}}$$

The scalar product between the solutions is still that of $(\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_{\phi})_{\text{diff}}$.

An operator \hat{L} in $(\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_{\phi})_{\text{diff}}$ defines a Dirac observable:

$$\mathcal{O}(\hat{L}) e^{i \int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)} \psi = e^{i \int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)} \hat{L} \psi \quad (8)$$

Those observables form an algebra

$$\mathcal{O}(\hat{L}) \mathcal{O}(\hat{L}') = \mathcal{O}(\hat{L} \hat{L}') \quad (9)$$

(suppose the operators \hat{L} are bounded).

The canonical LOOPS'13 – p.21

Gravity coupled to a potentialless scalar field

A, E - gravitational dof, ϕ, π - a real scalar field

The scalar constraint can be solved:

$$\pi(x) = h(A(x), E(x)), \quad \text{where}$$

- for non-rotating dust

$$h(x) = C^{gr}(x)$$

- for massless scalar field

$$h = \begin{cases} \sqrt{-\sqrt{q}C^{\text{gr}} + \sqrt{q}\sqrt{(C^{\text{gr}})^2 - q^{ab}C_a^{\text{gr}}C_b^{\text{gr}}}}, & \pi \geq 0, \pi^2 \geq q^{ab}\phi_{,a}\phi_{,b}q \\ \sqrt{-\sqrt{q}C^{\text{gr}} - \sqrt{q}\sqrt{(C^{\text{gr}})^2 - q^{ab}C_a^{\text{gr}}C_b^{\text{gr}}}}, & \pi \geq 0, \pi^2 \leq q^{ab}\phi_{,a}\phi_{,b}q \\ -\sqrt{-\sqrt{q}C^{\text{gr}} + \sqrt{q}\sqrt{(C^{\text{gr}})^2 - q^{ab}C_a^{\text{gr}}C_b^{\text{gr}}}}, & \pi \leq 0, \pi^2 \geq q^{ab}\phi_{,a}\phi_{,b}q \\ -\sqrt{-\sqrt{q}C^{\text{gr}} - \sqrt{q}\sqrt{(C^{\text{gr}})^2 - q^{ab}C_a^{\text{gr}}C_b^{\text{gr}}}}, & \pi \leq 0, \pi^2 \leq q^{ab}\phi_{,a}\phi_{,b}q \end{cases} \quad (3)$$

Kuchar-Romano 1995, Markopoulou 1996, Rovelli-Smolin 1997,
Thiemann 2006, Domagała, Giesel, Kamiński, L, Dziendzikowski
2010, 2012, Pawłowski, Husain 2011, Giesel, Thiemann 2013

The canonical LOOPS'13 – p.18

GR+scalar field: all the sectors

Actually, we have to consider 4 operators:

$$e^{\int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)}$$

and the physical states of the 4 types:

$$e^{i \int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)} \psi \in (\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_{\phi})_{\text{diff}}$$

The scalar product between the solutions is still that of $(\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_{\phi})_{\text{diff}}$.

An operator \hat{L} in $(\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_{\phi})_{\text{diff}}$ defines a Dirac observable:

$$\mathcal{O}(\hat{L}) e^{i \int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)} \psi = e^{i \int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)} \hat{L} \psi \quad (8)$$

Those observables form an algebra

$$\mathcal{O}(\hat{L}) \mathcal{O}(\hat{L}') = \mathcal{O}(\hat{L} \hat{L}') \quad (9)$$

(suppose the operators \hat{L} are bounded).

The canonical LOOPS'13 – p.21

GR+scalar field: all the sectors

Actually, we have to consider 4 operators:

$$e^{\int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)}$$

and the physical states of the 4 types:

$$e^{i \int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)} \psi \in (\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_{\phi})_{\text{diff}}$$

The scalar product between the solutions is still that of $(\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_{\phi})_{\text{diff}}$.

An operator \hat{L} in $(\mathcal{H}_{\text{gr}} \otimes \mathcal{H}_{\phi})_{\text{diff}}$ defines a Dirac observable:

$$\mathcal{O}(\hat{L}) e^{i \int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)} \psi = e^{i \int d^3x \widehat{\phi}(x) \widehat{h}_{\pm\pm}(x)} \hat{L} \psi \quad (8)$$

Those observables form an algebra

$$\mathcal{O}(\hat{L}) \mathcal{O}(\hat{L}') = \mathcal{O}(\hat{L} \hat{L}') \quad (9)$$

(suppose the operators \hat{L} are bounded).

The canonical LOOPS'13 - p.21

GR+scalar field: $C_a^{\text{gr}} = ?$

The operator we need is

$$e^{\pm i \int \phi \sqrt{-\sqrt{q} C^{\text{gr}}} \pm \widehat{\sqrt{q} \sqrt{(C^{\text{gr}})^2 - q^{ab} C_a^{\text{gr}} C_b^{\text{gr}}}}}$$

What do we do with the terms C_a^{gr} ? Assume the consistency with first solving the constraint classically, gauge fixing and next quantising. In the $+-$ and $--$ case, the “first solve” case is the Rovelli-Smolin model. For the consistency we need

$$e^{\pm i \int \phi \sqrt{-\sqrt{q} C^{\text{gr}}} \pm \widehat{\sqrt{q} \sqrt{(C^{\text{gr}})^2 - q^{ab} C_a^{\text{gr}} C_b^{\text{gr}}}}} \psi = e^{\pm i \int \phi \sqrt{-2\sqrt{q} C^{\text{gr}}}} \psi$$

This is equivalent to the ordering “ \hat{C}_a^{gr} to the right”, because ψ is diff invariant. For the other sectors we need to perform “first solve next quantise” to guess the consistent quantization.

GR+scalar field: $C_a^{\text{gr}} = ?$

The operator we need is

$$e^{\pm i \int \phi \sqrt{-\sqrt{q} C^{\text{gr}}} \pm \sqrt{q} \widehat{\sqrt{(C^{\text{gr}})^2 - q^{ab} C_a^{\text{gr}} C_b^{\text{gr}}}}}$$

What do we do with the terms C_a^{gr} ? Assume the consistency with first solving the constraint classically, gauge fixing and next quantising. In the $++$ and $--$ case, the “first solve” case is the Rovelli-Smolin model. For the consistency we need

$$e^{\pm i \int \phi \sqrt{-\sqrt{q} C^{\text{gr}}} \pm \sqrt{q} \widehat{\sqrt{(C^{\text{gr}})^2 - q^{ab} C_a^{\text{gr}} C_b^{\text{gr}}}}} \psi = e^{\pm i \int \phi \widehat{\sqrt{-2\sqrt{q} C^{\text{gr}}}}} \psi$$

This is equivalent to the ordering “ $\widehat{C}_a^{\text{gr}}$ to the right”, because ψ is diff invariant. For the other sectors we need to perform “first solve next quantise” to guess the consistent quantization.

GR+scalar field: $C_a^{\text{gr}} = ?$

The operator we need is

$$e^{\pm i \int \phi \sqrt{-\sqrt{q} C^{\text{gr}}} \pm \widehat{\sqrt{q} \sqrt{(C^{\text{gr}})^2 - q^{ab} C_a^{\text{gr}} C_b^{\text{gr}}}}}$$

What do we do with the terms C_a^{gr} ? Assume the consistency with first solving the constraint classically, gauge fixing and next quantising. In the $+-$ and $--$ case, the “first solve” case is the Rovelli-Smolin model. For the consistency we need

$$e^{\pm i \int \phi \sqrt{-\sqrt{q} C^{\text{gr}}} \pm \widehat{\sqrt{q} \sqrt{(C^{\text{gr}})^2 - q^{ab} C_a^{\text{gr}} C_b^{\text{gr}}}}} \psi = e^{\pm i \int \phi \sqrt{-2\sqrt{q} C^{\text{gr}}}} \psi$$

This is equivalent to the ordering “ \hat{C}_a^{gr} to the right”, because ψ is diff invariant. For the other sectors we need to perform “first solve next quantise” to guess the consistent quantization.

What diff invariant observables L ?

Observables can be constructed by quantising diffeo invariant kinematical observables $L(A, E)$. In LQG there are available quantum operators of:

$$\int d^3x \sqrt{q}, \int d^3x \pi, \frac{1}{2} \int d^3x \frac{\pi^2}{\sqrt{q}} + \phi_{,a}\phi_{,b} q^{ab} \sqrt{q}, \int \sqrt{q} R^{(3)}.$$

Can we do better? See Jędrzej Świerkowski's talk. Just briefly:

Suppose there is an observer at a fixed $x_0 \in \Sigma$; the observer has his sphere of directions parametrised by angles θ, φ .

For every (A, E, ϕ, π) use the geodesics of the corresponding q_{ab} to introduce in a neighborhood of x_0 in Σ coordinates $s^i = r, \theta, \varphi$:

$$r(x) = \text{distance}(x, x_0), \quad \theta(x), \quad \varphi(x).$$

Using the coordinates $(s^i)(E)$ define for example:

$$\mathcal{Q}^{\theta\theta}(r, \theta, \varphi) = q^{ab} \theta_{,a} \theta_{,b}(x(r, \theta, \varphi))$$

etc. $\mathcal{Q}^{\theta\theta}(r, \theta, \varphi)$ is invariant with respect to the diffeomorphisms preserving the observer: x_0 and its sphere of directions. We can take it for L :

$$L = \mathcal{Q}^{AB}(r, \theta, \varphi).$$

The canonical LOOPS'13 - p.23

Closing remarks

- ➊ advantage of the massless scalar field against the Husain-Pawlowski scalar field
- ➋ is the conclusion of the AQG correct?
- ➌ The relational observables: innocent mistakes in the literature, read Dapor, Kaminski, L, Swiezewski 2103
- ➍ $[C(N), C(M)]$ see Madhavan's talk
- ➎ ...

The canonical LOOPS'13 – p.24

$$F = dA + A \wedge A$$
$$\mathcal{O} = \{h(x), h(x')\}$$
$$\mathcal{O} = \sqrt{-2\sqrt{g}} C^{\mu} dx^{\nu}$$
$$\sqrt{-2\sqrt{g}} C^{\mu}(x)$$

$$F = dA + A \wedge A$$
$$\mathcal{O} = \underline{\{h(x), h(x')\}}$$

$$\rho = \sqrt{-2\sqrt{g} C^{\mu\nu}} dx^\mu$$

$$\sqrt{-2\sqrt{g} C^{\mu\nu}}(x)$$



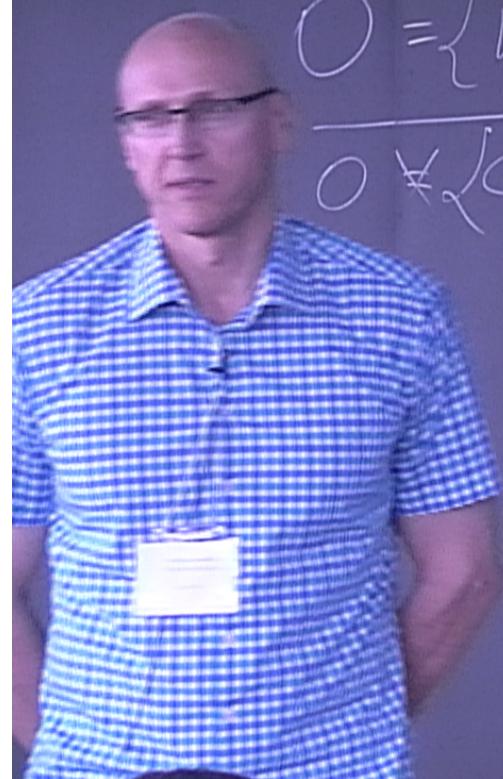
$$F = dA + A \wedge A$$
$$\mathcal{O} = \left\{ h(x), h(x') \right\}$$
$$\mathcal{O} = \left\{ \sqrt{-2\sqrt{g}} C^{\mu\nu} dx^\nu \right\}$$
$$\mathcal{O} = \left\{ \sqrt{-2\sqrt{g}} C^{\mu\nu} dx^\nu \right\}$$



$$F = dA + A \wedge A$$

$$\mathcal{O} = \left\{ h(x), h(x') \right\}$$
$$\frac{\mathcal{O}}{\mathcal{O} \times \left\{ C^{s_1}(x), C^{s_2}(x') \right\}}$$

$$\rho = \sqrt{-2\sqrt{q} C^{\nu}} dx$$
$$\sqrt{-2\sqrt{q} C^{\nu}}(x)$$



The unique state

The quantum *-algebra (abstract), defined by:

$$A_B^A \mapsto \hat{A}_B^A, \quad E_{S,f} \mapsto \hat{E}_{S,f}, \quad \{\cdot, \cdot\} \mapsto \frac{1}{i\hbar}[\cdot, \cdot]$$

and the natural relations satisfied by $E_{S,f}$,
admits a unique $\text{Diff}(M)$ invariant state

$$\omega : \hat{A}_{B_1 e_1}^{A_1}, \dots, \hat{A}_{B_n e_n}^{A_n} \hat{E}_{S_1, f_1}, \dots, \hat{E}_{S_m, f_m} \mapsto \mathbb{C}.$$

(L, Okolow, Sahlmann, Thiemann 2005)

It is defined as follows:

$$\omega(\dots E_{S,f}) = 0, \quad \omega(f(A_{e_1}, \dots, A_{e_n})) := \int f(A) d\mu(A).$$

Open questions:

- ➊ $\{E_{S,f}, E_{S',f'}\} \neq 0$
- ➋ diffeo covariant representations of the QHF algebra which do not admit a diffeo invariant state?

The canonical LOOPS'13 – p.8



Quantum matter in quantum space-time

Mikhail Kagan

Penn State Abington

and

Institute for Gravitation and the Cosmos, Penn State

in collaboration with

M. Bojowald, G. Hossain, C. Tomlin

Motivation

What we learned about quantum corrections:

- Anomaly free quantum corrected effective equations exist
- Non-trivial deformations of constraint algebra
- Gauge cannot be fixed prior to quantization
- Can't quantize matter without quantizing space-time

Motivation

What we learned about quantum corrections:

- Anomaly free quantum corrected effective equations exist
- Non-trivial deformations of constraint algebra
- Gauge cannot be fixed prior to quantization
- Can't quantize matter without quantizing space-time

Motivation

What we learned about quantum corrections:

- Anomaly free quantum corrected effective equations exist
- Non-trivial deformations of constraint algebra
- Gauge cannot be fixed prior to quantization
- Can't quantize matter without quantizing space-time

Conservation laws and space-time (ST) symmetries

- Hamilton equations (classical or quantum)
$$\hat{\vec{p}} = -i\hbar\vec{\nabla} \text{ and } \hat{E} = i\hbar\partial/\partial t$$
- Local energy conservation → local ST symmetries → general covariance
- Quantum corrections may affect structure of ST or form of energy conservation

3

Outline

1. Energy conservation in covariant form
2. Canonical gravity
 - Stress-energy tensor components
 - Space-time symmetries
3. Energy conservation in canonical form
4. Deformed energy conservation
5. Conclusions

Energy conservation in covariant form

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma_{\mu\rho}^\mu T^{\rho\nu} - \Gamma_{\mu\nu}^\rho T^{\mu\rho} = 0$$

With

$$T_{\mu\nu} = \frac{-2}{\sqrt{|\det g|}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}$$

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\sigma} \left(\frac{\partial g_{\sigma\mu}}{\partial x^\nu} + \frac{\partial g_{\sigma\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right)$$

Energy conservation in covariant form

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma_{\mu\rho}^\mu T^{\rho\nu} - \Gamma_{\mu\nu}^\rho T^{\mu\rho} = 0$$

With

$$T_{\mu\nu} = \frac{-2}{\sqrt{|\det g|}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}$$

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\sigma} \left(\frac{\partial g_{\sigma\mu}}{\partial x^\nu} + \frac{\partial g_{\sigma\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right)$$

For $\nu=0$

$$\sqrt{-\det g} \nabla_\mu T^{\mu 0} = \partial_\mu (\sqrt{-\det g} T^{\mu 0}) - \frac{1}{2} \frac{\partial g_{\mu\rho}}{\partial t} T^{\mu\rho}$$

Toward canonical formulation. Strategy

3+1 (ADM) decomposition

- Metric

$$g^{\mu\nu} \rightarrow N, N^a, h^{ab}$$

- Action

$$S \rightarrow \text{Constraints + Poisson Brackets}$$

Toward canonical formulation. Strategy

3+1 (ADM) decomposition

- Metric

$$g^{\mu\nu} \rightarrow N, N^a, h^{ab}$$

- Action

$$S \rightarrow \text{Constraints} + \text{Poisson Brackets}$$

E.g. for scalar field

$$S_{\text{matter}} = \int d^4x \left(\dot{\phi}_I p^I - N \mathcal{H}_{\text{matter}} - N^a \mathcal{D}_a^{\text{matter}} \right)$$

$$H_{\text{matter}}[N, N^a] = \int d^3x (N \mathcal{H}_{\text{matter}} + N^a \mathcal{D}_a^{\text{matter}})$$

Toward canonical formulation. Metric components

Covariant

$$g_{00} = -N^2 + h_{ab}N^a N^b$$

$$g_{0a} = h_{ab}N^b$$

$$g_{ab} = h_{ab}$$

Toward canonical formulation. Metric components

Covariant

$$g_{00} = -N^2 + h_{ab}N^a N^b$$

$$g_{0a} = h_{ab}N^b$$

$$g_{ab} = h_{ab}$$

Contravariant

$$g^{00} = -\frac{1}{N^2}$$

$$g^{0a} = \frac{N^a}{N^2}$$

$$g^{ab} = h^{ab} - \frac{N^a N^b}{N^2}$$

Determinant

$$\det g_{\mu\nu} = -N^2 \det h_{ab}$$

Toward canonical formulation. Metric components

Covariant

$$g_{00} = -N^2 + h_{ab}N^a N^b$$

$$g_{0a} = h_{ab}N^b$$

$$g_{ab} = h_{ab}$$

Contravariant

$$g^{00} = -\frac{1}{N^2}$$

$$g^{0a} = \frac{N^a}{N^2}$$

$$g^{ab} = h^{ab} - \frac{N^a N^b}{N^2}$$

Determinant

$$\det g_{\mu\nu} = -N^2 \det h_{ab}$$

Toward canonical formulation. Metric components

Covariant

$$g_{00} = -N^2 + h_{ab}N^a N^b$$

$$g_{0a} = h_{ab}N^b$$

$$g_{ab} = h_{ab}$$

Contravariant

$$g^{00} = -\frac{1}{N^2}$$

$$g^{0a} = \frac{N^a}{N^2}$$

$$g^{ab} = h^{ab} - \frac{N^a N^b}{N^2}$$

Variations

$$\delta g^{00} = \frac{2\delta N}{N^3}$$

$$\delta g^{0a} = \frac{\delta N^a}{N^2} - \frac{2N^a \delta N}{N^3}$$

$$\delta g^{ab} = \delta h^{ab} - \frac{1}{N^2} \left(N^a \delta N^b + N^b \delta N^a - \frac{2N^a N^b}{N} \delta N \right)$$

Determinant

$$\det g_{\mu\nu} = -N^2 \det h_{ab}$$

12

Canonical formulation. Stress-energy components

Variation of matter action

$$\begin{aligned}\delta S_{\text{matter}} &= -\frac{1}{2} \int d^4x \sqrt{-\det g} T_{\mu\nu} \delta g^{\mu\nu} \\ &= -\frac{1}{2} \int d^4x N \sqrt{\det h} (T_{00} \delta g^{00} + T_{a0} \delta g^{a0} + T_{0b} \delta g^{0b} + T_{ab} \delta g^{ab}) \\ &= - \int d^4x \sqrt{\det h} \left(\frac{\delta N}{N^2} (T_{00} - 2N^a T_{a0} + N^a N^b T_{ab}) \right. \\ &\quad \left. + \frac{\delta N^a}{N} (T_{a0} - N^b T_{ab}) + \frac{N}{2} \delta h^{ab} T_{ab} \right)\end{aligned}$$

Canonical formulation. Stress-energy components

Variation of matter action

$$\begin{aligned}\delta S_{\text{matter}} &= -\frac{1}{2} \int d^4x \sqrt{-\det g} T_{\mu\nu} \delta g^{\mu\nu} \\ &= -\frac{1}{2} \int d^4x N \sqrt{\det h} (T_{00} \delta g^{00} + T_{a0} \delta g^{a0} + T_{0b} \delta g^{0b} + T_{ab} \delta g^{ab}) \\ &= - \int d^4x \sqrt{\det h} \left(\frac{\delta N}{N^2} (T_{00} - 2N^a T_{a0} + N^a N^b T_{ab}) \right. \\ &\quad \left. + \frac{\delta N^a}{N} (T_{a0} - N^b T_{ab}) + \frac{N}{2} \delta h^{ab} T_{ab} \right)\end{aligned}$$

Stress-energy tensor components

$$\begin{aligned}T_{00} &= -\frac{N}{\sqrt{\det h}} \left(N \frac{\delta S_{\text{matter}}}{\delta N} + 2N^a \frac{\delta S_{\text{matter}}}{\delta N^a} + 2 \frac{N^a N^b}{N^2} \frac{\delta S_{\text{matter}}}{\delta h^{ab}} \right) \\ T_{0a} &= -\frac{N}{\sqrt{\det h}} \left(\frac{\delta S_{\text{matter}}}{\delta N^a} + 2 \frac{N^b}{N^2} \frac{\delta S_{\text{matter}}}{\delta h^{ab}} \right) \\ T_{ab} &= -\frac{2}{N \sqrt{\det h}} \frac{\delta S_{\text{matter}}}{\delta h^{ab}}\end{aligned}$$

14

Canonical formulation. Evolution equations

$$\dot{f} = \frac{\partial f}{\partial t} = \{f, H_{\text{total}}[N, N^a]\}$$

where

$$H_{\text{total}}[N, N^a] = H_{\text{total}}[N] + D_{\text{total}}[N^a]$$

Canonical formulation. Evolution equations

$$\dot{f} = \frac{\partial f}{\partial t} = \{f, H_{\text{total}}[N, N^a]\}$$

where

$$H_{\text{total}}[N, N^a] = H_{\text{total}}[N] + D_{\text{total}}[N^a]$$

Constraint algebra

$$\{D_{\text{total}}[M^a], D_{\text{total}}[N^a]\} = -D_{\text{total}}[N^b \partial_b M^a - M^b \partial_b N^a]$$

$$\{H_{\text{total}}[M], D_{\text{total}}[N^a]\} = -H_{\text{total}}[N^b \partial_b M]$$

$$\{H_{\text{total}}[M], H_{\text{total}}[N]\} = D_{\text{total}}[h^{ab}(M \nabla_b N - N \partial_b M)]$$

Energy conservation in canonical form.

$$\begin{aligned} N\sqrt{\det h}\nabla_\mu T^\mu{}_0 &= \partial_\mu(N\sqrt{\det h}T^\mu_0) + \frac{\partial g_{\mu\rho}}{\partial t} \frac{\delta H_{\text{matter}}}{\delta g_{\mu\rho}} \\ &= -\partial_0 \mathcal{C}_{\text{matter}}[N, N^a] + N^b \partial_b \mathcal{C}_{\text{matter}}[N, N^a] + (\partial_b N^b) \mathcal{C}_{\text{matter}}[N, N^a] \\ &\quad + \frac{\partial N}{\partial t} \mathcal{H}_{\text{matter}} + \frac{\partial N^a}{\partial t} \mathcal{D}_a^{\text{matter}} + \frac{\partial h_{ab}}{\partial t} \frac{\delta H_{\text{matter}}}{\delta h_{ab}} \\ &\quad + \partial_b \left(N^2 h^{ab} \frac{\delta H_{\text{matter}}}{\delta N^a} + 2N^c h^{ba} \frac{\delta H_{\text{matter}}}{\delta h^{ac}} \right) \end{aligned}$$

Energy conservation in canonical form.

$$\begin{aligned} N\sqrt{\det h}\nabla_\mu T^\mu{}_0 &= \partial_\mu(N\sqrt{\det h}T^\mu_0) + \frac{\partial g_{\mu\rho}}{\partial t} \frac{\delta H_{\text{matter}}}{\delta g_{\mu\rho}} \\ &= -\partial_0 \mathcal{C}_{\text{matter}}[N, N^a] + N^b \partial_b \mathcal{C}_{\text{matter}}[N, N^a] + (\partial_b N^b) \mathcal{C}_{\text{matter}}[N, N^a] \\ &\quad + \frac{\partial N}{\partial t} \mathcal{H}_{\text{matter}} + \frac{\partial N^a}{\partial t} \mathcal{D}_a^{\text{matter}} + \frac{\partial h_{ab}}{\partial t} \frac{\delta H_{\text{matter}}}{\delta h_{ab}} \\ &\quad + \partial_b \left(N^2 h^{ab} \frac{\delta H_{\text{matter}}}{\delta N^a} + 2N^c h^{ba} \frac{\delta H_{\text{matter}}}{\delta h^{ac}} \right) \end{aligned}$$

Energy conservation in canonical form.

$$\begin{aligned} N\sqrt{\det h}\nabla_\mu T^\mu{}_0 &= \partial_\mu(N\sqrt{\det h}T^\mu_0) + \frac{\partial g_{\mu\rho}}{\partial t} \frac{\delta H_{\text{matter}}}{\delta g_{\mu\rho}} \\ &= -\partial_0 \mathcal{C}_{\text{matter}}[N, N^a] + N^b \partial_b \mathcal{C}_{\text{matter}}[N, N^a] + (\partial_b N^b) \mathcal{C}_{\text{matter}}[N, N^a] \\ &\quad + \frac{\partial N}{\partial t} \mathcal{H}_{\text{matter}} + \frac{\partial N^a}{\partial t} \mathcal{D}_a^{\text{matter}} + \frac{\partial h_{ab}}{\partial t} \frac{\delta H_{\text{matter}}}{\delta h_{ab}} \\ &\quad + \partial_b \left(N^2 h^{ab} \frac{\delta H_{\text{matter}}}{\delta N^a} + 2N^c h^{ba} \frac{\delta H_{\text{matter}}}{\delta h^{ac}} \right) \end{aligned}$$

Time derivatives \rightarrow constraint algebra

$$\begin{aligned} \dot{\mathcal{H}}_{\text{matter}} &= \{\mathcal{H}_{\text{matter}}, H_{\text{total}}[N] + D_{\text{total}}[N^a]\} = \{\mathcal{H}_{\text{matter}}, H_{\text{total}}[N]\} + \mathcal{L}_{\vec{N}} \mathcal{H}_{\text{matter}} \\ &= \frac{1}{N} D^a (N^2 \mathcal{D}_a^{\text{matter}}) + \{h_{ab}, H_{\text{grav}}[N]\} \frac{\delta \mathcal{H}_{\text{matter}}}{\delta h_{ab}} + \mathcal{L}_{\vec{N}} \mathcal{H}_{\text{matter}} \end{aligned}$$

Energy conservation in canonical form.

$$\begin{aligned}
 N\sqrt{\det h} \nabla_\mu T^\mu{}_0 &= \partial_\mu(N\sqrt{\det h}T^\mu_0) + \frac{\partial g_{\mu\rho}}{\partial t} \frac{\delta H_{\text{matter}}}{\delta g_{\mu\rho}} \\
 &= -\partial_0 \mathcal{C}_{\text{matter}}[N, N^a] + N^b \partial_b \mathcal{C}_{\text{matter}}[N, N^a] + (\partial_b N^b) \mathcal{C}_{\text{matter}}[N, N^a] \\
 &\quad + \frac{\partial N}{\partial t} \mathcal{H}_{\text{matter}} + \frac{\partial N^a}{\partial t} \mathcal{D}_a^{\text{matter}} + \frac{\partial h_{ab}}{\partial t} \frac{\delta H_{\text{matter}}}{\delta h_{ab}} \\
 &\quad + \partial_b \left(N^2 h^{ab} \frac{\delta H_{\text{matter}}}{\delta N^a} + 2N^c h^{ba} \frac{\delta H_{\text{matter}}}{\delta h^{ac}} \right)
 \end{aligned}$$

Time derivatives \rightarrow constraint algebra

$$\begin{aligned}
 \dot{\mathcal{H}}_{\text{matter}} &= \{\mathcal{H}_{\text{matter}}, H_{\text{total}}[N] + D_{\text{total}}[N^a]\} = \{\mathcal{H}_{\text{matter}}, H_{\text{total}}[N]\} + \mathcal{L}_{\vec{N}} \mathcal{H}_{\text{matter}} \\
 &= \frac{1}{N} D^a (N^2 \mathcal{D}_a^{\text{matter}}) + \{h_{ab}, H_{\text{grav}}[N]\} \frac{\delta \mathcal{H}_{\text{matter}}}{\delta h_{ab}} + \mathcal{L}_{\vec{N}} \mathcal{H}_{\text{matter}}
 \end{aligned}$$

$$\begin{aligned}
 \dot{\mathcal{D}}_a^{\text{matter}} &= \{\mathcal{D}_a^{\text{matter}}, H_{\text{total}}[N] + D_{\text{total}}[N^a]\} = \{\mathcal{D}_a^{\text{matter}}, H_{\text{matter}}[N]\} + \mathcal{L}_{\vec{N}} \mathcal{D}_a^{\text{matter}} \\
 &= \mathcal{H}_{\text{matter}} \partial_a N + 2D^b \frac{\delta H_{\text{matter}}[N]}{\delta h^{ab}} + \mathcal{L}_{\vec{N}} \mathcal{D}_a^{\text{matter}}
 \end{aligned}$$

$$\dot{h}_{ab} = \{h_{ab}, H_{\text{total}}[N] + D_{\text{total}}[N^a]\} = \{h_{ab}, H_{\text{grav}}[N]\} - 2D_{(a} N_{b)}$$

Canonical formulation. Evolution equations

$$\dot{f} = \frac{\partial f}{\partial t} = \{f, H_{\text{total}}[N, N^a]\}$$

where

$$H_{\text{total}}[N, N^a] = H_{\text{total}}[N] + D_{\text{total}}[N^a]$$

Constraint algebra

$$\begin{aligned}\{D_{\text{total}}[M^a], D_{\text{total}}[N^a]\} &= -D_{\text{total}}[N^b \partial_b M^a - M^b \partial_b N^a] \\ \{H_{\text{total}}[M], D_{\text{total}}[N^a]\} &= -H_{\text{total}}[N^b \partial_b M] \\ \{H_{\text{total}}[M], H_{\text{total}}[N]\} &= D_{\text{total}}[h^{ab}(M \nabla_b N - N \partial_b M)]\end{aligned}$$

Canonical formulation. Evolution equations

$$\dot{f} = \frac{\partial f}{\partial t} = \{f, H_{\text{total}}[N, N^a]\}$$

where

$$H_{\text{total}}[N, N^a] = H_{\text{total}}[N] + D_{\text{total}}[N^a]$$

Constraint algebra

$$\{D_{\text{total}}[M^a], D_{\text{total}}[N^a]\} = -D_{\text{total}}[N^b \partial_b M^a - M^b \partial_b N^a]$$

$$\{H_{\text{total}}[M], D_{\text{total}}[N^a]\} = -H_{\text{total}}[N^b \partial_b M]$$

$$\{H_{\text{total}}[M], H_{\text{total}}[N]\} = D_{\text{total}}[h^{ab}(M \nabla_b N - N \partial_b M)]$$

$$\{H_{\text{total}}[M], H_{\text{total}}[N]\} = D_{\text{total}}[\beta h^{ab}(M \partial_b N - N \partial_b M)]$$

Energy conservation in canonical form.

$$\begin{aligned}
 N\sqrt{\det h} \nabla_\mu T^\mu{}_0 &= \partial_\mu(N\sqrt{\det h}T^\mu_0) + \frac{\partial g_{\mu\rho}}{\partial t} \frac{\delta H_{\text{matter}}}{\delta g_{\mu\rho}} \\
 &= -\partial_0 \mathcal{C}_{\text{matter}}[N, N^a] + N^b \partial_b \mathcal{C}_{\text{matter}}[N, N^a] + (\partial_b N^b) \mathcal{C}_{\text{matter}}[N, N^a] \\
 &\quad + \frac{\partial N}{\partial t} \mathcal{H}_{\text{matter}} + \frac{\partial N^a}{\partial t} \mathcal{D}_a^{\text{matter}} + \frac{\partial h_{ab}}{\partial t} \frac{\delta H_{\text{matter}}}{\delta h_{ab}} \\
 &\quad + \partial_b \left(N^2 h^{ab} \frac{\delta H_{\text{matter}}}{\delta N^a} + 2N^c h^{ba} \frac{\delta H_{\text{matter}}}{\delta h^{ac}} \right)
 \end{aligned}$$

Time derivatives \rightarrow constraint algebra

$$\begin{aligned}
 \dot{\mathcal{H}}_{\text{matter}} &= \{\mathcal{H}_{\text{matter}}, H_{\text{total}}[N] + D_{\text{total}}[N^a]\} = \{\mathcal{H}_{\text{matter}}, H_{\text{total}}[N]\} + \mathcal{L}_{\vec{N}} \mathcal{H}_{\text{matter}} \\
 &= \frac{1}{N} D^a (N^2 \mathcal{D}_a^{\text{matter}}) + \{h_{ab}, H_{\text{grav}}[N]\} \frac{\delta \mathcal{H}_{\text{matter}}}{\delta h_{ab}} + \mathcal{L}_{\vec{N}} \mathcal{H}_{\text{matter}}
 \end{aligned}$$

$$\begin{aligned}
 \dot{\mathcal{D}}_a^{\text{matter}} &= \{\mathcal{D}_a^{\text{matter}}, H_{\text{total}}[N] + D_{\text{total}}[N^a]\} = \{\mathcal{D}_a^{\text{matter}}, H_{\text{matter}}[N]\} + \mathcal{L}_{\vec{N}} \mathcal{D}_a^{\text{matter}} \\
 &= \mathcal{H}_{\text{matter}} \partial_a N + 2D^b \frac{\delta H_{\text{matter}}[N]}{\delta h^{ab}} + \mathcal{L}_{\vec{N}} \mathcal{D}_a^{\text{matter}}
 \end{aligned}$$

$$\dot{h}_{ab} = \{h_{ab}, H_{\text{total}}[N] + D_{\text{total}}[N^a]\} = \{h_{ab}, H_{\text{grav}}[N]\} - 2D_{(a} N_{b)}$$

Energy conservation in canonical form.

$$\begin{aligned}
 N\sqrt{\det h} \nabla_\mu T^\mu{}_0 &= \partial_\mu(N\sqrt{\det h}T^\mu_0) + \frac{\partial g_{\mu\rho}}{\partial t} \frac{\delta H_{\text{matter}}}{\delta g_{\mu\rho}} \\
 &= -\partial_0 \mathcal{C}_{\text{matter}}[N, N^a] + N^b \partial_b \mathcal{C}_{\text{matter}}[N, N^a] + (\partial_b N^b) \mathcal{C}_{\text{matter}}[N, N^a] \\
 &\quad + \frac{\partial N}{\partial t} \mathcal{H}_{\text{matter}} + \frac{\partial N^a}{\partial t} \mathcal{D}_a^{\text{matter}} + \frac{\partial h_{ab}}{\partial t} \frac{\delta H_{\text{matter}}}{\delta h_{ab}} \\
 &\quad + \partial_b \left(N^2 h^{ab} \frac{\delta H_{\text{matter}}}{\delta N^a} + 2N^c h^{ba} \frac{\delta H_{\text{matter}}}{\delta h^{ac}} \right)
 \end{aligned}$$

Time derivatives \rightarrow constraint algebra

$$\begin{aligned}
 \dot{\mathcal{H}}_{\text{matter}} &= \{\mathcal{H}_{\text{matter}}, H_{\text{total}}[N] + D_{\text{total}}[N^a]\} = \{\mathcal{H}_{\text{matter}}, H_{\text{total}}[N]\} + \mathcal{L}_{\vec{N}} \mathcal{H}_{\text{matter}} \\
 &= \frac{1}{N} D^a (N^2 \mathcal{D}_a^{\text{matter}}) + \{h_{ab}, H_{\text{grav}}[N]\} \frac{\delta \mathcal{H}_{\text{matter}}}{\delta h_{ab}} + \mathcal{L}_{\vec{N}} \mathcal{H}_{\text{matter}}
 \end{aligned}$$

$$\begin{aligned}
 \dot{\mathcal{D}}_a^{\text{matter}} &= \{\mathcal{D}_a^{\text{matter}}, H_{\text{total}}[N] + D_{\text{total}}[N^a]\} = \{\mathcal{D}_a^{\text{matter}}, H_{\text{matter}}[N]\} + \mathcal{L}_{\vec{N}} \mathcal{D}_a^{\text{matter}} \\
 &= \mathcal{H}_{\text{matter}} \partial_a N + 2D^b \frac{\delta H_{\text{matter}}[N]}{\delta h^{ab}} + \mathcal{L}_{\vec{N}} \mathcal{D}_a^{\text{matter}}
 \end{aligned}$$

$$\dot{h}_{ab} = \{h_{ab}, H_{\text{total}}[N] + D_{\text{total}}[N^a]\} = \{h_{ab}, H_{\text{grav}}[N]\} - 2D_{(a} N_{b)}$$

Summary with quantum corrections

1. Need to reconsider space-time structure and standard tensor calculus:
 - Space-time transformations generated by constraints
 - Can't use four-metric
 - Can't use covariant form of energy conservation

Summary with quantum corrections

1. Need to reconsider space-time structure and standard tensor calculus:
 - Space-time transformations generated by constraints
 - Can't use four-metric
 - Can't use covariant form of energy conservation
2. Can proceed canonically:
 - **Define** conservation of energy as closure of constraint algebra
 - **Define** energy density as combination of constraints
 - Potentially fruitful for cosmology

Summary with quantum corrections

1. Need to reconsider space-time structure and standard tensor calculus:
 - Space-time transformations generated by constraints
 - Can't use four-metric
 - Can't use covariant form of energy conservation
2. Can proceed canonically:
 - **Define** conservation of energy as closure of constraint algebra
 - **Define** energy density as combination of constraints
 - Potentially fruitful for cosmology



LOOPS '13



A quantum Ricci operator for LQG

E. Alesci, M. Assanioussi, J. Lewandowski
Faculty of Physics, University of Warsaw
[to appear]

Perimeter Institute, Waterloo 2013

Plan of the talk

- ➊ Motivation
- ➋ Classical theory
- ➌ Construction of the Ricci operator
- ➍ Properties of the Ricci operator
- ➎ Summary

Motivation

The Hamiltonian constraint in the Lorentzian case:

$$H = c_k^{ij} F_{ab}^k \frac{E_i^a E_j^b}{\sqrt{\det q}} - \sqrt{\det q} R$$

Motivation

The Hamiltonian constraint in the Lorentzian case:

$$H = c_k^{ij} F_{ab}^k \frac{E_i^a E_j^b}{\sqrt{\det q}} - \sqrt{\det q} R$$

Implementing Hamiltonian operators on the kinematical Hilbert space.

⇒ we need to define the quantum operator ($\widehat{\sqrt{\det q} R}$).

Regge Calculus

M an n -dimensional Riemannian manifold. Given a simplicial

decomposition Δ of M and considering that the curvature lies only on the *hinges* ($n - 2$ simplexes) of Δ :

$$S_{EH} = \int_M \sqrt{-g}R d^n x \quad \rightarrow \quad S_R = \frac{1}{2} \int_{\Delta} \sqrt{-g}R d^n x = \sum_i \epsilon_i V_i$$

Regge Calculus

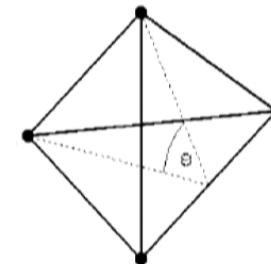
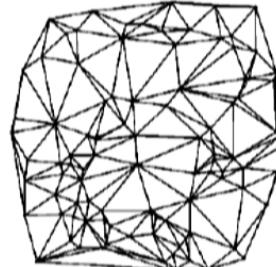
M an n -dimensional Riemannian manifold. Given a simplicial

decomposition Δ of M and considering that the curvature lies only on the *hinges* ($n - 2$ simplexes) of Δ :

$$S_{EH} = \int_M \sqrt{-g}R d^n x \rightarrow S_R = \frac{1}{2} \int_{\Delta} \sqrt{-g}R d^n x = \sum_i \epsilon_i V_i$$

For $n = 3$ the action can be expressed only in terms of the lengths of the segments:

$$S_R [L_i] = \sum_i L_i \cdot \epsilon_i(L_i) = \sum_s \sum_{i \in s} l_i^s \left(\frac{2\pi}{\alpha_i} - \theta_i^s \right)$$



Regge Calculus

- General piecewise flat cellular decompositions?

- General piecewise flat cellular decompositions?

A cellular decomposition Δ of a space Σ is a disjoint union (partition) of open cells of varying dimension satisfying the following conditions:

- An n -dimensional open cell is a topological space which is homeomorphic to the n -dimensional open ball.*
- The boundary of the closure of an n -dimensional cell is contained in a finite union of cells of lower dimension.*

Regge Calculus

- General piecewise flat cellular decompositions?

A cellular decomposition Δ of a space Σ is a disjoint union (partition) of open cells of varying dimension satisfying the following conditions:

- An n -dimensional open cell is a topological space which is homeomorphic to the n -dimensional open ball.*
- The boundary of the closure of an n -dimensional cell is contained in a finite union of cells of lower dimension.*

The same expression holds for arbitrary piecewise flat cellular decomposition

$$S_R [\Delta] = \sum_i L_i \cdot \epsilon_i = \sum_c \sum_{i \in c} l_i^c \left(\frac{2\pi}{\alpha_i} - \theta_i^s \right)$$

Regge Calculus

- General piecewise flat cellular decompositions?

A cellular decomposition Δ of a space Σ is a disjoint union (partition) of open cells of varying dimension satisfying the following conditions:

- An n -dimensional open cell is a topological space which is homeomorphic to the n -dimensional open ball.*
- The boundary of the closure of an n -dimensional cell is contained in a finite union of cells of lower dimension.*

The same expression holds for arbitrary piecewise flat cellular decomposition

$$S_R [\Delta] = \sum_i L_i \cdot \epsilon_i = \sum_c \sum_{i \in c} l_i^c \left(\frac{2\pi}{\alpha_i} - \theta_i^s \right)$$

- Convergence of Regge action?

$$\lim_{l \rightarrow 0} S_R = \frac{1}{2} S_{EH}$$

Construction of the Ricci operator

$$\widehat{\frac{1}{2}S_{EH}} = \lim_{\ell \rightarrow 0} \widehat{S_R}$$

Construction of the Ricci operator

$$\widehat{\frac{1}{2}S_{EH}} = \lim_{\ell \rightarrow 0} \widehat{S_R}$$

Classical expressions:

Given a curve γ embedded in a 3-manifold Σ

$$L(\gamma) = \int_0^1 ds \sqrt{\delta_{ij} G^i(s) G^j(s)}$$

where

$$G^i(s) = \frac{\frac{1}{2}\epsilon^{ijk}\epsilon_{abc}E_j^b E_k^c \dot{\gamma}^a(s)}{\sqrt{\frac{1}{3!}\epsilon^{ijk}\epsilon_{abc}E_i^a E_j^b E_k^c}}$$

Given two surfaces S^1 and S^2 intersecting in the curve γ

$$\theta_{12}(s) = \pi - \arccos \left[\frac{\delta^{jk} E_j^b n_b(S^1, s) E_k^c n_c(S^2, s)}{|E_j^b n_b(S^1, s)| |E_k^c n_c(S^2, s)|} \right]$$

where $|E_j^b n_b(S^k, s)| = \sqrt{\delta_{ij} E_i^b n_b(S^k, s) E_j^c n_c(S^k, s)}$ and $n_b(S^k, s)$ is the normal one form on the surface S_k .

- General piecewise flat cellular decompositions?
- ① A cellular decomposition Δ of a space Σ is a disjoint union (partition) of open cells of varying dimension satisfying the following conditions:
 - An n -dimensional open cell is a topological space which is homeomorphic to the n -dimensional open ball.*
 - The boundary of the closure of an n -dimensional cell is contained in a finite union of cells of lower dimension.*

- ③ Construction of the Ricci operator
The same expression holds for arbitrary piecewise flat cellular decomposition

- ④ Properties of $S_R[\Delta]$
$$S_R[\Delta] = \sum_i L_i \cdot \epsilon_i = \sum_c \sum_{i \in c} l_i^c \left(\frac{2\pi}{\alpha_i} - \theta_i^s \right)$$

- ⑤ Convergence
• Convergence of Regge action?

$$\lim_{l \rightarrow 0} S_R = \frac{1}{2} S_{EH}$$

Construction of the Ricci operator

$$\widehat{\frac{1}{2}S_{EH}} = \lim_{\ell \rightarrow 0} \widehat{S_R}$$

Classical expressions:

Given a curve γ embedded in a 3-manifold Σ

$$L(\gamma) = \int_0^1 ds \sqrt{\delta_{ij} G^i(s) G^j(s)}$$

where

$$G^i(s) = \frac{\frac{1}{2}\epsilon^{ijk}\epsilon_{abc}E_j^b E_k^c \dot{\gamma}^a(s)}{\sqrt{\frac{1}{3!}\epsilon^{ijk}\epsilon_{abc}E_i^a E_j^b E_k^c}}$$

Given two surfaces S^1 and S^2 intersecting in the curve γ

$$\theta_{12}(s) = \pi - \arccos \left[\frac{\delta^{jk} E_j^b n_b(S^1, s) E_k^c n_c(S^2, s)}{|E_j^b n_b(S^1, s)| |E_k^c n_c(S^2, s)|} \right]$$

where $|E_j^b n_b(S^k, s)| = \sqrt{\delta_{ij} E_i^b n_b(S^k, s) E_j^c n_c(S^k, s)}$ and $n_b(S^k, s)$ is the normal one form on the surface S_k .

Construction of the Ricci operator

To match Regge calculus context with LQG framework we invoke the dual picture of LQG.

Construction of the Ricci operator

To match Regge calculus context with LQG framework we invoke the dual picture of LQG.

A spin network graph → a cellular decomposition = *covering cellular decomposition*

A cellular decomposition Δ of a three-dimensional space Σ built on a graph Γ is said to be a covering cellular decomposition of Γ if:

- i) *Each 3-cell of Δ contains at most one vertex of Γ ;*
- ii) *Each 2-cell (face) of Δ is punctured at most by one edge of Γ and the intersection belongs to the interior of the edge;*
- iii) *Two 3-cells of Δ are glued such that the identified 2-cells match.*

Construction of the Ricci operator

To match Regge calculus context with LQG framework we invoke the dual picture of LQG.

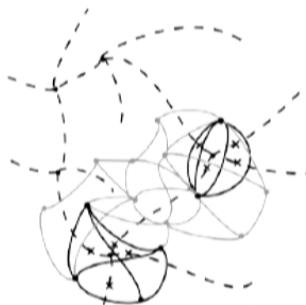
A spin network graph → a cellular decomposition = *covering cellular decomposition*

A cellular decomposition Δ of a three-dimensional space Σ built on a graph Γ is said to be a covering cellular decomposition of Γ if:

- i) *Each 3-cell of Δ contains at most one vertex of Γ ;*
- ii) *Each 2-cell (face) of Δ is punctured at most by one edge of Γ and the intersection belongs to the interior of the edge;*
- iii) *Two 3-cells of Δ are glued such that the identified 2-cells match.*

An additional requirement:

- iv) *If two 2-cells of the boundary of a 3-cell intersect, then their intersection is a connected 1-cell.*



7 / 22

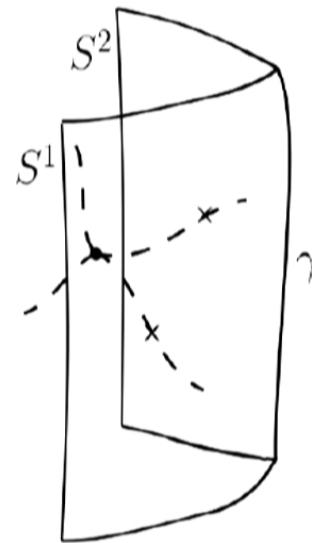
The length operator

Dual picture of LQG \Rightarrow Bianchi length operator

The length operator

Dual picture of LQG \Rightarrow Bianchi length operator

This length operator measures the length of the curve γ



[E. Bianchi, Nucl.Phys.B807:591-624 (2009). arXiv:0806.4710v2]

The length operator

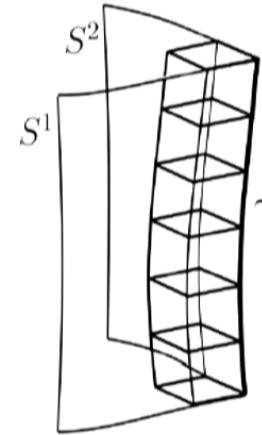
Classical expression:

$$L(\gamma) = \int_0^1 ds \sqrt{\delta_{ij} G^i(s) G^j(s)} \quad ; \quad G^i(s) = \frac{\frac{1}{2} \epsilon^{ijk} \epsilon_{abc} E_j^b E_k^c \dot{\gamma}^a(s)}{\sqrt{\frac{1}{3!} |\epsilon^{ijk} \epsilon_{abc} E_i^a E_j^b E_k^c|}}$$

The one-dimensional integral is replaced by the limit of a Riemann sum:

$$L(\gamma) = \lim_{\Delta s \rightarrow 0} \sum_I \Delta s \sqrt{\delta_{ij} G_{\Delta s}^i(s_I) G_{\Delta s}^j(s_I)}$$

with $s_I \in [I \Delta s; (I + 1) \Delta s]$.



$$G_{\Delta s}^i(s_I) = \frac{\frac{1}{2} \frac{1}{(\Delta s)^2} \int_{S_I^1} d^2 \sigma \int_{S_I^2} d^2 \sigma' V_{x_I}^{ijk}(\sigma, \sigma') E_j^a(\sigma) n_a(\sigma) E_k^b(\sigma') n_b(\sigma')}{\sqrt{\frac{1}{8 \cdot 3!} \frac{1}{(\Delta s)^6} \int_{S_I^1} d^2 \sigma \int_{S_I^2} d^2 \sigma' \int_{S_I^2} d^2 \sigma'' \left| T_{x_I}^{ijk}(\sigma, \sigma', \sigma'') E_i^a(\sigma) n_a(\sigma) E_j^b(\sigma') n_b(\sigma') E_k^c(\sigma'') n_c(\sigma'') \right|}}$$

$$n_a = \epsilon_{abc} \frac{\partial X^b}{\partial \sigma^1} \frac{\partial X^c}{\partial \sigma^2}$$

The length operator

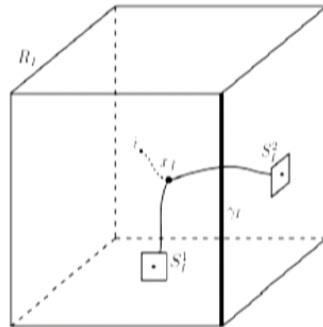
$$\left. \begin{aligned} V_{x_I}^{ijk}(\sigma, \sigma') &= \epsilon^{ij'k'} D^{(1)}(h_{\gamma_{x_I \sigma'}^1}[A])_{j'}{}^j D^{(1)}(h_{\gamma_{x_I \sigma''}^2}[A])_{k'}{}^k \\ T_{x_I}^{ijk}(\sigma, \sigma', \sigma'') &= \epsilon^{i'j'k'} D^{(1)}(h_{\gamma_{x_I \sigma}^1}[A])_{i'}{}^i D^{(1)}(h_{\gamma_{x_I \sigma'}^2}[A])_{j'}{}^j D^{(1)}(h_{\gamma_{x_I \sigma''}^3}[A])_{k'}{}^k \end{aligned} \right\} \begin{array}{l} SU(2) \\ \text{invariance} \end{array}$$

The length operator

$$\left. \begin{aligned} V_{x_I}^{ijk}(\sigma, \sigma') &= \epsilon^{ij'k'} D^{(1)}(h_{\gamma_{x_I \sigma'}^1}[A])_{j'}{}^j D^{(1)}(h_{\gamma_{x_I \sigma''}^2}[A])_{k'}{}^k \\ T_{x_I}^{ijk}(\sigma, \sigma', \sigma'') &= \epsilon^{i'j'k'} D^{(1)}(h_{\gamma_{x_I \sigma}^1}[A])_{i'}{}^i D^{(1)}(h_{\gamma_{x_I \sigma'}^2}[A])_{j'}{}^j D^{(1)}(h_{\gamma_{x_I \sigma''}^3}[A])_{k'}{}^k \end{aligned} \right\} \begin{array}{l} SU(2) \\ \text{invariance} \end{array}$$

we write the surface integrals in $G_{\Delta s}^i(s_I)$ as Riemann sums of fluxes:

$$G_I^i = \frac{\frac{1}{2} \sum_{\alpha, \beta} Y_{I\alpha\beta}^i}{\sqrt{\frac{1}{8 \cdot 3!} \sum_{\alpha, \beta, \rho} |Q_{I\alpha\beta\rho}|}} = \frac{\frac{1}{2} \sum_{\alpha, \beta} V_{x_I}^{ijk} F_j(S_{I\alpha}^1) F_k(S_{I\beta}^2)}{\sqrt{\frac{1}{8 \cdot 3!} \sum_{\alpha, \beta, \rho} |T_{x_I}^{ijk} F_i(S_{I\alpha}^1) F_j(S_{I\beta}^2) F_k(S_{I\rho}^2)|}}$$

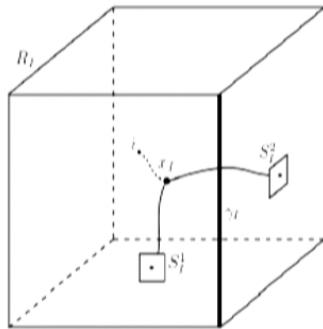


The length operator

$$\left. \begin{aligned} V_{x_I}^{ijk}(\sigma, \sigma') &= \epsilon^{ij'k'} D^{(1)}(h_{\gamma_{x_I \sigma'}^1}[A])_{j'}{}^j D^{(1)}(h_{\gamma_{x_I \sigma''}^2}[A])_{k'}{}^k \\ T_{x_I}^{ijk}(\sigma, \sigma', \sigma'') &= \epsilon^{i'j'k'} D^{(1)}(h_{\gamma_{x_I \sigma}^1}[A])_{i'}{}^i D^{(1)}(h_{\gamma_{x_I \sigma'}^2}[A])_{j'}{}^j D^{(1)}(h_{\gamma_{x_I \sigma''}^3}[A])_{k'}{}^k \end{aligned} \right\} \begin{array}{l} SU(2) \\ \text{invariance} \end{array}$$

we write the surface integrals in $G_{\Delta s}^i(s_I)$ as Riemann sums of fluxes:

$$G_I^i = \frac{\frac{1}{2} \sum_{\alpha, \beta} Y_{I\alpha\beta}^i}{\sqrt{\frac{1}{8 \cdot 3!} \sum_{\alpha, \beta, \rho} |Q_{I\alpha\beta\rho}|}} = \frac{\frac{1}{2} \sum_{\alpha, \beta} V_{x_I}^{ijk} F_j(S_{I\alpha}^1) F_k(S_{I\beta}^2)}{\sqrt{\frac{1}{8 \cdot 3!} \sum_{\alpha, \beta, \rho} |T_{x_I}^{ijk} F_i(S_{I\alpha}^1) F_j(S_{I\beta}^2) F_k(S_{I\rho}^2)|}}$$



$$F_i(S^k) := \int_{S^k} \epsilon_{abc} E_i^a dx^b \wedge dx^c$$

Then the length of the segment γ_I :

$$L_I = \sqrt{\delta_{ij} G_I^i G_I^j}$$

and therefore the length of γ :

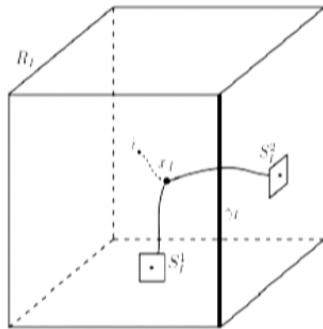
$$L(\gamma) = \lim_{\Delta s \rightarrow 0} \sum_I L_I$$

The length operator

$$\left. \begin{aligned} V_{x_I}^{ijk}(\sigma, \sigma') &= \epsilon^{ij'k'} D^{(1)}(h_{\gamma_{x_I \sigma'}^1}[A])_{j'}{}^j D^{(1)}(h_{\gamma_{x_I \sigma''}^2}[A])_{k'}{}^k \\ T_{x_I}^{ijk}(\sigma, \sigma', \sigma'') &= \epsilon^{i'j'k'} D^{(1)}(h_{\gamma_{x_I \sigma}^1}[A])_{i'}{}^i D^{(1)}(h_{\gamma_{x_I \sigma'}^2}[A])_{j'}{}^j D^{(1)}(h_{\gamma_{x_I \sigma''}^3}[A])_{k'}{}^k \end{aligned} \right\} \begin{array}{l} SU(2) \\ \text{invariance} \end{array}$$

we write the surface integrals in $G_{\Delta s}^i(s_I)$ as Riemann sums of fluxes:

$$G_I^i = \frac{\frac{1}{2} \sum_{\alpha, \beta} Y_{I\alpha\beta}^i}{\sqrt{\frac{1}{8 \cdot 3!} \sum_{\alpha, \beta, \rho} |Q_{I\alpha\beta\rho}|}} = \frac{\frac{1}{2} \sum_{\alpha, \beta} V_{x_I}^{ijk} F_j(S_{I\alpha}^1) F_k(S_{I\beta}^2)}{\sqrt{\frac{1}{8 \cdot 3!} \sum_{\alpha, \beta, \rho} |T_{x_I}^{ijk} F_i(S_{I\alpha}^1) F_j(S_{I\beta}^2) F_k(S_{I\rho}^2)|}}$$



$$F_i(S^k) := \int_{S^k} \epsilon_{abc} E_i^a dx^b \wedge dx^c$$

Then the length of the segment γ_I :

$$L_I = \sqrt{\delta_{ij} G_I^i G_I^j}$$

and therefore the length of γ :

$$L(\gamma) = \lim_{\Delta s \rightarrow 0} \sum_I L_I$$

The length operator

$$\hat{Y}^i(\gamma\omega) = \epsilon^{ij'k'} D^{(1)}(h_{\gamma_{x_I\sigma'}^1}[A])_{j'}{}^j D^{(1)}(h_{\gamma_{x_I\sigma''}^2}[A])_{k'}{}^k \hat{F}_j(S_{e1}) \hat{F}_k(S_{e2})$$

$$\hat{V}_n^{-1} = \lim_{\epsilon \rightarrow 0} (\hat{V}_n^2 + \epsilon^2 L_p^6)^{-1} \hat{V}_n$$

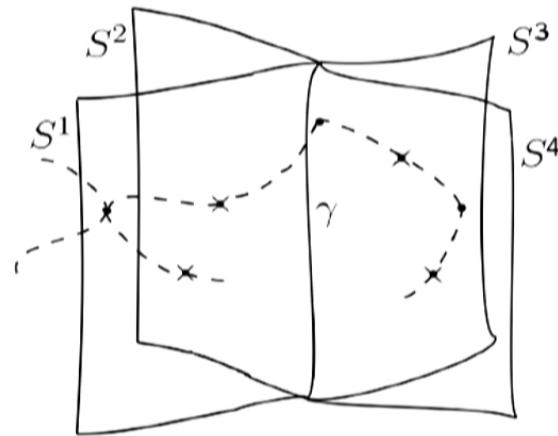
The length operator

$$\hat{Y}^i(\gamma_\omega) = \epsilon^{ij'k'} D^{(1)}(h_{\gamma_{x_I\sigma'}^1}[A])_{j'}{}^j D^{(1)}(h_{\gamma_{x_I\sigma''}^2}[A])_{k'}{}^k \hat{F}_j(S_{e1}) \hat{F}_k(S_{e2})$$

$$\hat{V}_n^{-1} = \lim_{\epsilon \rightarrow 0} (\hat{V}_n^2 + \epsilon^2 L_p^6)^{-1} \hat{V}_n$$

$\widehat{L(\gamma_\omega)}$ is associated to a wedge of the graph Γ :

$$\widehat{L(\gamma_\omega)} = \sqrt{\hat{V}_n^{-1} \delta_{ij} \hat{Y}^i(\gamma_\omega) \hat{Y}^j(\gamma_\omega) \hat{V}_n^{-1}}$$



11 / 22

The deficit angle operator

We proceed with the same scheme to regularize the expression of the dihedral angle as it was done for the length

$$\theta_{I\alpha}^{12} = \pi - \arccos \left[\frac{\delta'^{k'} D^{(1)}(h_{\gamma_{x_I\sigma'}^1}[A])_{i'} {}^i D^{(1)}(h_{\gamma_{x_I\sigma''}^2}[A])_{k'} {}^k F_i(S_{I\alpha}^1) F_k(S_{I\beta}^2)}{\sqrt{\delta'^{ij} F_i(S_{I\alpha}^1) F_j(S_{I\alpha}^1)} \sqrt{\delta'^{kl} F_k(S_{I\beta}^2) F_l(S_{I\beta}^2)}} \right]$$

The deficit angle operator

We proceed with the same scheme to regularize the expression of the dihedral angle as it was done for the length

$$\theta_{I\alpha}^{12} = \pi - \arccos \left[\frac{\delta^{i'k'} D^{(1)}(h_{\gamma_{x_I\sigma'}^1}[A])_{i'} {}^i D^{(1)}(h_{\gamma_{x_I\sigma''}^2}[A])_{k'} {}^k F_i(S_{I\alpha}^1) F_k(S_{I\beta}^2)}{\sqrt{\delta'^{ij} F_i(S_{I\alpha}^1) F_j(S_{I\alpha}^1)} \sqrt{\delta'^{kl} F_k(S_{I\beta}^2) F_l(S_{I\beta}^2)}} \right]$$

$\hat{\theta}_{ik}$ in the intertwiner basis:

$$\hat{\theta}_{ik} = \left(\pi - \arccos \left[\frac{j_{ik}(j_{ik} + 1) - j_i(j_i + 1) - j_k(j_k + 1)}{2\sqrt{j_i(j_i + 1)j_k(j_k + 1)}} \right] \right) |j_{ik}\rangle \langle j_{ik}|$$

The Ricci operator

The operator associated to a dual segment:

$$\left[L_i^c \left(\widehat{\frac{2\pi}{\alpha_i}} - \theta_i^c \right) \right] = \frac{2\pi}{\alpha_i} \hat{L}_i^c - \frac{1}{2} (\hat{L}_i^c \cdot \hat{\theta}_i^c + \hat{\theta}_i^c \cdot \hat{L}_i^c)$$

Where $\hat{L}_i^c = \widehat{L(\gamma_\omega)}$ and $\hat{\theta}_i^c = \widehat{\theta(\gamma_\omega)}$ with γ_ω the curve corresponding to the hinge i for the 3-cell c containing the wedge ω .

The Ricci operator

The operator associated to a dual segment:

$$\left[L_i^c \left(\widehat{\frac{2\pi}{\alpha_i}} - \theta_i^c \right) \right] = \frac{2\pi}{\alpha_i} \hat{L}_i^c - \frac{1}{2} (\hat{L}_i^c \cdot \hat{\theta}_i^c + \hat{\theta}_i^c \cdot \hat{L}_i^c)$$

Where $\hat{L}_i^c = \widehat{L(\gamma_\omega)}$ and $\hat{\theta}_i^c = \widehat{\theta(\gamma_\omega)}$ with γ_ω the curve corresponding to the hinge i for the 3-cell c containing the wedge ω .

$$\Rightarrow \hat{R}_\Delta := \sum_c \sum_{i \in c} \left[L_i^c \left(\widehat{\frac{2\pi}{\alpha_i}} - \theta_i^c \right) \right] = \sum_c \sum_{i \in c} \frac{2\pi}{\alpha_i} \hat{L}_i^c - \frac{1}{2} (\hat{L}_i^c \cdot \hat{\theta}_i^c + \hat{\theta}_i^c \cdot \hat{L}_i^c)$$

- hermitian
- depends on the choice of Δ .

We can also define an operator \hat{R}_c representing the action \hat{R}_Δ in the region contained in the 3-cell c

$$\hat{R}_c := \sum_{i \in c} \frac{2\pi}{\alpha_i} \hat{L}_i^c - \frac{1}{2} (\hat{L}_i^c \cdot \hat{\theta}_i^c + \hat{\theta}_i^c \cdot \hat{L}_i^c)$$

The Ricci operator

The operator associated to a dual segment:

$$\left[L_i^c \left(\widehat{\frac{2\pi}{\alpha_i}} - \theta_i^c \right) \right] = \frac{2\pi}{\alpha_i} \hat{L}_i^c - \frac{1}{2} (\hat{L}_i^c \cdot \hat{\theta}_i^c + \hat{\theta}_i^c \cdot \hat{L}_i^c)$$

Where $\hat{L}_i^c = \widehat{L(\gamma_\omega)}$ and $\hat{\theta}_i^c = \widehat{\theta(\gamma_\omega)}$ with γ_ω the curve corresponding to the hinge i for the 3-cell c containing the wedge ω .

$$\Rightarrow \hat{R}_\Delta := \sum_c \sum_{i \in c} \left[L_i^c \left(\widehat{\frac{2\pi}{\alpha_i}} - \theta_i^c \right) \right] = \sum_c \sum_{i \in c} \frac{2\pi}{\alpha_i} \hat{L}_i^c - \frac{1}{2} (\hat{L}_i^c \cdot \hat{\theta}_i^c + \hat{\theta}_i^c \cdot \hat{L}_i^c)$$

- hermitian
- depends on the choice of Δ .

We can also define an operator \hat{R}_c representing the action \hat{R}_Δ in the region contained in the 3-cell c

$$\hat{R}_c := \sum_{i \in c} \frac{2\pi}{\alpha_i} \hat{L}_i^c - \frac{1}{2} (\hat{L}_i^c \cdot \hat{\theta}_i^c + \hat{\theta}_i^c \cdot \hat{L}_i^c)$$

The Ricci operator

The action of \hat{R}_c on a cylindrical function $\Psi(\Gamma)$ gives zero unless the 3-cell c contains a node!

$$\Rightarrow \hat{R}_\Delta \Psi(\Gamma) = \sum_{n \in \Gamma} \sum_{\omega_n} \kappa(c, \omega_n) \left[\frac{2\pi}{\alpha_{\omega_n}} \hat{L}(\omega_n) - \frac{1}{2} (\hat{L}(\omega_n) \cdot \hat{\theta}(\omega_n) + \hat{\theta}(\omega_n) \cdot \hat{L}(\omega_n)) \right] \Psi(\Gamma)$$

The action of \hat{R}_Δ depends on the 3-cells containing the nodes of Γ (selecting the wedges) and the cells glued to them (fix the values of the coefficients α_{ω_n}). However its limit is well-defined as we shrink Δ

$$\lim_{\text{Volume}[c \in \Delta] \rightarrow 0} \hat{R}_\Delta \Psi(\Gamma) = \sum_{n \in \Gamma} \sum_{\omega_n} \kappa(c, \omega_n) \left[\frac{2\pi}{\alpha_{\omega_n}} \hat{L}(\omega_n) - \frac{1}{2} (\hat{L}(\omega_n) \cdot \hat{\theta}(\omega_n) + \hat{\theta}(\omega_n) \cdot \hat{L}(\omega_n)) \right] \Psi(\Gamma)$$

The Ricci operator

The action of \hat{R}_c on a cylindrical function $\Psi(\Gamma)$ gives zero unless the 3-cell c contains a node!

$$\Rightarrow \hat{R}_\Delta \Psi(\Gamma) = \sum_{n \in \Gamma} \sum_{\omega_n} \kappa(c, \omega_n) \left[\frac{2\pi}{\alpha_{\omega_n}} \hat{L}(\omega_n) - \frac{1}{2} (\hat{L}(\omega_n) \cdot \hat{\theta}(\omega_n) + \hat{\theta}(\omega_n) \cdot \hat{L}(\omega_n)) \right] \Psi(\Gamma)$$

The action of \hat{R}_Δ depends on the 3-cells containing the nodes of Γ (selecting the wedges) and the cells glued to them (fix the values of the coefficients α_{ω_n}). However its limit is well-defined as we shrink Δ

$$\lim_{\text{Volume}[c \in \Delta] \rightarrow 0} \hat{R}_\Delta \Psi(\Gamma) = \sum_{n \in \Gamma} \sum_{\omega_n} \kappa(c, \omega_n) \left[\frac{2\pi}{\alpha_{\omega_n}} \hat{L}(\omega_n) - \frac{1}{2} (\hat{L}(\omega_n) \cdot \hat{\theta}(\omega_n) + \hat{\theta}(\omega_n) \cdot \hat{L}(\omega_n)) \right] \Psi(\Gamma)$$

BUT it carries a memory of the choice of the covering decomposition Δ :

- $\kappa(c, \omega_n)$ → Averaging procedure
- α_{ω_n} → Free parameters

The Ricci operator

The dependence on $\kappa(c, \omega_n)$ rises directly from the choice of the 3-cells of Δ containing the nodes of Γ .

The Ricci operator

The dependence on $\kappa(c, \omega_n)$ rises directly from the choice of the 3-cells of Δ containing the nodes of Γ .

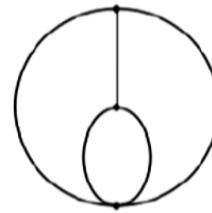
- Each 3-cell is isomorphic to a spherical polyhedron verifying requirement *iv*).
- For a fixed number of faces F , such spherical polyhedra regroup in a finite number of *classes*.
- Each class defines a different configuration.

\Rightarrow The number of inequivalent configurations $N_{\text{conf}}(F)$ is always finite

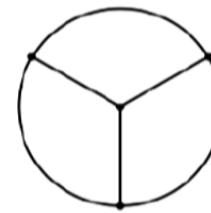
\Rightarrow Allows to define an averaging procedure over the different configurations associated to a 3-cell (F -valent node)



The 4-hosohedron



The class with 3 vertices



The spherical tetrahedron

The Ricci operator

A wedge $\omega \in \Gamma$ containing n has a number of appearances $N_{\text{app}}(F)$ in the set of configurations, thus:

$$\kappa(F_n) = \frac{N_{\text{app}}}{N_{\text{conf}}} \leq 1$$

$$\begin{aligned}\hat{R}_{\Delta}^{av} \Psi(\Gamma) &= \sum_{n \in \Gamma} \kappa(F_n) \sum_{\omega_n} \left[\frac{2\pi}{\alpha_{\omega_n}} \hat{L}(\omega_n) - \frac{1}{2} (\hat{L}(\omega_n) \cdot \hat{\theta}(\omega_n) + \hat{\theta}(\omega_n) \cdot \hat{L}(\omega_n)) \right] \Psi(\Gamma) \\ &= \sum_{\omega_n \in \Gamma} \kappa(F_n) \left[\frac{2\pi}{\alpha_{\omega_n}} \hat{L}(\omega_n) - \frac{1}{2} (\hat{L}(\omega_n) \cdot \hat{\theta}(\omega_n) + \hat{\theta}(\omega_n) \cdot \hat{L}(\omega_n)) \right] \Psi(\Gamma)\end{aligned}$$

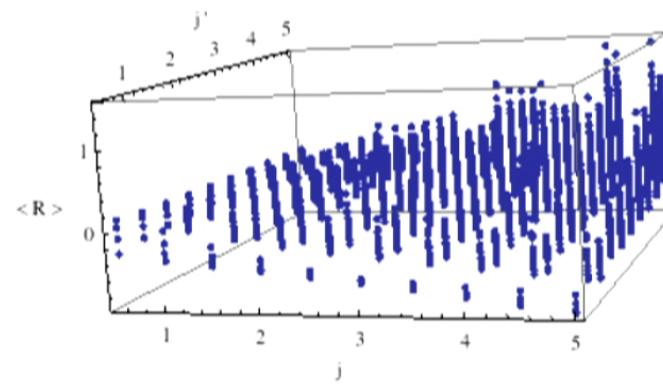
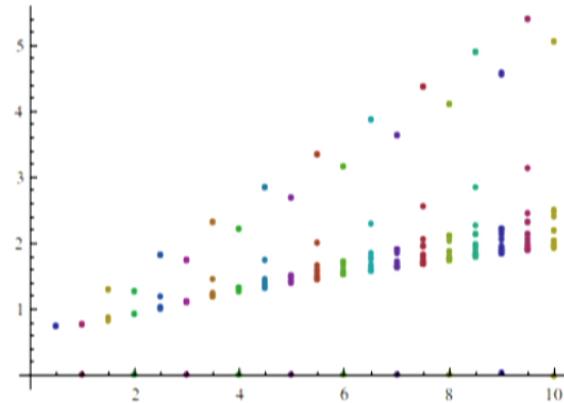
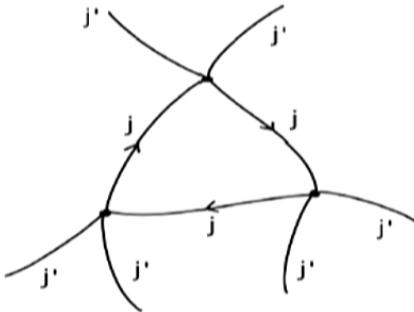
$\alpha_{\omega_n} = \alpha_i$ are totally arbitrary! \rightarrow choice of a prescription.

$$\boxed{\hat{R} \Psi(\Gamma) = \sum_{\omega_n \in \Gamma} \kappa(F_n) \left[\frac{2\pi}{\alpha_{\omega_n}} \hat{L}(\omega_n) - \frac{1}{2} (\hat{L}(\omega_n) \cdot \hat{\theta}(\omega_n) + \hat{\theta}(\omega_n) \cdot \hat{L}(\omega_n)) \right] \Psi(\Gamma)}$$

Properties of the Ricci operator

The Ricci operator \hat{R}

- $SU(2)$ gauge invariant
- Diffeomorphism invariant
- Discrete spectrum



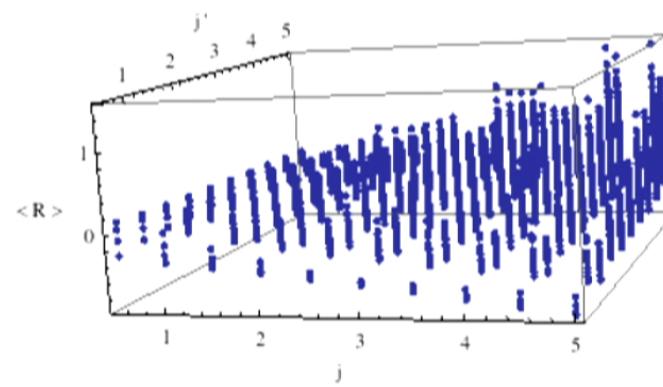
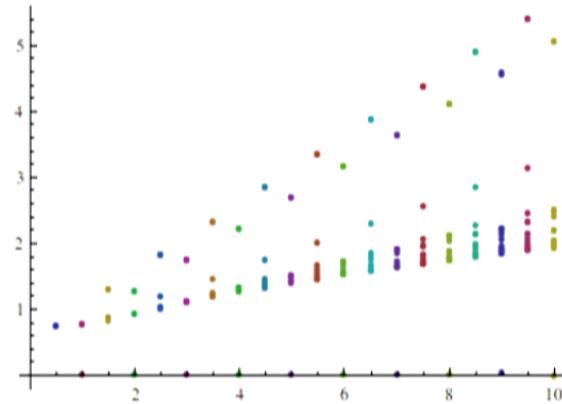
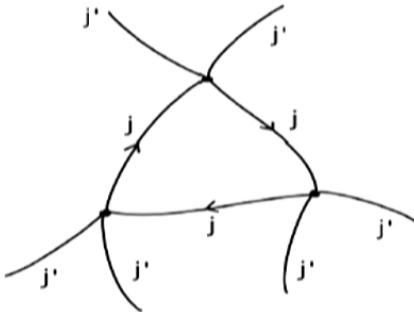
Units $(8\pi\gamma L_P^2)^{\frac{1}{2}}$ are used.

18 / 22

Properties of the Ricci operator

The Ricci operator \hat{R}

- $SU(2)$ gauge invariant
- Diffeomorphism invariant
- Discrete spectrum

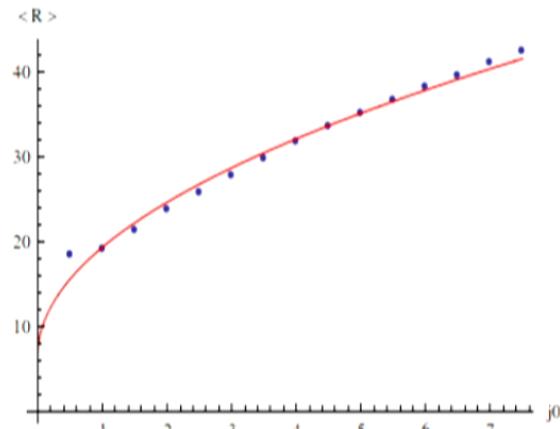


Units $(8\pi\gamma L_P^2)^{\frac{1}{2}}$ are used.

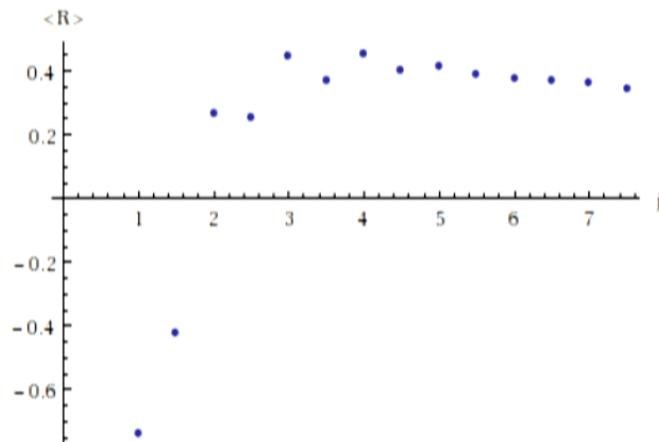
18 / 22

Properties of the Ricci operator

- “Rovelli-Speziale” semi-classical states:



Monochromatic four-valent node. Fit
 $6, 57 + 12, 87\sqrt{j_0}$



Internal geometry for a three four-valent nodes

Summary

$$\frac{1}{2} \int_{\Delta} \sqrt{-g} R \, d^n x = \sum_i \epsilon_i L_i$$

$$\rightarrow \boxed{\hat{R}\Psi(\Gamma) := \sum_{\omega_n \in \Gamma} \kappa(F_n) \left[\frac{2\pi}{\alpha_{\omega_n}} \hat{L}(\omega_n) - \frac{1}{2} (\hat{L}(\omega_n) \cdot \hat{\theta}(\omega_n) + \hat{\theta}(\omega_n) \cdot \hat{L}(\omega_n)) \right] \Psi(\Gamma)}$$

- Hermitian, not graph changing.
- Gauge invariant and diffeomorphic invariant
- Discrete spectrum. Eigenstates and eigenvalues can be computed algebraically and numerically.
- The appropriate semi-classical behavior on states peaked on classical geometry.



Thank you for your attention

Geometrical observables for General Relativity coupled to dust

Jędrzej Świeżewski

University of Warsaw

joint work with P. Duch, W. Kamiński and J. Lewandowski

Waterloo, 22.07.2013



Plan of the talk

- Introduction – a perspective
- Fermi coordinates
- Fermi observables and their problems
- Simplified approach
- Summary and outlook

Jędrzej Świeżewski, University of Warsaw

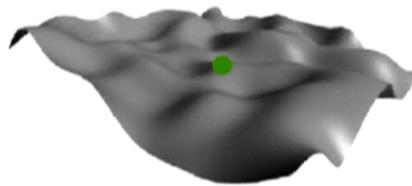
Introduction

physical quantities \rightsquigarrow diffeomorphism invariant observables

Jędrzej Świeżewski, University of Warsaw

Introduction

physical quantities \leadsto diffeomorphism invariant observables



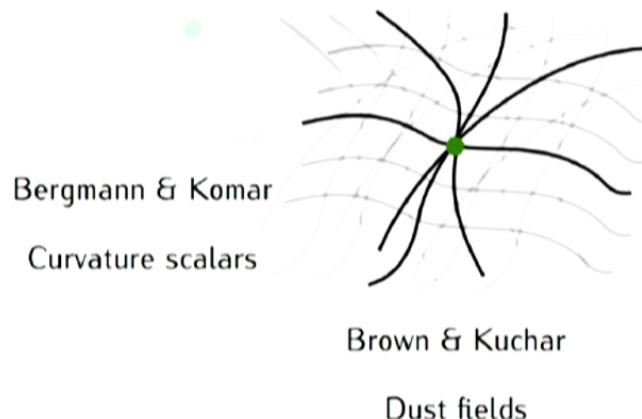
Bergmann & Komar

Curvature scalars

Jędrzej Świeżewski, University of Warsaw

Introduction

physical quantities \rightsquigarrow diffeomorphism invariant observables



Jędrzej Świeżewski, University of Warsaw

Introduction

physical quantities \leadsto diffeomorphism invariant observables

Bergmann & Komar

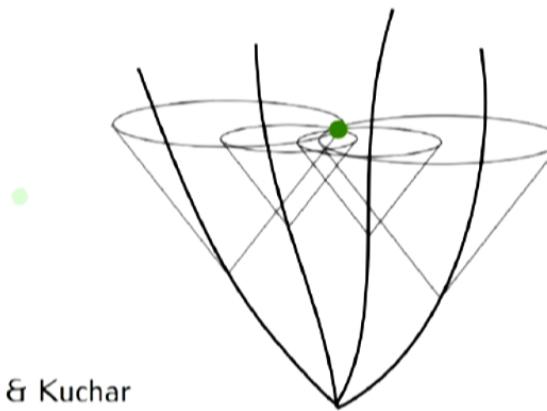
Curvature scalars

Brown & Kuchar

Dust fields

Rovelli

Satellites



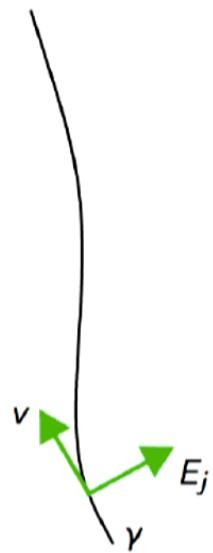
Jędrzej Świeżewski, University of Warsaw

Fermi coordinates



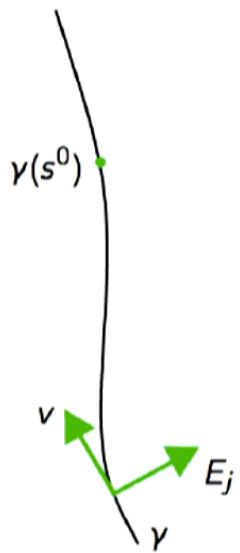
Jędrzej Świeżewski, University of Warsaw

Fermi coordinates



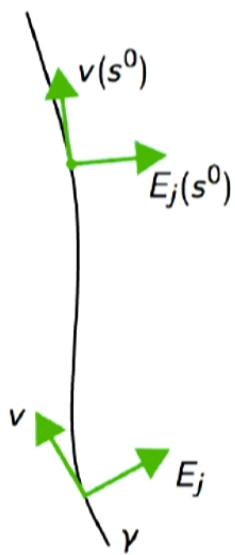
Jędrzej Świeżewski, University of Warsaw

Fermi coordinates



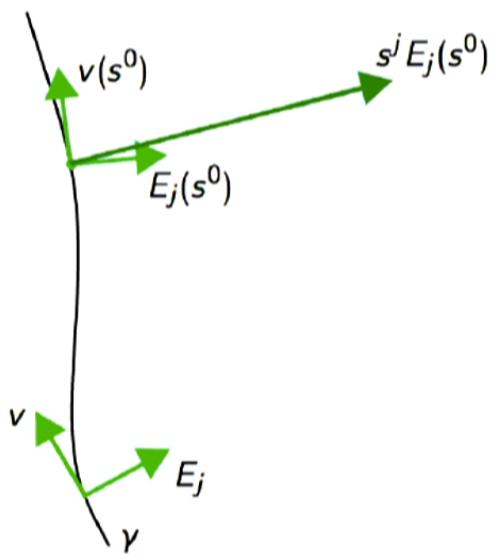
Jędrzej Świeżewski, University of Warsaw

Fermi coordinates



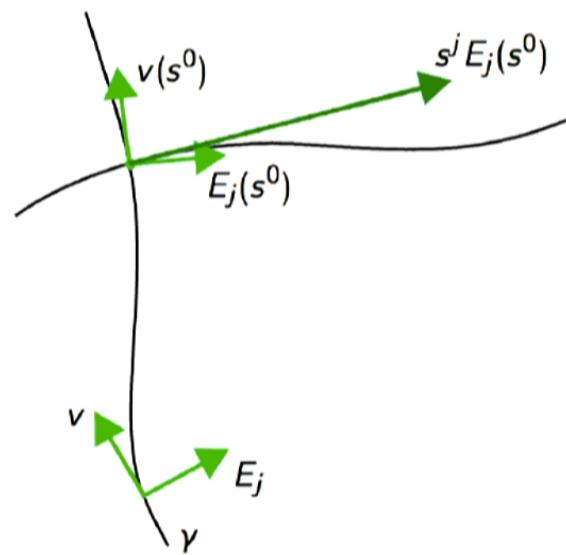
Jędrzej Świeżewski, University of Warsaw

Fermi coordinates



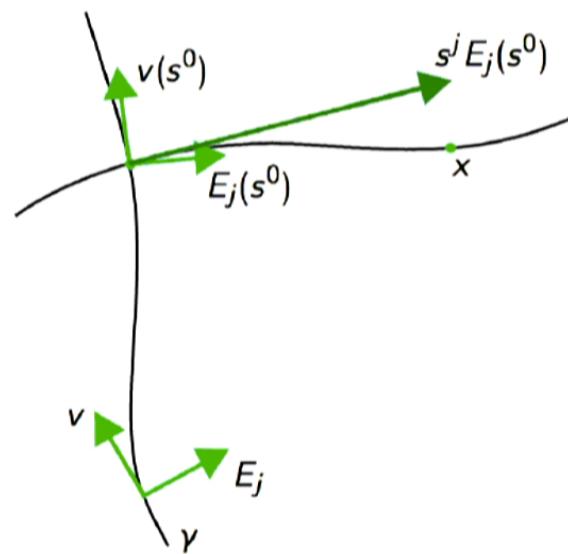
Jędrzej Świeżewski, University of Warsaw

Fermi coordinates



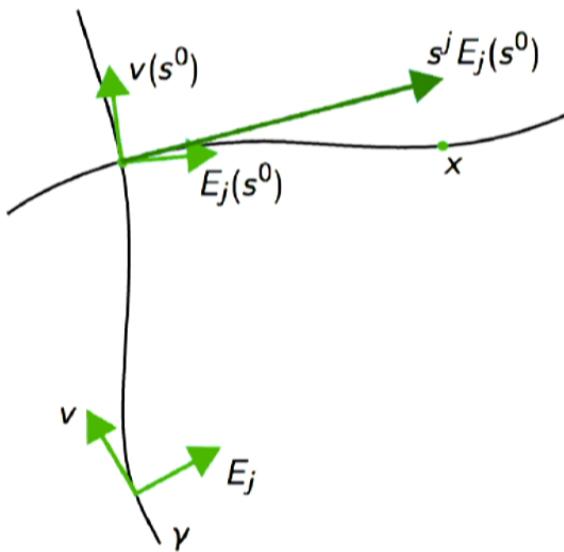
Jędrzej Świeżewski, University of Warsaw

Fermi coordinates



Jędrzej Świeżewski, University of Warsaw

Fermi coordinates



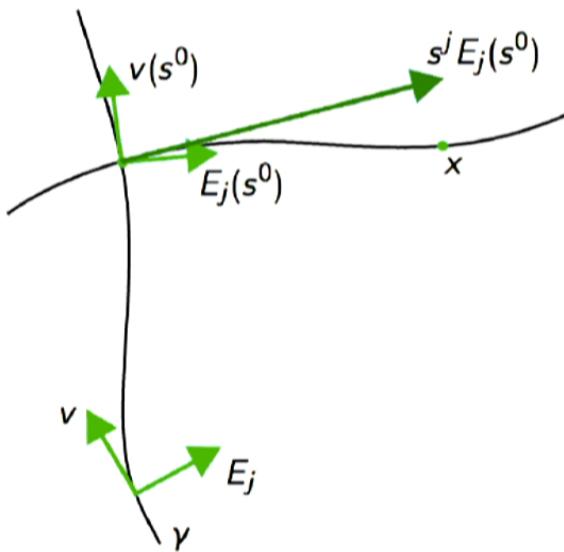
Mathematically speaking the definition of the Fermi parametrization is given by

$$x = f(s^0, s^j) = \exp_{\gamma(s^0)}(s^j E_j(s^0))$$

and depends on γ , E_j and $g_{\mu\nu}$.

Jędrzej Świeżewski, University of Warsaw

Fermi coordinates



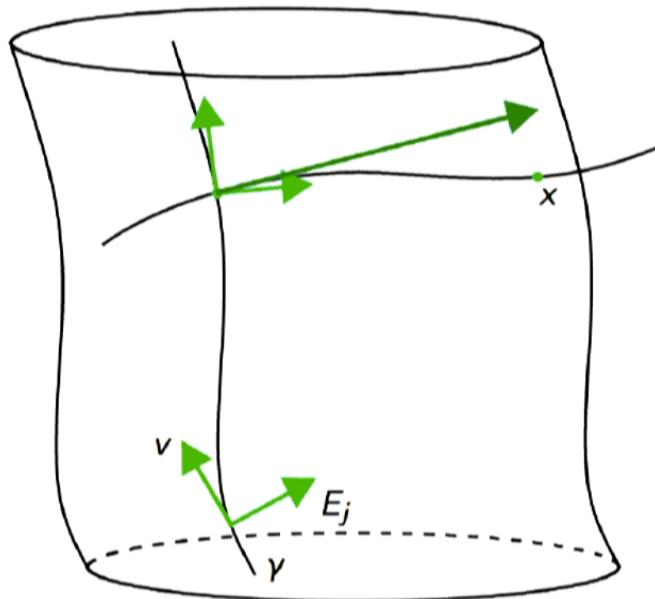
Mathematically speaking the definition of the Fermi parametrization is given by

$$x = f(s^0, s^j) = \exp_{\gamma(s^0)}(s^j E_j(s^0))$$

and depends on γ , E_j and $g_{\mu\nu}$.

Jędrzej Świeżewski, University of Warsaw

Fermi coordinates



Mathematically speaking the definition of the Fermi parametrization is given by

$$x = f(s^0, s^j) = \exp_{\gamma(s^0)}(s^j E_j(s^0))$$

and depends on γ , E_j and $g_{\mu\nu}$.

Jędrzej Świeżewski, University of Warsaw

Fermi observables

For any function of the metric consider its pullback by the Fermi parametrisation. The resulting function is a Dirac observable.

Jędrzej Świeżewski, University of Warsaw

Fermi observables

For any function of the metric consider its pullback by the Fermi parametrisation. The resulting function is a Dirac observable.

e.g.

$$g^{\mu\nu}(x) \rightsquigarrow \mathcal{O}_{g^{\mu\nu};f}^{IJ}(s^0, s^j) := s_{,\mu}^I(x)s_{,\nu}^J(x)g^{\mu\nu}(x) \Big|_{x \text{ s.t. } f(s^0, s^j)=x}$$

Fermi observables

For any function of the metric consider its pullback by the Fermi parametrisation. The resulting function is a Dirac observable.

e.g.

$$g^{\mu\nu}(x) \rightsquigarrow \mathcal{O}_{g^{\mu\nu};f}^{IJ}(s^0, s^j) := s_{,\mu}^I(x)s_{,\nu}^J(x)g^{\mu\nu}(x) \Big|_{x \text{ s.t. } f(s^0, s^j)=x}$$

Unfortunately, the Poisson bracket of two such observables is ill-defined.

spherical Fermi coordinates

Spherical Fermi coordinates (s^0, r, θ, ϕ) are given in terms of the one just introduced by the relations

$$\begin{cases} s^0 = s^0 \\ s^x = r \sin \theta \cos \phi \\ s^y = r \sin \theta \sin \phi \\ s^z = r \cos \theta \end{cases}$$

Jędrzej Świeżewski, University of Warsaw

spherical Fermi coordinates

Spherical Fermi coordinates (s^0, r, θ, ϕ) are given in terms of the one just introduced by the relations

$$\begin{cases} s^0 = s^0 \\ s^x = r \sin \theta \cos \phi \\ s^y = r \sin \theta \sin \phi \\ s^z = r \cos \theta \end{cases}$$

The canonical variables (h_{ab}, K_{ab}) have a very simple form in spherical Fermi coordinates

$$\begin{cases} h_{rr} = 1 \\ h_{r\theta} = 0 \\ h_{r\phi} = 0 \\ K_{rr} = 0 \end{cases}$$

Jędrzej Świeżewski, University of Warsaw

Gravity coupled to irrotational dust

We consider a theory given by the action [Kuchař, Torre '91, Husain, Pawłowski '12, Świeżewski '13]

$$S = \int d^4x \sqrt{-g}R - \frac{1}{2} \int d^4x \sqrt{-g}M (g^{\mu\nu}\partial_\mu T\partial_\nu T + 1)$$

Jędrzej Świeżewski, University of Warsaw

Gravity coupled to irrotational dust

We consider a theory given by the action [Kuchař, Torre '91, Husain, Pawłowski '12, Świeżewski '13]

$$S = \int d^4x \sqrt{-g}R - \frac{1}{2} \int d^4x \sqrt{-g}M (g^{\mu\nu}\partial_\mu T\partial_\nu T + 1)$$

After a the $3+1$ decomposition of the theory and in the time gauge ($t = T$), the constraints of the theory are

$$\begin{cases} \tilde{C} = C + p_T \\ \tilde{C}_a = C_a \end{cases}$$

Jędrzej Świeżewski, University of Warsaw

Gravity coupled to irrotational dust

We consider a theory given by the action [Kuchař, Torre '91, Husain, Pawłowski '12, Świeżewski '13]

$$S = \int d^4x \sqrt{-g}R - \frac{1}{2} \int d^4x \sqrt{-g}M (g^{\mu\nu}\partial_\mu T\partial_\nu T + 1)$$

After a the $3+1$ decomposition of the theory and in the time gauge ($t = T$), the constraints of the theory are

$$\begin{cases} \tilde{C} = C + p_T \\ \tilde{C}_a = C_a \end{cases}$$

The deparametrised theory is given by the action

$$S_{dep} = \int d^3x dt \left(\dot{h}_{ab}\pi^{ab} - C - N^a C_a \right)$$

Jędrzej Świeżewski, University of Warsaw

Gauge fixing

We introduce gauge fixing constraints

$$B(\vec{K}) = \int d^3x K^a B_a = \int d^3x K^a r (h_{ra} - \delta_{ra})$$

Jędrzej Świeżewski, University of Warsaw

Gauge fixing

We introduce gauge fixing constraints

$$B(\vec{K}) = \int d^3x K^a B_a = \int d^3x K^a r (h_{ra} - \delta_{ra})$$

For any functional on the phase space \mathcal{O} we take

$$\mathcal{O}_D = \mathcal{O} + C(\vec{N}_{\mathcal{O}}) + B(\vec{K}_{\mathcal{O}}) ,$$

where $\vec{N}_{\mathcal{O}}$ and $\vec{K}_{\mathcal{O}}$ are such that

$$\{\mathcal{O}_D, C_a\} \approx 0 \quad \{\mathcal{O}_D, B_a\} \approx 0$$

Gauge fixing

We introduce gauge fixing constraints

$$B(\vec{K}) = \int d^3x K^a B_a = \int d^3x K^a r (h_{ra} - \delta_{ra})$$

For any functional on the phase space \mathcal{O} we take

$$\mathcal{O}_D = \mathcal{O} + C(\vec{N}_{\mathcal{O}}) + B(\vec{K}_{\mathcal{O}}),$$

where $\vec{N}_{\mathcal{O}}$ and $\vec{K}_{\mathcal{O}}$ are such that

$$\{\mathcal{O}_D, C_a\} \approx 0 \quad \{\mathcal{O}_D, B_a\} \approx 0$$

The evolution of the Dirac observable \mathcal{O}_D is given by

$$\dot{\mathcal{O}}_D = \{\int d^3x C, \mathcal{O}_D\} \approx \dot{\mathcal{O}} + \int d^3x (2\pi^{ra} K_{\mathcal{O}a} - \pi^{ab} h_{ab} K_{\mathcal{O}}^r)$$

However...

... the observables we introduced are invariant under a large class of diffeomorphisms, but not all of them. There is still a residual, finite dimensional, gauge freedom we did not fix by our construction.

Jędrzej Świeżewski, University of Warsaw

Summary and outlook

Summary

- We introduced a new scheme for obtaining partial and complete^{*} observables for
 - vacuum General Relativity
 - General Relativity coupled to irrotational dust

Outlook

- How to make the observer (worldline and frame) dynamical?
- Reduced phase space approach?

Jędrzej Świeżewski, University of Warsaw

Summary and outlook

Summary

- We introduced a new scheme for obtaining partial and complete* observables for
 - vacuum General Relativity
 - General Relativity coupled to irrotational dust

Outlook

- How to make the observer (worldline and frame) dynamical?
- Reduced phase space approach?

$$\begin{bmatrix} 1 & 0 & 0 \\ & h_{\theta\theta} & h_{\theta\phi} \\ & & h_{\phi\phi} \end{bmatrix}$$

Jędrzej Świeżewski, University of Warsaw

Summary and outlook

Summary

- We introduced a new scheme for obtaining partial and complete* observables for
 - vacuum General Relativity
 - General Relativity coupled to irrotational dust

Outlook

- How to make the observer (worldline and frame) dynamical?
- Reduced phase space approach?

$$\begin{bmatrix} 1 & 0 & 0 \\ & h_{\theta\theta} & h_{\theta\phi} \\ & & h_{\phi\phi} \end{bmatrix} \quad \begin{bmatrix} \pi^{rr} & \pi^{r\theta} & \pi^{r\phi} \\ \pi^{\theta r} & \pi^{\theta\theta} & \pi^{\theta\phi} \\ & & \pi^{\phi\phi} \end{bmatrix}$$

Jędrzej Świeżewski, University of Warsaw

Thank you for your attention!

Jędrzej Świeżewski, University of Warsaw