Title: Spin Foams - 2

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Abstract:

Perimeter Institute for Theoretical Physics

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Spinors in Lorentzian Spin-Foam Theory

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spin-foam

- A reincarnation of Regge's idea, due to J.Barrett and L.Crane, based on discretizing the first order action: $S_{Holst} = \int J^{IJ} \wedge F_{IJ}$
- space-time gets sliced, each slice subdivided in 4-simplices.
- F_{IJ} replaced by the product of $g \in \mathrm{SL}(2,\mathbb{C})$ along the face dual to the triangle t.
- $J_t^{IJ} \in \mathrm{sl}(2,\mathbb{C})$ for each triangle t is a combination of the area tensor S_t^{IJ} and its dual $^*S_t^{IJ} = \frac{1}{2}\epsilon^{IJKL} S_{tKL}$ by a real parameter γ or an angle θ:

$$J_t^{IJ} := {}^*S_t^{IJ} - \frac{1}{\gamma} S_t^{IJ}; \quad \gamma = -i\,\frac{e^{i\theta}-1}{e^{i\theta}+1} \leftrightarrow e^{i\theta} = \frac{1+i\gamma}{1-i\gamma}$$

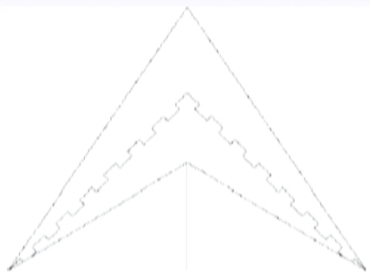
- The area tensors S_t^{IJ} are constrained to be 'simple', and to give closed tetrahedra: $\sum_{t\in\tau} S_t^{IJ} = 0$.

successful? so far, mostly in words. Somewhat unimaginative. The technical difficulties seem to have attracted most of the attention. Some brilliant ideas, e.g. imposing constraints à la Gupta-Bleurer. But more needed to make the theory 'practical' (competive?).

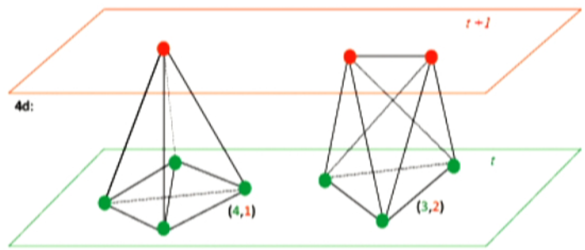
Difficulties: for large quantum numbers, $\sim$ cos($\mathrm{area\ x\ angle}$) instead of exp($\mathrm{i\,area\ x\ angle}$). I think this is serious, because it spoils causality (cfr. Livine-Oriti, gr-qc/0210064).

Almost all work on space-like tetrahedra. But can a triangulation be without time-like tetrahedra? I think NO. But the idea of specifying which are time-like and which space-like is ridicolous. Time-like tetrahedra are tricky (cfr. F. Conrady, arXiv:1003.5652).

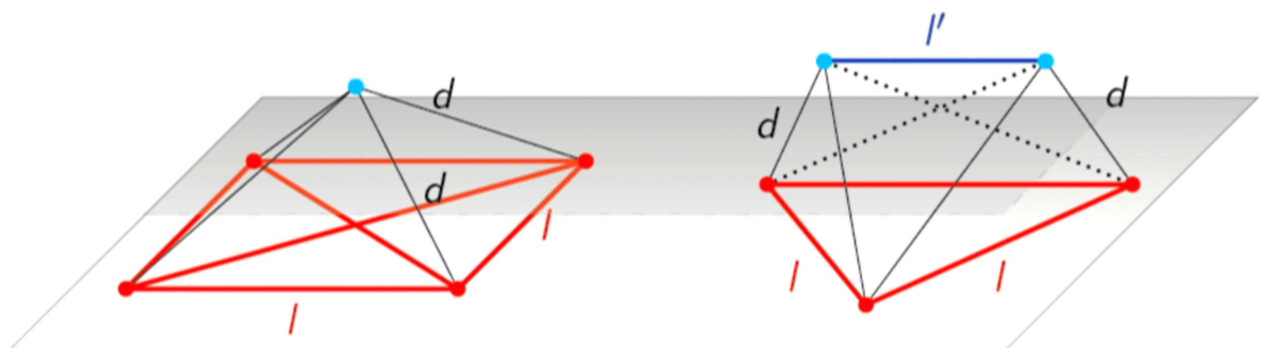
The competition: CDT is technically easier to set up, which has allowed vast simulations (none so far in SF). All CDT tetrahedra are time-like, sides $d^2 = -\alpha l^2$, violating Courant's criterium, related to causality;
this violation disturbs me (but nobody else).



meant to show Courant's criterium



CDT
SF



simplicity

The triangles $t, s, \ldots$ forming a tetrahedron τ must be 'simple', meaning that e.g. $S_t^{IJ} = a^I b^J - a^J b^I$, $\quad S_s^{IJ} = a^I c^J - a^J c^I$, or:

$$S_t^{IJ} {}^*S_{tIJ} = 0, \quad S_t^{IJ} {}^*S_{sIJ} = 0$$

or still, that for each tetrahedron there is a normal V_τ^I, such that:

$$V_{\tau M} S_t^{MI} = -\sin\frac{\theta}{2}\left(e^{i\frac{\theta}{2}} V_{\tau M} J_{t+}^{MI} + e^{-i\frac{\theta}{2}} V_{\tau M} J_{t-}^{MI}\right) = 0 \quad \forall\ t \in \tau.$$

for example, in two limit cases:

if $\quad V^I = (V^0, 0, 0, 0):\ L_1 + \frac{1}{\gamma} K_1 = L_2 + \frac{1}{\gamma} K_2 = L_3 + \frac{1}{\gamma} K_3 = 0$

if $\quad V^I = (0, 0, 0, V^3):\ L_1 - \gamma K_1 = L_2 - \gamma K_2 = L_3 + \frac{1}{\gamma} K_3 = 0$

can one forget the difference, and treat them together?

From an identity for ϵ^{IJKL}:

$${}^*S_t^{IJ} = V^I \frac{V_M {}^*S_t^{MJ}}{V^2} - \frac{V_M {}^*S_t^{MI}}{V^2} V^J := V^I N_t^J - V^J N_t^I$$

N_t^I normal to the triangle, such that: $N_{tI} V_\tau^I = N_{tI} S_t^{IJ} = 0$

spinors

cfr. M. Dupuis, L.Freidel, E.Livine, S.Speziale, arXiv:1107.5274)
two 2-spinors (u_t, t_t) for each triangle, with $\{t_\alpha, \bar{u}_\beta\} = -i\delta_{\alpha,\beta}$

$$J^L_{ta} = \frac{1}{2} t^\dagger_t \sigma_a u_t =, \frac{1}{2}(L_a + iK_a) \quad J^R_{ta} = \frac{1}{2} u^\dagger_t \sigma_a t_t = \frac{1}{2}(L_a + iK_a)$$

Under SL(2,$\mathbb{C}$) transformations $u \to gu$, $t \to g^{\dagger -1} t$

Closure constraints: $\sum_{t\in\tau} \bar{u}_{t\alpha} \sigma_{a\alpha\beta} t_{t\beta} = 0$ implies

$$\sum_{t\in\tau} \bar{u}_{t\alpha} t_{t\beta} = \frac{1}{2} \sum_{t\in\tau} (u^\dagger_t t_t) \delta_{\alpha\beta} := C_\tau e^{i\psi_\tau} \delta_{\alpha\beta}$$

Simplicity constraints: to go from $\pm$ to L, R use: $(\tilde{\sigma}_I = (1, \sigma_a))$

$$\tilde{\sigma}_I V_M J^{MI}_+ = i \tilde{\sigma}_I V^I \sigma_a J^L_a; \quad \tilde{\sigma}_I V_M J^{MI}_- = -i \sigma_a J^R_a \tilde{\sigma}_I V^I$$ to get:

$$e^{i\frac{\theta}{2}} \tilde{\sigma}_I V^I \sigma_a (t^\dagger_t \sigma_a u_t) - e^{-i\frac{\theta}{2}} \sigma_a (u^\dagger_t \sigma_a t_t) \tilde{\sigma}_I V^I = 0$$

'second class', but implying a relation between t_t and u_t:

$$t_{t\alpha} - \kappa_t \, e^{i\frac{\theta}{2}} \, \tilde{\sigma}_{I\alpha\beta} V^I \, u_{t\beta} = 0$$

$(\kappa_t = \pm 1$ or any real number) a set of 'first class' constraints.
This is just like R. Penrose's twistor equation (so what?).
They do not Poisson-commute with their complex conjugate.

Quite smart. As a consequence:

- $u_t^\dagger t_t = \kappa_t e^{i\frac{\theta}{2}}\, u_t^\dagger \tilde{\sigma}_I V_\tau^I\, u_t = e^{i\theta} t_t^\dagger u_t \;\rightarrow\; \frac{1}{2}\sum_{t\in\tau}(u_t^\dagger t_t) = C_\tau e^{i\frac{\theta}{2}}$
- for two triangles: $t_{t\alpha}\epsilon_{\alpha\beta}t_{s\beta} + \kappa_s\kappa_t e^{i\theta} V_\tau^2\, u_{t\alpha}\epsilon_{\alpha\beta}u_{s\beta} = 0$ (the 'holomorphic simplicity constraints' of DFLS).
- For non degenerate, non light-like tetrahedra, using $\tilde{\sigma}_I V^I \sigma_J V^J = V^2$, the normal to a tetrahedron is given by:

$$V_\tau^I = \frac{V_\tau^2}{2C_\tau}\sum_{t\in\tau}\kappa_t u_t^\dagger \tilde{\sigma}^I u_t$$

- For the normal to the triangle

$$\begin{aligned}\tilde{\sigma}_I N_t^I &= \frac{\tilde{\sigma}_I V_{\tau M}\, {}^*S_t^{MI}}{V_\tau^2} = \frac{e^{i\frac{\theta}{2}}\sin\frac{\theta}{2}}{V_\tau^2}\tilde{\sigma}_I V_\tau^I \sigma_a (t_t^\dagger \sigma_a u_t) = \\ &= \frac{\kappa_t \sin\frac{\theta}{2}}{V_\tau^2}(u_t^\dagger \tilde{\sigma}_I V_\tau^I \sigma_b V_\tau^b u_t,\; u_t^\dagger(V_\tau^a \tilde{\sigma}_I V_\tau^I - V_\tau^2 \sigma_a)u_t)\end{aligned}$$

- For time-like tetrahedra, the κ_t cannot be all of same sign.

For the classical theory this is all very nice, and takes care of J_t.
NOT much use for quantization, unless done replacing classical variable with operators, etc... That needs extension to F (Livine-Speziale-Tambornino arXiv1108.0369 ?)

Not what is usually done.

In a $J \wedge F$ theory one would sum over all rep.s of SL(2,$\mathbb{C}$).
The idea of Barrett&Crane, Engle-Pereira-Rovelli-Livine, ... is to use the constraints to limit the choice of rep.s.

Example: representations of SL(2,$\mathbb{C}$) are indexed by (n, ρ), n integer, ρ real, and have Casimirs:

$$(\mathbf{J}^L)^2 = \frac{1}{16}(n - i\rho)^2 - \frac{1}{4}; \quad (\mathbf{J}^R)^2 = \frac{1}{16}(n + i\rho)^2 - \frac{1}{4}$$

the 'diagonal constraint' requires (ignoring $\frac{1}{4}$):

$$e^{i\theta}(\mathbf{J}^L_t)^2 = e^{-i\theta}(\mathbf{J}^R_t)^2 \;\rightarrow\; \frac{(n+i\rho)^2}{(n-i\rho)^2} = e^{2i\theta} = \frac{(1+i\gamma)^2}{(1-i\gamma)^2}$$

two solutions: $\sqrt{e^{2i\theta}} = \pm e^{i\theta}$ or $\rho = n\gamma$, or $\rho = -\frac{n}{\gamma}$.

Spinors and all the previous discussion add nothing.

Unitary representations (n, ρ) of SL(2,$\mathbb{C}$) act on Hilbert space $\mathcal{H}^{(n,\rho)}$; in various subgroups are reduced to direct sum of representations. Constraints imposed on states à la Gupta-Bleuler: For **space-like tetrahedra** use 'injected' SU(2) states, $|jm> \rightarrow |(n,\rho)jm>$, the canonical basis of $\mathcal{H}^{(n,\rho)}$; $V^I = (V^0, 0, 0, 0) :< (n,\rho)jm|L_3 + \frac{1}{\gamma}K_3|(n,\rho)jm' >= 0$ gives $\boxed{\rho = \gamma j, n = 2j}$. The rest follows: $L_\pm + \frac{1}{\gamma}K_\pm = \pm[L_3 + \frac{1}{\gamma}K_3, L_\pm]$. In agreement wih area quantization of LQG, $\text{area}_t = \gamma\sqrt{j(j+1)}$. For **time-like tetrahedra**: SU(1,1) states (cfr. Conrady&Hnybida) the Casimir $Q = L_3^2 - K_1^2 - K_2^2$ has a discrete spectrum $Q > 0$ and a continous $Q < 0$. **If** $V^I = (0, 0, 0, V^3)$: $< .|L_3 + \frac{1}{\gamma}K_3|. >= 0$, the rest follows, by $K_\pm - \frac{1}{\gamma}L_\pm = [L_3 + \frac{1}{\gamma}K_3, K_\pm] = 0$. But then:

- space-like triangles: the discrete spectrum gives $\boxed{\rho = \gamma j}$ and $(\text{area}_t) = \gamma\sqrt{Q} = \gamma\sqrt{j(j-1)}$.
- time-like triangles: the continous spectrum needs $\boxed{\rho = -\frac{n}{\gamma}}$, $\frac{\rho}{2} = -\sqrt{s^2 + \frac{1}{4}}$, $(\text{area}_t) = \gamma\sqrt{-Q} = \gamma\sqrt{s^2 + \frac{1}{4}} = -\gamma\,\frac{\rho}{2} = \frac{n}{2}$.

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As I said , the trouble with this is:

- it only applies to space-like tetrahedra.
- in the limit $n \to \infty \sim \cos(\text{area x angle})$ instead of $\sim \exp(\mathrm{i}\,\text{area x angle})$.

Important? YES.
It implies that there is no causality built in the model, and that it applies only for very peculiar triangulations, if any.

So it should be modified/generalized, keeping only the $\sim \exp(\mathrm{i}\,\text{area x angle})$ (holomorphic?) part. How?

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alternatives?

This was a sketch/caricature of the state of the art. But if one finds absurd the a priori assignemt of space-like/time-like character to the tetrahedra of a triangulation, then one needs an approach that ignores this distinction.

- use the spinor formulation developed before, write group elements and Haar measure on SL(2,C) in terms of spinors (LST). (not attempted (yet)).
- use eigenstates of (L_3, K_3), i.e. the abelian subgroup they generate, and give up using states of definite areas, eigenstates of L_3 and of the Casimir of SU(2) or SU(1,1).

Very reluctant to abandon area quantization. Physically less clear: for a Rindler horizon K_3 is the 'Hamiltonian'
E.Bianchi, arXiv1204.5122, Frodden-Ghosh-Perez, arXiv1110.4055.
The matrix to go from the canonical to the (L_3, K_3) basis has been calculated (Bianchi, Huszar).

$\mathcal{H}^{(n,\rho)}$ as space of functions of light-like $P^I = u^\dagger \tilde{\sigma}^I u$.

Lomont-Moses 1962, Smorodinskii-Huszar, 1970, E.Bianchi.

$$P^I P_I = 0; \quad P_t^0 = -u_t^\dagger u_t \quad V_\tau^I P_{tI} = \kappa_t e^{-i\frac{\theta}{2}} u_t^\dagger t_t;$$
$$V_\tau^I = \frac{V_\tau^2}{2C_\tau} \sum_{t\in\tau} \kappa_t P_t^I; \quad N_t^I P_{tI} = \frac{\kappa_t \sin\frac{\theta}{2}}{V_\tau^2} e^{-i\theta} (u_t^\dagger t_t)^2$$
$$F(e^\alpha P) = e^{(i\frac{\rho}{2}-1)\alpha} F(P); \quad < F|G >= \int F^* G \frac{d^3 P}{2P}$$

space-like (SU(2)): $u = e^{\frac{\alpha}{2}}(\cos\frac{\theta}{2} e^{-i\frac{\varphi}{2}}, \sin\frac{\theta}{2} e^{i\frac{\varphi}{2}})$,

$$P_I = e^\alpha (1, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$
$$|(n\rho)jm> \leftrightarrow \Psi_{jm}^{(n\rho)}(\alpha,\theta,\varphi) = c_j e^{(i\frac{\rho}{2}-1)\alpha} e^{i(\frac{n}{2}+m)\varphi} d^j_{-m\frac{n}{2}}(\theta)$$

time-like (SU(1,1)): $u = e^{\frac{\alpha}{2}}(\cosh\frac{\beta}{2} e^{-i\frac{\varphi}{2}}, \sinh\frac{\beta}{2} e^{i\frac{\varphi}{2}})$,

$$P_I = e^\alpha (\cosh\beta, \, \sinh\beta\cos\varphi, \, \sinh\beta\sin\varphi, \, 1)$$

The alternative is to choose:

$$u = \frac{1}{\sqrt{2}} e^{\frac{\alpha}{2}} (e^{\frac{\beta - i\varphi}{2}}, e^{-\frac{\beta - i\varphi}{2}}),\ P_I = e^{\alpha}(\cosh\beta, \cos\varphi, \sin\varphi, \sinh\beta):$$

$$L_3 = -i\frac{\partial}{\partial\varphi} - \frac{n}{2}, \quad K_3 = -i\frac{\partial}{\partial\beta}, \quad P^I \frac{\partial}{\partial P^I} = \frac{\partial}{\partial\alpha}$$

states such that: $L_3 \Psi^{(n\rho)}_{m\nu} = m \Psi^{(n\rho)}_{m\nu}; \quad K_3 \Psi^{(n\rho)}_{m\nu} = \nu \Psi^{(n\rho)}_{m\nu}$

$$\Psi^{(n\rho)}_{m\nu}(\alpha, \beta, \varphi) = \frac{\sqrt{2}}{2\pi}\, e^{(i\frac{\rho}{2} - 1)\alpha} e^{i(\frac{n}{2} + m)\varphi} e^{i\nu\beta}$$

The program then would be to start from these to construct coherent states (minimal uncertainty states) and vertex functions. Some time.

thank you for the attention

Quantum Twisted Geometry

A new coherent and discrete basis of intertwiners

Who? **Jeff Hnybida** in collaboration with **Laurent Freidel**

When? Loops, 2013

Spinors and Twistors in LQG

- harmonic oscillators in LQG (Livine, Girelli)
- coherent intertwiners (Livine, Speziale)
- twisted geometry (Friedel, Speziale)
- U(N) intertwiners (Freidel, Livine)
- holomorphic simplicity constraints (Livine, Dupuis)
- generating functionals (Bonzom, Livine)
- spinors in LQG (Dupuis, Livine, Tambornino, Weiland)
- gluing conditions (Dittrich, Ryan)
- spin connection in twisted geometry (Haggard, Rovelli, Vidotto, Wieland)
- and more...

Plan:

1. Holomorphic representation of SU(2)
2. A new basis of intertwiners
3. 4-simplex Amplitude
4. Asymptotics
5. Classical action for Twisted Geometry

The Holomorphic Representation of SU(2)

Spinors

$$|z\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad |z] = \begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix} \qquad \alpha, \beta \in \mathbb{C}$$

Bargmann-Fock inner product

$$\langle f|g\rangle = \int_{\mathbb{C}^2} \overline{f(z)} g(z) \mathrm{d}\mu(z) \qquad \mathrm{d}\mu(z) = \frac{e^{-\langle z|z\rangle}}{\pi^2} \mathrm{d}^4 z$$

Orthonormal Basis

$$(z|j\, m\rangle = \frac{\alpha^{j+m}\beta^{j-m}}{\sqrt{(j+m)!(j-m)!}}$$

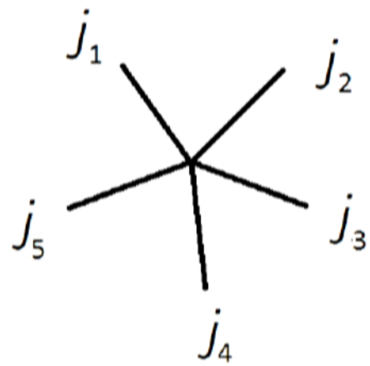
Intertwiners

Spin j subspace

$$V^j = \text{span}\{|j\,m\rangle : m = -j, ..., j\}$$

Intertwiners are invariant tensors

$$\mathcal{H}_{j_1,\ldots,j_n} \equiv \text{Inv}_{\text{SU}(2)}\left[V^{j_1} \otimes \cdots \otimes V^{j_n}\right]$$



A New Basis of Intertwiners

Suppose we have n spinors

$$z_1, \ldots, z_n$$

There are $n(n-1)/2$ holomorphic invariants

$$[z_i|z_j\rangle = \alpha_i\beta_j - \alpha_j\beta_i$$

New basis of intertwiners

$$\prod_{i<j} [z_i|z_j\rangle^{k_{ij}} \qquad k_{ij} = k_{ji} \in \mathbb{N}$$

having spins

$$\sum_{j \neq i} k_{ij} = 2j_i$$

The 3-valent case

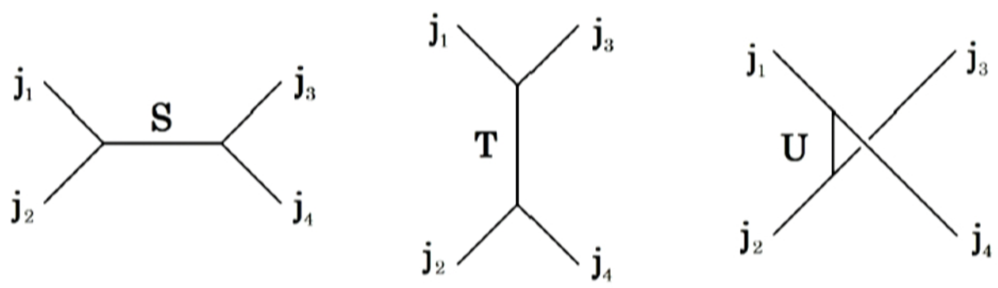
Unique solution of $\sum_{j \neq i} k_{ij} = 2j_i$ for $i = 1, 2, 3$

$$k_{12} = j_1 + j_2 - j_3 \quad k_{13} = j_1 - j_2 + j_3 \quad k_{23} = -j_1 + j_2 + j_3$$

Just the Wigner 3j symbol

$$\prod_{i<j} [z_i | z_j \rangle^{k_{ij}} \propto \sum_{m_1 m_2 m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \prod_{i=1}^{3} (z_i | j_i \, m_i \rangle$$

Combine to form an **orthonormal** basis of 4-valent intertwiners (as usual)



The 4-valent case

Non-unique solution of $\sum_{j\neq i} k_{ij} = 2j_i$

$$\begin{aligned} k_{12} &= j_1 + j_2 - S \qquad k_{34} = j_3 + j_4 - S \\ k_{13} &= j_1 + j_3 - T \qquad k_{24} = j_2 + j_4 - T \\ k_{14} &= j_1 + j_4 - U \qquad k_{23} = j_2 + j_3 - U \end{aligned}$$

Since $S + T + U = J$ parameterize $k_{ij} = k_{ij}(j_i, S, T)$

$$(z_i|S,T\rangle \equiv \prod_{i<j} [z_i|z_j\rangle^{k_{ij}(j_i,S,T)}$$

This basis is **overcomplete.** Nevertheless

$$1_{\mathcal{H}_{j_i}} = \sum_{S,T} \frac{|S,T\rangle\langle S,T|}{\|S,T\|^2_{j_i}} \qquad \|S,T\|^2_{j_i} \equiv \frac{(J+1)!}{\prod_{i<j} k_{ij}!}$$

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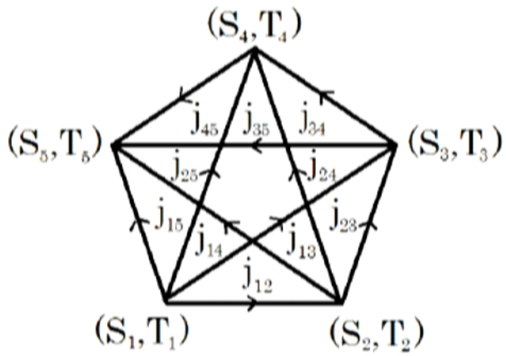
The 20j symbol

Contracting five $|S, T\rangle$ states



The 20j generates the 15j's

$$\sum_{T_i} \{20j\}_{S_i, T_i} = \{15j\}_{S_i}$$

and similarly for the other four non-equivalent 15j symbols

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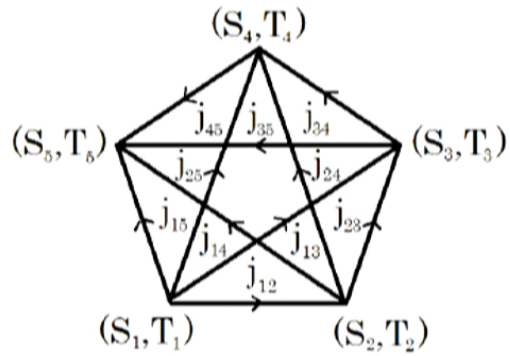
Contracting five $|S, T\rangle$ states



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Asymptotics of the 15j symbol

Inserting the resolution of identity $1 = \sum_{S'_i} \frac{|S'_i\rangle\langle S'_i|}{\|S'_i\|^2}$

$$\begin{aligned}
\{20j\}_{S_i,T_i} &= \sum_{S'_i} \{15j\}_{S'_i} \prod_i \frac{\langle S'_i|S,T\rangle}{\|S'_i\|^2} \\
&= \sum_{S'_i,T'_i} \{15j\}_{S'_i} \prod_i \frac{\langle S'_i,T'_i|S,T\rangle}{\|S'_i\|^2}
\end{aligned}$$

and using $\langle S',T'|S,T\rangle \sim \|S,T\|^2 \delta_{S,S'}\delta_{T,T'}$

$$\{20j\}_{S_i,T_i} \sim \{15j\}_{S_i} \prod_i \frac{\|S_i,T_i\|^2}{\|S_i\|^2}$$

Can we find the asymptotics of $\{20j\}_{S_i,T_i}$?

Asymptotics of the 20j symbol

The contraction as an integral

$$\{20j\}_{S_i,T_i} = \int \prod_{i \neq j} \frac{d^2 z_j^i}{\pi^2} e^{S(z_j^i)}$$

with the action

$$S = \sum_{i<j} [z_j^i | z_i^j\rangle + \sum_a \sum_{i<j} k_{ij}^a \ln [z_i^a | z_j^a\rangle$$

has saddle points with solutions describing oriented, framed tetrahedra, i.e. satisfying closure

$$\sum_i |z_i^a\rangle\langle z_i^a| = \frac{A}{2} 1 \qquad a = 1,..,5$$

with a $U(1)^4 \times \mathbb{Z}_2$ symmetry for each tetrahedron.

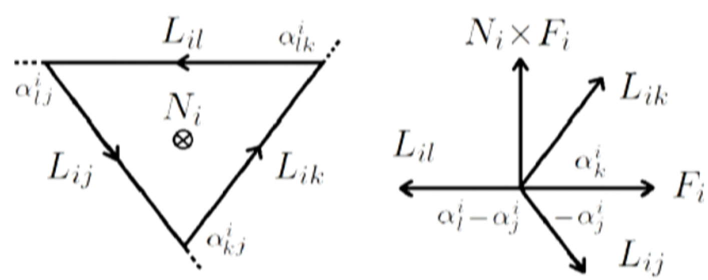
Geometrical Interpretation

Penrose null flag interpretation of the spinor

$$|z_i\rangle\langle z_i| - |z_i][z_i| = A_i N_i \cdot \sigma$$

$$|z_i\rangle[z_i| = iA_i\,(F_i + iN_i \times F_i) \cdot \sigma$$

where $N_i \cdot N_i = F_i \cdot F_i = 1$ and $N_i \cdot F_i = 0$.



Define edge vectors $L_{ij} = A_i A_j (N_i \times N_j)$ and angles

$$N_i \cdot N_j = \cos\theta_{ij} \qquad F_i \cdot L_{ij} = \alpha^i_j$$

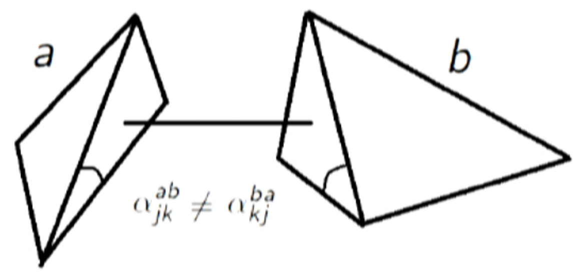
Gauge Invariant Data

Gauge transformation of frame vector (change of phase of spinor)

$$\alpha_j^{ai} \to \alpha_j^{ai} + \phi^{ai} \qquad \phi^{ai} = -\phi^{ia}$$

Gauge invariant variables

$$\alpha_{jk}^{ab} \equiv \alpha_j^{ab} - \alpha_k^{ab} \qquad \xi_i^{ab} \equiv \alpha_i^{ab} + \alpha_i^{ba}$$



Shape matching $\alpha_{ij}^{ab} = \alpha_{ji}^{ba} \quad \Rightarrow \quad \xi_i^{ab} = \xi_j^{ab}$.

Summary:

- **A new way to do SU(2) recoupling**
  - spinors, Gaussian integration, generating functionals
- **A new basis of n-valent intertwiners**
  - discrete: labelled by a set of spins
  - coherent: peaked on closed bounded tetrahedra
  - a bridge between coherent and orthonormal intertwiners
  - closely related to $U(N)$ intertwiners
- **A new {20j} symbol**
  - generates all five $\{15j\}$ symbols
  - gives asymptotics of $\{15j\}$ for the first time
  - Racah formula
- **Asymptotics**
  - simple generalization of Regge action
  - twisted geometric interpretation

Introduction

Is there a classical theory behind spinfoam gravity? Do we get general relativity (GR)? What is the relation to the canonical approach?

Two possible strategies to explore these questions:

- Start from the quantum theory, and study its semi-classical limit.
- Here: Start from the classical theory, canonically quantise and compare the resulting transition amplitudes with what we know from the spinfoam approach.

Outline

1. Continuum action for spinfoam gravity
2. Equations of motion: Hamiltonian formulation and relation to Regge calculus
3. Quantum theory: Inner product and transition amplitudes

Within the reduced setting of a fixed discretisation of space-time, I can show, that spinfoam gravity comes from a classical theory. This is a version of first-order Regge calculus, with spinors as the fundamental configuration variables.

1.1 Constrained BF-theory

Spinfoam gravity starts from a Holst-type topological action:

$$S_{\mathrm{top}}[\Sigma, A] = \int_M \underbrace{\frac{\mathrm{i}\hbar}{\ell_{\mathrm{P}}{}^2}\frac{\beta+\mathrm{i}}{\beta}\Sigma_{AB}}_{2\Pi_{AB}} \wedge F^{AB}[A] + \mathrm{cc.} \tag{1}$$

Adding the simplicity constraints implies geometricity and brings us break to GR. These are:

$$\Sigma_{(AB} \wedge \Sigma_{CD)} = 0, \quad \mathfrak{Re}\big(\Sigma_{AB} \wedge \Sigma^{AB}\big) = 0, \quad \Sigma_{AB} \wedge \bar{\Sigma}_{\bar{C}\bar{D}} = 0. \tag{2}$$

Peforming a 3+1 split we obtain the symplectic structure, e.g.:

$$\Pi_{AB} = -\frac{\hbar}{\ell_{\mathrm{P}}{}^2}\frac{\beta+\mathrm{i}}{2\mathrm{i}\beta}\Sigma_{AB} \ : \ \big\{\Pi_{CD}{}^a, A^{AB}{}_b(q)\big\} = \delta^{(A}_C \delta^{B)}_D \delta^a_b \delta(p,q). \tag{3}$$

Notation:

- $\Pi^A{}_B{}^a$ is an $\mathfrak{sl}(2,\mathbb{C})$-valued vector density (a 2-form).
- $A^A{}_{Ba}$ is the selfdual (Ashtekar–Sen) $SL(2,\mathbb{C})$ connection.
- $A, B, C, \dots = 0, 1$ are spinor indices, the complex conjugate representation carries a macron $\bar{A}, \bar{B}, \bar{C}, \dots = \bar{0}, \bar{1}$, and all indices are moved by ϵ_{AB}, $\bar{\epsilon}^{\bar{A}\bar{B}}, \dots$.
- $\Sigma_{\alpha\beta} = -\bar{\epsilon}_{\bar{A}\bar{B}}\Sigma_{AB} + \mathrm{cc.}$ is the Plebanski 2-form $e_\alpha \wedge e_\beta$.

1.3 Spinors for holonomies and fluxes

Spinors can diagonalise the flux. In the frame of the initial point:

$$\Pi_{AB} = -\frac{1}{2}\omega_{(A}\pi_{B)}. \tag{6}$$

The parallel transport maps the spinors into the frame of the final point:

$$\underset{\sim}{\pi}^A = h^A{}_B\pi^B, \quad \underset{\sim}{\omega}^A = h^A{}_B\omega^B. \tag{7}$$

Reversing the logic, we can start from spinors $(\pi^A, \omega^A, \underset{\sim}{\pi}^A, \underset{\sim}{\omega}^A)$ and get the holonomy:

$$h^A{}_B = \frac{\underset{\sim}{\omega}^A\pi_B - \underset{\sim}{\pi}^A\omega_B}{\sqrt{\pi\omega}\sqrt{\underset{\sim}{\pi}\underset{\sim}{\omega}}}. \tag{8}$$

This parametrisation is not unique, there is a discrete symmetry and a continuous gauge transformation. We also need constraints to recover flux and holonomy: These are $\pi_A\omega^A \neq 0$, and the

$$\text{area matching constraint}: \; C = \underset{\sim}{\pi}_A\underset{\sim}{\omega}^A - \pi_A\omega^A = 0. \tag{9}$$

We introduce $SL(2,\mathbb{C})$ invariant Poisson brackets:

$$\{\pi_A, \omega^B\} = \delta^B_A = -\{\underset{\sim}{\pi}_A, \underset{\sim}{\omega}^B\}, \qquad \{\bar{\pi}_{\bar{A}}, \bar{\omega}^{\bar{B}}\} = \delta^{\bar{B}}_{\bar{A}} = -\{\bar{\underset{\sim}{\pi}}_{\bar{A}}, \bar{\underset{\sim}{\omega}}^{\bar{B}}\}, \tag{10}$$

and symplectic reduction w.r.t. $C = 0$ brings us back to $T^*SL(2,\mathbb{C})$.

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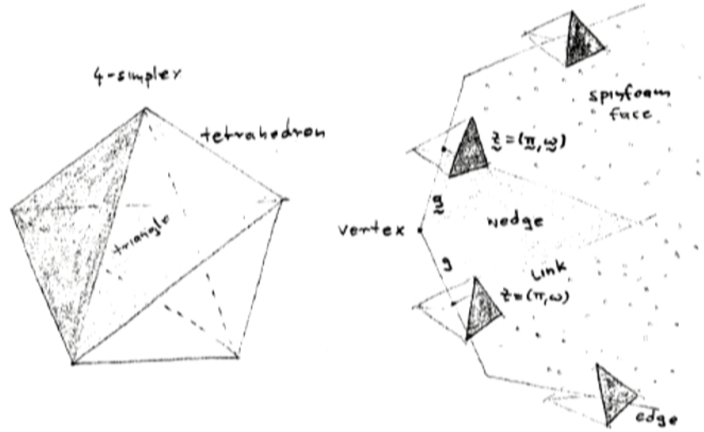
1.4 The topological action discretised

Let us now look at the four-dimensional discretisation.



We write the discretised topological action as a sum over wedges:

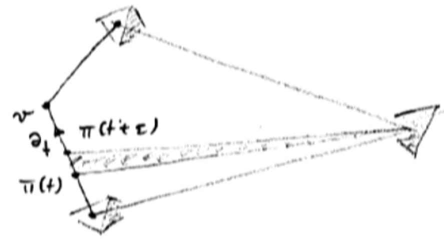
$$S_{\mathrm{top}}[\Sigma, A] = \int_M \underbrace{\frac{\mathrm{i}\hbar}{{\ell_{\mathrm{P}}}^2}\frac{\beta+\mathrm{i}}{\beta}\Sigma_{AB}}_{2\Pi_{AB}} \wedge F^{AB} + \mathrm{cc.}$$

$$\approx \frac{1}{4}\sum_{w:\mathrm{wedges}} \left(\Pi_{AB}[\tau_w]F^{AB}[w] + \Pi_{AB}[\tau_w^{-1}]F^{AB}[w^{-1}]\right) + \mathrm{cc.}$$

$$\approx \frac{1}{4}\sum_{w:\mathrm{wedges}} \left(\Pi_{AB}[\tau_w]h^{AB}[\partial w] + \Pi_{AB}[\tau_w^{-1}]h^{AB}[\partial w^{-1}]\right) + \mathrm{cc.} \equiv \sum_{w:\mathrm{wedges}} S^w_{\mathrm{top}}.$$

1.5 Partial continuum limit

- The first step is to write all configuration variables in terms of spinors (π_A, ω^A), and additional bulk holonomies $g \in SL(2,\mathbb{C})$.
- Splitting every wedge into N auxiliary wedges, we can take a continuum limit $N \to \infty$.



We end up with the following action on a wedge:

$$S^{w}_{\text{top}}[\pi,\omega,A] = \frac{1}{2}\int_e \mathrm{d}t\big(\omega_A \frac{\mathrm{D}}{\mathrm{d}t}\pi^A + \pi_A \frac{\mathrm{D}}{\mathrm{d}t}\omega^A\big) + \text{cc.} \tag{12}$$

- $\frac{\mathrm{D}}{\mathrm{d}t}\pi^A = \dot{\pi}^A + A^A{}_B(\dot{e})\pi^B$ is the $\mathfrak{sl}(2,\mathbb{C})$ covariant derivative into the direction of the edge.
- To get the full action we sum over all wedges, and impose boundary conditions such that all fluxes are continuous.

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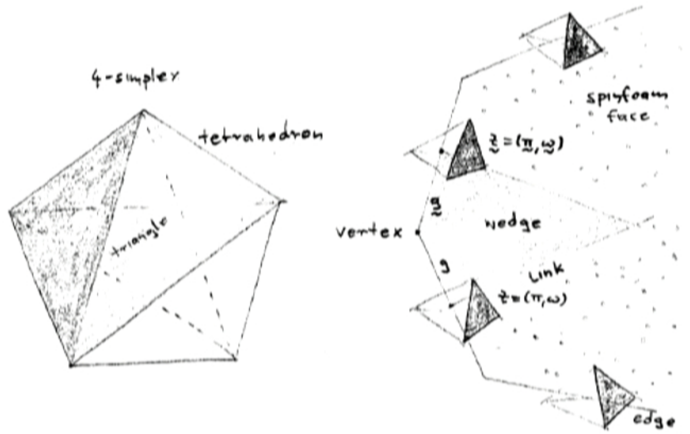

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$$\approx \frac{1}{4}\sum_{w:\mathrm{wedges}} \left(\Pi_{AB}[\tau_w]F^{AB}[w] + \Pi_{AB}[\tau_w^{-1}]F^{AB}[w^{-1}]\right) + \mathrm{cc.}$$

$$\approx \frac{1}{4}\sum_{w:\mathrm{wedges}} \left(\Pi_{AB}[\tau_w]h^{AB}[\partial w] + \Pi_{AB}[\tau_w^{-1}]h^{AB}[\partial w^{-1}]\right) + \mathrm{cc.} \equiv \sum_{w:\mathrm{wedges}} S^w_{\mathrm{top}}.$$

1.6 Adding the simplicity constraints to the theory

With spinors the linear constraints $\forall \tau \in T : n^\alpha(T)\Sigma_{\alpha\beta}[\tau] = 0$ turn into the following:

$$D = \frac{\mathrm{i}}{\beta + \mathrm{i}} \pi_A \omega^A + \mathrm{cc.} = 0, \tag{13a}$$

$$F_n = n^{A\bar{A}} \pi_A \bar{\omega}_{\bar{A}} = n^{A\bar{A}} m_{A\bar{A}} = 0. \tag{13b}$$

$D = 0$ is first-class, and guarantees that the area is real.
$F_n = 0$ is second-class, and generates an additional $\mathfrak{su}(2)$ algebra.
Introducing Lagrange multipliers we get the constrained wedge-action:

$$S^w_{\mathrm{cons}}[\pi, \omega, A; z, \lambda] = \frac{1}{2} \int_0^1 \mathrm{d}t \Big[\omega_A \frac{\mathrm{D}}{\mathrm{d}t} \pi^A + \pi_A \frac{\mathrm{D}}{\mathrm{d}t} \omega^A + \\ - 2z F_{n(t)}(\pi, \omega) - \lambda D(\pi, \omega) + \mathrm{cc.} \Big]. \tag{14}$$

Where $n^\alpha(t) \in \mathbb{R}^4$ is implicitly defined by parallel translation:

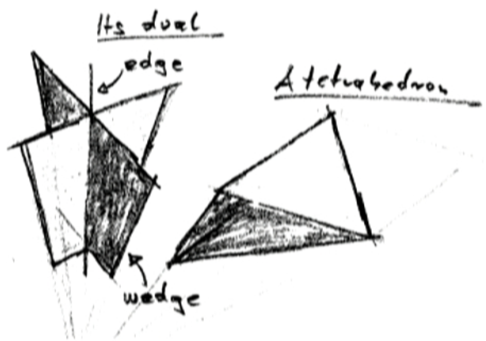
$$\text{locally:} \quad \frac{\mathrm{D}}{\mathrm{d}t} n^\alpha(t) = \dot{n}^\alpha(t) + A^\alpha{}_{\beta(t)} n^\beta(t) = 0. \tag{15}$$

The full action is the sum over all wedges.

2.1 Variation of the total action

Collecting all contributions coming from a given tetrahedron, we get the Edge action. (A tetrahedron is the dual of an edge.)

$$S_{\text{edge}}[\underline{\omega},\underline{\pi},\underline{z},\underline{\lambda},A^A{}_B]=\frac{1}{2}\sum_{I=1}^{4}\int_0^1 \mathrm{d}t\Big[\omega_A^{(I)}\frac{\mathrm{D}}{\mathrm{d}t}\pi^A_{(I)}+\pi_A^{(I)}\frac{\mathrm{D}}{\mathrm{d}t}\omega^A_{(I)}+ \\ -2z_{(I)}F_n(\pi_{(I)},\omega_{(I)})-\lambda_{(I)}D(\pi_{(I)},\omega_{(I)})\Big]+\mathrm{cc.} \qquad (16)$$



- Variation of the multipliers $A^A{}_B(t)$, $\lambda_{(I)}$ and $z_{(I)}$ gives Gauß's law and the simplicity constraints.
- The evolution equations come from $\delta\omega$ and $\delta\pi$.

First-class constraints: The rotational part of Gauß's law (generating $SU(2)$ rotations), and the D-constraint for each triangle (generating conformal transformations of the spinors).

2.2 Dirac analysis of the constraints

We can write the equations of motion for a single triangle in a Hamiltonian form:

$$\frac{\mathrm{D}}{\mathrm{d}t}\omega^A = \{H', \omega^A\}, \quad \frac{\mathrm{D}}{\mathrm{d}t}\pi^A = \{H', \pi^A\}. \tag{17a}$$

With the primary Hamiltonian:

$$H' = z(t)F_{n(t)}(\pi, \omega) + \frac{\lambda(t)}{2}D(\pi, \omega) + \mathrm{cc}. \tag{18}$$

Time evolution preserves the constraints (Gauß's law, simplicity constraints and area matching) provided $z = 0$.

This gives the secondary Hamiltonian:

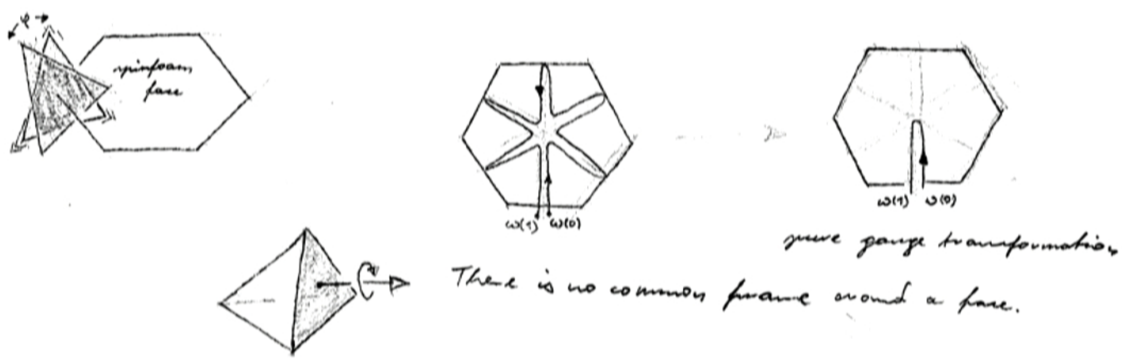
$$H'' = \lambda(t)D(\pi, \omega). \tag{19}$$

2.3 Boundary conditions and flatness problem

Twisted boundary conditions can avoid the flatness problem*

Take all fluxes to be continuous, but allow for gentle discontinuities in the spinors. These are those discontinuities that we can absorb into an $SU(2)$ transformation of the whole tetrahedron:

$$\exists \varphi \in [0, 4\pi) : (\pi^A(0), \omega^A(0)) = (\mathrm{e}^{\mathrm{i}\frac{\varphi}{2}} \pi^A(1), \mathrm{e}^{-\mathrm{i}\frac{\varphi}{2}} \omega^A(1)). \qquad (20)$$



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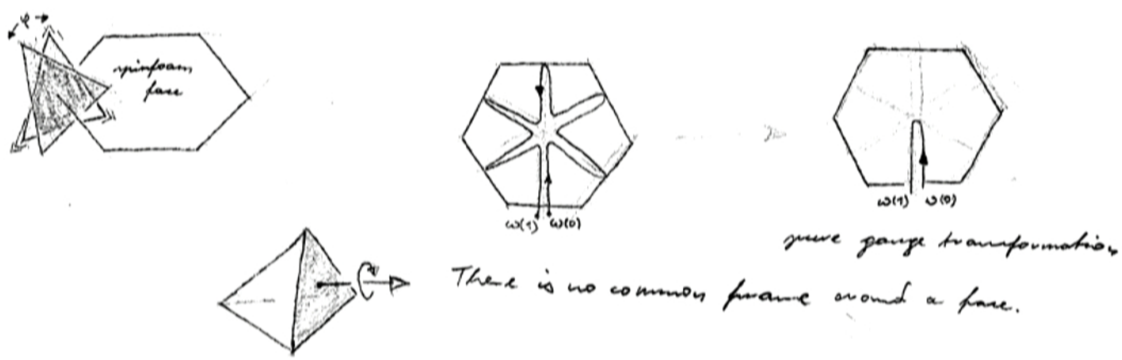

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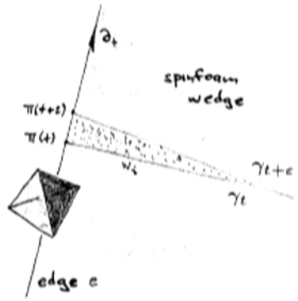
2.4 Is this discretised gravity?

The first indication comes from the analysis of Barrett et al*. The equations of motion imply that around each vertex there is a unique 4-simplex with "bones" $\ell^\alpha(ij) \in \mathbb{R}^4$ bounding all triangles, e.g.:

$$\Sigma_{\alpha\beta}[\tau_{12}, \underline{\ell}] = \ell_{[\alpha}(34)\ell_{\beta]}(45). \tag{21}$$

The length of a bone $\ell(ij)^2 = -\ell_\alpha(ij)\ell^\alpha(ij)$ is the same from whatever four-simplex we look at it.

The second indication is the presence of curvature in the model.



$$\frac{\mathrm{D}}{\mathrm{d}t}\pi^A \neq 0 \Rightarrow F^{AB} \neq 0, \tag{22a}$$

$$\mathrm{ch}(\Xi_{i,i+1}) = -n^\alpha(i) n_\alpha(i+1), \tag{22b}$$

$$\sum_i \Xi_i = \frac{2}{\beta^2+1}\Lambda = \frac{2}{\beta^2+1}\oint \mathrm{d}t\lambda. \tag{22c}$$

With the "good" boundary conditions we get the Regge holonomy:

$$h^A{}_B[\partial f] = (\pi_C\omega^C)^{-1}\big(\mathrm{e}^{-\frac{\Lambda}{\beta^2+1}}\omega^A\pi_B - \mathrm{e}^{+\frac{\Lambda}{\beta^2+1}}\pi^A\omega_B\big). \tag{23}$$

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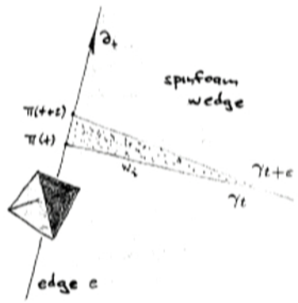

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3.1 Schrödinger quantisation

The primary phase space on a half link is $\mathbb{C}^2 \oplus \mathbb{C}^2 \ni (\pi^A, \omega^A)$, we take a position representation and define the auxiliary Hilbert space

$$\mathcal{H}_{\mathrm{aux}} := L^2(\mathbb{C}^2, d^4\omega) = \int_{\mathbb{R}}^{\oplus} \mathrm{d}\rho \sum_{k\in\mathbb{Z}} \mathcal{H}_{\rho,k}. \quad (24)$$

We use the canonical basis $\{f_{jm}^{(\rho,k)}\}$ simultaneously diagonalising the Casimirs $\vec{L}\vec{K}$, $\vec{L}^2 - \vec{K}^2$ of $SL(2,\mathbb{C})$ together with $\vec{L}^2$ and L_3.
The first-class constraint $D = 0$ is diagonal:

$$\hat{D} f_{jm}^{(\rho,k)} = \frac{2\hbar}{\beta^2+1}\big(\rho - \beta(k+1)\big) f_{jm}^{(\rho,k)}. \quad (25)$$

The second-class constraints $F_n = 0$ act like step operators for $\mathfrak{su}(2)$:

$$\hat{F}_{n_o} f_{jm}^{(\rho,k)} = -\frac{\hbar}{\sqrt{2}}\sqrt{(j-k)(j+k+1)}\, f_{jm}^{(\rho,(k+1))}, \quad (26a)$$

$$\hat{F}_{n_o}^{\dagger} f_{jm}^{(\rho,k)} = -\frac{\hbar}{\sqrt{2}}\sqrt{(j+k)(j-k+1)}\, f_{jm}^{(\rho,(k+1))}. \quad (26b)$$

3.2 Solution space and finite inner product

The D-constraint is first-class, we can impose it strongly, with the solution space spanned by functions

$$\mathcal{H}_D = \mathrm{span}\{ f^{(\beta(k+1),k)}_{jm} \;:\; k, j, m \}. \tag{27}$$

$f^{(\beta(k+1),k)}_{jm}$ are distributions in $\mathbb{C}^2$, but they are orthogonal and properly normalised with respect to the inner product on the orbits:

$$\boxed{\langle f, f' \rangle_{\mathbb{C}^2/D} \propto \int_{\mathbb{C}^2/D} X_D \lrcorner d^4\omega \, \bar{f} f' < \infty.} \tag{28}$$

The F-constraint is second-class. We search for $\mathcal{H}_{\mathrm{simpl}} \subset \mathcal{H}_{\mathrm{aux}}$ such that: $\hat{F}\mathcal{H}_{\mathrm{simpl}} = 0$, but $\hat{F}^\dagger \mathcal{H}_{\mathrm{simpl}} \perp \mathcal{H}_{\mathrm{simpl}}$. The resulting Hilbert space is:

$$\mathcal{H}_{\mathrm{simpl}} = \mathrm{span}\{ f^{(\beta(j+1),j)} \}_{j,m}. \tag{29}$$

The rotational part of the Gauß constraint is first-class, the other half holds already because of $F = 0 = D$. Imposing it strongly reveals the physical Hilbert space (of a quantised tetrahedron):

$$\Psi(\omega_{(1)}, \dots, \omega_{(4)}) \in \mathcal{H}_{\mathrm{phys}} = \mathrm{Inv}_{SU(2)}\Big(\bigotimes^4 \mathcal{H}_{\mathrm{simpl}} \Big). \tag{30}$$

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3.3 Transition amplitudes on a spinfoam face

- The area matching constraint glues the tetrahedra together, revealing the usual space of spin-network functions.
- What about the dynamics?

Time evolution along an edge is governed by the Hamilton equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}O_t = \Big\{\big(A^{AB}(t)\pi_A\omega_B + \mathrm{cc.}\big) + \lambda(t)D, O_t\Big\}. \tag{31}$$

In quantum theory this becomes the Schrödinger equation on an edge:

$$\boxed{\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = \big(A^{AB}(t)\hat{\pi}_A\hat{\omega}_B + \mathrm{h.c.}\big)\psi_t + \lambda(t)\hat{D}\psi_t.} \tag{32}$$

- The D-constraint annihilates $\mathcal{H}_{\mathrm{simpl}}$, therefore only the first term survives when acting on a physical state.
- The first term acts as an infinitesimal Lorentz generator, and matches Bianchi's* boundary Hamiltonian.

*E. Bianchi, Entropy of Non-Extremal Black Holes from Loop Gravity (2012), arXiv:1204.5122.
*S. Carlip and C. Teitelboim, The Off-Shell Black Hole, Class. Quant. Grav. 12 (1995), arXiv:gr-qc/9312002.

Conclusion

1. The EPRL proposal for the loop gravity transition amplitudes results from the canonical quantisation of a classical theory with a finite number of degrees of freedom.
2. I gave two arguments supporting the idea that the classical theory is a discretisation of GR: (i) the model has curvature, (ii) the equations of motion imply geometricity, which means that we can assign the unique length to any of the three bones bounding a triangle.
3. The spinorial framework allows to complete the canonical analysis. All constraints are preserved in the “time” variable around a spinfoam face. There are no secondary constraints.

Spinors are useful for the following reasons: (i) They are canonical Darboux coordinates taking care of the non-linearities of the loop gravity phase space. (ii) They transform covariantly under the local symmetry group of general relativity. (iii) Dynamics on a fixed discretisation of space-time simplifies.

Thanks for the attention!

This talk is based on the papers:

- WW., Twistorial phase space for complex Ashtekar variables, Class. Quantum Grav. 29 (2012), arXiv:1104.3683.
- S. Speziale and WW., The twistorial structure of loop-gravity transition amplitudes, Phys. Rev. D 86 (2012), arXiv:1207.6348.
- WW., Hamiltonian spinfoam gravity (2013), arXiv:1301.5859. *accepted for publication in:* Class. Quant. Grav.

See also:

- L. Freidel and S. Speziale, From twistors to twisted geometries; Phys. Rev. D 82 (2010), arXiv:1001.2748.
- M. Dupuis and E. Livine, Holomorphic Simplicity Constraints for 4d Spinfoam Models, Class. Quantum Grav. 28 (2011), arXiv:1001.2748.
- B. Dittrich and Höhn, Constraint analysis for variational discrete systems (2013), arXiv:1303.4294.

Spin-cube Models of Quantum Gravity

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Spin foams

- Problem with the classical limit: the effective action gives the area-Regge action [Miković and Vojinović; 2011]. It was also conjectured that the non-geometric configurations are exponentially supressed. No proof yet.
- Problem with matter: spinors couple to the edge lengths while a generic spin-foam configuration does not define a metric geometry.
- How to introduce the edge lengths (tetrads):
  1) AdS/dS BF theory: does not work [Martins and Miković, SIGMA, 2011].
  2) Poincare gauge theory: tetrads do not transform as connections.
  3) 2-groups

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  3) 2-groups

2-groups

- Category: objects and maps (1-morphisms)
- 2-Category: objects, maps and maps between maps (2-morphisms)
- Group = Category with one object and invertible 1-morphisms
- 2-Group = 2-Category with one object and invertible 1 and 2-morphisms
- 2-Group = Crossed module of groups: $(G, H, \partial, \triangleright)$ such that $g \triangleright h \in H$, $\partial h \in G$ and

$$\partial(g \triangleright h) = g(\partial h)g^{-1}\,, \quad (\partial h) \triangleright h' = hh'h^{-1}\,.$$

- G = 1-morphisms, $G \times_s H$ = 2-morphisms
- Poincare or Euclidean 2 group: $G = SO(1,3)$ or $G = SO(4)$, $H = \mathbf{R}^4$

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2-BF theory

- $(G, H, \partial, \triangleright) \longrightarrow (\mathfrak{g}, \mathfrak{h}, \partial, \triangleright)$ = differential crossed module
- $A \in \Omega_1(\mathfrak{g}) \longrightarrow (A, \beta) \in (\Omega_1(\mathfrak{g}), \Omega_2(\mathfrak{h}))$ = 2-connection
- 2-group gauge transformations: $g : M \longrightarrow G$ and $\eta : M \longrightarrow \Omega_1(\mathfrak{h})$

$$A \to g(A + d)g^{-1}\,, \quad \beta \to g^{-1} \triangleright \beta$$

$$A \to A + \partial\eta\,, \quad \beta \to \beta + d\eta + A \wedge^{\triangleright} \eta + \eta \wedge \eta$$

- 2-curvature

$$(\mathcal{F}, \mathcal{G}) = (F - \partial\beta, d\beta + A \wedge^{\triangleright} \beta)$$

where $F = dA + A \wedge A$.

BFCG action and GR

- BFCG action [Girelli, Pfeiffer and Popescu; 2008], [Martins and Miković; ATMP, 2011]

$$S_0 = \int_M \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}}$$

is invariant under the 2-group gauge transformations if

$$g: \quad B \to g^{-1}Bg\,, \quad C \to g \rhd C\,;$$

$$\eta: \quad B \to B - [C, \eta]\,, \quad C \to C\,.$$

- GR as a constrained BFCG theory for the Poincare 2-group [Miković and Vojinović; 2012]

$$S_{GR} = \int_M B^{ab} \wedge R_{ab} + e^a \wedge \nabla\beta_a - \lambda^{ab} \wedge (B_{ab} - \epsilon_{abcd} e^c \wedge e^d)\,,$$

where $R_{ab} = d\omega_{ab} + \omega_a^c \wedge \omega_{cb}$ and $\nabla\beta_a = d\beta_a + \omega_a^b \wedge \beta_b$.

State sum for BFCG

- Categorical and path-integral considerations imply that

$$Z_0 = \int_{L\in\tilde{\mathbf{R}}_+^E} \prod_{\epsilon=1}^{E} \mu(L_\epsilon) dL_\epsilon \sum_{\Lambda\in(Rep\, G)^F} \sum_{\iota\in(Itw\,\Lambda)^{E^*}} W_3(L,\Lambda,\iota)\,,$$

where $L_\epsilon, \Lambda_\Delta$ and ι_τ are labels for a Poincare/Euclidean 2-group representation, intertwiner and 2-intertwiner, respectively.

- In [Crane and Sheppeard; 2003] and [Baez, Baratin, Freidel and Wise; 2008] it was shown that there are irreps of Poincare/Euclidean 2-group labelled by $L_\epsilon \geq 0$. The corresponding intertwiners are the irreps of $SO(2)$ if L_ϵ form a triangle, and the 2-intertwiners ι_τ are trivial. Hence

$$Z_0 = \int_{L\in\tilde{\mathbf{R}}_+^E} \prod_{\epsilon=1}^{E} \mu(L_\epsilon) dL_\epsilon \sum_{m\in\mathbf{Z}^F} W_3(L,m)\,.$$

State sum for BFCG

- The results of [Baratin and Friedel; 2007] suggest that

$$W_3(L,m) \approx \prod_{\Delta=1}^{F} A_\Delta(L) \prod_{\sigma=1}^{V} \frac{\cos S_\sigma(L,m)}{V_\sigma(L)} ,$$

and $\mu(L) = L$. Here A_Δ is the area of a triangle Δ , V_σ is the volume of a 4-simplex σ and

$$S_\sigma = \sum_{\Delta\in\sigma} m_\Delta \theta_\Delta^{(\sigma)}(L) ,$$

where $\theta_\Delta^{(\sigma)}$ is the interior dihedral angle.

State sum for quantum GR

- Since GR can be considered as a constrained BFCG theory, one can try to impose a discretized analog of $B = (e \wedge e)^*$ constraint. A natural candidate is

$$\gamma m_\Delta = A_\Delta(L)\,.$$

$S_\sigma(m, L)$ then becomes proportional to the Regge action for σ.

- A good candidate is

$$Z_{GR} = \int_{L \in \tilde{\mathbf{R}}_+^E} \prod_\epsilon \mu(L_\epsilon) dL_\epsilon \sum_{m \in \mathbf{N}^F} \prod_\Delta \delta(\gamma m_\Delta - A_\Delta(L)) \prod_\sigma e^{iS_\sigma(m,L)},$$

where $\mu(L) \approx L^r$ for large L and $r < r_0$ in order to have finiteness [Miković and Vojinović; 2012].

Solving the GR constraint

- In a 4-manifold triangulation we have

$$F \geq \frac{4}{3}E > E$$

so that

$$L_\epsilon = \lambda_\epsilon(m_1, m_2, ..., m_E)\,, \quad \epsilon = 1, 2, ..., E\,;$$

and

$$m_k = \varphi_k(m_1, m_2, ..., m_E)\,, \quad k = E+1, E+2, ..., F\,,$$

where $\varphi_k(m) = A_k(\lambda(m))$.

- The Diofantine equation above is difficult to solve and may not have solutions, so we relax the GR constraint as

$$m_\epsilon = A_\epsilon(L)\,, \quad \epsilon = 1, 2, ..., E;$$

$$m_k = [A_k(L)]\,, \quad k = E+1, E+2, ..., F\,.$$


A Regge model
The GR constraint can be further relaxed as
This constraint has solutions for
The corresponding state sum is given by
where

Semi-classical limit

- Consider the effective action defined by

$$\Gamma(L) = Re\,\tilde{\Gamma}(L) + Im\,\tilde{\Gamma}(L)\,,$$

where $\tilde{\Gamma}(L)$ is a solution of

$$e^{i\tilde{\Gamma}(L)} = \int_{\mathbf{R}_+^E} d^E l \prod_{\epsilon=1}^{E} \mu(L_\epsilon + l_\epsilon) \exp\left(iS(L+l) - i\sum_{\epsilon=1}^{E} \tilde{\Gamma}'_\epsilon(L) l_\epsilon \right) .$$

- When $S(L) = S_R(L)$ or $S(L) = \tilde{S}_R(L)$ and $\mu(L) \approx L^r$ for $L \to \infty^E$, it is easy to show that

$$\Gamma(L) = S_R(L) + r\sum_{\epsilon=1}^{E} \ln L_\epsilon + \frac{1}{2} Tr\,\log S''_R(L) + O(L^{-k})\,,$$

for $L \to \infty^E$, where $k = 2$ for $S = S_R$ and $k = 0$ for $S = \tilde{S}_R$.

Conclussions

- Z_{GR} can give a constrained area-Regge model or a length-Regge model, depending on how the GR constraint is imposed.
- The effective action in the semi-classical approximation can be easilly calculated for the length-Regge model.
- Amplitude for matter coupling: $W_{matter} \propto e^{iS_{mR}(\psi,L)}$.
- Spin connection for fermions: $\omega_{ab} + e_{[a|}\bar{\psi}\gamma_{|b]}\psi$.
- Canonical quantization of 2-Poincare GR action
- Categorification of LQG
- Construction of 4-manifold invariants

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