

Title: A unifying approach to dark energy models

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Abstract: This talk will present an effective description of single field dark energy/modified gravity models, which encompasses most existing proposals. The starting point is a generic Lagrangian expressed in terms of the lapse and of the extrinsic and intrinsic curvature tensors of the uniform scalar field hypersurfaces. By expanding this Lagrangian up to quadratic order, one can describe the homogeneous background and the dynamics of linear perturbations. In particular, one can identify seven Lagrangian operators that lead to equations of motion containing at most second order derivatives, the time-dependent coefficients of three of these operators characterizing the background evolution. I will illustrate this approach with Horndeski's---or generalized Galileon---theories. Finally, I will discuss the link between this effective approach and observations.

A unifying description of Dark Energy (& modified gravity)

David Langlois
(APC, Paris)



Astroparticules
et Cosmologie

Outline

- Motivations
- ADM formulation & link with the EFT formalism
- Illustration: Horndeski's theories
- Link with observations

Based on J. Gleyzes, D.L., F. Piazza & F. Vernizzi, 1304.4840 [hep-th]

Introduction & motivations

- **Plethora of dark energy models:**
 - Dynamical dark energy: quintessence, K-essence
 - Modified gravity
- Large amount of data from future large scale cosmological surveys (BigBOSS, LSST, Euclid, etc...)
- Goal: **effective description** as a bridge between models and observations.
- Assumptions:
 - **Single scalar field** models
 - All matter fields minimally coupled to the same metric

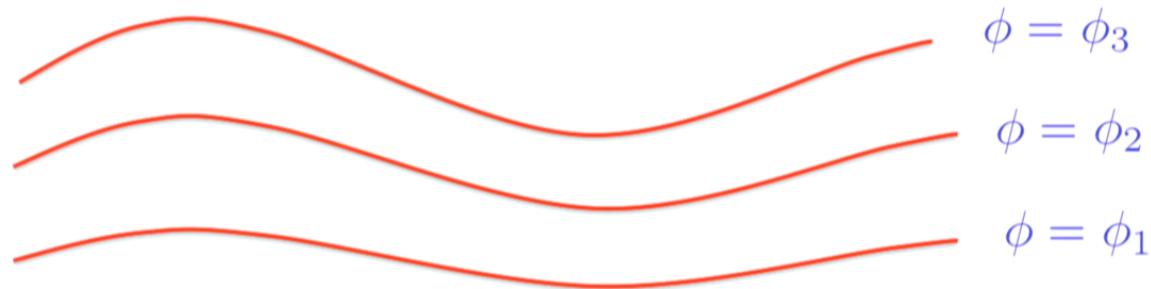
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ADM approach

- The scalar field defines a **preferred slicing**

Constant time hypersurfaces = uniform field hypersurfaces



- ADM decomposition based on this preferred slicing

ADM approach

- Basic ingredients:

$$n_\mu = -\frac{\partial_\mu \phi}{\sqrt{-(\partial\phi)^2}}$$

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

- Intrinsic curvature tensor ${}^{(3)}R_{\mu\nu}$

- Extrinsic curvature tensor $K_{\mu\nu} = h_{\mu\sigma} \nabla^\sigma n_\nu$

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- Generic action: $S = \int d^4x \sqrt{-g} L(t; N, K, \mathcal{S}, \mathcal{R}, \mathcal{Z}, \dots)$

$$K \equiv K^\mu_\mu, \quad \mathcal{S} \equiv K_{\mu\nu} K^{\mu\nu}, \quad \mathcal{R} \equiv {}^{(3)}R, \quad \mathcal{Z} \equiv {}^{(3)}R_{\mu\nu} {}^{(3)}R^{\mu\nu}$$

Note: dependence on ${}^{(3)}R^{\mu\nu} K_{\mu\nu}$ can be reabsorbed.

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Expansion of the Lagrangian

- Separation between FLRW background and perturbations

$$\delta K \equiv K - 3H, \quad \delta K_{\mu\nu} \equiv K_{\mu\nu} - H h_{\mu\nu}$$

which implies $\mathcal{S} = 3H^2 + 2H\delta K + \delta K^\mu_\nu \delta K^\nu_\mu$

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- **Lagrangian up to ... first order**

$$L(N, K, \mathcal{S}, \mathcal{R}, \mathcal{Z}) = \bar{L} + L_N \delta N + \mathcal{F} \delta K + L_{\mathcal{R}} \delta \mathcal{R} + \dots$$

$$\text{with } \mathcal{F} \equiv 2H\bar{L}_{\mathcal{S}} + \bar{L}_K$$

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Using $\mathcal{F} \delta K = \mathcal{F}(K - 3H)$

and $\int d^4x \sqrt{-g} \mathcal{F} K = - \int d^4x \sqrt{-g} n^\mu \nabla_\mu \mathcal{F} = - \int d^4x \sqrt{-g} \frac{\dot{\mathcal{F}}}{N}$

$$L(N, K, \mathcal{S}, \mathcal{R}, \mathcal{Z}) = \bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F} + (\dot{\mathcal{F}} + L_N) \delta N + L_{\mathcal{R}} \delta \mathcal{R} + \dots$$

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Expansion (up to second order)

Up to **quadratic order** in the perturbations, one finds

$$\begin{aligned} L(N, K, \mathcal{S}, \mathcal{R}, \mathcal{Z}) &= \bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F} + (\dot{\mathcal{F}} + L_N) \delta N + L_{\mathcal{R}} \delta \mathcal{R} \\ &+ \frac{\mathcal{A}}{2} \delta K^2 + L_{\mathcal{S}} \delta K^\mu_\nu \delta K^\nu_\mu + \left(\frac{1}{2} L_{NN} - \dot{\mathcal{F}} \right) \delta N^2 \\ &+ \frac{1}{2} L_{\mathcal{R}\mathcal{R}} \delta \mathcal{R}^2 + \mathcal{B} \delta K \delta N + \mathcal{C} \delta K \delta \mathcal{R} + L_{N\mathcal{R}} \delta N \delta \mathcal{R} + L_{\mathcal{Z}} \delta \mathcal{Z} \end{aligned}$$

with the coefficients

$$\begin{aligned} \mathcal{A} &\equiv 4H^2 L_{SS} + 4HL_{SK} + L_{KK} \\ \mathcal{B} &\equiv 2HL_{SN} + L_{KN}, \\ \mathcal{C} &\equiv 2HL_{SR} + L_{KR} \\ \mathcal{F} &\equiv 2HL_S + L_K \end{aligned}$$

Background equations

- FLRW metric: $ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j$
- Homogeneous action:

$$S_0 = \int dt d^3x N a^3 L \left[N, K = \frac{3H}{N}, \mathcal{S} = \frac{3H^2}{N^2}, \mathcal{R} = 0, \mathcal{Z} = 0 \right]$$

- Friedmann equations

$$\delta S_0 = \int dt d^3x \left[a^3 (\bar{L} + L_N - 3H\mathcal{F}) \delta N + (\bar{L} - 3H\mathcal{F} - \dot{\mathcal{F}}) \delta(a^3) \right]$$

$$\bar{L} + L_N - 3H\mathcal{F} = 0, \quad \bar{L} - 3H\mathcal{F} - \dot{\mathcal{F}} = 0$$

- Including matter

$$\bar{L} + L_N - 3H\mathcal{F} = \rho_m, \quad \bar{L} - 3H\mathcal{F} - \dot{\mathcal{F}} = -p_m$$

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Quadratic Lagrangian

- Explicit gauge choice: $h_{ij} = a^2(t)e^{2\zeta}\delta_{ij}$, $N_i = \partial_i\psi$
- Quadratic Lagrangian: $\mathcal{L}_2 = \mathcal{L}_2 [\zeta, \delta N, \partial^2\psi]$
$$\mathcal{L}_2 \supset \frac{1}{a} \left[\frac{1}{2} (\mathcal{A} + 2L_S) (\partial^2\psi)^2 + 4\mathcal{C} \partial^2\psi \partial^2\zeta + 2 (4L_{RR} + 3L_Z) (\partial^2\zeta)^2 \right]$$
- One can get rid of higher spatial derivatives by imposing

$$\mathcal{A} + 2L_S = 0, \quad \mathcal{C} = 0, \quad 4L_{RR} + 3L_Z = 0$$

The momentum constraint then reduces to: $\delta N = \frac{4L_S}{\mathcal{B} + 4HL_S} \dot{\zeta}$

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Quadratic Lagrangian

- One finally obtains the quadratic Lagrangian in the form

$$\mathcal{L}_2 = \frac{a^3}{2} \left[\mathcal{L}_{\dot{\zeta}\dot{\zeta}} \dot{\zeta}^2 + \mathcal{L}_{\partial_i \zeta \partial_i \zeta} \frac{(\partial_i \zeta)^2}{a^2} \right]$$

with the coefficients

$$\mathcal{L}_{\dot{\zeta}\dot{\zeta}} \equiv (L_{NN} + 2L_N - 6H\mathcal{B} - 12H^2 L_S) \mathcal{D}^2 + 12L_S$$

$$\mathcal{L}_{\partial_i \zeta \partial_i \zeta} \equiv 4 \left[L_{\mathcal{R}} - \frac{1}{a} \frac{d}{dt} (a \mathcal{D} (L_{\mathcal{R}} + L_{N\mathcal{R}})) \right] \quad \mathcal{D} \equiv \frac{4L_S}{\mathcal{B} + 4HL_S}$$

- Absence of ghosts: $\mathcal{L}_{\dot{\zeta}\dot{\zeta}} > 0$
- Effective speed of sound: $c_s^2 = -\frac{\mathcal{L}_{\partial_i \zeta \partial_i \zeta}}{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}}$

Simple example: K-essence

Armendariz-Picon et al. 00

- K-essence Lagrangian

$$L = P(\phi, X), \quad X \equiv \nabla_\mu \phi \nabla^\mu \phi$$

- Total Lagrangian in the ADM formulation

$$L = \frac{1}{2} (\mathcal{S} - K^2 + \mathcal{R}) + P(X, \phi) \quad X = -\frac{\dot{\phi}^2}{N^2}$$

- Background Lagrangian and its derivatives:

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Link with the EFT formalism

Creminelli et al. 06, Cheung et al. 07; Creminelli et al. 08; Gubitosi et al. 12

- Action up to quadratic order (with no higher derivatives)

$$\begin{aligned} L = & \frac{M_*^2}{2} f(t) R - \Lambda(t) - c(t) g^{00} + \frac{M_2^4(t)}{2} (\delta g^{00})^2 - \frac{m_3^3(t)}{2} \delta K \delta g^{00} \\ & - m_4^2(t) (\delta K^2 - \delta K^\mu_\nu \delta K^\nu_\mu) + \frac{\tilde{m}_4^2(t)}{2} {}^{(3)}R \delta g^{00} \end{aligned}$$

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- Background evolution described by only three functions $f(t), \Lambda(t), c(t)$

$$c + \Lambda = 3M_*^2 (f H^2 + \dot{f} H) , \quad \Lambda - c = M_*^2 (2f \dot{H} + 3f H^2 + 2\dot{f} H + \ddot{f})$$

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- Link between the ADM & EFT formulations

$$R = {}^{(3)}R + K_{\mu\nu} K^{\mu\nu} - K^2 + 2\nabla_\nu (n^\nu \nabla_\mu n^\mu - n^\mu \nabla_\mu n^\nu) \quad g^{00} = -\frac{1}{N^2}$$

Explicit dictionary for the EFT coefficients

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- Link between the ADM & EFT formulations

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Generalized Galileons

Horndeski 74; Nicolis et al. 08; Deffayet et al. 09 & 11

- Most general action for a scalar field leading to at most second order equations of motion (Horndeski '74)

Combination of the following four Lagrangians

$$L_2 = G_2(\phi, X) ,$$

with $X \equiv \phi^{;\mu} \phi_{;\mu}$

$$L_3 = G_3(\phi, X) \square \phi ,$$

$$L_4 = G_4(\phi, X) R - 2G_{4X}(\phi, X)(\square \phi^2 - \phi^{;\mu\nu} \phi_{;\mu\nu}) ,$$

$$L_5 = G_5(\phi, X) G_{\mu\nu} \phi^{;\mu\nu} + \frac{1}{3} G_{5X}(\phi, X) (\square \phi^3 - 3 \square \phi \phi_{;\mu\nu} \phi^{;\mu\nu} + 2 \phi_{;\mu\nu} \phi^{;\mu\sigma} \phi^{;\nu}_{;\sigma})$$

- **ADM formulation ?**
- Which operators appear in the quadratic Lagrangian ?

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Generalized Galileons: ADM form

Uniform scalar field slicing

- Unit normal vector $n_\mu = -\gamma \phi_{;\mu}$ with $\gamma = \frac{1}{\sqrt{-X}}$
- Induced metric $h_{\mu\nu} = n_\mu n_\nu + g_{\mu\nu}$
- Using $K_{\mu\nu} = h_\mu^\sigma n_{\nu;\sigma}$ $\dot{n}_\mu = n^\nu n_{\mu;\nu}$
one can write $n_{\nu;\mu} = K_{\mu\nu} - n_\mu \dot{n}_\nu$
 $\phi_{;\mu\nu} = -\gamma^{-1}(K_{\mu\nu} - n_\mu \dot{n}_\nu - n_\nu \dot{n}_\mu) + \frac{\gamma^2}{2} \phi^{;\lambda} X_{;\lambda} n_\mu n_\nu$
- The Gauss-Codazzi relations are also useful.

Generalized Galileons: ADM form

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Generalized Galileons

Horndeski 74; Nicolis et al. 08; Deffayet et al. 09 & 11

- Most general action for a scalar field leading to at most second order equations of motion (Horndeski '74)

Combination of the following four Lagrangians

$$L_2 = G_2(\phi, X) ,$$

with $X \equiv \phi^{;\mu} \phi_{;\mu}$

$$L_3 = G_3(\phi, X) \square \phi ,$$

$$L_4 = G_4(\phi, X) R - 2G_{4X}(\phi, X)(\square \phi^2 - \phi^{;\mu\nu} \phi_{;\mu\nu}) ,$$

$$L_5 = G_5(\phi, X) G_{\mu\nu} \phi^{;\mu\nu} + \frac{1}{3} G_{5X}(\phi, X) (\square \phi^3 - 3 \square \phi \phi_{;\mu\nu} \phi^{;\mu\nu} + 2 \phi_{;\mu\nu} \phi^{;\mu\sigma} \phi^{;\nu}_{;\sigma})$$

- **ADM formulation ?**
- Which operators appear in the quadratic Lagrangian ?

Generalized Galileons: ADM form

- Lagrangian $L_3 = G_3(\phi, X)\square\phi$

Introducing the auxiliary function $F_3(\phi, X)$ such that

$$G_3 \equiv F_3 + 2XF_{3X}$$

one finds

$$L_3 = 2(-X)^{3/2}F_{3X}K - XF_{3\phi}$$

- EFT coefficients:

$$\begin{aligned} L = & \cancel{\frac{M_*^2}{2} f(t)R - \Lambda(t) - c(t)g^{00}} + \frac{M_2^4(t)}{2}(\delta g^{00})^2 - \frac{m_3^3(t)}{2}\delta K\delta g^{00} \\ & - m_4^2(t)(\delta K^2 - \delta K^\mu_\nu \delta K^\nu_\mu) + \cancel{\frac{\tilde{m}_4^2(t)}{2}(^{(3)}R\delta g^{00})} \end{aligned}$$

Generalized Galileons: ADM form

- Lagrangian $L_4 = G_4(\phi, X)R - 2G_{4X}(\phi, X)(\square\phi^2 - \phi^{;\mu\nu}\phi_{;\mu\nu})$

Using the Gauss-Codazzi equations, one finally gets

$$L_4 = G_4 {}^{(3)}R + (2XG_{4X} - G_4)(K^2 - K_{\mu\nu}K^{\mu\nu}) - 2\sqrt{-X}G_{4\phi}K$$

- In the quadratic Lagrangian, all terms are present, but with $m_4^2 = \tilde{m}_4^2$

$$\begin{aligned} L = & \frac{M_*^2}{2}f(t)R - \Lambda(t) - c(t)g^{00} + \frac{M_2^4(t)}{2}(\delta g^{00})^2 - \frac{m_3^3(t)}{2}\delta K\delta g^{00} \\ & - \textcircled{-} m_4^2(t)(\delta K^2 - \delta K^\mu{}_\nu\delta K^\nu{}_\mu) + \textcircled{+} \frac{\tilde{m}_4^2(t)}{2} {}^{(3)}R\delta g^{00} \end{aligned}$$

Generalized Galileons: ADM form

$$L_5 = \frac{1}{3}G_{5X}(\phi, X)(\square\phi^3 - 3\square\phi\phi_{;\mu\nu}\phi^{;\mu\nu} + 2\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}_{;\sigma}) + G_5(\phi, X)G_{\mu\nu}\phi^{;\mu\nu}$$

Using the auxiliary function $F_5(\phi, X)$ such that $G_{5X} = F_{5X} + \frac{F_5}{2X}$ one finally gets

$$L_5 = \frac{1}{2}X(G_{5\phi} - F_{5\phi})^{(3)}R + \frac{1}{2}XG_{5\phi}(K^2 - K_{\mu\nu}K^{\mu\nu}) - \sqrt{-X}F_5\left(K^{\mu\nu}(3)R_{\mu\nu} - \frac{1}{2}K^{(3)}R\right) - \frac{1}{3}(-X)^{3/2}G_{5X}\mathcal{K}$$

$$\begin{aligned} \text{with } \mathcal{K} &\equiv K^3 - 3KK_{\mu\nu}K^{\mu\nu} + 2K_{\mu\nu}K^{\mu\sigma}K^{\nu}_{\sigma} \\ &= 6H^3 - 6H^2K + 3HK^2 - 3HS + \mathcal{O}(3) \end{aligned}$$

- All operators are present, again with the restriction $m_4^2 = \tilde{m}_4^2$

Generalized Galileons: ADM form

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Perturbations in an arbitrary gauge

- Description in an arbitrary slicing ?



- Transformation $t \rightarrow t + \pi(t, \vec{x})$

Perturbations in an arbitrary gauge

- Stueckelberg trick: $t \rightarrow t + \pi(t, \vec{x})$
- The new quadratic action can be derived via the substitutions:

$$\begin{aligned} f &\rightarrow f + \dot{f}\pi + \frac{1}{2}\ddot{f}\pi^2, \\ g^{00} &\rightarrow g^{00} + 2g^{0\mu}\partial_\mu\pi + g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi, \\ \delta K_{ij} &\rightarrow \delta K_{ij} - \dot{H}\pi h_{ij} - \partial_i\partial_j\pi, \\ \delta K &\rightarrow \delta K - 3\dot{H}\pi - \frac{1}{a^2}\partial^2\pi, \\ {}^{(3)}R_{ij} &\rightarrow {}^{(3)}R_{ij} + H(\partial_i\partial_j\pi + \delta_{ij}\partial^2\pi), \\ {}^{(3)}R &\rightarrow {}^{(3)}R + \frac{4}{a^2}H\partial^2\pi. \end{aligned}$$

Note: the 3-dim quantities on the right are defined with respect to the new time hypersurfaces.

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Perturbations in the Newtonian gauge

- Perturbed metric

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(t)(1 - 2\Psi)\delta_{ij}dx^i dx^j$$

- Einstein's equations

- Generalized Poisson equation

$$\begin{aligned} -\frac{k^2}{a^2} \left[(2fM_*^2 + 4\tilde{m}_4^2)\Psi - (\dot{f}M_*^2 - m_3^3 + 4Hm_4^2 - 4H\tilde{m}_4^2)\pi \right] + (6M_*^2 H^2 \dot{f} - 6Hc - \dot{c} - \dot{\Lambda} + 3m_3^3 \dot{H})\pi \\ -(2c + 4M_2^4)\dot{\pi} + (3M_*^2 H \dot{f} + 2c + 4M_2^4)\Phi - 3M_*^2 \dot{f}\dot{\Psi} + 3m_3^3(\dot{\Psi} + H\Phi) = \rho_m \Delta_m \end{aligned}$$

- Anisotropic equation

$$M_*^2 \left[f(\Phi - \Psi) + \dot{f}\pi \right] + 2 \left[m_4^2 \dot{\pi} + m_4^2 H\pi + (m_4^2)^\cdot \pi \right] + 2\tilde{m}_4^2(\Phi - \dot{\pi}) = 0$$

- Equation for π

Deviations from GR

- **Quasi-static approximation** for sub-Hubble scales
(time derivatives are neglected)

$$\mathcal{M}(t, k) \begin{pmatrix} \Phi \\ \Psi \\ \pi \end{pmatrix} \equiv \begin{pmatrix} D_\Phi & D_\Psi & D_\pi \\ E_\Phi & E_\Psi & E_\pi \\ F_\Phi & F_\Psi & F_\pi \end{pmatrix} \begin{pmatrix} \Phi \\ \Psi \\ \pi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \rho_m \Delta_m \end{pmatrix}$$

- **Effective Newton's constant**

$$-\frac{k^2}{a^2} \Phi \equiv 4\pi G_{\text{eff}}(t, k) \rho_m \Delta_m \quad 4\pi G_{\text{eff}}(t, k) = -\frac{k^2}{a^2} [\mathcal{M}^{-1}]_{13}$$

- Ratio between the two gravitational potentials

$$\Psi \equiv \gamma(t, k) \Phi \quad \gamma = [\text{com}(\mathcal{M})]_{32}/[\text{com}(\mathcal{M})]_{31}$$

Conclusions

- Unified treatment of single-field models of dark energy/modified gravity
- Models leading to *second-order* equations of motion for *linear* perturbations can be parametrized by 7 time-dependent functions (3 needed for the background)

$$\begin{aligned} L = & \frac{M_*^2}{2} f(t) R - \Lambda(t) - c(t) g^{00} + \frac{M_2^4(t)}{2} (\delta g^{00})^2 - \frac{m_3^3(t)}{2} \delta K \delta g^{00} \\ & - m_4^2(t) (\delta K^2 - \delta K^\mu_\nu \delta K^\nu_\mu) + \frac{\tilde{m}_4^2(t)}{2} {}^{(3)}R \delta g^{00} \end{aligned}$$

- Horndeski's theories: $m_4^2(t) = \tilde{m}_4^2(t)$
- Link with observations (effective Newton constant, ratio of the two gravitational potentials)



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