Title: Weak values: their meaning and uses in quantum foundations.

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Abstract: Weak values were introduced by Aharonov, Albert, and Vaidman 25 years ago, but it is only in the last 10 years that they have begun to enter into mainstream physics. I will introduce weak values as done by AAV, but then give them a modern definition in terms of generalized measurements. I will discuss their properties and their uses in experiment. Finally I will talk about what they have to contribute to quantum foundations.

Weak Values: Their meaning and uses in quantum foundations

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Quantum discord is Bohr's notion of non-mechanical disturbance introduced to answer EPR

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HIGHLIGHTS

- Both the EPR argument, and Bohr's reply, were technically correct.
- Their opposed conclusions came from different criteria for disturbance.
- Bohr's criterion works against even the simplified (one-variable) EPR argument.
- Bohr's criterion for disturbance is intimately related to "quantum discord".
- This illuminates the historical development of notions of quantum nonlocality.

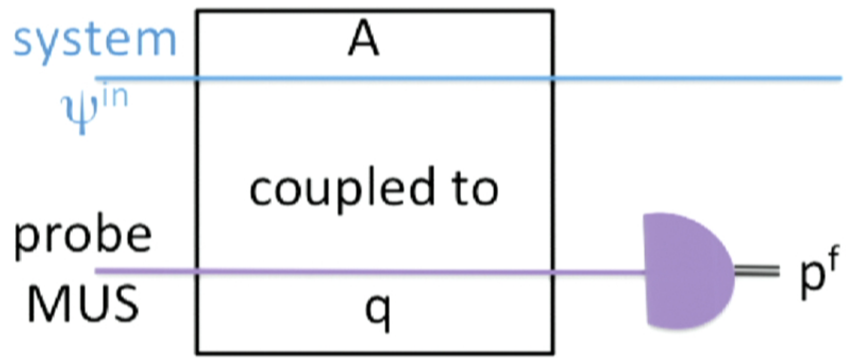
Outline

1. Defining weak values
2. Properties of Weak Values
3. Weak Values for exploring FQiQM in experiments
   - Tunneling time
   - Bell-nonlocal correlations
   - Measuring Bohmian-like trajectories
   - Disproving a naive measurement–disturbance relation
   - Other Examples
4. Conclusions

How the Result of a Measurement of a Component of the Spin of a Spin-$\frac{1}{2}$ Particle Can Turn Out to be 100

Yakir Aharonov, David Z. Albert, and Lev Vaidman

- PRL **60**, 1351 (1988).
- Consider an arbitrary system observable A.
- Assume a probe with $[\hat{q}, \hat{p}] = i$, initially in a MUS (minimum uncertainty state).



- The probe state is defined by $\sigma_p^{\rm in}$, $\bar{p}^{\rm in}$, and $\bar{q}^{\rm in} = 0$.
- Assume (von Neumann) $\hat{H} = \delta(t)\hat{A} \otimes \hat{q}$, so that $\hat{p}^{\rm f} - \hat{p}^{\rm in} = \hat{A}$.
- By measuring $p^{\rm f}$ we can **estimate** A as $A(p^{\rm f}) = p^{\rm f} - \bar{p}^{\rm in}$.

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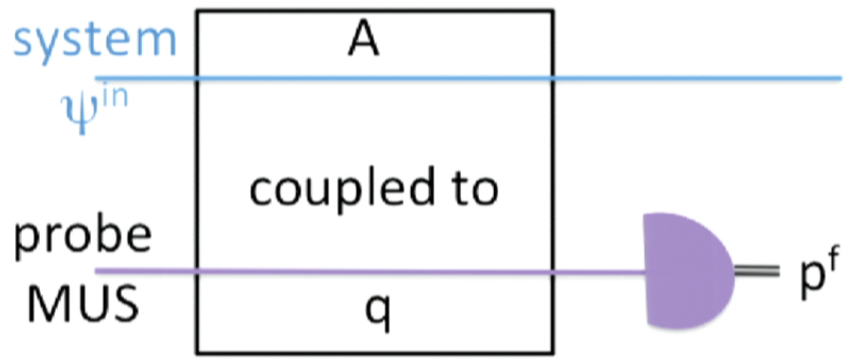

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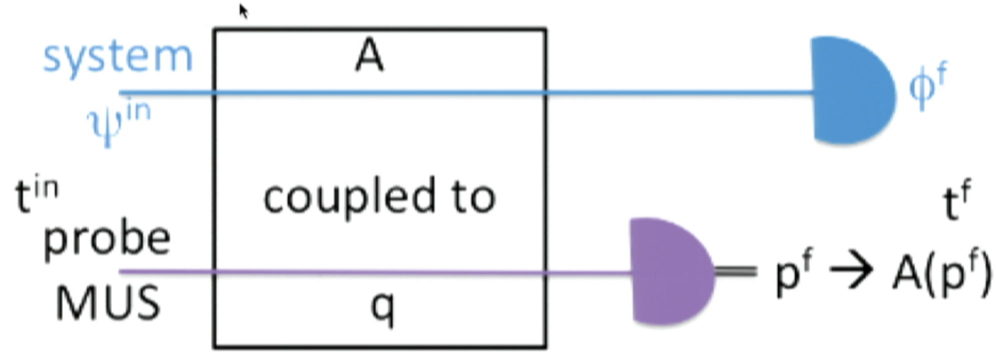
- By measuring $p^{\rm f}$ we can **estimate** A as $A(p^{\rm f}) = p^{\rm f} - \bar{p}^{\rm in}$.

Initial and Final States.

- For initial system state $|\psi^{\rm in}\rangle$, we can obtain, by repeating the experiment,
$$\mathrm{E}[A(p^{\rm f})|\psi^{\rm in}] = \langle\psi^{\rm in}|\hat{A}|\psi^{\rm in}\rangle.$$
- Now consider a final *strong* measurement on the system too.



- Consider the sub-ensemble where the final result corresponds to projecting onto state $|\phi^{\rm f}\rangle$.
- Then we can consider the *post-selected* average $\mathrm{E}[A(p^{\rm f})|\psi^{\rm in}, \phi^{\rm f}]$.

The Weak Measurement Limit

- In the **weak measurement limit**, $\sigma_p \to \infty$,

$$\mathrm{E}[A(p^{\mathrm{f}})|\psi^{\mathrm{in}}, \phi^{\mathrm{f}}] \to \mathrm{Re}\frac{\langle \phi^{\mathrm{f}}|\hat{A}|\psi^{\mathrm{in}}\rangle}{\langle \phi^{\mathrm{f}}|\psi^{\mathrm{in}}\rangle}.$$

Q Why is this the weak measurement limit?

A Because very little information in any individual result

$$A(\hat{p}^{\mathrm{f}}) = \hat{A} + (\hat{p}^{\mathrm{in}} - \bar{p}^{\mathrm{in}})$$

and $\langle (\hat{p}^{\mathrm{in}} - \bar{p}^{\mathrm{in}})^2 \rangle = \sigma_p^2 \to \infty$.

A Because weak (*not no*) disturbance:

$$\hat{s}^{\mathrm{f}} = \hat{s}^{\mathrm{in}} - i[\hat{s}^{\mathrm{in}}, \hat{A}] \otimes q^{\mathrm{in}}$$

and $\langle (q^{\mathrm{in}})^2 \rangle = 1/(2\sigma_p)^2 \to 0$ in this limit.

- Note: the weaker the measurement, the larger the number of repetitions required to obtain a reliable average.

AAV's weak values

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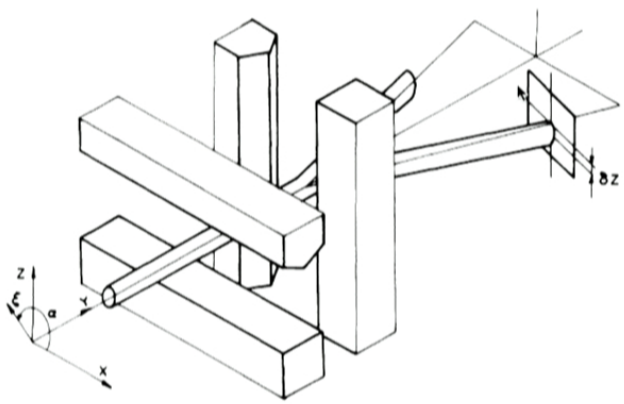

FIG. 1. The experimental device for measurement of the weak value of σ_z. The beam of particles with the spin pointed in the direction $\hat{\xi}$ passes through an inhomogeneous (in the z direction) *weak* magnetic field and is split by the strong magnet with an inhomogeneous field in the x direction. The beam of particles with $\sigma_x = 1$ comes toward the screen and the deflection of the spot on the screen in the z direction is proportional to the weak value of σ_z: $\sigma_{z_w} = (\delta z p_0 \mu / l)(\partial B_z / \partial z)^{-1}$.

- AAV call the *complex* fraction

$$\frac{\langle \phi^{\mathrm{f}} | \hat{A} | \psi^{\mathrm{in}} \rangle}{\langle \phi^{\mathrm{f}} | \psi^{\mathrm{in}} \rangle}$$

  the **weak value** of *A*.
- I (and many others) use this term for the real part:

$$_{\phi^{\mathrm{f}}}\langle A^{\mathrm{w}} \rangle_{\psi^{\mathrm{in}}} = \mathrm{Re} \frac{\langle \phi^{\mathrm{f}} | \hat{A} | \psi^{\mathrm{in}} \rangle}{\langle \phi^{\mathrm{f}} | \psi^{\mathrm{in}} \rangle}.$$

- The most interesting property it has is that it we **cannot** say

$$\lambda_{\min}(\hat{A}) \leq {}_{\phi^{\mathrm{f}}}\langle A^{\mathrm{w}} \rangle_{\psi^{\mathrm{in}}} \leq \lambda_{\max}(\hat{A}).$$

A Modern Treatment

- A *purity preserving* (PP) measurement is described by a set of *measurement operators* $\{\hat{M}_k\}$ with result-label k :

$$\sum_k \hat{E}_k = \hat{1},$$

where $\hat{E}_k = M_k^\dagger \hat{M}_k$. These specify the update rule for result k:

$$\rho \rightarrow \tilde{\rho}_k = \hat{M}_k \rho \hat{M}_k^\dagger,$$

where $\mathrm{Tr}[\tilde{\rho}_k] = \mathrm{Tr}[\rho \hat{E}_k]$ is the probability of outcome k occurring.
- A *weak family* of PP measurements $\{\hat{M}_k(\epsilon)\}$ is one with analytic dependence on ϵ such that $\forall k, \hat{M}_k(\epsilon) = \sqrt{w_k} + O(\epsilon)$.
- A *minimally disturbing* PP measurement is : $\forall k, \hat{M}_k^\dagger = \hat{M}_k$.
- A *measurement of* $\hat{A}$ is one : $\forall k, \hat{E}_k = E_k(\hat{A})$.

Dressel and Jordan's Formulation

- An $\hat{A}$-*estimating* measurement is : $\exists$ generalized eigenvalues α_k : $\sum_k \alpha_k \hat{E}_k = \hat{A}$, so that $\sum_k \alpha_k \Pr[k|\rho] = \mathrm{Tr}[\hat{A}\rho]$.
- Including post-selecting on the result with effect $\hat{E}^{\mathrm{f}}$, the mean is

$$\sum_k \alpha_k(\epsilon)\Pr[k|\rho^{\mathrm{in}}, \hat{E}^{\mathrm{f}}] = \sum_k \alpha_k(\epsilon) \frac{\mathrm{Tr}[\hat{E}^{\mathrm{f}} \hat{M}_k(\epsilon)\rho^{\mathrm{in}} \hat{M}_k^\dagger(\epsilon)]}{\sum_j \mathrm{Tr}[\hat{E}^{\mathrm{f}} \hat{M}_j(\epsilon)\rho^{\mathrm{in}} \hat{M}_j^\dagger(\epsilon)]}$$

which is complicated to work out in general. However,

Theorem (Dressel & Jordan, 2012)

For a weak family of $\hat{A}$-estimating PP minimally disturbing measurements of $\hat{A}$, it is **typically** *the case that post-selecting on the result with effect $\hat{E}^{\mathrm{f}}$ yields a mean value for the estimate of A of*

$$\lim_{\epsilon \to 0} \sum_k \alpha_k(\epsilon)\Pr[k|\rho^{\mathrm{in}}, \hat{E}^{\mathrm{f}}] = \mathrm{Re}\frac{\mathrm{Tr}[\hat{E}^{\mathrm{f}} \hat{A}\,\rho^{\mathrm{in}}]}{\mathrm{Tr}[\hat{E}^{\mathrm{f}} \rho^{\mathrm{in}}]} \equiv {}_{E^{\mathrm{f}}}\langle A^{\mathrm{w}} \rangle_{\rho^{\mathrm{in}}} .$$

Expected and Unexpected(?) Property of WVs.

- Expected property: **linearity** —

$$\hat{C} = \hat{A} + \hat{B} \implies {}_{E^{\mathrm{f}}}\langle C^{\mathrm{w}} \rangle_{\rho^{\mathrm{in}}} = {}_{E^{\mathrm{f}}}\langle A^{\mathrm{w}} \rangle_{\rho^{\mathrm{in}}} + {}_{E^{\mathrm{f}}}\langle B^{\mathrm{w}} \rangle_{\rho^{\mathrm{in}}} .$$

- Expected property: **consistency** with strong measurements — if, with pre- (ρ^{in}) and post- ($\hat{E}^{\mathrm{f}}$) selection a strong measurement of A *always* would yield the answer a, then ${}_{E^{\mathrm{f}}}\langle A^{\mathrm{w}} \rangle_{\rho^{\mathrm{in}}} = a$.

- Unexpected(?) property: **anomalous weak values** — it is **not** a theorem that

$$\lambda_{\min}(\hat{A}) \leq {}_{E^{\mathrm{f}}}\langle A^{\mathrm{w}} \rangle_{\rho^{\mathrm{in}}} \leq \lambda_{\max}(\hat{A}).$$

Q How is this possible?

A Because $\exists k$: for sufficiently small ϵ,

$$\exists k : \alpha_k(\epsilon) \notin [\lambda_{\min}(\hat{A}), \lambda_{\max}(\hat{A})].$$

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How interesting is this?

Q Isn't this just the sort of crazy weirdness we expect from QM with post-selection, unrelated to weak measurements?

A **No!** Because for *projective* measurements $\hat{M}_k = \hat{\Pi}_k$, $\alpha_k = \lambda_k(\hat{A})$,

$$\sum_k \lambda_k(\hat{A}) \frac{\mathrm{Tr}[\hat{E}^{\mathrm{f}} \hat{\Pi}_k \rho^{\mathrm{in}} \hat{\Pi}_k]}{\sum_j \mathrm{Tr}[\hat{E}^{\mathrm{f}} \hat{\Pi}_j \rho^{\mathrm{in}} \hat{\Pi}_j]} \in [\lambda_{\min}(\hat{A}), \lambda_{\max}(\hat{A})].$$

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Weak Probabilities

- Say $\hat{A} = \hat{\Pi}_b$, an eigenprojector for $\hat{B}$ (i.e. $\hat{B}\hat{\Pi}_b = b\hat{\Pi}_b = b\hat{\Pi}_b^2$).
- For a (weak or strong) measurement without post-selection, the mean value is a probability

$$\langle \Pi_b^{\mathrm{w}} \rangle_{\rho^{\mathrm{in}}} = \langle \Pi_b \rangle_{\rho^{\mathrm{in}}} = \wp(b|\rho^{\mathrm{in}}).$$

- With post-selection, I will call it a **weak probability**:

$$\wp_w(b|\rho^{\mathrm{in}}, E^{\mathrm{f}}) \equiv {}_{E^{\mathrm{f}}}\langle \Pi_b^{\mathrm{w}} \rangle_{\rho^{\mathrm{in}}}.$$

- Say $\hat{E}^{\mathrm{f}} = \hat{\Pi}_f$, an eigenprojector for $\hat{F}$. Then

$$\wp_w(f; b|\rho^{\mathrm{in}}) \equiv \wp_w(b|\rho^{\mathrm{in}}, \Pi_f^{\mathrm{f}}) \times \wp(f|\rho^{\mathrm{in}})$$

I will also call a weak probability. It is not confined to $[0, 1]$.

WVs with unknown evolution and preparation.

- In order to perform a weak measurement one must know the observable $\hat{A}$ to be estimated.
- But the initial state $\rho^{\rm in}$ may be unknown, as may be the evolution $\hat{V} \equiv \hat{U}(t^{\rm w}, t^{\rm in})$ up to the time of the weak measurement.
- In order to perform the post-selection one must know the effect $\hat{E}$.
- But it can be unknown in the sense that there could be unknown evolution $\hat{U} \equiv \hat{U}(t^{\rm f}, t^{\rm w})$ between the weak measurement and post-selection, so that the effect is really $\hat{E}(t^{\rm f}) = \hat{U}^\dagger \hat{E}(t^{\rm w})\hat{U}$.
- Thus the weak value can probe the unknowns:

$$ {}_{E^{\rm f}}\langle A^{\rm w}\rangle_{\rho^{\rm in}} = {\rm Re}\frac{{\rm Tr}[\hat{U}^\dagger E^{\rm f}\hat{U}\hat{A}\hat{V}\rho^{\rm in}\hat{V}^\dagger]}{{\rm Tr}[\hat{U}^\dagger E^{\rm f}\hat{U}\hat{V}\rho^{\rm in}\hat{V}^\dagger]} = {\rm Re}\frac{\left\langle \hat{E}^{\rm f}(t^{\rm f})\hat{A}(t^{\rm w})\right\rangle_{\rho^{\rm in}}}{\left\langle \hat{E}^{\rm f}(t^{\rm f})\right\rangle_{\rho^{\rm in}}}. $$

- I will return to this.

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2. Properties of Weak Values

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   - Tunneling time
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   - Disproving a naive measurement–disturbance relation
   - Other Examples

4. Conclusions

What (in my humble opinion) do Weak Values offer for "Fundamental Questions in Quantum Mechanics"?

- Many FQiQM have no answers in standard QM, but in many cases, Weak Values **do** offer answers, which
  - may be a new answer to the question,
  - or single out one answer out of a (possibly infinite) set of answers that had been proposed,
  - either of which may give new insights and prompt new research,
  - and if not, at least they often enable an experiment to be done,
  - which brings the issues to the attention of a broader audience (in a way that theory papers seldom do).
- Some FQiQM do have an answer in standard QM, but that answer may seem contrived and far removed from experiment. Weak Values may give a natural operational meaning to these, and in particular enable an experiment to be done (see above).

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How long does a particle spend "under the barrier"?

- A typical "nonsensical" question in QM.
- Nevertheless there were various answers given, including:
  - the dwell time,
    $$\tau_d = \int_{-\infty}^{\infty} dt \langle \psi(t) | \hat{\Pi}_B | \psi(t) \rangle,$$
    where $\hat{\Pi}_B$ is the projector onto the barrier region;
  - the Buttiker time τ_B, related to how much spin-rotation a transmitted particle suffers under a Hamiltonian $\propto \hat{\Pi}_B \hat{\sigma}_z$.
- Steinberg (PRL, 1995) suggested considering the *weak value* of $\hat{T} = \int_{-\infty}^{\infty} dt \hat{\Pi}_B$, post-selected on transmission, and found

$$\mathrm{Re} \frac{\langle \text{transmitted} | \hat{T} | \text{incident} \rangle}{\langle \text{transmitted} | \text{incident} \rangle} = \tau_d \; ; \quad \left| \frac{\langle \text{transmitted} | \hat{T} | \text{incident} \rangle}{\langle \text{transmitted} | \text{incident} \rangle} \right| = \tau_B \; .$$

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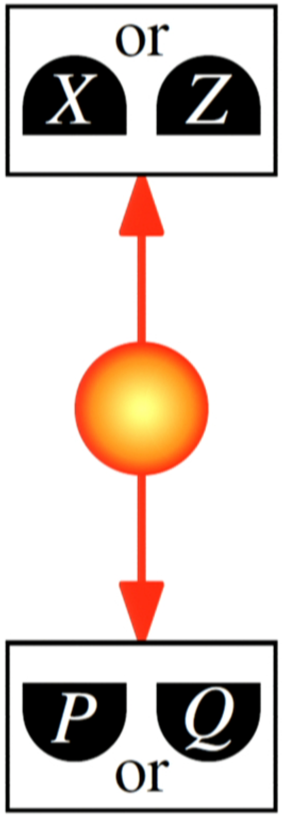
Feynman's "explanation" of Bell-nonlocal correlations



- Consider a CHSH test of Bell-nonlocality, where

$$S_{\mathrm{CHSH}} = \langle (X+Z)P + (X-Z)Q \rangle \,.$$

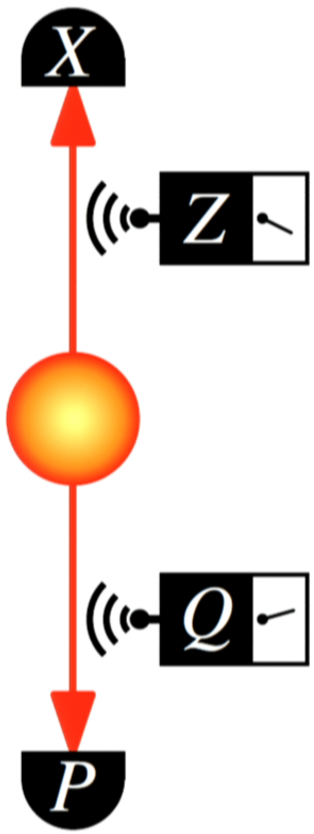
- In a LHV theory, $X = X(\lambda)$ etc, so there exists a joint distribution over these four variables, so

$$\langle S_{\mathrm{CHSH}} \rangle = \sum_{x,z,p,q} [(x+z)p + (x-z)q] \wp(x,z,p,q),$$

  which is ≤ 2, while QM allows $S_{\mathrm{CHSH}} = 2\sqrt{2}$.

- Feynman ("Negative Probabilities", 1991) pointed out that if $\wp(x,z,p,q)$ is not constrained to $[0,1]$ then we can have $S_{\mathrm{CHSH}} > 2$.
- But there are infinitely many possibilities even with the constraint $\sum_{x,p} \wp(x,z,p,q) = \wp(z,q|\psi)$ etc.

Weak Values Make Feynman's proposal Definite.

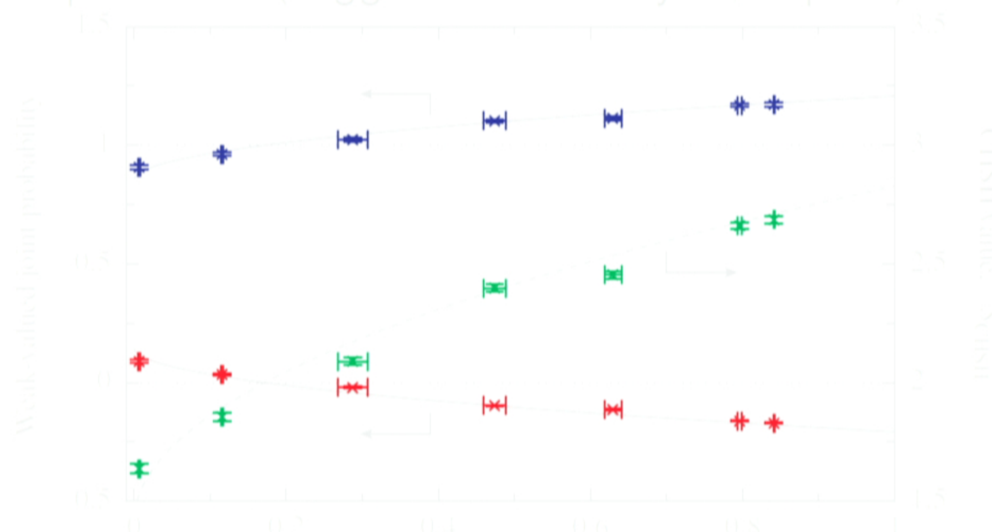


- Using a weak measurement of Π_z and Π_q, and post-selected as shown, we can measure

$$\wp_{\rm w}(z,q;x,p|\psi) \;\; [\, = \wp_{\rm w}(x,p;z,q|\psi) \;\; etc.\,]$$

and this gives $S_{\rm CHSH} \in [0, 2\sqrt{2}]$.

- Experiment! (Higgins & *al.* & Pryde, unpub.):



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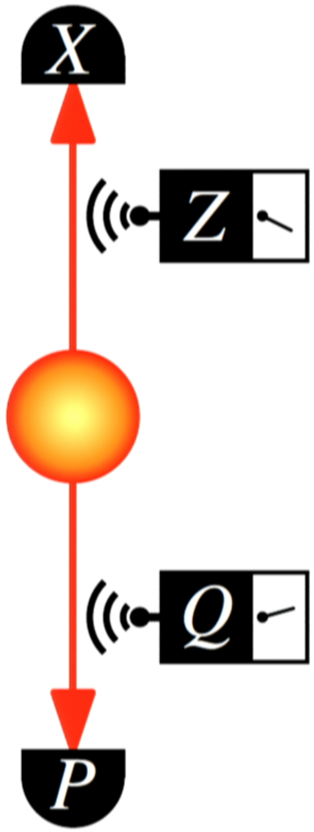

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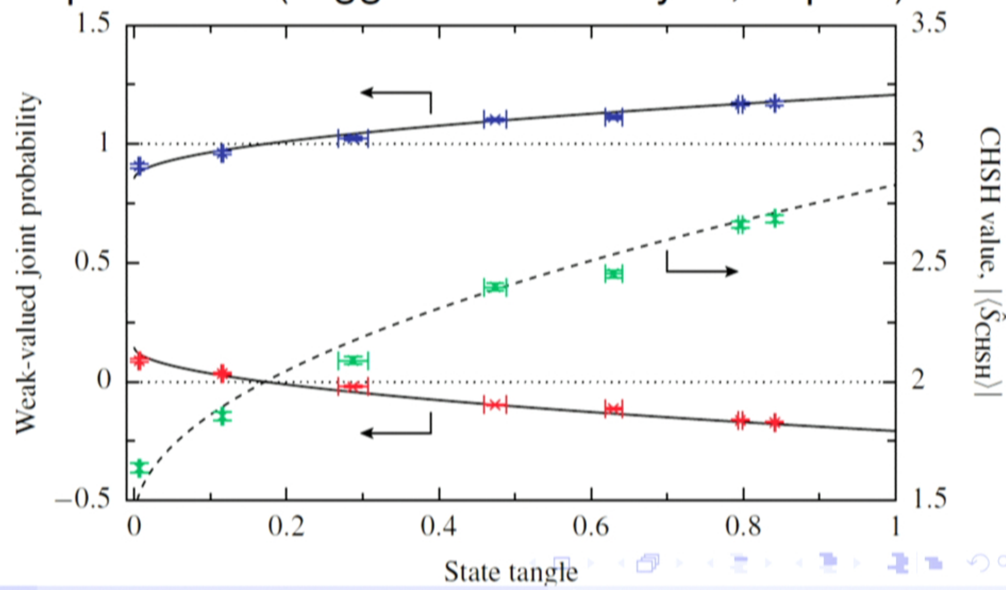

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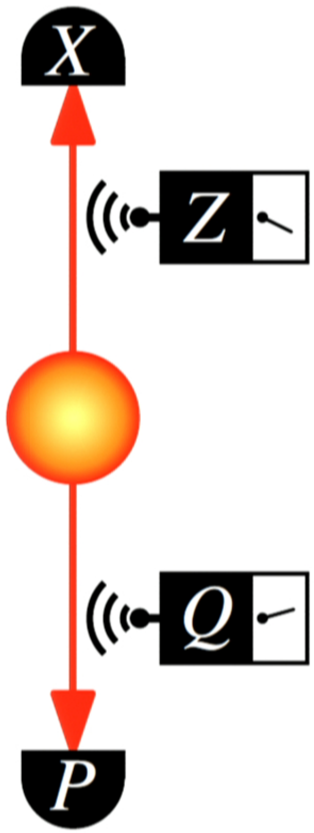

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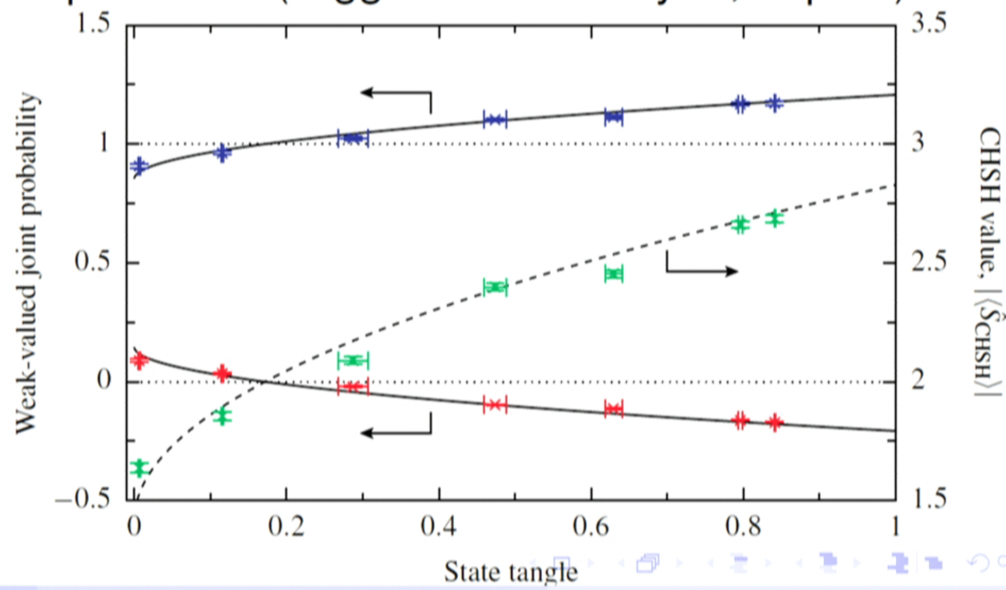

Outline

1. Defining weak values
2. Properties of Weak Values
3. Weak Values for exploring FQiQM in experiments
   - Tunneling time
   - Bell-nonlocal correlations
   - Measuring Bohmian-like trajectories
   - Disproving a naive measurement–disturbance relation
   - Other Examples
4. Conclusions

A unique Bohmian velocity

- Consider a Bohmian-(world-)particle with equation of motion

$$\dot{\mathbf{x}} = \mathbf{v}_{\psi(t)}(\mathbf{x})$$

- There are infinitely many functional expressions for $\mathbf{v}_{\bullet}(\bullet)$:

$$\partial P_{\psi(t)}(\mathbf{x})/\partial t + \nabla \cdot [P_{\psi(t)}(\mathbf{x};t)\mathbf{v}_{\psi(t)}(\mathbf{x})] = 0,$$

with $P_{\psi(t)}(\mathbf{x}) = \langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\psi(t)\rangle$.

- But if we define (HMW, NJP, 2007)

$$\begin{aligned} \mathbf{v}_{\psi(t)}(\mathbf{x}) &= \lim_{\tau\to 0} \tau^{-1}\,\mathrm{E}_{\psi(t)}[\mathbf{x}_{\rm strong}(t+\tau) - \mathbf{x}_{\rm weak}(t)|\mathbf{x}_{\rm strong}(t+\tau) = \mathbf{x}] \\ &= \lim_{\tau\to 0} \tau^{-1}\left[\mathbf{x} - {}_{\langle\mathbf{x}|\hat{U}(\tau)}\langle\hat{\mathbf{x}}^{\rm w}\rangle_{|\psi(t)\rangle}\right]. \end{aligned}$$

one gets the standard Bohmian expression for $\mathbf{v}_{\psi(t)}(\mathbf{x})$ ...

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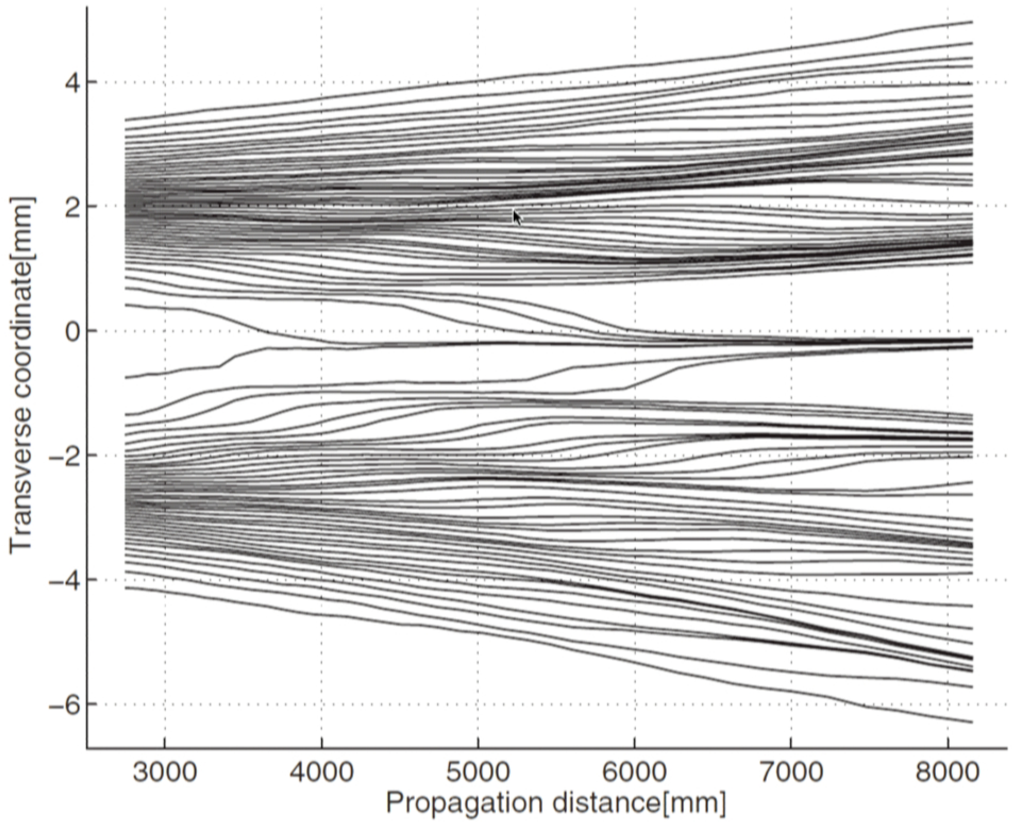
Experiment! Kocsis & *al.* & Steinberg (Science, 2011)

- ... and one can measure it (even as a "naive experimentalist")



Note that it is **not** possible to follow an individual particle.

These trajectories are created by patching together little increments inferred from the weak velocities.

A unique Bohmian ontology?

- The weak-valued velocity formula evaluates in general to

$$\mathbf{v}_{\psi(t)}(\mathbf{x}) = \mathrm{Re}\frac{\langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|i[\hat{H},\hat{\mathbf{x}}]|\psi(t)\rangle}{\hbar\langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\psi(t)\rangle}.$$

Q Is this always consistent with QM? i.e. Does

$$P_0(\mathbf{x}) = \langle\psi(0)|\mathbf{x}\rangle\langle\mathbf{x}|\psi(0)\rangle \rightarrow P_t(\mathbf{x}) = \langle\psi(t)|\mathbf{x}\rangle\langle\mathbf{x}|\psi(t)\rangle?$$

A Iff $\hat{H}$ is at most quadratic in operators canonically conjugate to $\hat{\mathbf{x}}$.

Q Isn't this a limitation of this approach?

A No! Because all physical Hamiltonians *are* so constrained *if we take* $\hat{\mathbf{x}}$ *to be the configuration operator* (as usual).
That is, this approach explains *why* HV $= \mathbf{x}$.

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Measurement–Disturbance Relations

- In 1927 Heisenberg proposed for a position measurement the following MDR (more or less)

$$\epsilon(q) \times \eta(p) \geq \hbar/2. \quad (1)$$

- However he rigorously defined neither the error $\epsilon(q)$ nor the disturbance $\eta(p)$, and (unlike $\sigma(q) \times \sigma(p) \geq \hbar/2$) never *proved* it.
- Ozawa (2003) proposed, for arbitrary observables A and B,

$$\epsilon^2(q) = \left\langle (\hat{A}^{\rm est} - \hat{A}^{\rm in})^2 \right\rangle \;;\; \eta^2(B) = \left\langle (\hat{B}^{\rm f} - \hat{B}^{\rm in})^2 \right\rangle \;.$$

Here $\hat{A}^{\rm est}$ is a final *meter* observable and so $[\hat{A}^{\rm est}, \hat{S}^{\rm f}] = 0$.

- He showed that the naive MDR (1) does not hold in general.
- However, Ozawa showed a different MDR does always hold,

$$\epsilon(A)\eta(B) + \epsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq \left| \left\langle [\hat{A}, \hat{B}] \right\rangle \right| , \quad (2)$$

where the $\langle \bullet \rangle$ and the σs apply to the initial system state.

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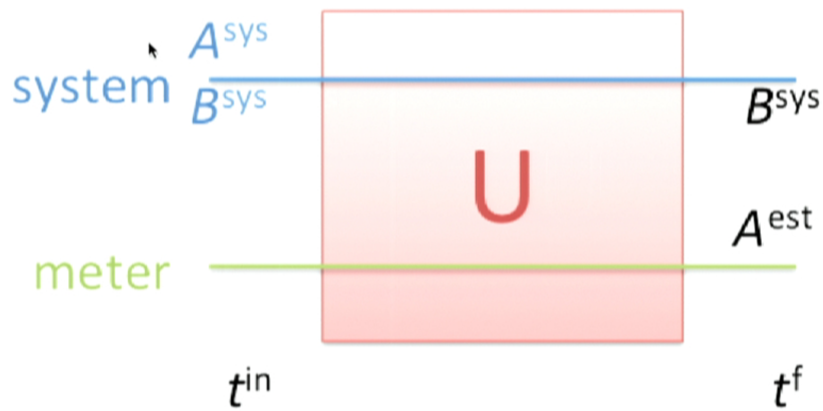
What do Ozawa's Quantities *Mean*?

- $\epsilon^2(A) = \left\langle (\hat{A}^{\rm est} - \hat{A}^{\rm in})^2 \right\rangle$ and $\eta^2(B) = \left\langle (\hat{B}^{\rm f} - \hat{B}^{\rm in})^2 \right\rangle$ both involve quantities at two different times:

$$\hat{A}^{\rm in} = \hat{A}^{\rm sys}(t^{\rm in})\ ;\ \ \hat{B}^{\rm in} = \hat{B}^{\rm sys}(t^{\rm in})\ ;\ \ \hat{A}^{\rm est} = \hat{A}^{\rm meter}(t^{\rm f})\ ;\ \ \hat{B}^{\rm f} = \hat{B}^{\rm sys}(t^{\rm f}).$$



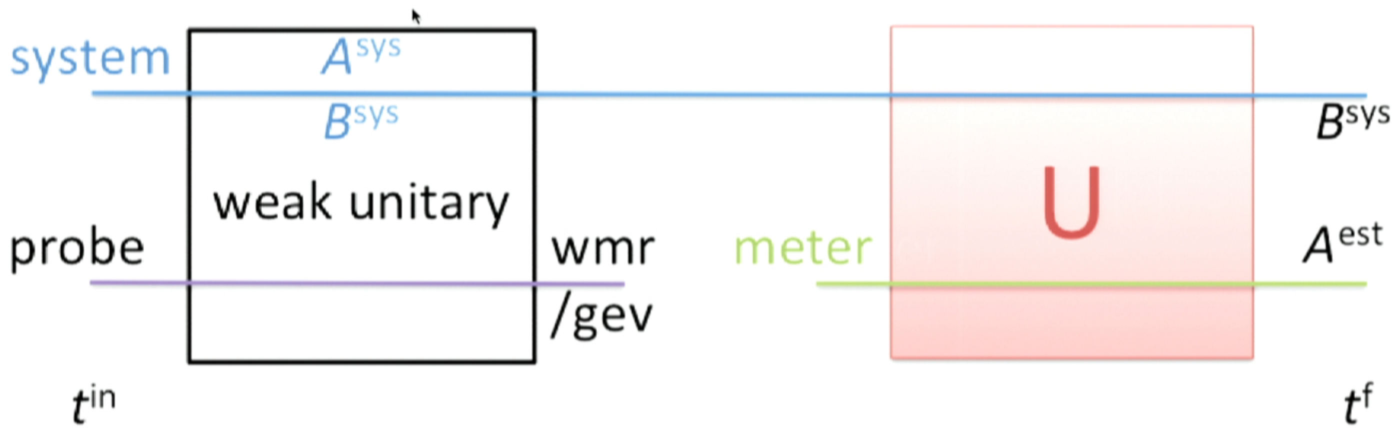
- Of course there exists an operator e.g. $\hat{B}^{\rm f} - \hat{B}^{\rm in} = \hat{U}^\dagger \hat{B}^{\rm in} \hat{U} - \hat{B}^{\rm in}$ that can be measured on the initial system+meter state.
- But doing this (and, hence determining $\eta^2(B)$) does not employ the *actual* measurement interaction, which may be unknown.

Measuring the Ozawa quantities (Lund & HMW, NJP, 2010)

- Recall that for a weak mst of Π_b followed by strong mst of Π_f

$$\sum_{b,f}(b-f)^n \wp_w(f;b|\rho^{\rm in}) = {\rm Tr}[(\hat{B}-\hat{F})^n\rho^{\rm in}], \text{ for } n = 0, 1, \text{ or } 2.$$

- We can apply that here, with **only the black elements known**



$$\epsilon^2(A) = \left\langle(\hat{A}^{\rm est}-\hat{A}^{\rm in})^2\right\rangle = \int da^{\rm in}_{\rm sys}\, da^{\rm f}_{\rm mtr}(a^{\rm in}_{\rm sys}-a^{\rm f}_{\rm mtr})^2\wp_w(a^{\rm in}_{\rm sys}; a^{\rm f}_{\rm mtr}|\rho^{\rm in}_{\rm sys}\otimes\rho^{\rm in}_{\rm mtr}).$$

$$\eta^2(B) = \left\langle(\hat{B}^{\rm f}-\hat{B}^{\rm in})^2\right\rangle = \int db^{\rm in}_{\rm sys}\, db^{\rm f}_{\rm sys}(b^{\rm in}_{\rm sys}-b^{\rm f}_{\rm sys})^2\wp_w(b^{\rm in}_{\rm sys}; b^{\rm f}_{\rm sys}|\rho^{\rm in}_{\rm sys}\otimes\rho^{\rm in}_{\rm mtr}).$$

Experiment!

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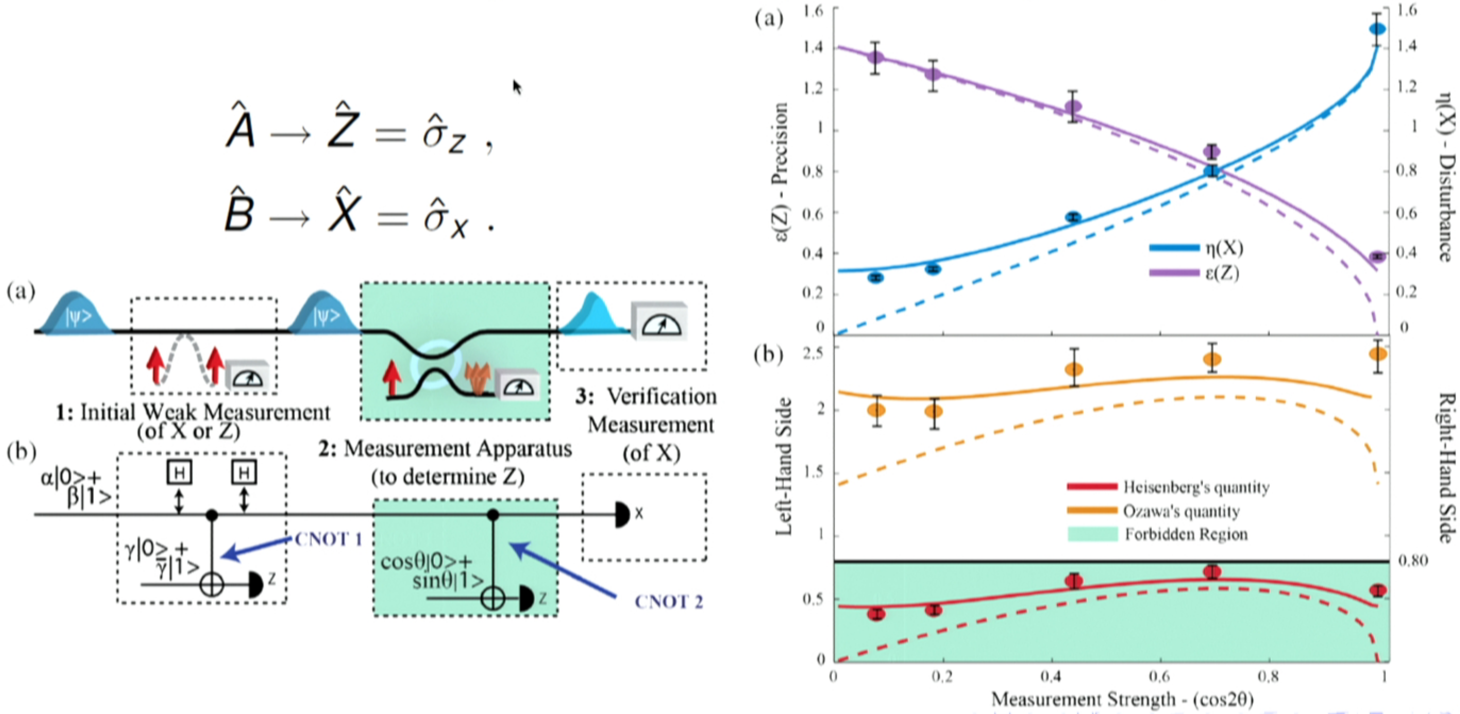
Violation of Heisenberg's Measurement-Disturbance Relationship by Weak Measurements

Lee A. Rozema, Ardavan Darabi, Dylan H. Mahler, Alex Hayat, Yasaman Soudagar, and Aephraim M. Steinberg

$$\hat{A} \rightarrow \hat{Z} = \hat{\sigma}_Z \ ,$$

$$\hat{B} \rightarrow \hat{X} = \hat{\sigma}_X \ .$$



Some Other Examples

- "Three-box paradox" (Vaidman, 1996) with experiment (Resch, Lundeen & Steinberg, 2004).
- Cherenkov radiation *in vacuo* by (weakly) superluminal particles (Rohrlich & Aharonov, 2002).
- Understanding previously observed puzzling phenomena:
  - Cavity Quantum Electrodynamics (HMW, 2002),
  - photonic fibre communication (Brunner & *al.*, 2003).
- Defining a momentum transfer probability distribution $\wp_{\rm w}(\Delta p)$ in *welcher Weg* measurements (HMW, 2003), with experiment (Mir, Steinberg, HMW & co-workers, 2007)
- Relation to the Legget-Garg inequality (Williams & Jordan, 2008)
- Testing universal *complementarity* relations (Weston, Hall, Palssen, HMW & Pryde, 2013)
- Detecting Bohmian nonlocality (Braverman & Simon, 2013)

Summary

- Weak values *per se* are not mysterious — they can be derived simply (and naturally) within standard quantum theory with non-projective measurements and post-selection.
- Weak values can be *anomalous* [e.g. ${}_{\phi^{\mathrm{f}}}\langle A^{\mathrm{w}}\rangle_{\psi^{\mathrm{in}}} > \lambda_{\max}(\hat{A})$], but nevertheless they follow certain logical principles.
- In particular, "weak probabilities" can replicate the Margenau-Hill distribution which gives the correct QM moments for quadratic functions of the weak and post-selecting observables.
- This allows unknown interactions on unknown initial states to be probed through two-time statistics.
- Weak values shed new light on fundamental questions in QM.
- In particular they allow one to *empirically* obtain a unique Bohmian velocity law, and thereby also single out the configuration as the unique Bohmian reality.

Some Other Examples

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How long does a particle spend "under the barrier"?

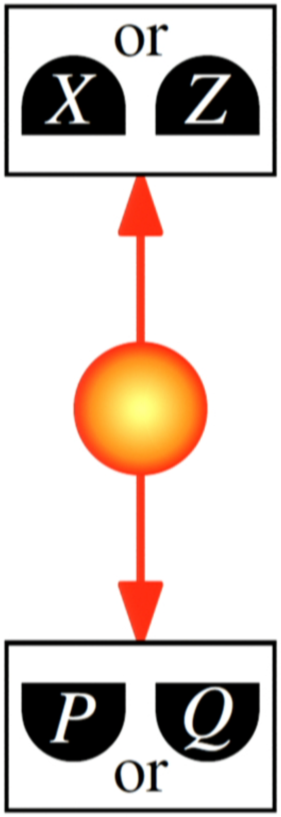
- A typical "nonsensical" question in QM.
- Nevertheless there were various answers given, including:
  - the dwell time,

$$\tau_d = \int_{-\infty}^{\infty} dt \langle \psi(t)|\hat{\Pi}_B|\psi(t)\rangle,$$

    where $\hat{\Pi}_B$ is the projector onto the barrier region;
  - the Buttiker time τ_B, related to how much spin-rotation a transmitted particle suffers under a Hamiltonian $\propto \hat{\Pi}_B \hat{\sigma}_z$.
- Steinberg (PRL, 1995) suggested considering the *weak value* of $\hat{T} = \int_{-\infty}^{\infty} dt \hat{\Pi}_B$, post-selected on transmission, and found

$$\mathrm{Re}\frac{\langle \text{transmitted}|\hat{T}|\text{incident}\rangle}{\langle \text{transmitted}|\text{incident}\rangle} = \tau_d \; ; \quad \left|\frac{\langle \text{transmitted}|\hat{T}|\text{incident}\rangle}{\langle \text{transmitted}|\text{incident}\rangle}\right| = \tau_B \; .$$

Feynman's "explanation" of Bell-nonlocal correlations



- Consider a CHSH test of Bell-nonlocality, where

$$S_{\text{CHSH}} = \langle (X+Z)P + (X-Z)Q \rangle \,.$$

- In a LHV theory, $X = X(\lambda)$ etc, so there exists a joint distribution over these four variables, so

$$\langle S_{\text{CHSH}} \rangle = \sum_{x,z,p,q} [(x+z)p + (x-z)q]\wp(x,z,p,q),$$

  which is ≤ 2, while QM allows $S_{\text{CHSH}} = 2\sqrt{2}$.
- Feynman ("Negative Probabilities", 1991) pointed out that if $\wp(x,z,p,q)$ is not constrained to $[0,1]$ then we can have $S_{\text{CHSH}} > 2$.
- But there are infinitely many possibilities even with the constraint $\sum_{x,p} \wp(x,z,p,q) = \wp(z,q|\psi)$ etc.